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GRAVITATIONAL WAVES

ICTP-SAIFR SCHOOL ON MODERN AMPLITUDE METHODS FOR GAUGE AND GRAVITY THEORIES, SÃO PAULO (SP), 24-28 JULY 2023

Contents

Notations

We use units $c = 1$, which means that 1 light-year (ly)=1 year $\simeq$ 3.16×10^7 sec $=9.46 \times 10^{15}$ m.

Another useful unit is the parsec, defined as the distance of an astropysical object having a parallax of $1''$ due to earth motion around the sun, givin $1pc = 1.5 \times 10^8 km/1'' \simeq 3.09 \times 10^{16}$ m $\simeq 3.26$ ly.

The mass of the sun is $M_\odot \simeq 1.99 \times 10^{33}$ g and its Schwarzschild radius $2G_N M_\odot \simeq 2.95$ km $\simeq 9.84 \times 10^{-6}$ sec.

We adopt the "mostly plus" ignature, i.e. the Minkowski metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)\,,$$

the Christoffel symbols is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha} \right)\,,$$

the Riemann tensor is

$$R^\mu{}_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}\,,$$

with symmetry properties

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\sigma\rho} = -R_{\mu\nu\sigma\rho}\,, \\ R^\mu_{\nu\rho\sigma} &+ R^\mu_{\rho\sigma\nu} + R^\mu_{\sigma\rho\nu} = 0\,, \end{aligned}$$

which gives $(4 \times 3/2)^2 - 4 \times 4 = 20$ independent components, which in $d+1$ dimensions turn out to be $\left(\frac{(d+1)d}{2}\right)^2 - \frac{(d+1)^2 d(d-1)}{6} = \frac{(d+1)^2 d(d+2)}{12}$.

The Ricci tensor is

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}\,,$$

the Ricci scalar

$$R = R_{\mu\nu} g^{\mu\nu}\,,$$

the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,,$$

the Weyl tensor

$$C^{\alpha\beta}_{\mu\nu} = R^{\alpha\beta}_{\mu\nu} - 2\delta^{[\alpha}_{[\mu} R^{\beta]}_{\nu]} + \frac{1}{3}\delta^{[\alpha}_{[\mu}\delta^{\beta]}_{\nu]} R\,,$$

which has 10 components in $3+1$ dimensions or $\frac{(d+1)^2 d(d+2)}{12} - \frac{(d+1)(d+2)}{2} = (d+2)(d+1)\frac{d^2+d-6}{12}$ in $d+1$ dimensions.

Greek indices $\alpha, \ldots \omega$ run over $d+1$ dimensions, Latin indices $a, \ldots i, j, \ldots$ over d spatial dimensions.

Fourier transform in d dimensions are defined as

$$\begin{aligned} \tilde{F}(\mathbf{k}) &= \int d^d x F(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \\ F(\mathbf{x}) &= \int \frac{d^d k}{(2\pi)^d} \tilde{F}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \int_{\mathbf{k}} \tilde{F}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\,. \end{aligned}$$

We will denote the modulus of a generic 3-vector $\mathbf{w}$ by $w \equiv |\mathbf{w}|$.

Fourier transform over time is defined as

$$\begin{aligned} \tilde{F}(\omega) &= \int dt F(t) e^{i\omega t}\,, \\ F(t) &= \int \frac{d\omega}{2\pi} \tilde{F}(\omega) e^{-i\omega t} = \int_{\omega} \tilde{F}(\omega) e^{-i\omega t}\,. \end{aligned}$$

1
Introduction

The existence of gravitational waves (GW) is an unavoidable prediction of General Relativity (GR): any change to a gravitating source must be communicated to distant observers no faster than the speed of light, c, leading to the existence of *gravitational radiation*, or GWs.

Before direct gravitational wave detection, the more precise evidence of a system emitting GWs comes from the celebrated "Hulse-Taylor" pulsar [1], where two neutron stars are tightly bound in a binary system with the observed decay rate of their orbit being in agreement with the GR prediction to about one part in a thousand [2] see Fig. 1.1 [3]. See also [4] for more examples of observed GW emission from pulsar and white dwarf binary systems and sec. 6.2 of [5] for a pedagogical discussion.

Today we observe ~ 100 galactic binary systems in which at least one of the components is a pulsar, which is about 6% of the total number of pulsars observed, that is $O(1500)$ [6]. In 18 cases one has *double* neutron star systems [7], with 12 of them having measurements of individual masses [8] and for the remarkable case of PSR J0737-3039 one has a *double pulsar* system.

On the other hand, earth-based, kilometer-sized gravitational wave observatories are currently taking data or under development. The two Laser Interferomenter Gravitational-Wave Observatories LIGO in the US, have been joined the Virgo interferometer in Italy (Virgo) at the end of second observation run, Another smaller detector, with reduced sensitivity, belonging to the network is the German-British Gravitational Wave Detector GEO600. The gravitational detector network has been joined recently by the Japanese (KAGRA) detector and by the end of this decade an additional interferometer in India (INDIGO) is expected to join the network.

At the time of delivering these lectures, the fourth observation run (O4) is ongoing, having started on May 24th, 2023 and it is planned to last for 20 months overall. O1 lasted from September 2015 to January 2016, O2 from December 2016 to Aug 2017 (Virgo joined

[1] R A Hulse and J H Taylor. Discovery of a pulsar in a binary system. *Astrophys. J.*, 195:L51, 1975

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[3] Joel M. Weisberg and Joseph H. Taylor. Relativistic binary pulsar B1913+16: Thirty years of observations and analysis. *ASP Conf. Ser.*, 328:25, 2005

[4] M. Burgay, N. D'Amico, A. Possenti, R. N. Manchester, A. G. Lyne, B. C. Joshi, M. A. McLaughlin, and M. Kramer *et al.* An increased estimate of the merger rate of double neutron stars from observations of a highly relativistic system. *Nature*, 426:531, 2003; M. Kramer and N. Wex. The double pulsar system: A unique laboratory for gravity. *Class. Quant. Grav.*, 26:073001, 2009; and J. J. Hermes, Mukremin Kilic, Warren R. Brown, D. E. Winget, Carlos Allende Prieto, A. Gianninas, Anjum S. Mukadam, Antonio Cabrera-Lavers, and Scott J. Kenyon. Rapid Orbital Decay in the 12.75-minute WD+WD Binary J0651+2844. *Astrophys. J. Lett.*, 757:L21, 2012. DOI: 10.1088/2041-8205/757/2/L21

[5] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

[6] R. N. Manchester, G. B. Hobbs, A. Teoh, and M. Hobbs. VizieR Online Data Catalog: ATNF Pulsar Catalog (Manchester+, 2005). *VizieR Online Data Catalog*, art. VII/245, August 2005

[7] Nicholas Farrow, Xing-Jiang Zhu, and Eric Thrane. The mass distribution of galactic double neutron stars. *The Astrophysical Journal*, 876(1):18, apr 2019. DOI: 10.3847/1538-4357/ab12e3. URL https://dx.doi.org/10.3847/1538-4357/ab12e3; and Gabriella Agazie et al. The Green Bank Northern

in August 2017), O3 from April 2019 to March 2020, and an O5 is foreseen to start around the year 2027 and collect data for a couple of years. A visual summary of confirmed detections so far is reported in fig. 1.2, and in fig. 1.3 the spectrum of 6 sample detections is displayed.

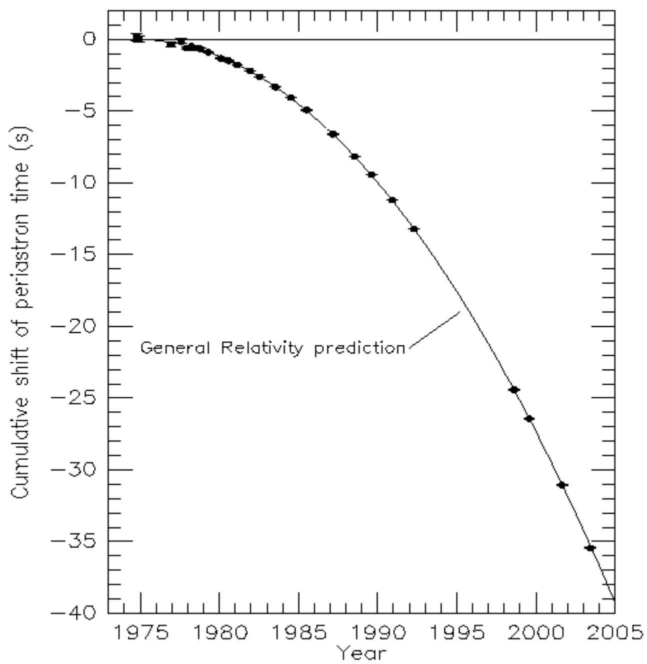

Figure 1.1: Deviation of the periastron time from a linear depence with time, showing the orbital decay of the binary pulsar system PSR B1913+16.

The output of GW observatories is particularly sensitive to the phase Φ of GW signals, and focusing on coalescing binary systems, it is possible to predict it via

$$\Phi(t) = 2 \int_{t_i}^{t} \omega(t')dt' \, , \tag{1.1}$$

where ω is the *orbital* angular velocity of the binary system and t_i stands for some initial time, like the time when the signal with increasing frequency enters the detector band-width. Note the factor 2 between the *GW* phase Φ of the *dominant* mode and the orbital angular velocity ω.

Binary orbits are in general eccentric, but at the frequency we are interested in ($> 10Hz$) binary system will have circuralized, see sec. 4.1.3 of [9] for a quantitative analysis of orbit circularization.

[9] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

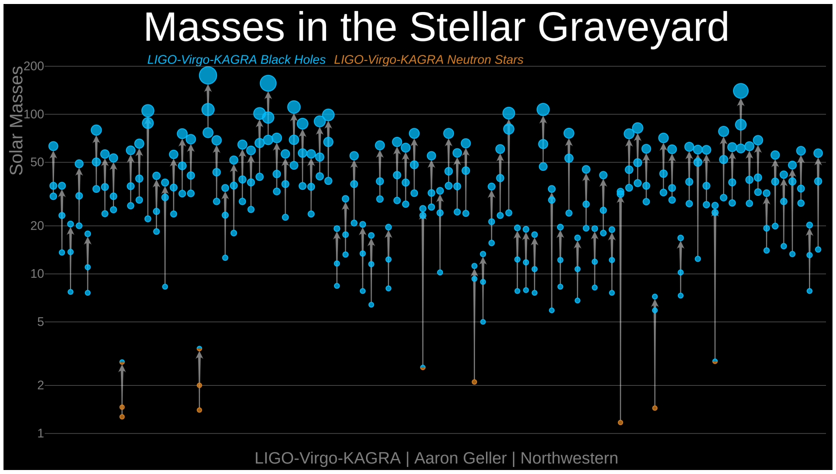

Figure 1.2: Summary of GW detections in chronological order, until O3 included.

For circular orbit the binding energy can be expressed in terms of a single parameter, say the relative velocity of the binary system components v, which by the virial theorem $v^2 \simeq G_N M/r$, being G_N the standard Newton's constant, M the total mass of the binary system, and r the orbital separation between its constituents. Note that the virial relationship, or its equivalent (on circular orbits) Kepler law $\omega^2 \simeq G_N M/r^3$, are not exact in GR, but holds only at Newtonian level.

For spin-less binary constituents the energy E of circular orbits can be expressed in terms of a series in $v \equiv (\pi G_N M f_{GW})^{1/3}$:

$$E(v) = -\frac{1}{2}\eta M v^2 \left(1 + e_{v^2}(\eta)v^2 + e_{v^4}(\eta)v^4 + \dots\right) , \tag{1.2}$$

where $\eta \equiv m_1 m_2/M^2$ is the symmetric mass ratio and the $e_{v^n}(\eta)$ coefficients stand for GR corrections to the Newtonian formula, and only even power of v are involved for the conservative energy. For the radiated flux $F(v)$ the leading term is the Einstein quadrupole formula, which we will derive in sec. 2.4, that in the circular orbit case reduces to

$$F(v) = \frac{dE}{dt} = \frac{32\eta^2}{5G_N} v^{10} \left(1 + f_{v^2}(\eta)v^2 + f_{v^3}(\eta)v^3 + \dots\right) , \tag{1.3}$$

where also post-Newtonian corrections are indicated. Note that using our definition of v we have $v^3 = \omega G_N M$ allowing to re-write (note that during the coalescence v increases monotonically) eq. (1.1) as

$$\Phi(v) \simeq \frac{2}{G_N M} \int_{v_i}^{v} v^3 \frac{dE/dv}{dE/dt} dv = \frac{5}{16\eta} \int_{v_i}^{v} \frac{1}{v^6} \left(1 + p_{v^2}v^2 + p_{v^3}v^3 + \dots\right) dv \, , \tag{1.4}$$

where we used $dE/dt = -F$. Since the phase has to be matched with $O(1)$ precision, corrections at least $O(v^6)$ must be considered.

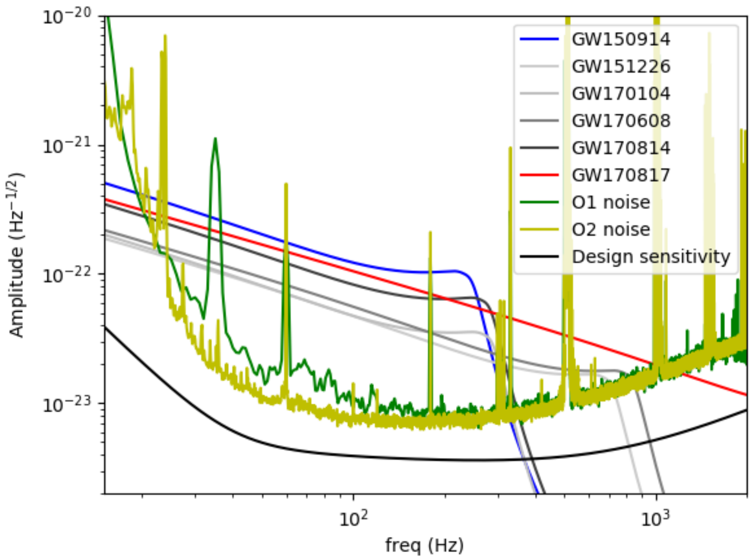

Figure 1.3: The spectrum of 6 sample gravitational events compared to the real O1 noise and the Advanced LIGO design sensitivity.

Note that before direct GW detections only the leading and 1PN terms of the energy function had been tested, and only the leading order of the flux formula, respectively with solar system observations, where $v \sim 10^{-4}$ e.g. for the Earth-Sun system, and with binary pulsars, for which $v \sim 1.5 \times 10^3$ for the flux formula [10].

All signals detected by terrestrial detectors so far have been interpreted as emitted by coalescing binaries located well outside our galaxy. The typical (dimension-less) strength h of a GW signal is given by the kinetic energy of the source divided by the source-observer distance:

$$h \sim \frac{G_N M_\odot v^2}{100 Mpc} \simeq \frac{km}{Mpc} \simeq 10^{-22} \,. \tag{1.5}$$

The typical density of Milky Way-type galaxies, which has a mass $M_{MW} \sim 3 \times 10^{11} M_\odot$, in the local universe is $\sim 5 \times 10^{-3}$ Mpc^{-3} [11], one can then infer the average galaxy-galaxy distance of $\sim$ 6 Mpc. The nearest galaxy to us is Adromeda ($M_{Andromeda} \sim 10^{12} M_\odot$) at 0.7Mpc, with which the Milky Way forms the Local Group.

Recently the NANOGrav collaboration announced the detection of a *stochastic* GW background at frequency $\sim 10^{-8} Hz$ [12], observing correlated distorsions (or *residulas*) in the time of arrival of the EM signals of 67 pulsars monitored over 15 years. The GW fingerprint is given by the specific shape of pulsar signal correlation, reproducing the expected Hellings-Downs effect [13].

The aim of this course is to give an overview of the science that

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[11] Gergely Dálya, Gábor Galgóczi, László Dobos, Zsolt Frei, Ik Siong Heng, Ronaldas Macas, Christopher Messenger, Péter Raffai, and Rafael S. de Souza. GLADE: A galaxy catalogue for multimessenger searches in the advanced gravitational-wave detector era. *Mon. Not. Roy. Astron. Soc.*, 479(2):2374–2381, 2018. DOI: 10.1093/mnras/sty1703

[12] Gabriella Agazie et al. The NANOGrav 15 yr Data Set: Evidence for a Gravitational-wave Background. *Astrophys. J. Lett.*, 951(1):L8, 2023. DOI: 10.3847/2041-8213/acdac6

[13] R. w. Hellings and G. s. Downs. UPPER LIMITS ON THE ISOTROPIC GRAVITATIONAL RADIATION BACKGROUND FROM PULSAR TIMING ANALYSIS. *Astrophys. J. Lett.*, 265:L39–L42, 1983. DOI: 10.1086/183954

the rise of GW astronomy has allowed, with a focus on fundamental physics. An overview of an analytic perturbative treatment of the two-body problem will be given, the post-Newtonian approximation to General Relativity (GR), whose small parameter of expansion around Minkowski space is the aformentioned v. Gravitationally bound binary systems exhibit a clear separation of scales: the size of the compact objects r_s, like black holes and/or neutron stars, the orbital separation r and the gravitational wave-length λ. Using again the virial theorem the hierarchy $r_s < r \sim r_s/v^2 < \lambda \sim r/v$ can be established, which is at the hearth of the PN expansion, for which excellent reviews exist, see e.g. [14] for general PN, or [15] and [16] for effective field theory approach. Finally a hint to comological applications of GW detections will be given, following the original idea by [17]

These are the notes of a lecture series going to be held in July 2023 at the Giambiagi Winter School School on Cosmology held in Buenos Aires, and they are not meant in any way to replace or improve the extensive literature existent on the topic, but rather to collect in single document the material relevant for this course, which could otherwise be found scattered in different places.

In particular this lecture series will give an overview of the basics of GW-detector interaction and GW data analysis. Books and reviews exist on the subject, see see e.g. [18], for an astrophysics-oriented review on GWs, [19] for a data-analysis oriented review.

[14] Luc Blanchet. Gravitational radiation from post-newtonian sources and inspiralling compact binaries. *Living Reviews in Relativity*, 9(4), 2006. URL `http://www.livingreviews.org/lrr-2006-4`

[15] W. D. Goldberger. Les houches lectures on effective field theories and gravitational radiation. In *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime*, 2007

[16] Stefano Foffa and Riccardo Sturani. Effective field theory methods to model compact binaries. *Class.Quant.Grav.*, 31 (4):043001, 2014. DOI: 10.1088/0264-9381/31/4/043001

[17] Bernard F. Schutz. Determining the Hubble Constant from Gravitational Wave Observations. *Nature*, 323:310–311, 1986. DOI: 10.1038/323310a0

[18] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

[19] B.S. Sathyaprakash and B.F. Schutz. Physics, Astrophysics and Cosmology with Gravitational Waves. *Living Rev.Rel.*, 12:2, 2009

Problems

***** Exercise 1 Newtonian gravitational waveform**

Using eqs. (1.2,1.3,1.4) compute numerically the "Newtonian" gravitational waveform (for semplicity assume constant amplitude).

***** Exercise 2 Period variation of binary pulsars**

Using the quadrupole emission formula for GWs, derive the rate of variation of the orbital period of a binary system made of equally massive objects $\sim M_\odot$ at a reletive velocity $c \sim 10^{-3}$.

***** Exercise 3 Period variation of binary pulsars**

Using the quadrupole emission formula for GWs

$$\dot{E} \sim \frac{G}{5}\dddot{Q}^2 ,$$

where Q is the quadrupole magnitude, derive how many *gravitational quanta* would be emitted per orbital period by the earth-sun system and by a lab system experiment (masses $\sim$ kg, frequency $\sim$ Hz, size $\sim$ m). *Result for earth-sun:* For the system earth-sun $\eta = 3 \times 10^{-7}$, $M = 2 \times 10^{33}$ gr, $R = 1.5 \times 10^{8}$km, $\omega = 2\pi/(\pi \times 10^{7}\text{sec})$ one finds a number of emitted gravitons per orbit equal to 1.6×10^{48}.

2

General Theory of GWs

2.1 *Expansion around Minkowski*

We start by recalling the Einstein's eqations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \,, \tag{2.1}$$

however it will be useful for our purposes to work also at the level of the action

$$\begin{aligned} S_{EH} &= \frac{1}{16\pi G_N} \int dt d^d x \sqrt{-g}\, R \,, &(2.2)\\ S_m \to \delta S_m &= \frac{1}{2} \int dt d^d x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} \,. &(2.3) \end{aligned}$$

Here we focus on an expansion around the Minkowski space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \,, \tag{2.4}$$

and we are interested in a systematic expansion in powers of $h_{\mu\nu}$. As in the binary system case the metric perturbation $|h_{\mu\nu}| \sim G_N m/r \sim v^2$, a suitable velocity expansion will has to be considered.

GR admits invariance under general coordinate transformations

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \,, \tag{2.5}$$

which change the metric according to

$$\begin{aligned} g_{\mu\nu}(x) \to g'_{\mu\nu}(x') &= g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \,, \\ h_{\mu\nu}(x) \to h'_{\mu\nu}(x') &= h_{\mu\nu}(x) - \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu\right) \,. \end{aligned} \tag{2.6}$$

At linear order around a Minkoswki background we have ($h \equiv \eta_{\mu\nu} h^{\mu\nu}$)

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \frac{1}{2} \left(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}\right) \,, \\ R_{\mu\nu} &= \frac{1}{2} \left(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h\right) \,, \\ R &= \partial_\mu \partial_\nu h^{\mu\nu} - \Box h \,, \\ G_{\mu\nu} &= \frac{1}{2} \left(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu} \Box h\right) \,, \end{aligned} \tag{2.7}$$

and we remind that with the metric signature adopted here $\Box = \partial_i\partial^i - \partial_t^2$.

The expression for $G_{\mu\nu}$ can be used to write the Einstein equations as

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma} - \partial_\mu\partial_\rho\bar{h}^\rho_\nu - \partial_\nu\partial_\rho\bar{h}^\rho_\mu = -16\pi G_N T_{\mu\nu}\,, \tag{2.8}$$

where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ which transforms as

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\alpha_\alpha\,.$$

Let us be specific about the matter action and assume that it is given by the world-line action

$$\begin{aligned} S &= -m\int d\tau \\ &= -m\int d\tau\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \\ &= -m\int dt\left[-g_{00} - 2g_{0i}\frac{dx^i}{dt} - g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right]^{1/2}\,. \end{aligned} \tag{2.9}$$

It is convenient to expand the "bulk" dynamics of the gravitational degrees of freedom is given by the standard Einstein-Hilbert action, to quadratic order

$$S_{EH2} = -\frac{1}{64\pi G_N}\int dt d\mathbf{x}\left[\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h\partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu\right] \tag{2.10}$$

The relative Lagrangean density in Fourier space is

$$\mathcal{L}_2 = -\frac{1}{64\pi G_N}\left(A^{\mu\nu\rho\sigma}(k_\alpha)h_{\mu\nu}(k)h_{\rho\sigma}(-k)\right) - \frac{1}{2}h_{\mu\nu}(k)T^{\mu\nu}(-k)\,, \tag{2.11}$$

with

$$\begin{aligned} A_{\mu\nu\rho\sigma} = \; & \frac{1}{2}k^2\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}\right) \\ & -k^2\eta_{\mu\nu}\eta_{\rho\sigma} \\ & +k_\mu k_\nu\eta_{\rho\sigma} + k_\rho k_\sigma\eta_{\mu\nu} \\ & -\frac{1}{2}\left(k_\mu k_\rho\eta_{\nu\sigma} + k_\mu k_\sigma\eta_{\nu\rho} + k_\nu k_\rho\eta_{\mu\sigma} + k_\nu k_\sigma\eta_{\mu\rho}\right)\,, \end{aligned} \tag{2.12}$$

leading to the equations of motion

$$A^{\mu\nu\rho\sigma}(k)h_{\rho\sigma} = -16\pi G_N T^{\mu\nu}\,,$$

which is a coupled system of linear differential equations. Individual polarizations of the gravitational field can be solved for if and only if A is invertible, i.e. if an operator $B = A^{-1}$ such that

$$B_{\alpha\beta\gamma\delta}A^{\gamma\delta\rho\sigma} = \frac{1}{2}\left(\delta^\rho_\alpha\delta^\sigma_\beta + \delta^\sigma_\alpha\delta^\rho_\beta\right)\,,$$

exists. It turns out that A is not ivertible as it can be verified by observing that

$$A^{\alpha\beta\rho\sigma}\left(k_\rho\xi_\sigma + k_\sigma\xi_\rho\right) = 0\,,$$

as it should be as a gauge transformation cannot affect the dynamics.

We can however *add a gauge-fixing term* to the Lagrangean, which correspond to finding the GR solutions in a specific gauge. Here we add the gauge-fixing *Lorentz* term

$$S_{GF} = -\frac{1}{32\pi G_N}\int d^4x \left(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h\right)^2 , \tag{2.13}$$

to the Einstein-Hilbert action to obtain

$$\begin{aligned} A'_{\mu\nu\rho\sigma} &= A_{\mu\nu\rho\sigma} - \left(k_\mu k_\nu \eta_{\rho\sigma} + \eta_{\mu\nu} k_\rho k_\sigma\right) \\ &\quad + \frac{1}{2}\left(k_\mu k_\rho \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\rho} + k_\nu k_\rho \eta_{\mu\sigma} + k_\nu k_\sigma \eta_{\mu\rho}\right) + \\ &\quad + \frac{1}{2}k^2 \eta_{\mu\nu}\eta_{\rho\sigma} \\ &= \tfrac{1}{2}k^2\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right) , \end{aligned} \tag{2.14}$$

which makes A' easily invertible by noting that

$$A'_{\mu\nu\alpha\beta} A'^{\alpha\beta\rho\sigma} = \frac{1}{2}\left(\delta_\mu^\rho \delta_\nu^\sigma + \delta_\mu^\sigma \delta_\nu^\rho\right) . \tag{2.15}$$

The new quadratic lagrangian is

$$\mathcal{L}_{GFEH2} = -\frac{1}{64\pi G_N} A'^{\mu\nu\rho\sigma} h_{\mu\nu} h_{\rho\sigma} , \tag{2.16}$$

so that the equations of motion derived from the new Lagrangean are

$$\Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu} , \tag{2.17}$$

which is equivalent to (2.8) under the condition

$$\partial^\mu \bar{h}_{\mu\nu} = 0 , \tag{2.18}$$

A consequence of this gauge choice is that all polarizations of the gravitational tensor *seem* to satisfy a wave-like equation.

Note that in this way we have not completely fixed the gauge *in vacuum*. Indeed one can perform a transformation with $\Box\xi_\mu = 0$ and still be in the Lorentz gauge class, as long as $T_{\mu\nu}$ is vanishing, as it will be shown momentarily.

It is useful to solve eq. (2.17) by using the Green's function G with appropriate boundary conditions, where the G is defined as

$$\Box_x G(x-y) = \delta^{(4)}(x-y) . \tag{2.19}$$

The explicit direct space form of the Green's functions with retarded and and advanced boundary conditions are respectively

$$\begin{aligned} G_{ret}(t,\mathbf{x}) &= -\delta(t-r)\frac{1}{4\pi r} , \\ G_{adv}(t,\mathbf{x}) &= -\delta(t+r)\frac{1}{4\pi r} , \end{aligned} \tag{2.20}$$

where $r \equiv |\mathbf{x}| > 0$ (note that $G_{ret}(t,\mathbf{x})) = G_{adv}(-t,\mathbf{x})$. By solving ex. 8,9 one can show that these are indeed Green functions for the eq. (2.19). For the Feynman prescription of the Green function G_F see ex. 10, where the motivated student is asked to demonstrate that the G_F ensures pure incoming wave at past infinity and pure outgoing wave at future infinity.

We have shown that in the Lorentz gauge class all gravitational field polarizations satisfy a wave equation: we now show that the correct physical statement is that only 2 degrees of freedom (polarizations) are physical and *radiative*, additional 4 are physical and *non-radiative* and finally 4 are pure gauge, and can be killed by the Lorentz condition in eq. (2.18), for instance.

Let us now focus on the wave eq. (2.17) in the vacuum case

$$\Box \bar{h}_{\mu\nu} = 0 \tag{2.21}$$

and define

$$\xi_{\mu\nu} \equiv \partial_\nu \xi_\mu + \partial_\mu \xi_\nu - \eta_{\mu\nu} \partial^\alpha \xi_\alpha \,. \tag{2.22}$$

Using that $\delta\bar{h}_{\mu\nu} = -\xi_{\mu\nu}$, making a specific coordindate transformation satisfying $\Box\xi_\mu^{(B)} = 0$ implies that both $\partial^\mu \bar{h}_{\mu\nu} = 0 = \partial^\mu \bar{h}'_{\mu\nu}$ as $\partial^\mu \xi_{\mu\nu}^{(B)} = 0$.

Since the free wave equation is satisfied by both the gravitational field $\bar{h}_{\mu\nu}$ and by the residual gauge transformation parametrized by $\xi_\mu^{(B)}$ that preserves the Lorentz condition, we can use the four available $\xi_\mu^{(B)}$ to set four conditions on $\bar{h}_{\mu\nu}$. In particular ξ_0 can be used to make h vanish (so that $\bar{h}_{\mu\nu} = h_{\mu\nu}$) and the three ξ_i can be used to make the three $\bar{h}_{0i}$ vanish. The Lorentz condition eq. (2.18) for $\mu = 0$ will now look like $\partial^0 h_{00} = 0$, which means that h_{00} is constant in time, hence not contributing to any GW. In conclusion, in vacuum and with the Lorentz condition holding, one can set

$$h_{0\mu} = h = \partial^i h_{ij} = 0 \,, \tag{2.23}$$

defining the *transverse-traceless*, or TT gauge, which then describe only the physical GW propagating in vacuum. For a wave propagating along the $\mu = 3$ axis, for instance, the wave eq. (2.17) admits the solution $h_{\mu\nu}(t-z)$ and the gauge condition $\partial^j h_{ij} = 0$ reads $h_{i3} = 0$. In terms of the tensor components we have

$$h_{\mu\nu}^{(TT,\hat{z})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.24}$$

Given a generic gravitational perturbation h_{kl}, which is in the Lorentz gauge, but not necessarily TT, its TT form can be obtained by applying the projector $\Lambda_{ij,kl}$ defined as

$$\begin{aligned} \Lambda_{ij,kl}(\hat{n}) &= \frac{1}{2}\left[P_{ik}P_{jl} + P_{il}P_{jk} - P_{ij}P_{kl}\right] , \\ P_{ij}(\hat{n}) &= \delta_{ij} - \hat{n}_i\hat{n}_j , \end{aligned} \tag{2.25}$$

according to

$$h_{ij}^{(TT,\hat{n})} = \Lambda_{ij,kl}(\hat{n})h_{kl} . \tag{2.26}$$

The Λ projector ensures transversality and tracelessness of the resulting tensor (starting from a tensor in the Lorentz gauge!).

The TT gauge cannot be imposed where $T_{\mu\nu} \neq 0$, as we cannot set to 0 any component of $\bar{h}_{\mu\nu}$ which satisfies a $\Box\bar{h}_{\mu\nu} \neq 0$ equation by using a ξ_μ which satisfies a $\Box\xi_\mu = 0$ equation, which is necessary not to move away from the Lorentz condition.

2.2 *Radiative degrees of freedom of $h_{\mu\nu}$*

The argument in the previous section shows how to individuate the radiative degrees of freedom of the gravitational tensor. It is possible however to individuate also the longitudinal ones, i.e. the ones which are not radiative, and its demonstration is reported here, faithful to the original one of sec. 2.2 of [1]. In particular what follows will show that the gravitational perturbation $h_{\mu\nu}$ has 6 physical degrees of freedom: 4 constrained and 2 radiative. The argument is based on a Minkowski equivalent of Bardeen's gauge-invariant cosmological perturbation formalism. The reader interested only in the result but not in the derivation may skip the rest of this section, but will be highly recommended to separate radiative and longitudinal degrees of freedom in the technically simpler case of the electromagnetic field, see ex.14.

[1] Eanna E. Flanagan and Scott A. Hughes. The basics of gravitational wave theory. *New J. Phys.*, 7:204, 2005

We begin by defining the decomposition of the metric perturbation $h_{\mu\nu}$, in any gauge, into irreducible pieces. Assuming that $h_{\mu\nu} \to 0$ as $r \to \infty$, we decompose $h_{\mu\nu}$ into irreducible quantities ϕ, β_i, γ, H, ε_i, λ and $h_{ij}^{(\mathrm{TT})}$ via the definitions

$$h_{00} = 2\phi , \tag{2.27}$$

$$h_{0i} = \beta_i + \partial_i\gamma , \tag{2.28}$$

$$h_{ij} = h_{ij}^{(\mathrm{TT})} + \frac{1}{3}H\delta_{ij} + \partial_{(i}\varepsilon_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda , \tag{2.29}$$

together with the constraints

$$\partial_i \beta_i = 0 \quad \text{(1 constraint)} \tag{2.30}$$
$$\partial_i \varepsilon_i = 0 \quad \text{(1 constraint)} \tag{2.31}$$
$$\partial_i h_{ij}^{(\mathrm{TT})} = 0 \quad \text{(3 constraints)} \tag{2.32}$$
$$\delta^{ij} h_{ij}^{(\mathrm{TT})} = 0 \quad \text{(1 constraint)} \tag{2.33}$$

and boundary conditions

$$\gamma \to 0, \quad \varepsilon_i \to 0, \quad \lambda \to 0, \quad \nabla^2 \lambda \to 0 \tag{2.34}$$

as $r \to \infty$. Here $H \equiv \delta^{ij} h_{ij}$ is the trace of the *spatial* portion of the metric perturbation. The spatial tensor $h_{ij}^{(\mathrm{TT})}$ is transverse and traceless, and is the TT piece of the metric discussed above which contains the physical radiative degrees of freedom. The quantities β_i and $\partial_i \gamma$ are the transverse and longitudinal pieces of h_{ti}. The uniqueness of this decomposition follows from taking a divergence of Eq. (2.28) giving $\nabla^2 \gamma = \partial_i h_{ti}$, which has a unique solution by the boundary condition (2.34). Similarly, taking two derivatives of Eq. (2.29) yields the equation $2\nabla^2 \nabla^2 \lambda = 3\partial_i \partial_j h_{ij} - \nabla^2 H$, which has a unique solution by Eq. (2.34). Having solved for λ, one can obtain a unique ε_i by solving $3\nabla^2 \varepsilon_i = 6\partial_j h_{ij} - 2\partial_i H - 4\partial_i \nabla^2 \lambda$.

The total number of free functions in the parameterization (2.27) – (2.29) of the metric is 16: 4 scalars (ϕ, γ, H, and λ), 6 vector components (β_i and ε_i), and 6 symmetric tensor components ($h_{ij}^{(\mathrm{TT})}$). The number of constraints (2.30) – (2.33) is 6, so the number of independent variables in the parameterization is 10, consistent with a symmetric 4×4 tensor.

We next discuss how the variables ϕ, β_i, γ, H, ε_i, λ and $h_{ij}^{(\mathrm{TT})}$ transform under gauge transformations ξ^a with $\xi^a \to 0$ as $r \to \infty$. We parameterize such gauge transformation as

$$\xi_\mu = (\xi_t, \xi_i) \equiv (A, B_i + \partial_i C) \,, \tag{2.35}$$

where $\partial_i B_i = 0$ and $C \to 0$ as $r \to \infty$; thus B_i and $\partial_i C$ are the transverse and longitudinal pieces of the spatial gauge transformation. Decomposing this transformed metric into its irreducible pieces yields the transformation laws

$$\phi \to \phi - \dot{A} \,, \tag{2.36}$$
$$\beta_i \to \beta_i - \dot{B}_i \,, \tag{2.37}$$
$$\gamma \to \gamma - A - \dot{C} \,, \tag{2.38}$$
$$H \to H - 2\nabla^2 C \,, \tag{2.39}$$
$$\lambda \to \lambda - 2C \,, \tag{2.40}$$
$$\varepsilon_i \to \varepsilon_i - 2B_i \,, \tag{2.41}$$
$$h_{ij}^{(\mathrm{TT})} \to h_{ij}^{(\mathrm{TT})} \,. \tag{2.42}$$

Gathering terms, we see that the following combinations of these functions are gauge invariant:

$$\Phi \equiv -\phi + \dot{\gamma} - \frac{1}{2}\ddot{\lambda} \,, \tag{2.43}$$

$$\Theta \equiv \frac{1}{3}\left(H - \nabla^2\lambda\right) \,, \tag{2.44}$$

$$\Xi_i \equiv \beta_i - \frac{1}{2}\dot{\varepsilon}_i \,; \tag{2.45}$$

$h_{ij}^{(\mathrm{TT})}$ is gauge-invariant without any further manipulation. In the Newtonian limit Φ reduces to the Newtonian potential Φ_N, while $\Theta = -2\Phi_N$. The total number of free, gauge-invariant functions is 6: 1 function Θ; 1 function Φ; 3 functions Ξ_i, minus 1 due to the constraint $\partial_i\Xi_i = 0$; and 6 functions $h_{ij}^{(\mathrm{TT})}$, minus 3 due to the constraints $\partial_i h_{ij}^{(\mathrm{TT})} = 0$, minus 1 due to the constraint $\delta^{ij}h_{ij}^{(\mathrm{TT})} = 0$. This is in keeping with the fact that in general the 10 metric functions contain 6 physical and 4 gauge degrees of freedom.

We would now like to enforce Einstein equation. Before doing so, it is useful to first decompose the stress energy tensor in a manner similar to that of our decomposition of the metric. We define the quantities ρ, S_i, S, P, σ_{ij}, σ_i and σ via the definitions

$$T_{00} = \rho \,, \tag{2.46}$$

$$T_{0i} = S_i + \partial_i S \,, \tag{2.47}$$

$$T_{ij} = P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma, \tag{2.48}$$

together with the constraints

$$\partial_i S_i = 0 \,, \tag{2.49}$$

$$\partial_i \sigma_i = 0 \,, \tag{2.50}$$

$$\partial_i \sigma_{ij} = 0 \,, \tag{2.51}$$

$$\delta^{ij}\sigma_{ij} = 0, \tag{2.52}$$

and boundary conditions

$$S \to 0, \quad \sigma_i \to 0, \quad \sigma \to 0, \quad \nabla^2\sigma \to 0 \tag{2.53}$$

as $r \to \infty$. These quantities are not all independent. The variables ρ, P, S_i and σ_{ij} can be specified arbitrarily; stress-energy conservation ($\partial^a T_{ab} = 0$) then determines the remaining variables S, σ, and σ_i via

$$\nabla^2 S = \dot{\rho} \,, \tag{2.54}$$

$$\nabla^2 \sigma = -\frac{3}{2}P + \frac{3}{2}\dot{S} \,, \tag{2.55}$$

$$\nabla^2 \sigma_i = 2\dot{S}_i \,. \tag{2.56}$$

We now compute the Einstein tensor from the metric (2.27) – (2.29). The result can be expressed in terms of the gauge invariant observables:

$$\begin{aligned} G_{00} &= -\nabla^2\Theta\,, && (2.57)\\ G_{0i} &= -\frac{1}{2}\nabla^2\Xi_i - \partial_i\dot{\Theta}\,, && (2.58)\\ G_{ij} &= -\frac{1}{2}\Box h^{(\mathrm{TT})}_{ij} - \partial_{(i}\dot{\Xi}_{j)} - \frac{1}{2}\partial_i\partial_j\left(2\Phi+\Theta\right)\\ &\quad +\delta_{ij}\left[\frac{1}{2}\nabla^2\left(2\Phi+\Theta\right) - \ddot{\Theta}\right]\,. && (2.59)\end{aligned}$$

We finally enforce Einstein equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and simplify using the conservation relations (2.54) – (2.56); this leads to the following field equations:

$$\begin{aligned} \nabla^2\Theta &= -8\pi\rho\,, && (2.60)\\ \nabla^2\Phi &= 4\pi\left(\rho + 3P - 3\dot{S}\right)\,, && (2.61)\\ \nabla^2\Xi_i &= -16\pi S_i\,, && (2.62)\\ \Box h^{(\mathrm{TT})}_{ij} &= -16\pi\sigma_{ij}\,. && (2.63)\end{aligned}$$

Notice that **only the metric components $h^{(\mathrm{TT})}_{ij}$ obey a wave-like equation**. The other variables Θ, Φ and Ξ_i are determined by Poisson-type equations. Indeed, in a purely vacuum spacetime, the field equations reduce to five Laplace equations and a wave equation:

$$\begin{aligned} \nabla^2\Theta^{\mathrm{vac}} &= 0\,, && (2.64)\\ \nabla^2\Phi^{\mathrm{vac}} &= 0\,, && (2.65)\\ \nabla^2\Xi^{\mathrm{vac}}_i &= 0\,, && (2.66)\\ \Box h^{(\mathrm{TT}),\mathrm{vac}}_{ij} &= 0\,. && (2.67)\end{aligned}$$

This manifestly demonstrates that only the $h^{(\mathrm{TT})}_{ij}$ metric components — the transverse, traceless degrees of freedom of the metric perturbation — characterize the radiative degrees of freedom in the spacetime. Although it is possible to pick a gauge in which other metric components *appear* to be radiative, they are not: their radiative character is an illusion arising due to the choice of gauge or coordinates.

The field equations (2.60) – (2.63) also demonstrate that, far from a dynamic, radiating source, the time-varying portion of the physical degrees of freedom in the metric is dominated by $h^{(\mathrm{TT})}_{ij}$. If we expand the gauge invariant fields Φ, Θ, Ξ_i and $h^{(\mathrm{TT})}_{ij}$ in powers of $1/r$, then, at sufficiently large distances, the leading-order $O(1/r)$ terms will dominate. For the fields Θ, Φ and Ξ_i, the coefficients of the $1/r$ pieces are combinations of the conserved quantities given by the mass $\int d^3x T_{00}$, the linear momentum $\int d^3 T_{0i}$ and the angular momentum $\int d^3x(x_iT_{0j} - x_jT_{0i})$. Thus, the only time-varying piece of the physical degrees of freedom in the metric perturbation at order $O(1/r)$ is the TT piece $h_{ij}^{(\mathrm{TT})}$.

Although the variables Φ, Θ, Ξ_i and $h_{ij}^{(\mathrm{TT})}$ have the advantage of being gauge invariant, they have the disadvantage of being *non-local* ($h_{ij}^{(\mathrm{TT})}$ apart, computation of these variables at a point requires knowledge of the metric perturbation $h_{\mu\nu}$ everywhere). This non-locality obscures the fact that the physical, non-radiative degrees of freedom are causal, a fact which is explicit in Lorentz gauge. One way to see that the gauge invariant degrees of freedom are causal is to combine the vacuum wave equation eq. (2.21) for the metric perturbation with the expression (2.7) for the gauge-invariant Riemann tensor. Moreover, observations that seek to detect GWs are sensitive only to the value of the Riemann tensor at a given point in space (see sec. **??**). For example, the Riemann tensor components R_{itjt} are given in terms of the gauge invariant variables as

$$R_{i0j0} = -\frac{1}{2}\ddot{h}_{ij}^{(\mathrm{TT})} + \Phi_{,ij} + \dot{\Xi}_{(i,j)} - \frac{1}{2}\ddot{\Theta}\delta_{ij}. \tag{2.68}$$

Thus, at least certain combinations of the gauge invariant variables are locally observable.

2.3 *Energy of GWs*

Only the TT part of the metric is actually a radiative degree of freedom, i.e. a GW, which is capable of transporting energy, momentum and angular momentum from the source emitting them. To derive the expression for such quantities we follow here sec. 1.4 of [2].

[2] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

In principle it is not unambigous to separate the background metric from the perturbation, but a natural splitting between spacetime background and GWs arise when there is a clear separation of scales: if the variation scale of $h_{\mu\nu}$ is λ and the variation scale of the background is $L_B \gg \lambda$ a separation is possible. We can e.g. average over a time scale $\bar{t} >> \lambda$ and obtain

$$\bar{R}_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T \right) - \langle R_{\mu\nu}^{(2)} \rangle \,, \tag{2.69}$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor computed on the background metric (and then vanishing in an expansion over Minkowski background), and $R_{\mu\nu}^{(2)}$ is the part of the Ricci tensor quadratic in the GW perturbation (no part linear in the perturbation survives after averaging). The $-\langle R^{(2)} \rangle$ in the above equation can be interpreted as giving contribution to an effective energy momentum tensor of the GWs $\tau_{\mu\nu}$ given by

$$\tau_{\mu\nu} = -\frac{1}{8\pi G_N}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)}\rangle \qquad (2.70)$$

(and $\tau = \langle R^{(2)}\rangle/(8\pi G_N)$). The expression for $R^{(2)}_{\mu\nu}$ is quite lengthy, see eq. (1.131) of [3], but using the Lorentz condition, $h = 0$ and neglecting terms which vanish on the equation of motion $\Box\bar{h}_{\mu\nu} = 0$ we have

[3] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu h_{\alpha\beta}h^{\alpha\beta}\partial_\nu\rangle\,, \qquad (2.71)$$

$$\tau_{\mu\nu} = \frac{1}{32\pi G_N}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle\,, \qquad (2.72)$$

This effective energy-momentum tensor is gauge invariant and thus depend only on $h^{(TT)}_{ij}$, giving

$$\tau^{00} = \frac{1}{16\pi G_N}\langle\dot{h}^2_+ + \dot{h}^2_\times\rangle\,.$$

For a plane wave travelling along the z direction we have $\tau_{01} = 0 = \tau_{02}$ and $\partial_z h^{(TT)}_{ij} = \partial^0 h^{(TT)}_{ij}$ and then $\tau^{03} = \tau^{00}$. For a spherical wave $\partial_r h^{(TT)}_{ij} = \partial^0 h^{(TT)}_{ij} + O(1/r^2)$, so similarly $\tau^{0r} = \tau^{00}$.

The time derivative of the GW energy E_V (or energy flux dE_V/dt) can be written as

$$\begin{aligned}\frac{dE_V}{dt} &= \int_V d^3x\partial_0\tau^{00}\\ &= -\int_V d^3x\partial_i\tau^{0i}\\ &= -\int_S dAn_i\tau^{0i} =\\ &= -\int_S dA\tau^{00}\end{aligned} \qquad (2.73)$$

The outward propagating GW carries then an energy flux $F = -\frac{dE}{dt}$

$$F = \frac{r^2}{32\pi G_N}\int d\Omega\langle\dot{h}^{(TT)}_{ij}\dot{h}^{(TT)}_{ij}\rangle \qquad (2.74)$$

or equivalently

$$F = \frac{1}{16\pi G_N}\langle\dot{h}^2_+ + \dot{h}^2_\times\rangle\,. \qquad (2.75)$$

The linear momentum P^i_V of GW is

$$P^i_V = \int d^3x\tau^{0i}\,. \qquad (2.76)$$

Considering a GW propagating radially outward, we have

$$\dot{P}^i_V = -\int_S \tau^{0i}\,, \qquad (2.77)$$

hence the momentum carried away by the outward-propagating wave is $\frac{dP^i}{dAdt} = t^{0i}$. In terms of the GW amplitude it is

$$\frac{dP^i}{dt} = -\frac{r^2}{32\pi G_N} \int d\Omega \langle \dot{h}^{(TT)}_{jk} \partial^i h^{(TT)}_{jk} \rangle \,. \qquad (2.78)$$

Note that, if τ^{0k} is odd under a parity transformation $\mathbf{x} \to -\mathbf{x}$, then the angular integral of eq. (2.78) vanishes. For the angular momentum of GW we refer to sec. 2.1.3 of [4].

[4] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

2.4 *Multipole expansion*

Given that the variation scale of the energy momentum tensor and of the radiation field are respectively r_{source} and λ, by Taylor-expanding the wave solution[5]

$$h^{(TT)}_{ij} = \frac{4G_N}{r} \Lambda_{ij;kl} \int d^3x' T_{kl} \left(t - |\vec{r} - \vec{r}'|, x' \right) \,. \qquad (2.79)$$

[5] Note that substituing T_{kl} with $\int_{\mathbf{k}} d\omega/(2\pi) \tilde{T}_{kl}(\omega, \vec{k})$ one gets
$h^{(TT)}_{ij} = \frac{4G_N}{r} \Lambda_{ij,kl}(\hat{n}) \int \frac{d\omega}{2\pi} \tilde{T}_{kl}(\omega, \omega\hat{n}) e^{-i\omega(t-r)}$

one has[6]

$$\begin{aligned} &T_{kl}(t - |\vec{r} - \vec{r}'|, \vec{x}') \simeq T_{kl}(t - r + \vec{x}' \cdot \hat{n}, \vec{x}') \\ &\simeq T_{kl}(t - r, \vec{x}') + \vec{x}' \cdot \hat{n} \dot{T}_{kl}(t - r, \vec{x}') + \tfrac{1}{2}(\vec{x}' \cdot \hat{n})^2 \ddot{T}_{kl}(t - r, \vec{x}') + \ldots \,, \end{aligned} \qquad (2.80)$$

[6] We use r to denote both the binary separation and the source-observer distance, hopefully this will not cause too much confusion.

i.e. we obtain a series in r_{source}/λ, which for binary systems gives $r_{source} = r \ll \lambda = r/v$.

Inserting the expansion (2.80) into eq. (2.79) one obtains source moments of increasing order that, following standard procedures can be traded for mass and velocity multipoles. E.g. , the integrated moment of the energy momentum tensor can be traded for the mass quadrupole

$$Q^{ij}(t) \equiv \int \mathrm{d}^d x \, T^{00}(t, x) x^i x^j \,, \qquad (2.81)$$

by repeatedly using the equations of motion under the form $T^{\mu\nu}_{\;\;\;,\nu} = 0$:

$$\begin{aligned} \int \mathrm{d}^d x \left[T^{0i} x^j + T^{0j} x^i \right] &= \int \mathrm{d}^d x \, T^{0k} \left(x^i x^j \right)_{,k} \\ &= -\int \mathrm{d}^d x \, T^{0k}_{\;\;\;,k} \, x^i x^j \\ &= \int \mathrm{d}^d x \, \dot{T}^{00} \, x^i x^j = \dot{Q}^{ij} \end{aligned} \qquad (2.82)$$

$$\begin{aligned} 2 \int \mathrm{d}^d x \, T^{ij} &= \int \mathrm{d}^d x \left[T^{ik} \, x^j_{\;,k} + T^{kj} \, x^i_{\;,k} \right] \\ &= \int \mathrm{d}^d x \left[\dot{T}^{0i} x^j + \dot{T}^{0j} x^i \right] \\ &= \int \mathrm{d}^d x \, \ddot{T}^{00} x^i x^j = \ddot{Q}^{ij} \,. \end{aligned} \qquad (2.83)$$

The above equations also show that as for a composite binary system $T_{00} \sim O(v^0)$, then $T_{0i} \sim O(v^1)$ and $T_{ij} \sim O(v^2)$.

Let us consider the multipolar expansion at the level of the action. At $O(v^0)$ one has

$$\mathcal{S}_{ext}|_{v^0} = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^d x\, T^{00}|_{v^0}\, h_{00} = \frac{M}{2}\int \mathrm{d}t\, h_{00}\,, \qquad (2.84)$$

where in the last passage the explicit expression

$$T^{00}(t,x)|_{v^0} = \sum_A m_A \delta^{(3)}(x - x_A(t))\,, \qquad (2.85)$$

has been inserted. At order v one needs to add the contribution from the first order derivative of h_{00} and the leading order of T_{0i}, giving

$$\mathcal{S}_{ext}|_{v} = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^d x\, \left(T^{00}|_{v^0}\, x^i h_{00,i} + 2T^{0i}|_v\, h_{0i}\right)\,, \qquad (2.86)$$

with

$$T^{0i}(t,x)|_v = \sum m_A v_{Ai} \delta^{(3)}(x - x_A(t))\,, \qquad (2.87)$$

and neither T_{00} nor T_{ij} contain terms linear in v. Since the total mass appearing in eq. (2.84) is conserved (at this order) and given that in the center of mass frame $\sum_A m_A x_{Ai} = 0 = \sum_A m_A v_{Ai}$, there is no radiation up to order v.

Using a trick similar to (2.82) one has

$$\begin{aligned} \int \mathrm{d}^d x T^{0i} &= \int \mathrm{d}^d x T^{0j} x^i_{,j} \\ &= -\int \mathrm{d}^d x T^{0j}_{,j} x^i \\ &= \int \mathrm{d}^d x \dot{T}^{00} x^i\,, \end{aligned} \qquad (2.88)$$

one can rewrite (2.86) as

$$\mathcal{S}_{ext}|_{v} = -\frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^d x\, D^i\left(-h_{00,i} + 2\dot{h}_{0i}\right)\,, \qquad (2.89)$$

where $D^i \equiv \int \mathrm{d}^d x T^{00} x^i$, and it can be interpreted as the coupling of the dipole to the linearized Γ^i_{00}.

From order v^2 on, following a standard procedure, see e.g. [7], it is useful to decompose the source coupling to the gravitational fields in irreducible representations of the $SO(3)$ rotation group, to obtain

[7] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

$$\begin{aligned} S_{ext}|^{\mathbf{1}}_{v^2} &= \frac{1}{4\Lambda}\int \mathrm{d}t\,\mathrm{d}^d x\, T^{0i}|_v\, x^j (h_{0i,j} - h_{0j,i})\,, \\ S_{ext}|^{\mathbf{0+2}}_{v^2} &= \frac{1}{4\Lambda}\int \mathrm{d}t\,\mathrm{d}^d x\left(2T^{ij}h_{ij} + T^{0i}x^j\left(h_{0i,j} + h_{0j,i}\right) + T_{00}x^i x^j h_{00,ij}\right) \\ &= \frac{1}{4\Lambda}\int \mathrm{d}t\, Q^{ij}|_{v^0}\left(\ddot{h}_{ij} + h_{00,ij} - \dot{h}_{0i,j} - \dot{h}_{0j,i}\right) \\ &= -\frac{1}{2\Lambda}\int \mathrm{d}t\left(QR_{00} + Q_{ij}R_{0i0j}\right)\,, \end{aligned} \qquad (2.90)$$

where eqs. (2.82,2.83) and integration by parts have been used. The superscripts $\mathbf{0}, \mathbf{1}, \mathbf{2}$ stand for the scalar, vector and symmetric-traceless representations of $SO(3)$.

The **0** part includes the mass coupling and the quadrupole trace, both non-radiaitive, the **1** part describes the angular momentum coupling, which is also non-radiative, and the **2** part descrive the first radiative multipole. Additional radiative multipoles will be present at any odd-dimensional representation of $SO(3)$. Overall the multipolar action looks like

$$\mathcal{S}_{ext} \supset \frac{1}{2}\int d\tau \left(Mh_{00} - L_{ab}h_{0a,b} - c_Q I^{ij}E_{ij} - c_O I^{ijk}\partial_i E_{jk} - c_J J^{ij}B_{ij} + \ldots \right) \quad (2.91)$$

angular momentum, while the electric (magnetic) tensor E_{ij} (B_{ij}) is defined by

$$\begin{aligned} E_{ij} &= C_{\mu i \nu j} u^\mu u^\nu \,, \\ B_{ij} &= -\frac{1}{2}\epsilon_{i\mu\nu\rho}\, u^\rho C^{\mu\nu}_{\ \ \ j\sigma}\, u^\sigma \,. \end{aligned} \quad (2.92)$$

Theya are obtained by decomposing the Weyl tensor $C_{\mu\alpha\nu\beta}$ analogously to the electric and magnetic decomposition of the standard electromagnetic tensor $F_{\mu\nu}$. The $\mathbf{0}+\mathbf{2}$ term in eq. (2.90) reproduces at linear order the coupling $Q_{ij}R_{0i0j}$ term in eq. (2.91), allowing to identify I_{ij} with Q_{ij} at lowest v-order, and sets $c_Q = 1/2$.

This amounts to decompose the source motion in terms of the world-line of its center of mass and moments describing its internal dynamics. The I^{ij}, I^{ijk}, J^{ij} tensors are the lowest order in an infinite series of source moments, the 2^{nth} electric (magnetic) moment in the above action scales at leading order as mr^n_{source} (mvr^n_{source}), and they couple to $n-2$ gradients of E_{ij} (B_{ij}) which scales as λ^{-n}, showing that the above multipole expansion is an expansion in terms of r_{source}/λ.

Note that the multipoles, beside being intrinsic, can also be induced by the tidal gravitational field or by the intrinsic angular momentum (spin) of the source. For quadrupole moments the tidal induced quadrupole moments $I_{ij}, J_{ij}|_{tidal} \propto E_{ij}, B_{ij}$ give rise to the following terms in the effective action

$$\mathcal{S}_{tidal} = \int d\tau \left[c_E E_{ij}E^{ij} + c_B B_{ij}B^{ij} \right] \,. \quad (2.93)$$

This is also in full analogy with electromagnetism, where for instance particles with no permanent electric dipole experience a quadratic coupling to an external electric field. Eq. (2.93) can be used to describe a single, spin-less compact object in the field of its binary system companion. Considering that the Riemann tensor generated at a distance r by a source of mass m goes as m/r^3, the finite size effect given by the $E_{ij}E^{ij}$ term goes as $c_E m^2/r^6$. For dimensional reasons $c_E \sim G_N r^5_{source}$ [8], thus showing that the finite size effects of a spherical symmetric body in the binary potential are $O(Gm/r)^5$ times the Newtonian potential, a well known result which goes under the

[8] Walter D. Goldberger and Ira Z. Rothstein. An Effective field theory of gravity for extended objects. *Phys.Rev.*, D73:104029, 2006

name of *effacement principle* [9] (the coefficient c_E actually vanishes for black holes in $3+1$ dimensions [10]).

Going to higher multipoles, to simplify calculations we choose to work in the transverse-traceless (TT) gauge, in which the only relevant radiation field is the traceless and transverse part of h_{ij}, thus we forget the other polarizations h_{00}, h_{0i}, whose presence can be inferred by requiring gauge invariance.

At order v^3 beyond leadin, one has

$$S_{ext}|_{v^3} = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^d x\, T^{ij}|_{v^2}\, x^k\, h_{ij,k} \tag{2.94}$$

and using the decomposition see ex. 18

$$\begin{aligned}\int \mathrm{d}^d x\, T^{ij} x^k = & \frac{1}{6}\int \mathrm{d}^d x\, \ddot{T}^{00} x^i x^j x^k \\ & +\frac{1}{3}\int \mathrm{d}^d x\, \left(\dot{T}^{0i} x^j x^k + \dot{T}^{0j} x^i x^k - 2\dot{T}^{0k} x^i x^j\right),\end{aligned} \tag{2.95}$$

one can re-write, see ex. 19

$$S_{ext}|_{v^3} = -\int \mathrm{d}t \left(\frac{1}{6} Q^{ijk} E_{ij,k} + \frac{2}{3} P^{ij} B_{ij}\right) \tag{2.96}$$

where

$$Q^{ijk} = \int \mathrm{d}^d x\, T^{00}\, x^i x^j x^k\,, \tag{2.97}$$

and

$$P^{ij} = \frac{1}{2}\int \mathrm{d}^d x\, \left(\epsilon^{ikl} x_k T^0_{\ l} x^j + \epsilon^{jkl} x_k T^0_{\ l} x^i\right), \tag{2.98}$$

allowing to identify $I^{ijk} \leftrightarrow Q^{ijk}$ and $J^{ij} \leftrightarrow P^{ij}$ at leading order, with $c_O = 1/3$ and $c_J = 4/3$.

For the systematics at higher orders we refer to the standard textbook [11].

To obtain the total radiated energy one has to solve for h^{TT}

$$h_{ij}^{(TT)}(t, r\hat{n}) = \frac{2G_N}{r}\Lambda_{ijkl}(\hat{n})\left(\ddot{I}_{kl}(t) + \frac{1}{6}\dddot{I}^{klm} n^m + \frac{2}{3}\ddot{J}^{kl} n^m\right), \tag{2.99}$$

and then use the flux formula (2.74) to obtain

$$\begin{aligned} F &= \frac{G_N}{8}\left[4\dddot{I}_{ij}\dddot{I}_{kl}\int\frac{\mathrm{d}\Omega}{4\pi}\Lambda_{ijkl}(\hat{n}) \right.\\ &\quad +\frac{4}{9}\ddddot{I}_{ijk}\ddddot{I}_{lmn}\int\frac{\mathrm{d}\Omega}{4\pi}\Lambda_{ijlm}(\hat{n})\hat{n}^k\hat{n}^n \\ &\quad \left. +\frac{64}{9}\epsilon_{kip}\dddot{J}_{pj}\epsilon_{nlq}\dddot{J}_{qm}\int\frac{\mathrm{d}\Omega}{4\pi}\Lambda_{ijlm}(\hat{n})\hat{n}^k\hat{n}^n\right] \\ &= G_N\left[\frac{1}{5}\dddot{I}^2_{ij} + \frac{16}{45}\dddot{J}^2_{ij} + \frac{1}{189}\ddddot{I}^2_{ijk} + \dots\right].\end{aligned} \tag{2.100}$$

[9] T. Damour. *Gravitational radiation and the motion of compact bodies*, pages 59–144. North-Holland, Amsterdam, 1983b

[10] T. Damour. Gravitational radiation and the motion of compact bodies. In N. Deruelle and T. Piran, editors, *Gravitational Radiation*, pages 59–144. North-Holland, Amsterdam, 1983a; and Barak Kol and Michael Smolkin. Black hole stereotyping: Induced gravito-static polarization. *JHEP*, 1202:010, 2012. DOI: 10.1007/JHEP02(2012)010

[11] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

To obtain the above one needs the following

$$\begin{aligned}\int \frac{d\Omega}{4\pi}\,\hat{n}_i\hat{n}_j &= \frac{1}{3}\delta_{ij}\,,\\ \int \frac{d\Omega}{4\pi}\,\hat{n}_{a_1}\hat{n}_{a_2}\ldots\hat{n}_{a_{2n}} &= \frac{1}{(2n+1)!!}\left(\delta_{a_1a_2}\ldots\delta_{a_{2n-1}a_{2n}}+perms\right)\,.\end{aligned} \tag{2.101}$$

Higher order corrections derive from PN corrections to the multipoles, e.g. $I^{ij} = Q^{ij} + O(v^2)$, and by interaction of GWs with other gravitational modes emitted by the same source, which can be either longitudinal (source by M and L_{ab}) or radiative (sourced by radiative multipoles), as exemplified in fig. 2.1

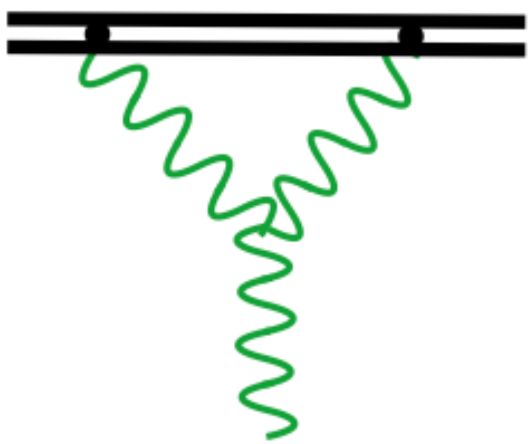

Figure 2.1: Example of non-linear interaction of the source with emitted GW.

2.5 Conservative dynamics

Starting from the linearized equations for the gravitational field (2.17) we have already derived the solution

$$\bar{h}^{(N)}_{\mu\nu} = 4G_N \int dt d^3\mathbf{x}'\; G_{Ret}(t-t',\mathbf{x}-\mathbf{x}')T_{\mu\nu}(t',\mathbf{x}')\,. \tag{2.102}$$

Finally plugging this solution back into the $O(h^2)$ Einstein's equation

$$\Box h^{(1PN)}_{\mu\nu} \simeq \partial^2\left(h^{(N)}_{\mu\nu}\right)^2\,, \tag{2.103}$$

one can solve perturbatively for the gravitational field, but we are now going to perform this computation more efficiently to derive how the longitudinal degrees of freedom of gravity rule the binary motion.

We can use the Lagrangian construction to solve for the h field, as in ex. 23, but here we want to show how powerful the effective action method is in determining the dynamics of the 2-body system, "integrating out" the gravitational degrees of freedom.

Let us pause briefly to introduce some technicalities about *Gaussian integrals* that will be helpful later. The basic formula we will need is

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2+Jx}dx = \left(\frac{2\pi}{a}\right)^{1/2} \exp\left(\frac{J^2}{2a}\right) . \tag{2.104}$$

or its multi-dimensional generalization

$$\int e^{-\frac{1}{2}x^i A_{ij} x^j + J^i x_i} dx_1 \ldots dx_n = \frac{(2\pi)^{n/2}}{(detA)^{1/2}} \exp\left(\frac{1}{2}J^t A^{-1} J\right) . \tag{2.105}$$

Other useful formulae are

$$\int x^k e^{-\frac{1}{2}ax^2+Jx}dx = \left(\frac{2\pi}{a}\right)^{1/2} \left(\frac{d}{dJ}\right)^k \exp\left(\frac{J^2}{2a}\right) \tag{2.106}$$

from which it follows that

$$\begin{aligned} \int x^{2n} e^{-\frac{1}{2}ax^2} dx &= \left(\frac{2\pi}{a}\right)^{1/2} \left(\frac{d}{dJ}\right)^{2n} \exp\left(\frac{J^2}{2a}\right)\Bigg|_{J=0} \\ &= \frac{(2n-1)!!}{a^n} \left(\frac{2\pi}{a}\right)^{1/2} , \end{aligned} \tag{2.107}$$

which also admit a natural generalization in case of x is not a real number but an element of a vector space. Let us see how this will be useful in the toy model of massless, non self-interacting scalar field Φ interacting with a source J:

$$\begin{aligned} S_{toy} &= \int dt d^d x \left[-\frac{1}{2}\left(\partial\Phi(t,\mathbf{x})\right)^2 + J(t,\mathbf{x})\Phi(t,\mathbf{x})\right] \\ &= \int_{\mathbf{k}} \frac{dk_0}{2\pi} \left[\Phi(k_0,\mathbf{k})\Phi^*(k_0,\mathbf{k})\left(k_0^2-k^2\right) + J(k_0,\mathbf{k})\Phi^*(k_0,\mathbf{k})\right] \end{aligned} \tag{2.108}$$

and apply the above eqs.(2.104–2.107), with two differences:

- Here the integration variable is Φ, depending on 2 continous indices $(k_0, \mathbf{k})$, instead of the discrete index $i \in 1 \ldots n$

- the Gaussian integrand is actually turned into a complex one, as we are taking at the exponent

$$\begin{aligned} Z_0[J] \equiv \int \mathcal{D}\Phi \exp\Bigg\{ i \int_{\mathbf{k}} \frac{dk_0}{2\pi} \Bigg[\frac{1}{2}\left(k_0^2-k^2\right)\Phi(k_0,\mathbf{k})\Phi(-k_0,-\mathbf{k}) + \\ J(k_0,\mathbf{k})\Phi(-k_0,-\mathbf{k}) + i\epsilon|\Phi(k_0,\mathbf{k})|^2 \Bigg] \Bigg\} , \end{aligned} \tag{2.109}$$

where the ϵ has been added term ensure convergence for $|\Phi| \to \infty$.

We can now perform the Gaussian integral by using the new variable

$$\Phi'(k_0,\mathbf{k}) = \Phi(k_0,\mathbf{k}) + (k_0^2 - k^2 + i\epsilon)J(k_0,\mathbf{k})$$

that allows to rewrite eq. (2.109) as

$$\begin{aligned} Z_0[J] = & \exp\left[-\frac{i}{2}\int_{\mathbf{k}}\frac{dk_0}{2\pi}\frac{J(k_0,\mathbf{k})J^*(k_0,\mathbf{k})}{k_0^2-k^2+i\epsilon}\right] \\ & \times\int\mathcal{D}\Phi'\exp\left\{i\int_{\mathbf{k}}\frac{dk_0}{2\pi}\left[\frac{1}{2}\left(k_0^2-k^2\right)\Phi'(k_0,\mathbf{k})\Phi^{*'}(k_0,\mathbf{k})+i\epsilon|\Phi|^2\right]\right\}. \end{aligned} \tag{2.110}$$

The integral over Φ' gives an uninteresting normalization factor $\mathcal{N}$, thus we can write the result of the functional integration as

$$\begin{aligned} Z_0[J] &= \mathcal{N}\exp\left[-\frac{i}{2}\int_{\mathbf{k}}\frac{dk_0}{2\pi}\frac{J(k_0,\mathbf{k})J(-k_0,-\mathbf{k})}{k_0^2-k^2+i\epsilon}\right] \\ &= \mathcal{N}\exp\left[-\frac{1}{2}\int dtd^3x\, G_F(t-t',\mathbf{x}-\mathbf{x}')J(t,\mathbf{x})J(t',\mathbf{x}')\right], \end{aligned} \tag{2.111}$$

The $Z_0[J]$ functional is the main ingredient allowing to compute the *effective action* describing the dynamics of the sources of the field we are integrating over and the dynamics of the extra field we are *not* integrating over. For instance starting from S_{toy} defined in eq. (2.108) we would obtain the effective action for the sources J from

$$S_{eff}[J] = -i\log Z_0[J]\,, \tag{2.112}$$

where we can safely discard the normalization constant $\mathcal{N}$. For instance substituting in S_{toy} $J\phi \to J_0 + J\Phi$, with

$$J_0(t,\mathbf{x}) + J(t,\mathbf{x})\Phi(t,\mathbf{x}) = -\sum_A m_A\delta^{(3)}(\mathbf{x}-\mathbf{x}_A)(1+\Phi(t,\mathbf{x}))\,, \tag{2.113}$$

one would obtain the effective action

$$\begin{aligned} S_{eff}(x_A) = & \sum_A\left[-m_A\int d\tau_A + \right. \\ & \left.\frac{i}{2}\sum_B m_A m_B\int d\tau_A d\tau_B G_F\left(t_A-t_B,\mathbf{x}_A(t_A)-\mathbf{x}_B(t_B)\right)\right]. \end{aligned} \tag{2.114}$$

Taking the Green function in Fourier domain in its quasi-static limit

$$\begin{aligned} & S_{eff} \supset m_a m_b\int dt_a dt_b\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t_a-t_b)+i\mathbf{k}(\mathbf{x}_a-\mathbf{x}_b)}}{k^2-\omega^2+i\epsilon} \\ \simeq & \int dt_a dt_b\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t_a-t_b)+i\mathbf{k}(\mathbf{x}_a-\mathbf{x}_b)}}{k^2}\left(1+\frac{\omega^2}{k^2}+\dots\right) \\ \simeq & \int dt_a dt_b\delta(t_a-t_b)\int_{\mathbf{k}}\frac{e^{i\mathbf{k}(\mathbf{x}_a-\mathbf{x}_b)}}{\mathbf{k}^2}\left(1+\frac{\partial_{t_a}\partial_{t_b}}{\mathbf{k}^2}\right) \\ = & \int dt\left[\frac{1}{4\pi|\mathbf{x}|}+\frac{O(v^2)}{|\mathbf{x}|}\right] \end{aligned} \tag{2.115}$$

one recover the instantaneous $1/r$, Newtonian interaction (plus $O(v^2)$ corrections). Note that we have implemented the substitution $\omega = -i\partial_{t_1} = i\partial_{t_2}$ to work out the systematic expansion in v. This is justified by observing that the wave-number $k^\mu \equiv (k^0, \mathbf{k})$ of the gravitational modes mediating this interaction have $(k^0 \sim v/r, k \sim 1/r)$, so in order to have manifest power counting it is necessary to Taylor expand the propagator.

The individual particles can also exchange *radiative* modes (with $k_0 \simeq k \sim v/r$), but such processes give sub-leading contributions to the effective potential in the PN expansion. In other words we are not integrating out the entire gravity field, but the specific off-shell modes in the kinematic region $k_0 \ll k$.

If there are interaction terms which cannot be written with terms linear or quadratic in the field, the Gaussian integral cannot be done analytically, so the rule to follow is to separate the quadratic action $S_{quad}[\Phi]$ of the field (its kinetic term) and Taylor expand all the rest: for an action $S = S_{quad} + (S - S_{quad})$ one would write

$$\begin{aligned} Z[J] &= \int \mathcal{D}\Phi e^{iS+i\int J\Phi} \\ &= \int \mathcal{D}\Phi e^{iS_{quad}+i\int J\Phi} \left[1 + i\left(S - S_{quad}\right) - \frac{\left(S - S_{quad}\right)^2}{2} + \dots\right], \end{aligned} \quad (2.116)$$

where J is now an auxiliary source, the physical source term will be Taylor expanded in the $S - S_{quad}$ term. As long as the $S - S_{quad}$ contains only polynomials of the field which is integrated over, the integral can be *perturbatively* performed analytically, inheriting the rule from eqs. (2.106,2.107): roughly speaking fields have to be paired up, each pair is going to be substitued by a Green function.

Our perturbative expansion admits a nice and powerful representation in terms of Feynman diagrams. Incoming and outgoing particle world-lines are represented by horizontal lines, Green functions by dashed lines connecting points, see e.g. fig. 2.2 the Feynman diagram accounting for the Newtonian potential, which is obtained by pairing the fields connected in the following expression

$$\begin{aligned} e^{iS_{eff}} &= Z[J, x_A]|_{J=0} = \int \mathcal{D}\Phi e^{iS_{quad}} \times \{1 \\ &- \frac{1}{2}\left[\sum_A m_A \int dt_A \underline{\Phi\left(t_A, \mathbf{x}_A(t_A)\right)}\right]\left[\sum_B m_B \int dt_B \underline{\Phi}\left(t_B, \mathbf{x}_B(t_B)\right)\right] + \dots\Bigg\} \end{aligned} \quad (2.117)$$

If following Green function lines all the vertices can be connected the diagram is said *connected*, otherwise it is said *disconnected*: only connected diagrams contribute to the effective action. We will not demonstrate this last statement, but its proof relies on the following argument. Taking the logarithm of eq. (2.117) we get

$$S_{eff} = -i \log Z_0[0] - i \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} Z_0^{-1}[0] \left(Z[0] - Z_0[0]\right)^n . \quad (2.118)$$

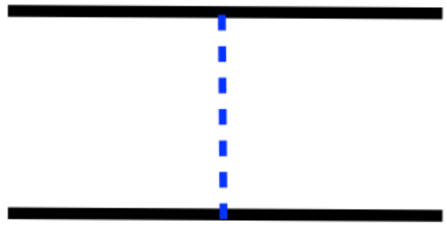

Figure 2.2: Feynman graph accounting for the Newtonian potential.

All terms with $n > 1$ describe disconnected diagrams, and some disconnected diagrams can also be present in the $n = 1$ term. However the $n = 1$ disconnected contribution is precisely canceled by the $n = 2$ terms.

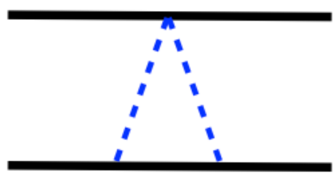

Figure 2.3: Graph giving a G_N^2 contribution to the 1PN potential via ϕ propagators.

2.6 *Power counting*

Path integral and Feynman diagrams are tools which have been invented for analyzing quantum processes, here on the other hand we are interested in classical gravity. Quantum corrections are suppressed with respect to classical terms by terms of the order $\hbar/L$, where L is the typical angular momentum of the systems, whic in our case is

$$L \sim mvr \sim 10^{77}\hbar \left(\frac{m}{M_\odot}\right)^2 \left(\frac{v}{0.1}\right)^{-1} . \tag{2.119}$$

Intermediate massive object lines, (like the ones in fig. 2.3) have no propagator associated, as they represent a static source (or sink) of gravitational modes.

The $\hbar$ counting of the diagrams can be obtained by restoring the proper normalization in the functional action definition eq. (2.116), implying that in the expansion we have $[(S - S_{quad})/\hbar]^n$ and that each Green function, being the inverse of the quadratic operator acting on the fields, brings a $\hbar^{-1}$ factor. Note that we consider that the classical sources do not recoil when interacting with via "field pairing". This is indeed consistent with neglecting quantum effects, as the wavenumber **k** of the exchanged gravitational mode has $k \sim 1/r$ and thus momentum $\hbar/r$.

Applying the previous rules one finds e.g. that the diagram in fig. 2.4 scales with one power more of $\hbar$ than the standard Newtonian graph in fig. 2.2.

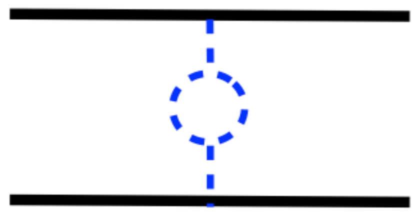

Figure 2.4: Quantum contribution to the 2-body potential.

Note that instead of computing the efective potential we can also compute the effective energy momentum tensor of an isolated source by considering the the effective action with one external gravitational mode $H_{\mu\nu}(t, x)$ *not* to be integrated over:

$$\begin{aligned} iS_{eff-1g}(m, H_{\mu\nu}) &= \log \int \mathcal{D}\phi \mathcal{D}A \mathcal{D}\sigma e^{iS_{EH+GF}(h_{\mu\nu}+H_{\mu\nu})} \\ &\times \left(1 - m \int d\tau (h_{\mu\nu} + H_{\mu\nu}) + \ldots\right) |_{H^1_{\mu\nu}} \\ &= \frac{i}{2} \int dt d^3x T^{(eff)}_{\mu\nu} H_{\mu\nu} , \end{aligned} \tag{2.120}$$

where the computation is made as usual by performing a Gaussian integration on all gravity field variables and the result will be linear in the external field which instead of being integrated over, is the $H_{\mu\nu}$ is the one we want to find what it is coupled to. Alternatively one can take eq. (2.117) in the presence of physical sources $J \sim -m \int d\tau$ and compute perturbatively the Feynman integral to obtain

$$\langle H_{\mu\nu} \rangle = \int \mathcal{D}\phi \mathcal{D}A \mathcal{D}\sigma H_{\mu\nu} e^{iS(h+H)} . \tag{2.121}$$

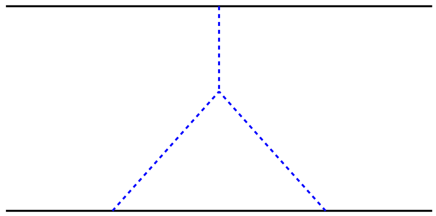

Figure 2.5: Example of $O(G^2)$ diagram which is IR divergent at 4PN order.

Problems

** Exercise 4 Coordinate transformation

Derive the second of eq. (2.6) by assuming (2.5).

** Exercise 5 Linearized Riemann, Ricci and Einstein tensors

Using that the Christoffel symbols at linear level are

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h^{\alpha}_{\nu} + \partial_\nu h^{\alpha}_{\mu} - \partial^{\alpha} h_{\mu\nu}\right)$$

derive eqs.(2.7)

* Exercise 6 Retarded Green function I

Show that the Green functions in eqs. (2.19) satisfy eq. (2.20) *Hint:* use that in spherical coordinates

$$\nabla^2\psi(r) = \frac{1}{r}\partial_r^2(r\psi(r))\,.$$

*** Exercise 7 Retarded Green function II

Show that the two representation of the retarded Green function given by eq. (2.20) and

$$G_{ret}(t,x) = -i\theta(t)\left(\Delta_+(t,x) - \Delta_-(t,x)\right)\,,$$

where

$$\Delta_\pm(t,x) \equiv \int_{\mathbf{k}} e^{\mp ikt}\frac{e^{i\mathbf{kx}}}{2k}$$

are equivalent. *Hint:*use that

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx} = \delta(x)\,,$$

and that

$$\theta(t)\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(t+r)} = 0 \quad \text{for } r \geq 0\,.$$

*** Exercise 8 Retarded Green function III

Use the representation of the G_{ret} obtained in the previous exercise to show that

$$G_{ret}(t,\mathbf{x}) = -\int_{\mathbf{k}} \frac{d\omega}{2\pi} \frac{e^{-i\omega t + i\mathbf{kx}}}{k^2 - (\omega + i\epsilon)^2},$$

where ϵ is an arbitrarily small positive quantity. Hint: use that

$$\theta(\pm t) = \mp \frac{1}{2\pi i} \int \frac{e^{-i\omega t}}{\omega \pm i\epsilon}.$$

Show that G_{ret} is real.

*** Exercise 9 Advanced Green function

Same as the two exercises above for G_{adv}, with

$$\begin{aligned} G_{adv}(t,\mathbf{x}) &= i\theta(-t)\left(\Delta_+(t,x) - \Delta_-(t,x)\right), \\ G_{adv}(t,\mathbf{x}) &= -\int_{\mathbf{k}} \frac{d\omega}{2\pi} \frac{e^{-i\omega t + i\mathbf{kx}}}{k^2 - (\omega - i\epsilon)^2}. \end{aligned}$$

Show that G_{adv} is real.

** Exercise 10 Feynman Green function I

Show that the G_F defined by

$$G_F(t,\mathbf{x}) = \int_{\mathbf{k}} \frac{d\omega}{2\pi} \frac{e^{-i\omega t + i\mathbf{kx}}}{\omega^2 - \mathbf{k}^2 + i\epsilon}$$

is equivalent to

$$G_F(t,\mathbf{x}) = -i\theta(t)\Delta_+(t,\mathbf{x}) + \theta(-t)\Delta_-(t,\mathbf{x}).$$

Derive the relationship

$$G_F(t,\mathbf{x}) = \frac{1}{2}\left(G_{adv}(t,\mathbf{x}) + G_{ret}(t,\mathbf{x})\right) + \frac{i}{2}\left(\Delta_+(t,\mathbf{x}) + \Delta_-(t,\mathbf{x})\right).$$

*** Exercise 11 Feynman Green function II

By integrating over $\mathbf{k}$ the G_F in the $\sim 1/(k^2 - \omega^2)$ representation, show that G_F implements boundary conditions giving rise to field h behaving as

$$h(t,\mathbf{x}) \sim \int d\omega e^{-i\omega t + i|\omega| r},$$

corresponding to out-going (in-going) wave for $\omega > (<) 0$. Since an $\omega < 0$ solution is equivalent to a $\omega > 0$ solution propagating backward in time, this result can be interpreted by saying that using G_F results into having pure out-going (in-going) wave for $t \to \pm\infty$.

* Exercise 12 TT gauge

Show that the projectors defined in eq. 2.25 satisfy the relationships

$$P_{ij}P_{jk} = P_{ik}$$
$$\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}\,,$$

which charecterize projectors operator.

***** Exercise 13 Energy of circular orbits in a Schwarzschild metric

Consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2G_N M}{r}\right)} + r^2 d\Omega^2\,. \qquad (2.122)$$

The dynamics of a point particle with mass m moving in such a background can be described by the action

$$S = -m\int d\tau = -m\int d\lambda\,\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

for any coordinate λ parametrizing the particle world-line. Using $S = \int d\lambda\, L$, we can write

$$L = -m\left[\left(1 - \frac{2G_N M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{\left(\frac{dr}{d\tau}\right)^2}{\left(1 - \frac{2G_N M}{r}\right)} - r^2\left(\frac{d\phi}{d\tau}\right)^2\right]^{1/2}\,.$$

Verify that L has cyclic variables t and ϕ and derive the corresponding conserved momenta.
(Hint: use $g_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = -1$. Result: $E = m(dt/d\tau)(1 - 2G_N M/r) \equiv m\,e$ and $L = mr^2(d\phi/d\tau) \equiv m\,l$).
By expressing $dt/d\tau$ and $d\phi/d\tau$ in terms of e and l, derive the relationship

$$e^2 = (1 - 2G_N M/r)\left(1 + l^2/r^2\right) + \left(\frac{dr}{d\tau}\right)^2\,.$$

From the circular orbit conditions ($\frac{de}{dr} = 0 = dr/d\tau = 0$), derive the relationship between l and r for circular orbits.
(Result: $l^2 = Mr/(1 - 3M/r)$).
Substitute into the energy function e and find the circular orbit energy

$$e(x) = \frac{1 - 2x}{\sqrt{1 - 3x}}\,,$$

where $x \equiv (G_N M\dot{\phi})^{2/3}$ is an observable quantity as it is related to the GW frequency f_{GW} by $x = (G_N M\pi f_{GW})^{2/3}$.
(Hint: Use

$$\dot{\phi} = \frac{d\phi}{d\tau}\dot{\tau} = \frac{d\phi}{d\tau}\frac{1 - 2M/r}{e}$$

to find that on circular orbits $(G_N M\dot{\phi})^2 = (G_N M/r)^3$, an overdot stands for derivative with respect to t.)
Derive the relationships for the Inner-most stable circular orbit

$$\begin{aligned} r_{ISCO} &= 6G_N M = 4.4\text{km}\left(\frac{M}{M_\odot}\right) \\ f_{ISCO} &= \frac{1}{6^{3/2}}\frac{1}{G_N M\pi} \simeq 8.8\text{kHz}\left(\frac{M}{M_\odot}\right)^{-1} \\ v_{ISCO} &= \frac{1}{\sqrt{6}} \simeq 0.41 \end{aligned}$$

***** Exercise 14 Degrees of freedom of electromagnetic field

Separate the degrees of freedom of the electromagnetic field into gauge, longitudinal and transverse, analogously to the gravitational case in eqs. (2.60-2.63).

Hint: split the equation

$$\partial_\nu F^{\nu\mu} = 4\pi J^\mu$$

into its 0 and its 3 spatial components

$$\begin{aligned} \nabla^2 A^0 + \dot{A}^i_{\ ,i} &= 4\pi\rho \\ \Box A^i + \dot{A}_{0,i} + A_{k,i}^{\ \ k} &= 4\pi J^i\,. \end{aligned}$$

Now decompose spatial vectors into a sum of a pure gradient and a divergence-free part:

$$\begin{aligned} A^i &= \bar{A}^i + \partial^i \mathcal{A}\,, \\ J^i &= \bar{J}^i + \partial^i \mathcal{J}\,, \end{aligned}$$

where $\nabla_i \bar{A}^i = 0 = \nabla_i \bar{J}^i$, and $\mathcal{A} = \frac{\partial_i A^i}{\nabla^2}$ and $\mathcal{J} = \frac{\partial_i J^i}{\nabla^2}$.

Note that under a gauge transformation $A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda$, hence $A_0 \to A'_0 = A_0 + \dot{\Lambda}$, $\mathcal{A} \to \mathcal{A}' = \mathcal{A} + \Lambda$ and $\bar{A}_i \to \bar{A}'_i = \bar{A}_i$, implying that $A_0 - \dot{\mathcal{A}}$ is invariant. One can then rewrite the equations as

$$\begin{aligned} \nabla^2(A^0 + \dot{\mathcal{A}}) &= 4\pi\rho\,, \\ \Box \bar{A}^i &= 4\pi \bar{J}^i\,, \\ -\dot{A}^0 - \ddot{\mathcal{A}} &= 4\pi\mathcal{J}\,, \end{aligned}$$

which are compatible $\iff \dot{\rho} + \partial_i J^i = \dot{\rho} + \nabla^2 \mathcal{J} = 0$, which is the continuity equation for the electromagnetic source.

Note that the combination $A^0 + \dot{\mathcal{A}}$ is gauge invariant and is fixed by a Poisson equation to ρ.

** Exercise 15 Locality of electric and magnetic field

Veirify that while the field $A^0 + \dot{\mathcal{A}}$ depends instantanously from the source, it is not directly observable, while the electric and magnetic fields, which are observable, depend casually on the sources.

** Exercise 16 Gauge fixing of electromagnetic field

Consider the electromagnetic action

$$-\frac{1}{2}\left(k^2\eta_{\mu\nu} - k^\mu k^\nu\right) A_\mu A_\nu$$

ad try to inver the kinetic operator. What goes wrong?

Add to the Lagrangean an arbitrary term $\frac{1}{\xi}k^\mu k^\nu A_\mu A_\nu$ and derive the inverse of the kinetic operator

** Exercise 17 Lorentz gauge

Show that a coordinate transformation characterized by $\Box\xi^\mu = 0$ does not invalidate the Lorentz gauge condition $\partial_\mu h^\mu_\nu = \frac{1}{2}\partial_\nu h$.

**** Exercise 18 Multipole reduction

Derive eq. (2.95).

Hint: use a trick analog to eq. (2.83) (i.e. $\int d^d x (T^{il} x^j x^k)_{,l} = 0$) to derive

$$\int d^3x \left[T_{ij}x_k + T_{ki}x_j + T_{jk}x_i\right] = \frac{1}{2}\int d^3x \left[\dot{T}_{i0}x_jx_k + \dot{T}_{k0}x_ix_j + \dot{T}_{k0}x_ix_j\right]$$

and then use

$$\begin{aligned}
\int d^3x \left[T_{ij}x_k - T_{ik}x_j\right] &= \int d^3x \left[T_{il}x_{j,l}x_k - T_{kl}x_{i,l}x_j\right] \\
&= \int d^3x \left[\dot{T}_{i0}x_jx_k - T_{ik}x_j - \dot{T}_{k0}x_ix_j + T_{kj}x_i\right] \\
\Longrightarrow \int d^3x \left[T_{ij}x_k - T_{jk}x_i\right] &= \int d^3x \left[\dot{T}_{i0}x_jx_k - \dot{T}_{k0}x_ix_j\right] \\
\Longrightarrow \int d^3x \left[T_{ij}x_k - T_{ki}x_j\right] &= \int d^3x \left[\dot{T}_{j0}x_ix_k - \dot{T}_{k0}x_ix_j\right]
\end{aligned}$$

Now combine the 3 equations involving $T_{ij}x_k$ to derive the decomposition into a completely symmetric part ($l = 3$ of $SO(3)$) and a triple of indices with mixed symmetry (symmetric under $i \leftrightarrow j$ and antsymmetrc under $i \leftrightarrow k$, $l = 2$).

**** Exercise 19 Electric octupole and magnetic quadrupole

For the comletely symmetric piece use $\int d^d x (T^{mn} x^i x^j x^k)_{,mn} = 0$ to derive

$$\begin{aligned}
\int d^d x \left(T^{ij}x^k + T^{ki}x^j + T^{jk}x^i\right) &= -\frac{1}{2}\int d^d x \left[T^{mn}_{,mn}x^ix^jx^k + 2T^{mn}_{,m}(x^ix^jx^k)_{,n}\right] \\
&= -\frac{1}{2}\int d^d x \left[T^{00}x^ix^jx^k - 2\left(\dot{T}^{0i}x^jx^k + \dot{T}^{0j}x^ix^k + \dot{T}^{0k}x^ix^j\right)\right],
\end{aligned}$$

and using again $\int d^d x (T^{0m} x^i x^j x^k)_{,m} = 0$ to derive

$$\begin{aligned}
&\int d^d x T^{0m}(x^ix^jx^k)_{,m} = \\
&= \int d^d x \left(T^{0i}x^jx^k + T^{0k}x^ix^j + T^{0j}x^kx^i\right) \\
&= -\int d^d x \dot{T}^{0m}_{,m}x^ix^jx^k, \\
&= \int d^d x \dot{T}^{00}x^ix^jx^k,
\end{aligned}$$

allows to write

$$\int \mathrm{d}^d x \left(T^{ij} x^k + T^{ki} x^j + T^{jk} x^i \right) = \frac{1}{2} \int \mathrm{d}^d x \ddot{T}^{00} x^i x^j x^k \,.$$

From eq. 2.92 one can derive the linearized, TT part of the Riemann tensor

$$E_{ij} = -\frac{1}{2} \ddot{h}_{ij} \,,$$

and obtain the first term in eq. 2.96.

For the combination with mixed symmetry in eq. 2.95

$$\begin{aligned}
&\dot{T}^{0i} x^j x^k + \dot{T}^{0j} x^i x^k - 2\dot{T}^{0k} x^i x^j h_{ij,k} \\
&= \dot{T}^{0i} x^j x^k \left(\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}\right) h_{jm,n} + \dot{T}^{0j} x^i x^k \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) h_{im,n} \\
&= \epsilon_{aik} \dot{T}^{0i} x^j x^k \epsilon_{amn} h_{jm,n} + \epsilon_{ajk} \dot{T}^{0j} x^i x^k \epsilon_{amn} h_{im,n} \,.
\end{aligned}$$

Now using the definition (2.98) of the source magnetic quadrupole and the linearized, TT expression of the magnetic part of the Riemann tensor

$$B_{ij} = \frac{1}{4} \epsilon_{ikl} \left(\dot{h}_{kj,l} - \dot{h}_{lj,k} \right) \,,$$

one gets

$$\int \mathrm{d}^d x \; T^{ij} x^k \Big|_{magnetic} h_{ij,k} = \frac{4}{3} P^{ij} B_{ij} \,,$$

to finally obtain the magnetic part of (2.96).

**** Exercise 20 Energy momentum conservation

Veirify that energy-momentum conservation

$$T^{\mu\nu}_{\ \ ,\nu} = 0$$

is euivalent at coupling linear in $h_{\mu\nu}$ by $\dot{M} = 0 = \dot{L}_{ab}$.

*** Exercise 21 Geodesic Equation

Derive the geodesic equation from the world line action (2.9).

*** Exercise 22 Gauge fixed quadratic action

Derive eq. (2.16) from eq. (2.10) and the gauge fixing term (2.18).

*** Exercise 23 Schwarzschild solution at Newtonian order

Starting from action (2.16) derive the eq. (2.17). Expand (2.9) so to obtain

$$S_{m-static} = -m \int dt' \left(1 - \frac{h_{00}(t', \mathbf{x}_p)}{2} \right)$$

and then derive

$$T^{(static)}_{\mu\nu}(t, \mathbf{x}) = 2 \frac{\delta S_{static}}{\delta g^{\mu\nu}(t, \mathbf{x})} = m \delta^{(3)}(\mathbf{x} - \mathbf{x}_p) \delta_{\mu 0} \delta_{\nu 0}$$

Use eq. (2.15) to obtain from the variation of the action eq. (2.17) and plug in the specific form of the retarded Green function to obtain

$$\begin{aligned}\bar{h}_{\mu\nu} &= (-16\pi G_N)\left(-\frac{1}{4\pi}\right)\delta_{\mu 0}\delta_{\nu 0} m \int dt' d\mathbf{x}' \frac{\delta\left(t-t'-|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|}\delta^{(3)}(\mathbf{x}'-\mathbf{x}_p) \\ &= \frac{4G_N m}{r}\delta_{\mu 0}\delta_{\nu 0}\,.\end{aligned}$$

Using that

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}_{\mu\nu}$$

and that on the above solution

$$\bar{h} = -4\frac{G_N m}{r}\,,$$

find the final result

$$\begin{aligned}h_{00} &= \bar{h}_{00} + \frac{1}{2}\bar{h} = 2\frac{G_N m}{r} \\ h_{xx} &= \bar{h}_{xx} - \frac{1}{2}\bar{h} = 2\frac{G_N m}{r}\,.\end{aligned}$$

* Exercise 24 Gaussian integrals

Derive eq. (2.104) from

$$\int e^{-\frac{1}{2}ax^2} dx = \left(\frac{2\pi}{a}\right)^{1/2}\,.$$

Derive the above from

$$\left[\int dx e^{-\frac{1}{2}ax^2}\right]^2 = 2\pi \int \rho e^{-\frac{1}{2}a\rho^2} d\rho\,.$$

** Exercise 25 Power $\hbar$

Derive the $\hbar$ scaling of graphs: $\hbar^{I-V}$, where I stands for the number of internal lines (propagators) and V for the numer of vertices. All classical diagrams are homogeneous in $\hbar^{-1}$

**** Exercise 26 Quantum correction to classical potential

Derive the quantum correction to the classical potential given by the term proportional to $G_N m_1 m_2 G_F^2(\mathbf{x}_1 - \mathbf{x}_2)$ originated from the expansion of $S_{quad} - S$ at linear order.

***** Exercise 27 Newtonian potential as graviton exchange

In non-relativistic quantum mechanics a one-particle state with momentum $\hbar\mathbf{p}$ in the coordinate representation is given by a plane wave

$$\psi_{\mathbf{p}}(\mathbf{x}) = C \exp\left(i\mathbf{p}\mathbf{x}\right)\,,$$

where the normalization constant C can be found by imposing

$$\int_V |\psi_{\mathbf{p}}|^2 = 1 \implies C = \frac{1}{\sqrt{V}}\,.$$

Since we want to trace the powers of M and $\hbar$ in the amplitude, it is necessary to avoid ambiguities: variables $\mathbf{p}_i$ are wave-numbers and ω is a frequency, so that $\hbar\omega$ is an energy and $\hbar\mathbf{p}$ is a momentum.

We define the non-relativistic scalar product

$$\langle \mathbf{p}_1|\mathbf{p}_2\rangle^{\rm NR} = \int d^3x\, \psi^*_{\mathbf{p}_1}(\mathbf{x})\psi_{\mathbf{p}_2}(\mathbf{x}) = \delta_{\mathbf{p}_1,\mathbf{p}_2}$$

which differs from the relativistic normalization used in quantum field theory for the scalar product $\langle \mathbf{p}_1|\mathbf{p}_2\rangle^R$ according to:

$$\langle \mathbf{p}_1|\mathbf{p}_2\rangle^{\rm R} = \frac{2\omega_{\mathbf{p}_1}}{\hbar} V \langle \mathbf{p}_1|\mathbf{p}_2\rangle^{\rm NR}\,,$$

which implies

$$|\mathbf{p}\rangle^{NR} = \left(\frac{\hbar}{2\omega_{\mathbf{p}}V}\right)^{1/2} |\mathbf{p}\rangle^R\,.$$

Note that a scalar field in the relativistic normalization has the *relativistic* expansion in terms of creator and annihilator operators

$$\psi(t,\mathbf{x}) = \int_{\mathbf{k}} \frac{1}{\sqrt{\omega_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i\omega_k t + i\mathbf{k}\mathbf{x}} + a^\dagger_{\mathbf{k}} e^{i\omega_k t - i\mathbf{k}\mathbf{x}}\right] \qquad (2.123)$$

with

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = i\hbar(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,.$$

We want to study the relativistic analog of the process described in fig. 2.2, assuming the coupling of gravity with a scalar particle given by the second quantized action

$$S_{pp-quant} = \frac{1}{2}\int dt d^3x \left\{\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}\eta_{\mu\nu}\left[(\partial_\alpha\psi\partial^\alpha\psi) + \frac{M^2}{\hbar^2}\right]\right\}\frac{h^{\mu\nu}}{\Lambda}\,,$$

where the Λ factor has been introduced to have a canonically normalized gravity field.

Derive the quantum amplitude $A(p_1, p_2, p_1+k, p_2-k)$ for the 1-graviton exchange between two particles with incoming momenta $p_{1,2}$ and outgoing momenta $p_{1,2}\pm k$

$$\begin{aligned} iA(p_1,p_2,p_1+k,p_2-k) &= A'_{\alpha\beta\mu\nu}\frac{i\hbar}{k^2-k_0} \\ &\times \frac{i^2}{\hbar^2\Lambda^2}\langle \mathbf{p}_1+\mathbf{k}|T_{\mu\nu}|\mathbf{p}_1\rangle^{NR}\langle \mathbf{p}_2-\mathbf{k}|T_{\alpha\beta}|\mathbf{p}_2\rangle^{NR} \end{aligned}$$

where $A'_{\alpha\beta\mu\nu}$ is defined in eq.(2.14).

Hint: from $S_{p-quant}$ above derive

$$\begin{aligned} T_{00} &= \tfrac{1}{2}\left[\dot\psi^2 + (\partial_i\psi)^2 + \tfrac{M^2}{\hbar^2}\psi^2\right], \\ T_{0i} &= \partial_0\psi\partial_i\psi, \\ T_{ij} &= \partial_i\psi\partial_j\psi - \tfrac{1}{2}\delta_{ij}\left[-\dot\psi^2 + \partial_i\psi\partial^i\psi + \tfrac{M^2}{\hbar^2}\psi^2\right], \end{aligned}$$

show that in the non-relativistic limit the term the A_{0000} term dominates and use the above equation to express $|\mathbf{q}\rangle^R$ in terms of $|\mathbf{q}\rangle^{NR}$. Use also that in the NR limit $p^\mu = \delta^{\mu 0}\omega_p = \delta^{\mu 0}M/\hbar$.

******* Exercise 28 Classical corrections to the Newtonian potential in a quantum set-up

Consider the process in fig. 2.5, write down its amplitude

$$\begin{aligned} iA_{fig.2.5} &= \int dt \left(\frac{i}{\hbar}\right)^3 \langle \mathbf{p}_1 + \mathbf{k}|T_{00}|\mathbf{p}_1\rangle^{NR}\langle \mathbf{p}_2 + \mathbf{q}|T_{00}|\mathbf{p}_2\rangle^{NR}\langle \mathbf{p}_2 - \mathbf{k}|T_{00}|\mathbf{p}_2\rangle^{NR} \\ &\times \int_{\mathbf{k},\mathbf{q}} \frac{d\omega_2}{2\pi} e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)} \frac{i\hbar}{k^2}\frac{i\hbar}{(\mathbf{k}+\mathbf{q})^2}\frac{i\hbar}{q^2}\frac{i\hbar}{(\mathbf{p}_2+\mathbf{q})^2 + M^2/\hbar^2 - \omega_2^2 - i\epsilon}. \end{aligned}$$

After writing the massive propagator as

$$\frac{1}{\sqrt{(\mathbf{p}_2+\mathbf{q})^2 + M^2/\hbar^2} + \omega_2}\frac{1}{\sqrt{(\mathbf{p}_2+\mathbf{q})^2 + M^2/\hbar^2} - \omega_2)} \sim \frac{\hbar/M}{\sqrt{(\mathbf{p}_2+\mathbf{q})^2 + M^2/\hbar^2} - \omega_2}$$

Perform the integral first in ω_2 and take the limit $\hbar \to 0$ and $M \to \infty$ to recover the result in the non-relativistic classical theory.

******* Exercise 29 Lorentz invariance in the non-relativistic limit

Derive the non-relativistic limit of the Lorentz transformation

$$\begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} = (1-w^2)^{-1/2}\begin{pmatrix} 1 & -\mathbf{w} \\ -\mathbf{w} & 1 \end{pmatrix}\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$$

Result:

$$\begin{aligned} \mathbf{x}_a &\to \mathbf{x}'_a = \mathbf{x} - \mathbf{w}t + \mathbf{v}_a(\mathbf{w}\cdot\mathbf{x}_a) \\ t &\to t' = t \end{aligned}$$

where $\mathbf{w}$ is the boost velocity and $\mathbf{w}_a = \dot{\mathbf{x}}_a$. From the Lagrangian $L(x_a, v_a)$, let us define

$$\frac{\delta L}{\delta x_a^i} \equiv -\frac{d}{dt}\left(\frac{\partial L}{\partial v_a^i}\right) + \frac{\partial L}{\partial x_a}.$$

and on the equations of motion $\delta L/\delta x_a^i = 0$. Show that for *any* transformation $\delta x_a^i = x_a^{i'} - x_a^i$ one can write the Lagrangian variation as

$$\delta L = \frac{dQ}{dt} + \sum_a \frac{\delta L}{\delta x_a^i}\delta x_a^i + O(\delta x_a^2),$$

with

$$Q \equiv \sum_a \frac{\partial L}{\partial v_a^i}\delta x_a^i = \sum_a p_a^i \delta x_a^i.$$

If the transformation δx_a^i is a symmetry, the Lagrangian transforms as a total derivative, that is, in the case of boosts, $\delta L = w^i dZ^i/dt + O(w^2)$ for some function Z^i. Apply the above equation for δL in terms of Q to derive that invariance under boosts implies conservation of the quantity

$$G^i - \sum_a p_a^i t \equiv -Z^i + \sum_a x_a^i (\mathbf{p}_a \cdot \mathbf{v}_a) - \sum_a p_a^i t$$

and find the specific form of Z^i at Newtonian level and at 1PN level. Interpret G^i as the center of mass position, by imposing $G^i = 0$ and $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{r}$ find the expressions of $x_{1,2}^i$ in terms of the relative coordinate r^i.

Find the energy E of the system

$$E = \sum_a v_a^i p_a^i - L\,.$$

at Newtonian and 1PN level. Use the result of the previous step to express v_a in terms of the relative velocity v. Use the result of the previous exercise to find the energy of the circular orbit in terms of x and compare with the Schwarzschild result of ex. 13, the result being

$$E(x) = -\mu\frac{x}{2}\left[1 + x\left(-\frac{3}{4} - \frac{1}{12}\eta)\right)\right]\,,$$

with $\eta \equiv m_1 m_2/(m_1+m_2)^2$.

****** Exercise 30 Feynman trick

$$\frac{1}{k^{2a}} = \frac{1}{\Gamma(a)}\int_0^\infty ds s^{a-1} e^{-sk^2}\,,$$

to show that

$$\int \frac{d^3 q}{(2\pi)^3}\frac{1}{(k-q)^{2a} q^{2b}} = \frac{(k^2)^{d/2-a-b}}{(4\pi)^{d/2}}\frac{\Gamma(d/2-a)\Gamma(d/2-b)\Gamma(a+b-d/2)}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}$$

Hint:

Use twice the Feynman trick (introducing s_1 for $(k-q)^2$ and s_2 for q^2) and then multiply the result by

$$\int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{s_1+s_2}{\lambda}\right) = 1\,.$$

Then change the variable $(s_1, s_2) \to (x_1\lambda, x_2\lambda)$, with the resulting x_1 varying from 0 to 1 and $q \to q' = q - xk$ The perform the integral of $d^3 q'$ first, remembering that

$$\int d\Omega^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,,$$

and that

$$\int_0^\infty dx e^{-\lambda x^2} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}},$$

then the one over λ, and finally integrate over x by using that

$$\int_0^1 dx x^a (1-x)^b = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}.$$

********** **Exercise 31 Einstein-Infeld-Hoffman potential**

Using the expanded version of the point-particle world-line coupling to gravity

$$S_{pp} = -m_a \int \mathrm{d}t \left[\frac{1}{2}h_{00}\left(1+\frac{1}{2}v_a^2\right) + \frac{1}{8}h_{00}^2 + h_{0i}v^i + \frac{1}{2}h_{ij}v^i v^j + \dots\right],$$

derive the contribution to the 1PN potential due to the exchange of one h_{00} mode:

$$\begin{aligned} iS_{eff}|_{h_{00}-h_{00}} &= -i^3 \frac{32\pi G m_1 m_2}{8} \int dt_1\, dt_2 \delta(t_1-t_2) \int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_1(t_1)-\mathbf{x}_2(t_2))}}{k^2} \times \\ & \left[1+\frac{1}{2}\left(v_1^2+v_2^2\right)\right]\left(1+\frac{\partial_{t_1}\partial_{t_2}}{k^2}\right) \\ &= i4\pi G m_1 m_2 \int \mathrm{d}t \int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_1(t)-\mathbf{x}_2(t))}}{k^2} \times \\ & \left[1+\frac{1}{2}\left(v_1^2+v_2^2\right)\right]\left(1+\frac{v_1^i v_2^j k_i k_j}{k^2}\right) \\ &\simeq i\frac{G_N m_1 m_2}{r}\left[1+\frac{1}{2}\left(v_1^2+v_2^2\right)+\frac{1}{2}\left(v_1 v_2 - (v_1\hat{r})(v_2\hat{r})\right)\right], \end{aligned}$$

where we have used

$$\int_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \frac{k^i k^j}{k^{2\alpha}} = \left[\frac{1}{2}\delta^{ij} - \left(\frac{d}{2}-\alpha+1\right)\hat{r}^i\hat{r}^j\right] \frac{\Gamma(d/2-\alpha+1)}{(4\pi)^{d/2}\Gamma(\alpha)} \left(\frac{r}{2}\right)^{2\alpha-d-2}.$$

2.1 Contribution from the $h_{00}-h_{ii}$ exchange:

$$\begin{aligned} iS_{eff}|_{h_{00}-h_{ii}} &= -i^3 \frac{32\pi G m_1 m_2}{8} \int dt_1\, dt_2 \delta(t_1-t_2) \int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_1(t_1)-\mathbf{x}_2(t_2))}}{k^2} \left(v_1^2+v_2^2\right) \\ &= i\frac{G m_1 m_2}{r}\left(v_1^2+v_2^2\right). \end{aligned}$$

2.2 Contribution from one h_{0i} exchange:

$$iS_{eff}|_{h_{0i}} = i^3 \frac{32\pi G m_1 m_2}{2} \int dt \int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)}}{k^2} \delta_{ij} v_1^i v_2^j = -i\frac{4G_N m_1 m_2}{r} \vec{v}_1\cdot\vec{v}_2,$$

2.3 Contribution from two h_{00} exchange

$$iS_{eff}|_{h_{00}^2} = i^5 \frac{(32\pi G)^2 m_1^2 m_2}{128} \int dt \left(\int_{\mathbf{k}} \frac{e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)}}{k^2}\right)^2 = i\frac{G_N^2 m_1^2 m_2}{2r^2},$$

2.4 Collect all the contributions to find the Einstein-Infled-Hoffman potential

$$V_{EIH}=-\frac{G_N m_1 m_2}{2r}\left[3\left(v_1^2+v_2^2\right)-7v_1v_2-(v_1\hat{r})(v_2\hat{r})\right]+\frac{G_N^2 m_1m_2(m_1+m_2)}{2r^2}\,,$$

(the Lagrangean is minus the potential!). To compute the 1PN gravitational 2-body potential one needs correction to the Newtonian potential both at v^2 and GM/r order, since for a bound orbit $v^2\sim GM/r$. Verify that one needs to add the exchange of a h_{00} mode (with v^2 corrections coming from the Taylor expansion of the Green's function and from), a h_{0i} one, the mixed $h_{00}-h_{ii}$, the contribution from h_{00}^2 coupling to the world-line, the contribution from a $\partial^2 h^3$ bulk interaction. Gathering all of these contributions

$$\begin{aligned}
V_{h_{00}-h_{00}} &= -G_N\frac{m_1m_2}{r}\left[1+\frac{1}{2}\left(v_1\cdot v_2-(\hat{n}\cdot v_1)\left(\hat{n}\cdot v_2\right)\right)+\frac{3}{2}\left(v_1^2+v_2^2\right)\right]\,,\\
V_{h_{0i}-h_{0i}} &= 4G_N\frac{m_1m_2}{r}v_1\cdot v_2\,,\\
V_{h_{00}-h_{ii}} &= -G_N\frac{m_1m_2}{r}\left(v_1^2+v_2^2\right)\,,\\
V_{h_{00}^2} &= -G_N^2\frac{m_1m_2(m_1+m_2)}{2r^2}\,,\\
V_{\partial^2h^3} &= G_N^2\frac{m_1m_2(m_1+m_2)}{r^2}\,,
\end{aligned}$$

where for $V_{\partial^2 h^3}$ it is useful to know that the Gaussian integral of

$$\int \mathrm{d}\phi e^{iS_{quad}}\left[\frac{1}{2}\left(\int\partial^2h^3\right)\left(\int J_1h_{00}\right)\left(\int J_2h_{00}\right)\left(\int J_2h_{00}\right)\right]=$$
$$-i\int\mathrm{d}t\int_{\mathbf{k},\mathbf{q}}\frac{k^2+q^2+(k-q)^2}{k^2q^2(k-q)^2}\,.$$

wherel all Green's functions have been approximated according to $1/(k^2-\omega^2)\simeq 1/k^2$, which finally gives the contribution $V_{\partial h^3}$ above.

******* Exercise 32 Appearance of IR singularities**

Show that at G^2 order the near zone computation exhibits IR singularities.

*H*int:

E.g. the diagram in fig. 2.5

$$\int dt\, d^3p\, e^{i\vec{p}(\vec{x}_1-\vec{x}_2)}\frac{p^ip^j}{|\mathbf{p}^2|}\int d^3k\frac{1}{|\mathbf{k}|^2|\mathbf{p}-\mathbf{k}|^2}\left(1+\ldots+\frac{\omega^4}{|\mathbf{k}|^4}+\ldots\right)$$
$$\simeq\int dt\, d^3p e^{i\vec{p}\cdot\vec{x}_{12}}\frac{p^ip^j}{|\mathbf{p}|^3}\left\{1+\ldots+\frac{1}{|\mathbf{p}|^4}\left[(\vec{p}\cdot\vec{v}_1)^3(\vec{p}\cdot\vec{v}_2)^3+\ldots+\vec{p}\cdot\dot{\vec{a}}_1\vec{p}\cdot\dot{\vec{a}}_2\right]\right\}$$

******** Exercise 33 Classical limit from quantum amplitude**

Treating the diagram in fig. 2.5 with relativistic sources involves a loop integral

$$\int_{\mathbf{k}} \frac{d\omega}{2\pi} \frac{1}{\omega^2 - \mathbf{k}^2 + i\epsilon} \frac{1}{\omega^2 - (\mathbf{q}-\mathbf{k})^2 + i\epsilon} \frac{1}{(E+\omega)^2 - (\mathbf{p}+\mathbf{k})^2 - m^2/\hbar^2 + i\epsilon} \quad (2.124)$$

where $(0, \mathbf{q})$ is the overall wave-number exchanged between the two objects, and the massive body with two gravitational insertions has initial wave number $(E, \mathbf{p})$. In the non-relativistic limit $E \sim m/\hbar$, $\hbar\mathbf{p}/m \to 0$ gives [12]

$$\frac{\hbar}{2m} \int_{\mathbf{k}} \frac{d\omega}{2\pi} \frac{1}{\omega^2 - \mathbf{k}^2 + i\epsilon} \frac{1}{\omega^2 - (\mathbf{q}-\mathbf{k})^2 + i\epsilon} \frac{1}{\omega + i\epsilon} \left[1 + O\left(\frac{\hbar\mathbf{q}}{m}\right)\right] \quad (2.125)$$

which can be integrated by closing the integration contour in the upper half-plane to pick the contribution from the residues at the poles $\omega = -|\mathbf{k}| + i\epsilon$ and $\omega = -|\mathbf{k}+\mathbf{q}| + i\epsilon$ to get

$$-i\frac{\hbar}{4m} \int_{\mathbf{k}} \frac{1}{\mathbf{k}^2} \frac{1}{(\mathbf{k}+\mathbf{q})^2} \left[1 + O\left(\frac{\hbar\mathbf{q}}{m}\right)\right] . \quad (2.126)$$

This result is proportional to the loop integral in the potential region of momenta, and it is understood that the region of integration is $|\mathbf{k}| \ll m/\hbar$.

[12] N. E. J. Bjerrum-Bohr, Poul H. Damgaard, Guido Festuccia, Ludovic Planté, and Pierre Vanhove. General Relativity from Scattering Amplitudes. *Phys. Rev. Lett.*, 121(17):171601, 2018. DOI: 10.1103/PhysRevLett.121.171601

** Exercise 34 Near-Far

A simplified example of how the near/far splitting works can be understood as follows, see [13] for a proper demonstation. Let us define

$$\begin{aligned} I &\equiv e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{\mathbf{k}^2 - \omega^2} , \\ N &\equiv e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{\mathbf{k}^2} \sum_{m\geq 0} \left(\frac{\omega^2}{\mathbf{k}^2}\right)^m , \\ F &\equiv \sum_{n\geq 0} \frac{(i\mathbf{k}\cdot\mathbf{r})^n}{n!} \frac{1}{\mathbf{k}^2 - \omega^2} , \end{aligned} \quad (2.127)$$

and rewrite the original $\mathbf{k}$ integral of (**??**) as the trivial identity

$$\begin{aligned} \int_{\mathbf{k}} I &= \int_{\mathbf{k}} F + \int_{\mathbf{k}} N \\ &+ \int_{\mathbf{k}} \theta(k - \bar{k}) \, (I - N - F) + \int_{\mathbf{k}} \theta(\bar{k} - k) \, (I - N - F) , \end{aligned} \quad (2.128)$$

being θ the usual Heaviside step-function. Show that the second line in eq. (2.128) vanish.

Hint: Since for $k > \bar{k}$ (near zone) one has $k > \omega$ and for $k < \bar{k}$ (far zone) $kr < 1$, the full integral in each region can be written as

$$\begin{aligned} \int_{\mathbf{k}} \theta(k-\bar{k}) I &= \int_{\mathbf{k}} \theta(k-\bar{k}) N , \\ \int_{\mathbf{k}} \theta(\bar{k}-k) I &= \int_{\mathbf{k}} \theta(\bar{k}-k) F . \end{aligned} \quad (2.129)$$

[13]

To reach our conclusion we just need to show that

$$\int_{\mathbf{k}} \theta(k-\bar{k})F + \int_{\mathbf{k}} \theta(\bar{k}-k)N = 0\,, \tag{2.130}$$

which holds because both integrands of F at small k and of N at high k admit the same parametrization

$$\int_{\mathbf{k}} \theta(k-\bar{k})F = \int_{\mathbf{k}} \frac{1}{k^2} \sum_{m,n\geq 0} \frac{(ikr)^n}{n!} \left(\frac{\omega^2}{k^2}\right)^m = \int_{\mathbf{k}} \theta(\bar{k}-k)N\,, \tag{2.131}$$

which vanishes in dimensional regularization.

3
GW Phenomenology

3.1 Interaction of GWs with interferometric detectors

We now describe the interaction of a GW with a simplified exeperimental apparatus, following the discussion of sec. 1.3.3 of [1]. In a typical interferometer light goes back and forth in orthogonal arms and by recombining the photons after different trajectories very precise length measurements can be performed.

[1] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

The dynamics of a massive point particle can be inferred from the world-line action (2.9), whose variation with respect to the particle trajectory x^μ gives the geodesic equation of motion

$$\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}g^{\mu\alpha}\left(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - \frac{1}{2}g_{\nu\rho,\alpha}\right)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\,. \tag{3.1}$$

Let us apply the above equation to the motion of a mirror, at rest at $\tau = 0$ in the position $\mathbf{x}_m = (L, 0, 0)$, in a laser interferometer under the influence of a GW propagating along the z direction:

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = -\,\Gamma^i_{00}\left.\left(\frac{dx^0}{d\tau}\right)^2\right|_{\tau=0}\,. \tag{3.2}$$

In the TT gauge $\Gamma^i_{00} = 0$, showing that an object initially at rest will remain at rest even during the passage of a GW. This does not mean that the GW will have no effect, as the *physical* distance l between the the mirror and the beam splitter, say, at $\mathbf{x}_{bs} = (0, 0, 0)$ is given by

$$l = \int_0^L \sqrt{g_{xx}}dx \simeq \left(1 + \frac{1}{2}h_+(t)\right)L\,,$$

which shows how physical relative distance chage with time (we have assumed that h_+ does not depend on x, which is correct for $\lambda_{GW} \gg L$).

The trajectory $x_0^\mu(\tau) = \delta^{\mu 0}\tau$ is clearly a geodesic, and we can consider the *geodesic deviation* equation, which gives the time evolution separation of two nearby geodesics $x_0^\mu(\tau)$ and $x_0^\mu(\tau) + \xi^\mu(\tau)$

$$\frac{D^2\xi^i}{d\tau^2} = -R^i{}_{0j0}\xi^j \left(\frac{dx^0}{d\tau}\right)^2 . \tag{3.3}$$

The geodesic deviation equation can be recast into

$$\ddot{\xi} = \frac{1}{2}\ddot{h}_{ij}\xi^j , \tag{3.4}$$

where an overdot stands for a derivative with respect to t and terms of order h^2 have been neglected. This means that the effect of GWs on a point particle of mass m placed at coordinate ξ can be described in terms of a Newtonian force F_i

$$F_i = \frac{m}{2}\ddot{h}_{ij}^{(TT)}(t)\xi^j \tag{3.5}$$

(we neglect again the space dependence of h_+ as typically GW wavelength $\lambda \gg L$).

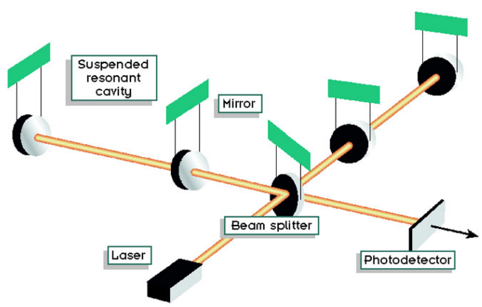

Figure 3.1: Interferometer scheme. Light emitted from the laser is shared by the two orthogonal arms after going through the beam splitter. After bouncing at the end mirrors it is recombined at the photo-detector.

The laser light travels in two orthogonal arms of the interferometer and the electric fields are finally recombined on the photo-detector. The reflection off a 50-50 beam splitter can be modeled by multiplying the amplitude of the incoming field by $1/\sqrt{2}$ for reflection on one side and $-1/\sqrt{2}$ for reflection on the other, while transmission multiplies it by $1/\sqrt{2}$ and reflection by the end mirrors by -1, see sec. 2.4.1 of [2] for details.

Setting the mirrors at positions $(L_x, 0, 0)$ and $(0, L_y, 0)$ and the beam splitter at the origin of the coordinates, we can compute the phase change of the laser electric field moving in the x and y cavity which are eventually recombined on the photo-multiplier. For the light propagating along the x-axis, using the TT metric (2.24) for a $z-$propagating GW, one has

$$\begin{aligned} L_x &= (t_{1x} - t_{0x}) - \frac{1}{2}\int_{t_{0x}}^{t_{1x}} h_+(t')dt' , \\ L_x &= (t_{2x} - t_{1x}) - \frac{1}{2}\int_{t_{1x}}^{t_{2x}} h_+(t')dt' , \end{aligned} \tag{3.6}$$

[2] Andreas Freise and Kenneth A. Strain. Interferometer techniques for gravitational-wave detection. *Living Reviews in Relativity*, 13(1), 2010. DOI: 10.12942/lrr-2010-1. URL http://www.livingreviews.org/lrr-2010-1

where $t_{0,1,2}$ stand respectively for the time when the laser leaves the beam splitter, bounces off the mirror, returns to the beam splitter. The time at which laser from the two arms is combined at the beam splitter is common: $t_{2x} = t_{2y} = t_2$, and using $h_+(t) = h_0 \cos(\omega_{GW} t)$ we get

$$\begin{aligned} t_2 &= t_{0x} + 2L_x + \frac{h_0}{2\omega_{GW}} \left[\sin\left(\omega_{GW}(t_0 + 2L)\right) - \sin\left(\omega_{GW} t_0\right)\right] \\ &= t_{0x} + 2L_x + h_0 L_x \frac{\sin\left(\omega_{GW} L_x\right)}{\omega_{GW} L_x} \cos\left(\omega_{GW}(t_0 + L_x)\right) \end{aligned} \tag{3.7}$$

where the trigonometric identity $\sin(\alpha + 2\beta) = \sin(\alpha) + 2\cos(\alpha + \beta)\sin\beta$ has been used. Using that the phase of the laser field x (y) at recombination time t_2 is the same it had at the time it left the beam splitter at time t_{0x} (t_{0y}), we can write

$$\begin{aligned} E^{(x)}(t_2) &= -\frac{1}{2} E_0 e^{-i\omega_L t_{0x}} = \\ &= -\frac{1}{2} E_0 e^{-i\omega_L(t_2 - 2L) + i\phi_0 + i\Delta\phi_x} \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} L &\equiv \frac{L_x + L_y}{2} \\ \phi_0 &= \omega_L (L_X - L_y) \\ \Delta\phi_x &= h_0 \omega_L L \frac{\sin(\omega_{GW} L)}{\omega_{GW} L} \cos(\omega_{GW}(t_2 - L)) \,. \end{aligned} \tag{3.9}$$

where in the terms $O(h)$ we have set $L_x \simeq L_y \simeq L$. Analogously for the field that traveled through the $y-$ arm

$$\begin{aligned} E^{(y)}(t_2) &= \frac{1}{2} E_0 e^{-i\omega_L t_{0y}} = \\ &= \tfrac{1}{2} E_0 e^{-i\omega_L(t_2 - 2L) - i\phi_0 + i\Delta\phi_y} \end{aligned} \tag{3.10}$$

with $\Delta\phi_y = -\Delta\phi_x$. The fields E_{pd} recombined at the photo-detector gives

$$\begin{aligned} E_{pd}(t) &= E^{(x)}(t) + E^{(y)}(t) \\ &= -iE_0 e^{-i\omega_L(t - 2L)} \sin\left(\phi_0 + \Delta\phi_x\right) \,, \end{aligned} \tag{3.11}$$

with a total power $P = P_0 \sin^2\left(\phi_0 + \Delta\phi_x(t)\right)$. Note that at the other output of the beam splitter, towards the laser, $E_L = E^{(x)} - E^{(y)}$ so that energy is conserved. The optimal length giving the highest $\Delta\phi_x$ is

$$L = \frac{\pi}{2\omega_{GW}} \simeq 750\text{km} \left(\frac{f_{GW}}{100Hz}\right)^{-1} \,. \tag{3.12}$$

Actually real interferometers include Fabry-Perot cavities, where the laser beam goes back and forth several times before being recombined at the beam splitter, allowing the *actual* of the photon travel path to be $\sim$ 100 km. For discussion of real interferometers with Fabry-Perot cavities see e.g. sec. 9.2 of [3], with the result that the sensitivity is enhanced by a factor $2F/\pi$ where F is the finesse of the Fabry-Perot cavity and typcally $F \sim O(100)$, giving a measured phase-shift of the order

[3] M. Maggiore. *Gravitational Waves*. Oxford University Press, 2008

$$\Delta\phi_{FB} \sim \frac{4F}{\pi}\omega_L L h_0 \simeq \frac{8F}{\pi}\omega_L \Delta L \tag{3.13}$$

The typical amplitude h_0 that can be measured is of order 10^{-20} which at the best sensitive frequency gives for the Michelson interferometer $\delta L \sim 10^{-15}$ km!

The typical GW amplitude emitted by a binary system is

$$h \sim G_N M v^2/r \simeq 2.4 \times 10^{-22} \left(\frac{M}{M_\odot}\right) \left(\frac{v}{0.1}\right)^2 \left(\frac{r}{Mpc}\right)^{-1} \tag{3.14}$$

many order of magnitudes small than the earth gravitational field. It is its peculiar time oscillating behaviour that makes possible its detection. We will see however in sec. **??** that rather then the instantaneous aamplitude of the signal, its integrated vallue will be of interest for GW detection.

3.2 *Numerology*

Interferometric detectors are very precise and rapidly responsive ruler, they can detect the change of an arm length down to values of 10^{-15} m, however not at all frequency scales. At very low frequency ($f_{GW} \lesssim 10$ Hz) the noise from seismic activity and generic vibrations degrade the sensitivity of the instrument, whereas at high frequency laser shot noise does not allow to detect signal with frequency larger than few kHz. Considering binary systems, which emit according to the flux given in eq. (1.3), what is the typical lentgth, mass, distance scale of the source? Using

$$v \equiv (\pi f_{GW} G_N M)^{1/3} \, , \tag{3.15}$$

we obtain

$$\begin{aligned} v = & \; (G_N M \pi f_{GW})^{1/3} & \simeq & \; 0.054 \left(\frac{M}{M_\odot}\right)^{1/3} \left(\frac{f_{GW}}{10\text{Hz}}\right)^{1/3} , \\ r = & \; G_N M (G_N M \pi f_{GW})^{-2/3} & \simeq & \; 6.4\text{Km} \left(\frac{M}{M_\odot}\right)^{1/3} \left(\frac{f_{GW}}{10\text{Hz}}\right)^{-2/3} . \end{aligned} \tag{3.16}$$

It can also be interesiting to estimate how long it will take to for a coalescence to take place. Using the lowest order expression for energy and flux, one has

$$\begin{aligned} -\eta M v \frac{dv}{dt} &= -\frac{32}{5G_N}\eta^2 v^{10} \implies \\ \frac{dv}{v^9} &= \frac{32\eta}{5G_N M} dt \implies \\ \frac{1}{v_i^8} - \frac{1}{v_f^8} &= \frac{256\eta}{5G_N M}\Delta t\,, \end{aligned} \tag{3.17}$$

for the time Δt taken for the inspiralling system to move from v_i to v_f. If $v_i \ll v_f$ we can estimate

$$\Delta t \simeq \frac{5G_N M}{256\eta} v_i^{-8} \simeq 1.4 \times 10^4 \text{sec} \frac{1}{\eta} \left(\frac{M}{M_\odot}\right)^{-5/3} \left(\frac{f_{iGW}}{10Hz}\right)^{-8/3} . \tag{3.18}$$

Note that v_f can be comparable to v_i for very massive systems, which enter the detector sensitivity band when $v_i \lesssim 1$. For an estimate of the maximum relative binary velocity during the inspiral, we can take the inner-most stable circular orbit v_{ISCO} of the Schwarzschild case, which gives

$$v_{ISCO} = \frac{1}{\sqrt{6}} \simeq 0.41\,. \tag{3.19}$$

The number of cycles N the GW spends in the detector sensitivity band can be derived by noting that

$$E = -\frac{1}{2}\eta M \left(G_N M \pi f_{GW}\right)^{2/3} \tag{3.20}$$

and from eq. (1.4)

$$\begin{aligned} N(t) &= \int_{t_i}^{t} f_{GW}(t) dt' \implies \\ N(f_{GW}) &\simeq \int_{f_{iGW}}^{f_{GW}} f \frac{dE/df}{dE/dt} df \\ &\simeq \frac{5G_N M}{96\eta} \int_{f_{iGW}}^{f_{GW}} \left(G_N M \pi f\right)^{-8/3} df \\ &= \frac{1}{32\pi\eta} \left(G_N M \pi\right)^{-5/3} \left(\frac{1}{f_{iGW}^{-5/3}} - \frac{1}{f_{GW}^{-5/3}}\right) \\ &\simeq 1.5 \times 10^5 \frac{1}{\eta} \left(\frac{M}{M_\odot}\right)^{-5/3} \left(\frac{f_{iGW}}{10\text{Hz}}\right)^{-5/3} \end{aligned} \tag{3.21}$$

We can finally obtain the time evolution of the GW frequency

$$\dot{f}_{GW} = \frac{96}{5}\pi^{8/3}\eta \left(G_N M\right)^{5/3} f_{GW}^{11/3} \tag{3.22}$$

which has solution

$$\begin{aligned} f_{GW}(t) &= \frac{1}{\eta^{3/8}\pi} \left(\frac{5}{256}\frac{1}{|t|}\right)^{3/8} G_N M^{-5/8} \\ &= 151\text{Hz} \frac{1}{\eta^{3/8}} \left(\frac{M}{M_\odot}\right)^{-5/8} \left(\frac{|t|}{1\text{sec}}\right)^{-3/8} , \end{aligned} \tag{3.23}$$

which can be inverted to give

$$|t_*(f)| = \frac{5}{256\pi\eta}\left(\frac{1}{\pi G_N M\pi}\right)^{5/3} . \tag{3.24}$$

Starting from the GW expression in term of the source quadrupole

$$h_{ij}^{TT}(t,\mathbf{x}) = \frac{2G_N}{d}\Lambda_{ij;kl}(\hat{\mathbf{n}})\ddot{Q}_{kl}(t) \tag{3.25}$$

For $\hat{\mathbf{n}} = \hat{\mathbf{z}}$

$$\Lambda_{ij;kl}M_{kl} = \begin{pmatrix} (Q_{xx}-Q_{yy})/2 & Q_{12} & 0 \\ Q_{12} & (Q_{yy}-Q_{xx})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.26}$$

we have

$$\begin{aligned} h_+ &= G_N\frac{\ddot{Q}_{xx}-\ddot{Q}_{yy}}{d} , \\ h_+ &= \frac{2G_N\ddot{Q}_{xy}}{d} . \end{aligned} \tag{3.27}$$

Assuming the sources are in circular motion, their relative distance can be parametrized as for

$$\begin{aligned} x(t) &= r\sin(\omega_s t) , \\ y(t) &= -r\cos(\omega_s t) , \\ z(t) &= 0 , \end{aligned} \tag{3.28}$$

hence yielding to

$$\begin{aligned} \ddot{Q}_{xx} &= -\ddot{M}_{yy} = 2\mu r^2\omega_S^2\cos(2\omega_s t) \\ \ddot{Q}_{xy} &= 2\mu r^2\omega_S^2\sin(2\omega_s t) \end{aligned} \tag{3.29}$$

When the orbital planes is inclined by an angle ι with respect to the propagation direction (conventionally kept along the z-axis) one has to compute the rotated projected quadrupole tensor according to

$$Q'_{ij} = R^{(y)}(\iota)_{ii'}Q_{i'j'}\left(R^{(y)}\right)^{-1}_{j'j}(\iota) \tag{3.30}$$

and then project it with Λ to obtain

$$\Lambda_{ij,kl}Q'_{kl} = \begin{pmatrix} 1/2\cos^2\iota(Q_{xx}-Q_{yy}) & \cos\iota Q_{xy} & 0 \\ \cos\iota Q_{xy} & 1/2\cos^2\iota(Q_{yy}-Q_{xx}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.31}$$

Now using the explicit expressions eqs. (**??**) one finds

$$\begin{aligned} h_+ &= \frac{1}{d}4G_N\mu\omega_S^2R^2\left(\frac{1+\cos^2\iota}{2}\right)\cos(2\omega_s t) , \\ h_\times &= \frac{1}{d}4G_N\mu\omega_S^2R^2\cos\iota\sin(2\omega_s t) . \end{aligned} \tag{3.32}$$

Note that the rotation (??) is not the most generic rotation setting the orbital plane at an angle ι with the view direction $(0,0,1)$: an additional rotation by ψ around the z axis is permitted.

Using $f_{GW} = \omega_s/\pi$ and the explicit expression $f_{GW}(t)$ eq. (??) one has

$$\begin{aligned} h_+(t) &= \frac{1}{d}(G_N M_c)^{5/4}\left(\frac{5}{\tau}\right)^{1/4}\left(\frac{1+\cos^2\iota}{2}\right)\cos\Phi(\tau) \\ h_\times(t) &= \frac{1}{d}(G_N M_c)^{5/4}\left(\frac{5}{\tau}\right)^{1/4}(\cos\iota)\sin\Phi(\tau) \end{aligned} \tag{3.33}$$

For data analysis we actually need this expression in frequency space, so let us compute it for the + polarization

$$\tilde{h}_+(f) = \frac{1}{2}\int dt A(t)\left(e^{i(2\pi f t+\Phi(\tau(t)))} + e^{i(2\pi f t-\Phi(\tau(t)))}\right), \tag{3.34}$$

where $A(t)$ is defined by comparison with the (??). The above integral can be computed in the *stationary phase approximation* by expanding the exponent in the integrand around the stationary point t_* characterized by

$$2\pi f - \dot{\Phi}(\tau(t_*)) = 2\pi f + \left.\frac{d\Phi(\tau)}{d\tau}\right|_{\tau=t_c-t_*} = 0 \tag{3.35}$$

as follows:

$$e^{2\pi i f t - i\Phi(t)} \to e^{2\pi i f t_* - i\Phi(t_*)}\exp\left(-i\ddot{\Phi}(t_*)\frac{(t-t_*)^2}{2}\right) \tag{3.36}$$

then performing the resulting Gaussian integral as

$$\begin{aligned} \tilde{h}_+(f) &= \frac{1}{2}A(t_*)e^{i(2\pi f t_*-\Phi(t_*))}\int dt\, e^{-i\ddot{\Phi}(t_*)(t-t_*)^2/2} \\ &= \frac{1}{2}A(t_*)e^{i(2\pi f t_*-\Phi(t_*)-\pi/4)}\left(\frac{2\pi}{\ddot{\Phi}(t_*)}\right)^{1/2}. \end{aligned} \tag{3.37}$$

Time t_* can be expressed in terms of f_{GW} by inverting (??) and eq. (??) enable the identification between the Fourier transform argument f and f_{GW}, thus allowing to write

$$\tilde{h}_+(t_*(f)) = \left(\frac{5}{24}\right)^{1/2}\frac{\pi^{-2/3}}{d}(G_N M_c)^{5/6} f^{-7/6}\left(\frac{1+\cos^2\theta}{2}\right)e^{i(2\pi i t_* f-\Phi(t_*)-\pi/4)} \tag{3.38}$$

It is useful to introduce

$$v \equiv (\pi G_N M f_{GW})^{1/3} = (G_N M\omega)^{1/3}. \tag{3.39}$$

The most commonly used approximant is defined in the frequency domain as *TaylorF2*:

$$\begin{aligned}
\psi(f) &= 2\pi f t_* + \phi_{ref} - 2\int^f \omega \frac{dt}{df} df \\
&= \phi_{ref} + \frac{2}{G_N M}\int^f (v_f^3 - v^3(f'))\frac{v(f')}{f'}\frac{dE/dv}{dE/dt}\frac{f'}{v(f')}\frac{dv}{df'}df' \\
&= 2\pi \int^f (v_f^3 v^{-2} - v)\left[\frac{-\eta M v}{-32/(G_N 5)\eta^2 v^{10}}\right]\frac{1}{3}df' \\
&= \frac{5\pi G_N M}{48\eta}\int^f (v_f^3 v^{-11} - v^{-8})\left(1 + c_{1PN}v^2 + \dots\right)df' \\
&= \frac{5}{48\eta}(\pi G_N M f)^{-8/3}\left[\left(-\frac{3}{8}+\frac{3}{5}\right) + \left(-\frac{1}{2}+1\right)c_{1PN}\dots\right] \\
&= \frac{3}{128\eta v^5(f)}\left(1 + \frac{20}{9}c_{1PN}v^2(f) + \dots\right)
\end{aligned} \tag{3.40}$$

It may also be useful to have an expression of the phase in time-domain. Let us now relate the phase of the waveform to the dynamics of the sources by defining

$$\begin{aligned}
\Delta\phi(t) &= 2\pi\int_{t_0}^t f_{GW}(t')dt' = 2\int_{v(t_0)}^{v(t)} \omega(v)\frac{dE/dv}{dE/dt}dv \\
&= \frac{2}{G_N M}\int_{v(t_0)}^{v(t)} v^3 \frac{dE/dv}{dE/dt}dv \\
&= \frac{5}{16\eta}\int_{v(t_0)}^{v(t)} \frac{1}{v^6}\frac{1 + e_{v^2}v^2 + \dots}{1 + f_{v^3}v^3 + f_{v^4}v^4 + \dots}dv\,,
\end{aligned} \tag{3.41}$$

where we have inserted the formal Taylor expansion of the energy and flux as functions of v and we have substituted $v = (G_N M\omega)^{1/3}$ (that is valid for circular orbits). We now see that we have different possibilities to compute the phase

- Truncate the v-series at some order both in the numerator and the denominator gives rise to the *TaylorT1* expression of the phase
- Expand the fraction in the above formula and truncate at some finite v-order $\to$ *TaylorT4*
- expand the inverse of the fraction appearing in the integrand $\to$ *TaylorT1*.

3.3 *Elements of data analysis*

The output of the detector $o(t)$ is a scalar time domain function, which in general will result of the addition of a part $h(t)$ linear in the impinging GW $h_{ij}(t, \mathbf{x}_D)$ at the location detector $\mathbf{x}_D$ and the instrumental noise $n(t)$. Usually the output of the detector is linearly related to the GW amplitude locally in the frequency space, i.e.

$$\tilde{h}(f) = \tilde{T}_{ij}(f)\tilde{h}_{ij}(f) \tag{3.42}$$

where $T_{ij}(f)$ is the *transfer function* of the system and

$$\tilde{o}(f) = \tilde{h}(f) + \tilde{n}(f)\,. \tag{3.43}$$

If the noise is *stationary* then different Fourier components are uncorrelated (see derivation below) and we can write

$$\langle \tilde{n}^*(f)\tilde{n}(f')\rangle = \delta(f-f')\frac{1}{2}S_n(f)\,, \tag{3.44}$$

which defines the *noise correlation function* $S_n(f)$, or *noise power spectral density*, with dimensions Hz^{-1}. Note also that $S_n(-f) = S_n(f)$.

The average in eq. (**??**) is taken over many noise realizations, but we have only one detectors, so it should be actually replaced over different time span averages:

$$\langle \tilde{n}^*(f)\tilde{n}(f')\rangle = \frac{1}{N}\sum_{i=1}^{N}\tilde{n}_i(f)\tilde{n}_i^*(f')\,,$$

where

$$\tilde{n}_i(f) = \int_{t_i-T/2}^{t_i+T/2} n(t)e^{2\pi i f t}dt\,,$$

and $t_i = (\ldots, -2T, -T, 0, T, 2T, \ldots)$. We define the Fourier Transform of a function defined on an interval as

$$\tilde{n}(f) = \int_{t_i-T/2}^{t_i+T/2} n(t)e^{2\pi i f t}\,dt \qquad \text{with } n(t+T) = n(t) \tag{3.45}$$

with the periodicity requirement implying the Fourier transform $\tilde{n}(f)$ be discrete, with support at $f_n = n/T$, i.e. with frequency resolution $\Delta f = 1/T$. This discreteness condition on the frequency automatically ensure that the inverse Fourier transform returns a periodic function[4]:

$$n(t) = \frac{1}{T}\sum_{n\in\mathbb{N}}\tilde{n}(f)e^{-2\pi i n t/T}\,. \tag{3.47}$$

A very welcome by-product of the definition (**??**), restricting the integration domain to a finite interval and imposing periodicity condition on $n(t)$, is

$$\frac{1}{T}\int_{t_i-T/2}^{t_i+T/2} e^{2i\pi f t}dt = \frac{e^{2i\pi f t_i}}{2\pi i f T}\left(e^{2\pi i f T/2} - e^{-2\pi i f T/2}\right) = e^{2\pi i f t_i}\frac{\sin(\pi f T)}{\pi f T} = \delta_{f,0} \tag{3.48}$$

which is reminiscent of the standard $\int e^{2i\pi f t}dt = \delta(f)$ valid for functions defined on the entire real axis. The only difference between the finite segment and the entire real axis case is that here we deal with a dimension-less, discrete Kronecker delta rather than a delta function with dimension of time, so that the identification

$$\delta(f) \to T\delta_{f,0} \tag{3.49}$$

[4] Note that the definition (**??**) for a function defined on an interval is *not* equivalent to

$$\tilde{n}(f) \neq \int_{-\infty}^{\infty} n(t)e^{2\pi i f t}\theta(t - t_i + T/2)\theta(t_i + T/2 - t)dt\,.$$

Had we used this definition one would have dealt with a non-analytic integrand in the time integral, that would have had discontinuities in his derivatives. As the discontinuities in the direct space functions are related to "bad" behaviour at infinity via

$$\frac{d^m n(t)}{dt^m} = \int df(2\pi i f)^m\tilde{n}(f)e^{2\pi i f t} \tag{3.46}$$

one would expect in general a non-physical high power in $\tilde{n}(f)$ for large $|f|$, or that $f^m\tilde{n}(f)$ must not be integrable, that is $\lim_{f\to\infty}|f^{m+1}n(f)|^2 \neq 0$. It is a general rule that the the large f behaviour of $\tilde{n}(f)$ depends on the continuity property of $n(t)$.

can be made.

Let us consider than the definition (**??**) and exploit the stationarity property of the noise:

$$\begin{aligned} \langle \tilde{n}(f)\tilde{n}(f')\rangle &= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \langle n(t)n(t')\rangle e^{2\pi i(ft+f't')} \\ &= \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \langle n(t+T_s)n(t'+T_s)\rangle e^{2\pi i(ft+f't'+T_s(f+f'))} \\ &= \delta_{f+f',0} \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \langle n(t)n(t')\rangle e^{2\pi i f(t-t')} \end{aligned} \quad (3.50)$$

where we have averaged over a time coordinate T_s and used stationarity to substitute $n(t+T_s) \to n(t)$ inside the average. Short-circuiting the above with the definition (**??**) and the correspondence (**??**) we obtain

$$S(f) = 2\frac{\langle|\tilde{n}(f)|^2\rangle}{T} = 2\langle|\tilde{n}(f)|^2\rangle\Delta f\,. \quad (3.51)$$

We can make the further connection to the noise auto-correlation function

$$R(\tau) \equiv \langle n(t+\tau)n(t)\rangle\,, \quad (3.52)$$

white noise corresponding to $R(\tau) \propto \delta(\tau)$. By noting that

$$\begin{aligned} &\langle \int df' \int df\, n(f)n(f')e^{-2\pi i(f(t+\tau)+f't)} \rangle \\ = &\ \frac{1}{2}\int df S_n(f)e^{-2\pi i f\tau} \end{aligned} \quad (3.53)$$

we are enabled to interpret the noise spectral density function as the Fourier transform of the noise correlation function

$$\frac{1}{2}S_n(f) = \int d\tau R(\tau)e^{2\pi i f\tau} \quad (3.54)$$

and hence

$$R(\tau) = \frac{1}{T}\sum_{n\in\mathbb{N}} e^{-2\pi i f\tau}\frac{S_n(f)}{2}\,. \quad (3.55)$$

The factor $1/2$ in the definition is conventionally inserted as

$$\begin{aligned} \langle n^2(t)\rangle &= \frac{1}{T^2}\sum_n\sum_{n'}\langle n(f)n(f')\rangle e^{2\pi i(ft+f't')} \\ &= \frac{1}{2T}\sum_{n\in\mathbb{N}} S_n(f) = \frac{1}{T}\sum_{n\geq 0} S_n(f) \end{aligned} \quad (3.56)$$

and the factor $1/2$ disappears when sum is taken over positive frequencies only (we neglect subtleties about the $n = 0$ mode). The power spectral density of white noise is f-independent.

Actually in practice the time domain noise function will be discrete as well, implying that what we'll be really using are

$$\begin{aligned} \tilde{n}(f = k/T) &= \Delta t \sum_{j=0}^{N-1} e^{2\pi ijk\Delta t/T} \\ n(t = j\Delta t) &= \Delta f \sum_{j=0}^{N-1} e^{2\pi ijk\Delta t/T} \end{aligned} \tag{3.57}$$

with $\Delta f \Delta t = 1/N$ with $T/\Delta t = N$. With a finite sampling size there has to be a maximum frequency, called the *Nyquist frequency*

$$f_{Nyquist} = \frac{1}{2\Delta t} \tag{3.58}$$

so that we have N points for $n(t)$ and $N/2 + 1$ for $\tilde{n}(f)$ if N is even, as we will assume. Note that $\tilde{n}(f = 0)$ is real as well as $n(f = N/(2T))$ so that the information stored in the N real numbers of $n(t)$ is fully equivalent to the information stored in the $N/2 + 1$ numbers making up $\tilde{n}(f)$, 2 of which are real and $N/2 - 1$ of which are complex.

In particular the Parseval identity has a discrete counterpart

$$\Delta f \sum_{k=0}^{N/2} |\tilde{n}(k/T)|^2 = \Delta t \sum_{j=0}^{N-1} n^2\left(j\Delta t\right) . \tag{3.59}$$

Matched filtering

The signal amplitude is much smaller than the noise, just think of the earth gravitational field that is responsible for $h \sim 10^{-9} \gg 10^{-21}$. However if the signal is known in advance, we can correlate detector's output $o(t)$ with our expectation and dig it out of the noise floor. We thus have to *filter* the detector output to highlight the signal. An important quantity for any experiment is the *signal-to-noise ratio (SNR)* we are going to define now. It must involve a ratio S/N of a quantity linear in the signal h possibly filtered to enhance it and a quantity representative of the noise. We want to choose the filter function so to maximize the SNR, i.e. the filter has to *match* the signal. We can assume that by linearly filtering the detector output we can pick only the signal part $h(t)$ in $o(t)$ and we can tentatively define the numerator of the SNR as

$$S = \int dt \langle o(t)\rangle K(t)\,, \tag{3.60}$$

and since $\langle n(t)\rangle = 0$ we have

$$S = \int dt \langle h(t)\rangle K(t) = \int df \tilde{h}(f)\tilde{K}^*(f)\,, \tag{3.61}$$

where for simplicity we have moved back to continuum time-frequency space. For the SNR denominator N we want an estimator of the noise. A reasonable guess would be the root mean square of the detector output in absence of the signal, i.e.

$$N^2 \stackrel{?}{=} \langle o^2(t)\rangle - \langle o(t)\rangle^2|_{h=0} = \langle n(t)\rangle^2 \tag{3.62}$$

but we want the overall filter scale to drop out of the SNR, so we'd better define

$$\begin{aligned} N^2 &= \int dt\, dt'\, K(t)K(t')\langle n(t)n(t')\rangle \\ &= \int df\, df' \langle n(f)n(f')\rangle e^{2\pi i(ft+ft')}\tilde{K}(f)\tilde{K}(f') \\ &= \frac{1}{2}\int df S_n(f)|\tilde{K}(f)|^2 . \end{aligned} \tag{3.63}$$

We have constructed then our SNR as

$$\frac{S}{N} = \frac{2^{1/2}\int_{-\infty}^{\infty} df \tilde{h}(f)\tilde{K}^*(f)}{\left(\int_{-\infty}^{\infty} df S_n(f)|\tilde{K}(f)|^2\right)^{1/2}} . \tag{3.64}$$

To find the filter function maximizing the SNR we define a positive definite scalar product

$$(A|B) \equiv 2\int df \frac{A(f)B^*(f)}{S_n(f)} \tag{3.65}$$

which is real if we assume that $A^*(f) = A(-f)$ and $B^*(f) = B(-f)$, as it is for the Fourier transform of real functions. We can now re-write the SNR as

$$\frac{S}{N} = \frac{(\tilde{u}|\tilde{h})}{(\tilde{u}|\tilde{u})^{1/2}} \tag{3.66}$$

with $\tilde{u}(f) \equiv 1/2 S_n(f)\tilde{K}(f)$. We are thus searching for the normalized vector u that maximizes its scalar product with h, clearly the solution can only be $\tilde{u} \propto \tilde{h}$, i.e.

$$\tilde{K}(f) = \frac{\tilde{h}(f)}{S_n(f)} \tag{3.67}$$

up to an inessential f-independent constant. Substituting in eq. (**??**) we finally obtain

$$\frac{S}{N} = \left[2\int_{-\infty}^{\infty} df \frac{|\tilde{h}(f)|^2}{S_n(f)}\right]^{1/2} . \tag{3.68}$$

So far we have assumed perfect knowledge of the signal. What if we do not know the exact *time location* of the $h(t)$? By considering a function $h(t)$ and its time-shifted $h_{t_0}(t) \equiv h(t - t_0)$ the relationship among their Fourier transform is

$$\tilde{h}(f) = \tilde{h}_{t_0}(f)\, e^{2\pi i f t_0} . \tag{3.69}$$

If we try to match data $\tilde{h}(f)$ with a *template* signal $\tilde{h}_{tmplt}(f)$ allowing for a generic time-shift t, we obtain an SNR time series

$$\begin{aligned} SNR(t) &= \sqrt{2}\frac{\int_{-\infty}^{\infty} df\, \frac{\tilde{h}(f)\tilde{h}^*_{tmplt}(f)}{S_n(f)} e^{2\pi i f t}}{\left(\int_{-\infty}^{\infty} df\, |\tilde{h}_{tmplt}(f)|^2/S_n(f)\right)^{1/2}} = \\ &= \frac{\int_0^{\infty} df\, \left(\tilde{h}(f)\tilde{h}^*_{tmplt}(f)e^{2\pi i f t} + \tilde{h}^*(f)\tilde{h}_{tmplt}(f)e^{-2\pi i f t}\right)/S_n(f)}{\left(\int_0^{\infty} df\, |\tilde{h}_{tmplt}(f)|^2/S_n(f)\right)^{1/2}} \end{aligned} \quad (3.70)$$

and the use of a Fast Fourier Transform allows to search for all time values efficiently.

Let us consider now the case of a constant phase offset between the template and the signal and let us concentrate on the simplified situation of a signal $h(t) \propto \cos(2\pi f_0 t)$, i.e. $\tilde{h}(f) \propto \delta(f - f_0) + \delta(f + f_0)$.

Taking the correlator with a filter function of the type $K(t) \propto \cos(2\pi f_1 t + \bar{\phi})$ we would have a non-null result in case $f_0 = f_1$ proportional to $\cos(\phi)$. Clearly the filter *is* matching the signal but our ignorance on the right ϕ value may suppress the matched filter output. What one would like is the SNR time series to be the maximum as $\bar{\phi}$ varies. It turns out that it is in fact possible to maximize over $\bar{\phi}$ analytically. Let us fix $t = 0$ for simplicity:

$$\begin{aligned} \left(\frac{1}{2}\int_{-\infty}^{\infty} df |h_t(f)|^2/S_n(f)\right)^{1/2} \frac{S}{N}(t=0) &= \int_{-\infty}^{\infty} \tilde{h}(f)\tilde{h}_t^*(f)e^{i\bar{\phi}(f)} df \\ = \int_0^{\infty} \left(\tilde{h}(f)\tilde{h}_t^*(f)e^{i\bar{\phi}} + \tilde{h}^*(f)\tilde{h}_t(f)e^{-i\bar{\phi}}\right) df &= \mathcal{R}\cos\bar{\phi} - \mathcal{I}\sin\bar{\phi}\,, \end{aligned} \quad (3.71)$$

where the real quantities $\mathcal{R}$ and $\mathcal{I}$ are defined as

$$\begin{aligned} \mathcal{R} &\equiv \int_0^{\infty} \left(\tilde{h}(f)\tilde{h}_t^*(f) + \tilde{h}^*(f)\tilde{h}_t^*(f)\right) df &= 2Re\int_0^{\infty} \tilde{h}(f)\tilde{h}_t(f) df \\ \mathcal{I} &\equiv i\int_0^{\infty} \left(h(f)h_t^*(f) - h^*(f)h_t(f)\right) df &= -2Im\int_0^{\infty} \tilde{h}(f)\tilde{h}_t^*(f) df\,. \end{aligned} \quad (3.72)$$

The output of the matched filter depends on $\bar{\phi}$, but analytically maximizing over $\bar{\phi}$ is possible:

$$\frac{dS/N}{d\phi_0} = 0 \implies \cos\bar{\phi} = \frac{\mathcal{R}}{\sqrt{\mathcal{R}^2 + \mathcal{I}^2}}\,, \qquad \sin\bar{\phi} = -\frac{\mathcal{I}}{\sqrt{\mathcal{R}^2 + \mathcal{I}^2}}\,, \quad (3.73)$$

which give the SNR maximized over $\bar{\phi}$

$$\left(\stackrel{Max}{\phi_0} \left.\frac{S}{N}\right|_{t=0}\right)^2 = 2\frac{\mathcal{R}^2 + \mathcal{I}^2}{\int_{-\infty}^{\infty} |h_t(f)|^2/S_n(f) df}\,. \quad (3.74)$$

Note that there is an efficient way to compute both the quantities $\mathcal{R}$ and $\mathcal{I}$: it is by computing the *complex inverse Fourier transform*

$$\rho(t) \equiv \sqrt{2}(\tilde{h}_t|\tilde{h}_t)^{-1/2}\int_0^{\infty} \frac{\tilde{h}(f)\tilde{h}_t^*(f)}{S_n(f)} e^{2\pi i f t} df \quad (3.75)$$

and we have

$$\overset{Max}{\phi_0} \frac{S}{N}(t) = 2\sqrt{2}|\rho(t)|\,. \tag{3.76}$$

Templates differing by a phase shift like $\tilde{h}_{tmplt}(f) \to \tilde{h}'_{tmplt}(f) = \tilde{h}_{tmplt}(f)e^{i\phi_0}$ (for $f > 0$ and $\tilde{h}'_{tmplt}(f) = \tilde{h}_{tmplt}(f)e^{-i\phi_0}$ for $f < 0$) will give rise to different SNRs, but the modulus of ρ obtained with them will be the same.

3.4 *Cosmology*

In standard cosmology the cosmic expansion history at background level can be studied by relating the luminosity distance to the redshift.

Given a background Friedmann-Lemaître-Robertson-Walker metric

$$ds^2 = a^2(\eta)\left(-d\eta^2 + dx^{i^2}\right) = -dt^2 + a^2(t)dx^{i^2}\,, \tag{3.77}$$

where $a(\eta)$ is the cosmological scale factor. In standard cosmology $a(t)$ is a monotonically growing function, hence one can trade time for the redshift variable $1 + z \equiv 1/a(t)$, with $a(t_0)$ conventionally set to 1. Using the 00 Einstein equation for the unperturbed FRW metric (**??**)

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G_N \rho\,, \tag{3.78}$$

where ρ is the energy density in the univese, and short-circuiting with the continuity equation for matter:

$$\dot{\rho} = 3H(\rho + p)\,, \tag{3.79}$$

where p is the pressure, one determines the background cosmological model.

Light propagation is trivial in the η, x plane, hence it is straighforward to define a comoving coordinate distance d_c between the origin and a point with coordinate x as

$$d_c = x = \int_{t_e}^{t_o} \frac{\mathrm{d}t}{a(t)} = \int_{a_e}^{a_o} \frac{\mathrm{d}a}{a^2}\frac{a}{\dot{a}} = \int_0^{z_e} \frac{\mathrm{d}z}{H(z)}\,. \tag{3.80}$$

As an operational way to define distances, it is common to use the luminosity distance d_L defined as

$$d_L \equiv \left(\frac{\dot{E}}{Flux}\right)^{1/2}\,, \tag{3.81}$$

which can be measured in terms of the measured flux if the intrinsic luminosity $\dot{E}$ of the source is known. Given the redshift of the source of radiation and the cosmological background dynamics which is fixed by the matter content of the universe, the luminosity distance can be expressed as

$$d_L = \frac{1+z}{H_0}\int_0^z \frac{dz'}{\left[\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}(1+z)^{3(1+\omega_\Lambda)}\right]^{1/2}}\,, \tag{3.82}$$

where we have assumed two, non interacting components: dark matter with negligible pressure, and dark energy with $\omega_\Lambda \equiv \frac{p_\Lambda}{\rho_\Lambda} \simeq -1$. The quantity Ω_i is a convenient way to parametrize the present energy density of the i-th species:

$$\Omega_i \equiv \frac{8\pi G\rho_i}{3H_0^2}\,, \tag{3.83}$$

with $\sum_i \Omega_i = 1$. At lowest order one has a linear relationship between luminosity distance and redshift, as $d_L H_0 = z + O(z^2)$, and a full determination of d_L at all redshifts would fix the background cosmological evolution of the universe.

With GWs it is possible to measure the luminosity distance as

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} = \begin{pmatrix} \frac{1+\cos^2\iota}{2} \\ \cos\iota \end{pmatrix} \frac{\eta M_c v^2}{d_c} \begin{pmatrix} \cos\phi(t_s/M_c, \eta, S_i^2/m_i^4, S_2 S_2/m_1^2 m_2^2) \\ \cos\phi(t_s/M_c, \eta, S_i^2/m_i^4 \ldots) \end{pmatrix} \tag{3.84}$$

When expressing the phase in terms of the observer time the waveform fit will return the *redshifted* chirp mass $\mathcal{M}_c(z) \equiv M_c(1+z)$ which will appear in the amplitude in the combination $\mathcal{M}_c/d_L$, enabling the measure of d_L if ι is known.

In fig. **??** we report the present measure of H_0 with GW170817 done in [5] with the only GW multimessenger observation so far, and also with the possibility that sky localizations from GW detection without EM counterparts may point to a number of galaxies with known redshift [6].

[5] R. Abbott et al. Constraints on the Cosmic Expansion History from GWTC–3. *Astrophys. J.*, 949(2):76, 2023. DOI: 10.3847/1538-4357/ac74bb

[6] Bernard F. Schutz. Determining the Hubble Constant from Gravitational Wave Observations. *Nature*, 323:310–311, 1986. DOI: 10.1038/323310a0

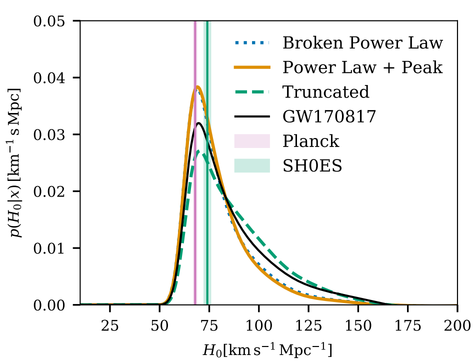

Figure 3.2: H_0 determination from one EM bright standard candle and 46 dark ones, short-circuiting with galaxy survey catalog GLADE+.

In fig. **??** we report a forecast of the accuracy of the measure of H_0 as a function of observation of standard sirens [7].

[7] Rachel Gray et al. Cosmological inference using gravitational wave standard sirens: A mock data analysis. *Phys. Rev. D*, 101(12):122001, 2020. DOI: 10.1103/PhysRevD.101.122001

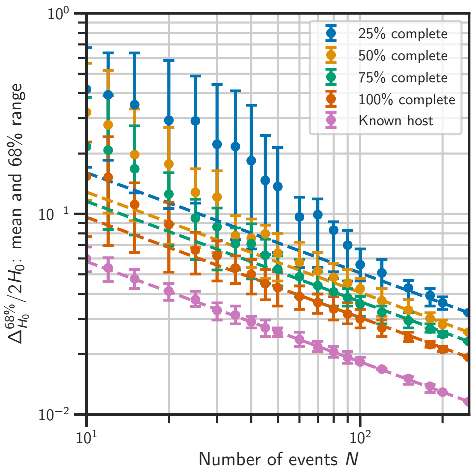

Figure 3.3: Forecast for H_0 determination as a function of detections of dark and bright standard sirens, with galaxy catalogs partially complete.

Problems

** Exercise 35 Newtonian force exerced by GWs

Derive eq. (**??**) from eq. (**??**)

**** Exercise 36 Energy realeased by GWs

Derive the work done on the experimental apparatus by the GW Newtonian force of eq. (**??**).

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