Introduction to AdS-CFT

lectures by

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Abstract

These lectures present an introduction to AdS-CFT, and is intended both for begining and more advanced graduate students, which are familiar with quantum field theory and have a working knowledge of their basic methods. Familiarity with supersymmetry, general relativity and string theory is helpful, but not necessary, as the course intends to be as self-contained as possible. I will introduce the needed elements of field and gauge theory, general relativity, supersymmetry, supergravity, strings and conformal field theory. Then I describe the basic AdS-CFT scenario, of $\mathcal{N} = 4$ Super Yang-Mills's relation to string theory in $AdS_5 \times S_5$, and applications that can be derived from it: 3-point functions, quark-antiquark potential, finite temperature and scattering processes, the pp wave correspondence and spin chains.

1 Elements of quantum field theory and gauge theory

Here I will review some elements of quantum field theory and gauge theory that will be needed in the following.

The Fenyman path integral and Feynman diagrams

Conventions: throughout this course, I will use theorist's conventions, where $\hbar = c = 1$. To reintroduce $\hbar$ and c one can use dimensional analysis. In this conventions, there is only one dimensionful unit, $mass = 1/length = energy = 1/time = ...$ and when I speak of dimension of a quantity I refer to mass dimension, i.e. the mass dimension of d^4x, $[d^4x]$, is -4. The Minkowski metric $\eta^{\mu\nu}$ will have signature $(-+++)$, thus $\eta^{\mu\nu} = diag(-1,+1,+1,+1)$.

I will use the example of the scalar field $\phi(x)$, that transforms as $\phi'(x') = \phi(x)$ under a coordinate transformation $x_\mu \to x'_\mu$. The action of such a field is of the type

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian density.

Classically, one varies this action with respect to $\phi(x)$ to give the classical equations of motion for $\phi(x)$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \tag{1.2}$$

Quantum mechanically, the field $\phi(x)$ is not observable anymore, and instead one must use the vacuum expectation value (VEV) of the scalar field quantum operator instead, which is given as a "path integral"

$$< 0|\hat{\phi}(x_1)|0 >= \int \mathcal{D}\phi e^{iS[\phi]} \phi(x_1) \tag{1.3}$$

Here the symbol $\mathcal{D}\phi$ represents a discretization of spacetime followed by integration of the field at each discrete point:

$$\mathcal{D}\phi(x) = \prod_i \int d\phi(x_i) \tag{1.4}$$

A generalization of this object is the correlation function or n-point function

$$G_n(x_1, ..., x_n) =< 0|T\{\hat{\phi}(x_1)...\hat{\phi}(x_n)\}|0 > \tag{1.5}$$

The generating function of the correlation functions is called the partition function,

$$Z[J] = \int \mathcal{D}\phi e^{iS[\phi] + i\int d^4x J(x)\phi(x)} \tag{1.6}$$

It turns out to be convenient to write quantum field theory in Euclidean signature, and go between the Minkowski signature $(-+++)$ and the Euclidean signature $(++++)$ via a Wick rotation, $t = -it_E$ and $iS \to -S_E$, where t_E is Euclidean time (with positive metric) and S_E is the Euclidean action.

The partition function in Euclidean space is

$$Z_E[J] = \int \mathcal{D}\phi e^{-S_E[\phi]+\int d^4x J(x)\phi(x)} \tag{1.7}$$

and the correlation functions

$$G_n(x_1, ..., x_n) = \int \mathcal{D}\phi e^{-S_E[\phi]}\phi(x_1)...\phi(x_n) \tag{1.8}$$

are given by differentiation of the partition function

$$G_n(x_1, ..., x_n) = \frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)} \int \mathcal{D}\phi e^{-S_E[\phi]+\int d^4x J(x)\phi(x)}|_{J=0} \tag{1.9}$$

This formula can be calculated in perturbation theory, using the so called "Feynman diagrams". To exemplify it, we will use a scalar field Euclidean action

$$S_E[\phi] = \int d^4x[\frac{1}{2}(\partial_\mu\phi)^2 + m^2\phi^2 + V(\phi)] \tag{1.10}$$

Here I have used the notation

$$(\partial_\mu\phi)^2 = \partial_\mu\phi\partial^\mu\phi = \partial_\mu\phi\partial_\nu\phi\eta^{\mu\nu} = -\dot{\phi}^2 + (\vec{\nabla}\phi)^2 \tag{1.11}$$

Moreover, for concreteness, I will use $V = \lambda\phi^4$.

Then, the **Feynman diagram in x space** is obtained as follows. One draws a diagram, in the example in Fig.1a) it is the so-called "setting Sun" diagram.

A line between point x and point y represent the propagator

$$\Delta(x, y) = [-\partial_\mu\partial^\mu + m^2]^{-1} = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2 + m^2} = \frac{1}{(x-y)^2} \tag{1.12}$$

A 4-vertex at point x represents the vertex

$$\int d^4x(-\lambda) \tag{1.13}$$

And then the value of the Feynman diagram, $F_D^{(N)}(x_1, ...x_n)$ is obtained by multiplying all the above elements, and the value of the n-point function is obtained by summing over diagrams, and over the number of 4-vertices N with a weight factor:

$$G_n(x_1, ..., x_n) = \sum_{N\geq 0}\frac{1}{N!}\sum_{diag\ \ D} F_D^{(N)}(x_1, ..., x_n) \tag{1.14}$$

(Equivalently, one can use a $\lambda\phi^4/4!$ potential and construct only *topologically inequivalent* diagrams and the vertices are still $\int d^4x(-\lambda)$, but we now multiply each inequivalent diagram by a statistical weight factor).

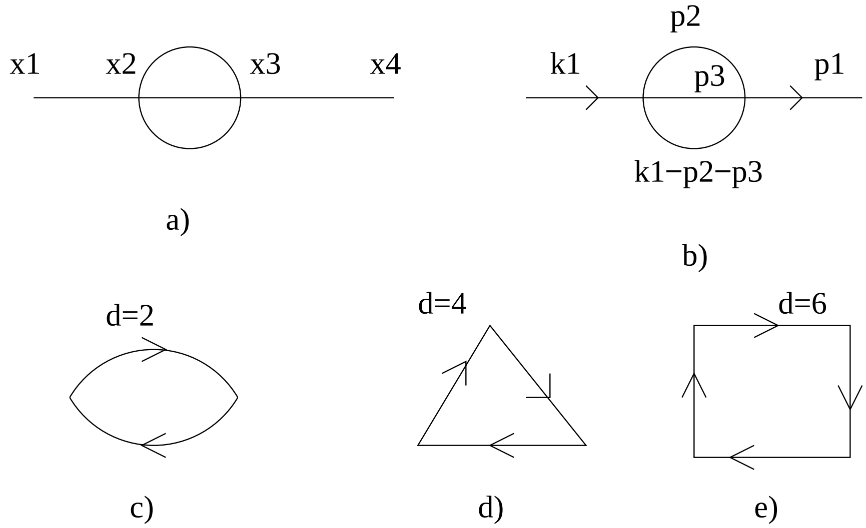

Figure 1: a) "Setting sun" diagram in x-space. b) "Setting sun" diagram in momentum space. c)anomalous diagram in 2 dimensions; d)anomalous diagram (triangle) in 4 dimensions; e)anomalous diagram (box) in 6 dimensions.

We mentioned that the VEV of the scalar field operator is an obsevable. In fact, the normalized VEV in the presence of a source $J(x)$,

$$\phi(x;J) = \frac{{}_J<0|\hat{\phi}(x)|0>_J}{{}_J<0|0>_J} = \frac{1}{Z[J]}\int \mathcal{D}\phi e^{-S[\phi]+J\cdot\phi}\phi(x) = \frac{\delta}{\delta J}\ln Z[J] \tag{1.15}$$

is called the classical field and satisfies an analog (quantum version) of the classical field equation.

S matrices

For real scattering, one constructs incoming and outgoing wavefunctions, representing actual states, in terms of the idealized states of fixed (external) momenta $\vec{k}$.

Then one treats the scattering of these idealized states and at the end one convolutes with the wavefunctions. The S matrix defines the transition amplitude between the idealized states by

$$<\vec{p}_1,\vec{p}_2,...|S|\vec{k}_1,\vec{k}_2,...> \tag{1.16}$$

The value of this S matrix transition amplitude is given in terms of Feynman diagrams in momentum space. The diagrams are of a restricted type: connected (doesn't contain disconnected pieces) and amputated (which means that one does not use propagators for the external lines).

For instance, the setting sun diagram with external momenta k_1 and p_1 and internal momenta p_2, p_3 and $k_1 - p_2 - p_3$ in Fig.1b) is

$$\delta^4(k_1-p_1)\int d^4p_2d^4p_3\lambda^2\frac{1}{p_2^2+m^2}\frac{1}{p_3^2+m^2}\frac{1}{(k_1-p_2-p_3)^2+m_4^2} \tag{1.17}$$

The **LSZ formulation** relates S matrices in Minkowski space with correlation functions as follows. The Fourier transformed $n+m$-point function near the physical poles $P_I^2 = M^2$ behaves as

$$\tilde{G}_{n+m}(p_1,...,p_n)(x_1,...,x_n) \sim \left(\prod_{i=1}^{n} \frac{\sqrt{Z}i}{p_i^2 - m^2 + i\epsilon}\right)\left(\prod_{j+1}^{m} \frac{\sqrt{Z}i}{k_j^2 - m^2 + i\epsilon}\right) < p_1,...,p_n|S|k_1,..,k_m > \tag{1.18}$$

For this reason, the study of correlation functions, which is easier, is preferred, since any physical process can be extracted from them as above.

If the external states are not states of a single field, but of a composite field $\mathcal{O}(x)$, e.g.

$$\mathcal{O}_{\mu\nu}(x) = (\partial_\mu \phi \partial_\nu \phi)(x)(+...) \tag{1.19}$$

it is useful to define Euclidean space correlation functions for these operators

$$\begin{aligned} < \mathcal{O}(x_1)...\mathcal{O}(x_n) >_{Eucl} &= \int \mathcal{D}\phi e^{-S_E} \mathcal{O}(x_1)...\mathcal{O}(x_n) \\ &= \frac{\delta^n}{\delta J(x_1)...\delta J(x_n)} \int \mathcal{D}\phi e^{-S_E + \int d^4x \mathcal{O}(x) J(x)}|_{J=0} \end{aligned} \tag{1.20}$$

which can be obtained from the generating functional

$$Z_{\mathcal{O}}[J] = \int \mathcal{D}\phi e^{-S_E + \int d^4x \mathcal{O}(x) J(x)} \tag{1.21}$$

Yang-Mills theory and gauge groups

Electromagnetism

In electromagnetism we have a gauge field

$$A_\mu(x) = (\phi(\vec{x},t), \vec{A}(\vec{x},t)) \tag{1.22}$$

with the field strength (containg the $\vec{E}$ and $\vec{B}$ fields)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu} A_{\nu]} \tag{1.23}$$

The observables like $\vec{E}$ and $\vec{B}$ are defined in terms of $F_{\mu\nu}$ and as such the theory has a gauge symmetry under a U(1) group, that leaves $F_{\mu\nu}$ invariant

$$\delta A_\mu = \partial_\mu \lambda; \quad \delta F_{\mu\nu} = 2\partial_{[\mu}\partial_{\nu]}\lambda = 0 \tag{1.24}$$

The Minkowski space action is

$$S_{Mink} = -\frac{1}{4}\int d^4x F_{\mu\nu}^2 \tag{1.25}$$

which becomes in Euclidean space

$$S_E = \frac{1}{4}\int d^4x (F_{\mu\nu})^2 = \frac{1}{4}\int d^4x F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho}\eta^{\nu\sigma} \tag{1.26}$$

The coupling of electromagnetism to a scalar field ϕ and a fermion field ϕ is obtained as follows

$$S_E^{total} = S_{E,A} + \int d^4x[\bar{\psi}(\not{D} + m)\psi + (D_\mu\phi)^* D^\mu \phi]$$
$$\not{D} = D_\mu \gamma^\mu; \quad D_\mu = \partial_\mu - ieA_\mu \tag{1.27}$$

This is known as the minimal coupling. Then there is a U(1) local symmetry that extends the above gauge symmetry, namely

$$\psi' = e^{ie\lambda(x)}\psi; \quad \phi' = e^{ie\lambda(x)}\phi \tag{1.28}$$

under which $D_\mu\psi$ transforms as $e^{ie\lambda}D_\mu\psi$, i.e transforms *covariantly*, as does $D_\mu\phi$.

The reverse is also possible, namely we can start with the action for ϕ and ψ only, with ∂_μ instead of D_μ. It will have the symmetry in (1.28), except with a global parameter only. If we want to promote the global symmetry to a local one, we need to introduce a gauge field and minimal coupling as above.

In the following, I will sometimes replace e by ie, thus $D_\mu = \partial_\mu + eA_\mu$.

Yang-Mills fields

Yang-Mills fields A_μ^a are self-interacting gauge fields, where a is an index belonging to a nonabelian gauge group. There is thus a 3-point self-interaction of the gauge fields A_μ^a, A_ν^b, A_ρ^c, that is defined by the constants $f^a{}_{bc}$.

The gauge group G has generators $(T^a)_{ij}$ in the representation R. T^a satisfy the Lie algebra of the group,

$$[T_a, T_b] = f_{ab}{}^c T_c \tag{1.29}$$

The group G is usually $SU(N), SO(N)$. The adjoint representation is defined by $(T^a)_{bc} = f^a{}_{bc}$. Then the gauge fields live in the adjoint representation and the field strength is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^a{}_{bc} A_\mu^b A_\nu^c \tag{1.30}$$

One can define $A = A^a T_a$ and $F_{\mu\nu} = F_{\mu\nu}^a T_a$ in terms of which we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \tag{1.31}$$

(If one further defines the forms $F = 1/2 F_{\mu\nu} dx^\mu \wedge dx^\nu$ and $A = A_\mu dx^\nu$ where wedge $\wedge$ denotes antisymmetrization, one has $F = dA + gA \wedge A$).

The generators T^a are taken to be antihermitian, their normalization being defined by their trace in the fundamental representation,

$$tr T^a T^b = -\frac{1}{2}\delta^{ab} \tag{1.32}$$

and here group indices are raised and lowered with δ^{ab}.

The local symmetry under the group G or gauge symmetry has now the infinitesimal form

$$\delta A_\mu^a = (D_\mu \epsilon)^a \tag{1.33}$$

where

$$(D_\mu \epsilon)^a = \partial_\mu \epsilon^a + g f^a{}_{bc} A^b_\mu \epsilon^c \tag{1.34}$$

The finite form of the transformation is

$$A^U_\mu(x) = U^{-1}(x) A_\mu(x) U(x) + U^{-1}\partial_\mu U(x); \quad U = e^{\lambda^a T_a} = e^\lambda \tag{1.35}$$

and if $\lambda^a = \epsilon^a$ =small, we get the previous. This transformation leaves invariant the Euclidean action

$$S_E = -\frac{1}{2}\int d^4x tr(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{4}\int d^4x F^a_{\mu\nu}F^{b,\mu\nu}\delta_{ab} \tag{1.36}$$

whereas the fields stregth transforms covariantly, i.e.

$$F'_{\mu\nu} = U^{-1}(x) F_{\mu\nu} U(x) \tag{1.37}$$

Coupling with other fields is done again by using the covariant derivative. In representation R, the covariant derivative D_μ (that also transforms covariantly) is

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu + g(T^a)_{ij} A^a_\mu(x) \tag{1.38}$$

and one replaces ∂_μ by D_μ, e.g. for a fermion, $\bar\psi \partial\!\!\!/ \psi \to \bar\psi D\!\!\!\!/ \psi$.

Symmetry currents and anomalies

The Noether theorem states that a global classical symmetry corresponds to a conserved current (on-shell), i.e.

$$\delta_{symm.}\mathcal{L} = \epsilon^a \partial_\mu j^{\mu,a} \tag{1.39}$$

so that a classical symmetry corresponds to having the Noether current $j^{\mu,a}$ conserved, i.e. $\partial_\mu j^{\mu,a}$=0. If the transformation is

$$\delta\phi^i = \epsilon^a (T^a)_{ij}\phi^j \tag{1.40}$$

then the Noether current is

$$j^{\mu,a} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^i)} T^a_{ij}\phi^j \tag{1.41}$$

Quantum mechanically however, the current can have an *anomaly*, i.e. $< \partial_\mu j^{\mu,a} > \neq 0$. In momentum space, this will be $p_\mu < j^{\mu,a} > \neq 0$.

As an example, take the Lagrangian

$$\mathcal{L} = \bar\psi^i \gamma^\mu D_\mu \psi_i \tag{1.42}$$

with $\delta\psi^i = \epsilon^a (T^a)_{ij}\psi^j$. It gives the symmetry (Noether) current

$$j^{\mu,a} = \bar\psi^i \gamma^\mu T^a_{ij}\psi^j \tag{1.43}$$

Some observations can be made about this example. First, j^a_μ is a composite operator. Second, if ψ^i has also some gauge (local symmetry) indices, then $j^{\mu,a}$ is gauge invariant, so it can represent a physical state.

One can use the formalism for composite operators and define the correlator

$$< j^{\mu_1,a_1}(x_1), ...j^{\mu_n,a_n}(x_n) >= \frac{\delta^n}{\delta A^{a_1}_{\mu_1}(x_1)...\delta A^{a_n}_{\nu_n}(x_n)} \int \mathcal{D}[fields]e^{-S+\int d^4x j^{\mu,a}(x)A^a_\mu(x)} \quad (1.44)$$

which will then be a correlator of some external physical states (observables).

We will see that this kind of correlators are obtained from AdS-CFT. The current anomaly can manifest itself also in this correlator. $j^{\mu,a}$ is inserted inside the quantum average, thus in momentum space, we could a priori have the anomaly

$$p_{\mu_1} < j^{\mu_1,a_1}, ...j^{\mu_n,a_n} >\neq 0 \quad (1.45)$$

In general, the anomaly is 1-loop only, and is given by polygon graphs, i.e. a 1-loop contribution (a 1-loop Feynman diagram) to an n-point current correlator that looks like a n-polygon with vertices= the $x_1, ..x_n$ points. In d=2, only the 2-point correlator is anomalous by the Feynman diagram in Fig.1c, in d=4, the 3-point, by a triangle Feynman diagram, as in Fig.1d, in d=6 the 4-point, by a box (square) graph, as in Fig.1e, etc.

Therefore, in d=4, the anomaly is called triangle anomaly.

Important concepts to remember

- Correlation functions are given by a Feynman diagram expansion and appear as derivatives of the partition function
- S matrices defining physical scatterings are obtained via the LSZ formalism from the poles of the correlation functions
- Correlation functions of composite operators are obtained from a partition function with sources coupling to the operators
- Coupling of fields to electromagnetism is done via minimal coupling, replacing the derivatives d with the covariant derivatives $D = d - ieA$.
- Yang-Mills fields are self-coupled. Both the covariant derivative and the field strength transform covariantly.
- Classically, the Noether theorem associates every symmetry with a conserved current.
- Quantum mechanically, global symmetries can have an anomaly, i.e the current is not conserved, when inserted inside a quantum average.
- The anomaly comes only from 1-loop Feynman diagrams. In d=4, it comes from a triangle, thus only affects the 3-point function.
- In a gauge theory, the current of a global symmetry is gauge invariant, thus corresponds to some physical state.

Exercises, Section 1

1. If we have the partition function

$$Z[J] = \exp\{-\int d^4x[(\int d^4x_0 K(x,x_0)J(x_0))(-\frac{\Box_x}{2})(\int d^4y_0 K(x,y_0)J(y_0)) \\ +\lambda(\int d^4x_0 K(x,x_0)(J(x_0)))^3]\} \quad (1.46)$$

write an expression for $G_2(x,y)$ and $G_3(x,y)$.

2. If we have the Euclidean action

$$S_E = \int d^4x[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2\phi^2}{2} + \lambda\phi^3] \quad (1.47)$$

write down the integral for the Feynman diagram in Fig.2a.

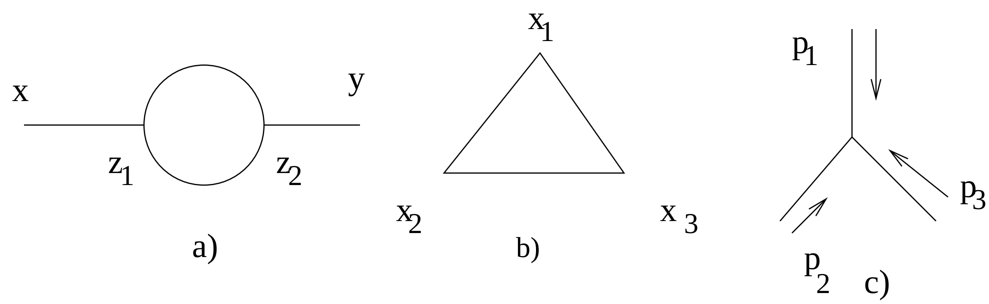

Figure 2: a) Setting sun diagram in x space; b) Triangle diagram in x space; c) Star diagram in p space

3. Show that the Fourier transform of the triangle diagram in x space in Fig.2b is the star diagram in p space in Fig.2c.

4. Derive the Hamiltonian $H(\vec{E},\vec{B})$ for the electromagnetic field by putting $A_0 = 0$, from $S_M = -\int F_{\mu\nu}^2/4$.

5. Show that $F_{\mu\nu} = [D_\mu, D_\nu]/g$. What is the infinitesimal transformation of $F_{\mu\nu}$? For SO(d) groups, the adjoint representation is antisymmetric, (ab). Calculate $f^{(ab)}{}_{(cd)(ef)}$ and write down $F_{\mu\nu}^{ab}$.

6. Consider the action

$$S = -\frac{1}{4}\int F_{\mu\nu}^2 + \frac{1}{2}\int \bar{\psi}(\not{D}+m)\psi + \frac{1}{2}\int (D_\mu\phi)^2 D^\mu\phi \quad (1.48)$$

and the U(1) electromagnetic transformation. Calculate the Noether current.

2 Basics of general relativity; Anti de Sitter space.

Curved spacetime and geometry

In **special relativity**, one postulates that the speed of light is constant in all inertial reference frames, i.e. $c = 1$. As a result, the line element

$$ds^2 = -dt^2 + d\vec{x}^2 = \eta_{ij} dx^i dx^j \tag{2.1}$$

is invariant, and is called the invariant distance. Here $\eta_{ij} = diag(-1, 1, ..., 1)$. Therefore the symmetry group of general relativity is the group that leaves the above line element invariant, namely SO(1,3), or in general SO(1,d-1).

This *Lorentz group* is a generalized rotation group: The rotation group SO(3) is the group that leaves the 3 dimensional length $d\vec{x}^2$ invariant. The Lorentz transformation is a generalized rotation

$$x'^i = \Lambda^i{}_j x^j; \quad \Lambda^i{}_j \in SO(1,3) \tag{2.2}$$

Therefore the statement of special relativity is that physics is Lorentz invariant (invariant under the Lorentz group SO(1,3) of generalized rotations).

In **general relativity**, one considers a more general spacetime, specifically a curved spacetime, defined by the distance between two points, or line element,

$$ds^2 = g_{ij}(x) dx^i dx^j \tag{2.3}$$

where $g_{ij}(x)$ are arbitrary functions called *the metric* (sometimes one refers to ds^2 as the metric). This situation is depicted in Fig.3a.

As we can see from the definition, the metric $g_{ij}(x)$ is a symmetric matrix.

To understand this, let us take the example of the sphere, specifically the familiar example of a 2-sphere embedded in 3 dimensional space. Then the metric in the embedding space is the usual Euclidean distance

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^3 \tag{2.4}$$

but if we are on a two-sphere we have the constraint

$$\begin{aligned} & x_1^2 + x_2^2 + x_3^2 = R^2 \Rightarrow 2(x_1 dx_1 + x_2 dx_2 + x_3 dx_3) = 0 \\ & \Rightarrow dx_3 = -\frac{x_1 dx_1 + x_2 dx_2}{x_3} = -\frac{x_1}{\sqrt{R^2 - x_1^2 - x_2^2}} dx_1 - \frac{x_2}{\sqrt{R^2 - x_1^2 - x_2^2}} dx_2 \end{aligned} \tag{2.5}$$

which therefore gives the induced metric (line element) on the sphere

$$ds^2 = dx_1^2 \left(1 + \frac{x_1^2}{R^2 - x_1^2 - x_2^2}\right) + dx_2^2 \left(1 + \frac{x_2^2}{R^2 - x_1^2 - x_2^2}\right) + 2dx_1 dx_2 \frac{x_1 x_2}{R^2 - x_1^2 - x_2^2} = g_{ij} dx^i dx^j \tag{2.6}$$

So this is an example of a curved d-dimensional space which is obtained by emebedding it into a flat (Euclidean or Minkowski) d+1 dimensional space. But if the metric $g_{ij}(x)$ are arbitrary functions, then one cannot in general embed such a space in flat d+1 dimensional space: there are $d(d+1)/2$ functions $g_{ij}(x)$ to be obtained and only one function (in the above

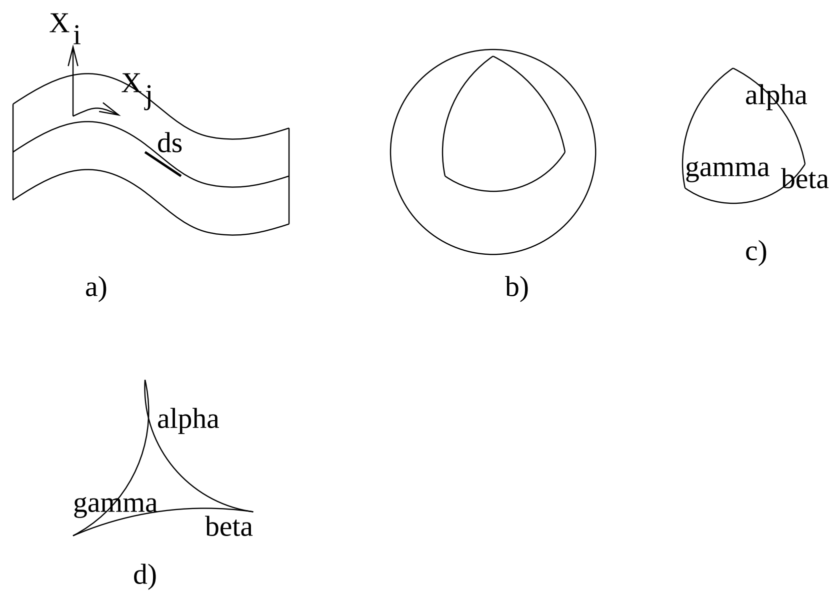

Figure 3: a) curved space. The functional form of the distance between 2 points depends on local coordinates. b) A triangle on a sphere, made from two meridian lines and a seqment of the equator has two angles of 90^0 ($\pi/2$). c) The same triangle, drawn for a general curved space of positive curvature, emphasizing that the sum of the angles of the triangle exceeds 180^0 (π). d) In a space of negative curvature, the sum of the angles of the triangle is below 180^0 (π).

example, the function $x_3(x_1, x_2)$), together with d coordinate transformations $x'_i = x'_i(x_j)$ available for the embedding. In fact, we will see that the problem is even more complicated in general due to the signature of the metric (signs on the diagonal of the diagonalized matrix g_{ij}). Thus, even though a 2 dimensional metric has 3 components, equal to the 3 functions available for a 3 dimensional embedding, to embed a metric of Euclidean signature in 3d one needs to consider both 3d Euclidean and 3d Minkowski space.

That means that a general space can be *intrinsically curved*, defined not by embedding in a flat space, but by the arbitrary functions $g_{ij}(x)$ (the metric). In a general space, we define the *geodesic* as the line of shortest distance $\int_a^b ds$ between two points a and b.

In a curved space, the triangle made by 3 geodesics has an unusual property: the sum of the angles of the triangle, $\alpha+\beta+\gamma$ is not equal to π. For example, if we make a triangle from geodesics on the sphere as in Fig.3b, we can easily convince ourselves that $\alpha + \beta + \gamma > \pi$. In fact, by taking a vertex on the North Pole and two vertices on the Equator, we get $\beta = \gamma = \pi/2$ and $\alpha > 0$. This is the situation for a space with positive curvature, $R > 0$: two parallel geodesics converge to a point (Fig.3c). In the example given, the two parallel geodesics are the lines between the North Pole and the Equator: both lines are perpendicular to the equator, therefore are parallel by definition, yet they converge at the North Pole.

But one can have also a space with negative curvature, $R < 0$, for which $\alpha + \beta + \gamma < \pi$ and two parallel geodesics diverge, as in Fig.3d. Such a space is for instance the so-called *Lobachevski space*, which is a two dimensional space of Euclidean signature (like the two dimensional sphere), i.e. the diagonalized metric has positive numbers on the diagonal. However, this metric cannot be obtained as an embedding in a Euclidean 3d space, but rather an embedding in a Minkowski 3 dimensional space, by

$$ds^2 = dx^2 + dy^2 - dz^2; \quad x^2 + y^2 - z^2 = -R^2 \tag{2.7}$$

Einstein's theory of general relativity makes two physical assumptions

- gravity is geometry: matter follows geodesics in a curved space, and the resulting motion appears to us as the effect of gravity. AND
- matter sources gravity: matter curves space, i.e. the source of spacetime curvature (and thus of gravity) is a matter distribution.

We can translate these assumptions into two mathematically well defined physical principles and an equation for the dynamics of gravity (Einstein's equation). The physical principles are

- Physics is invariant under general coordinate transformations

$$x'_i = x'_i(x_j) \Rightarrow ds^2 = g_{ij}(x)dx^i dx^j = g'_{ij}(x')dx'^i dx'^j \tag{2.8}$$

- The Equivalence principle, which can be stated as "there is no difference between acceleration and gravity" OR "if you are in a free falling elevator you cannot distinguish it from being weightless (without gravity)". This is only a *local* statement: for example, if you are falling towards a black hole, tidal forces will pull you apart before you reach it (gravity acts slightly differently at different points). The quantitative way to write this principle is

$$m_i = m_g \text{ where } \vec{F} = m_i\vec{a} \text{ (Newton's law) and } \vec{F}_g = m_g\vec{g} \text{ (gravitational force)} \tag{2.9}$$

In other words, both gravity and acceleration are manifestations of the curvature of space.

Before describing the dynamics of gravity (Einstein's equation), we must define the kinematics (objects used to describe gravity).

As we saw, the metric $g_{\mu\nu}$ changes when we make a coordinate transformation, thus different metrics can describe the same space. In fact, since the metric is symmetric, it has $d(d+1)/2$ components. But there are d coordinate transformations $x'_\mu(x_\nu)$ one can make that leave the physics invariant, thus we have $d(d-1)/2$ degrees of freedom that describe the curvature of space (different physics).

We need other objects besides the metric that can describe the space in a more invariant manner. The basic such object is called the Riemann tensor, $R^\mu{}_{\nu\rho\sigma}$. To define it, we first define the inverse metric, $g^{\mu\nu} = (g^{-1})_{\mu\nu}$ (matrix inverse), i.e. $g_{\mu\rho}g^{\rho\sigma} = \delta^\sigma_\mu$. Then we define an object that plays the role of "gauge field of gravity", the Christoffel symbol

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho}) \tag{2.10}$$

Then the Riemann tensor is like the "field strength of the gravity gauge field", in that its definition can be written as to mimic the definition of the field strength of an SO(n) gauge group,

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + A^{ac}_\mu A^{cb}_\nu - A^{ac}_\nu A^{cb}_\mu \tag{2.11}$$

where a, b, c are fundamental SO(n) indices, i.e. ab (antisymmetric) is an adjoint index. We put brackets in the definition of the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ to emphasize the similarity with the above:

$$(R^\mu{}_\nu)_{\rho\sigma}(\Gamma) = \partial_\rho(\Gamma^\mu{}_\nu)_\sigma - \partial_\sigma(\Gamma^\mu{}_\nu)_\rho + (\Gamma^\mu{}_\lambda)_\rho(\Gamma^\lambda{}_\nu)_\sigma - (\Gamma^\mu{}_\lambda)_\sigma(\Gamma^\lambda{}_\nu)_\rho \tag{2.12}$$

the only difference being that here both "gauge" and "spacetime" indices are the same.

From the Riemann tensor we construct by contraction the Ricci tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \tag{2.13}$$

and the Ricci scalar $R = R_{\mu\nu}g^{\mu\nu}$. The Ricci scalar is coordinate invariant, so it is truly an invariant measure of the curvature of space. The Riemann and Ricci tensors are examples of tensors. A contravariant tensor A^μ transforms as dx^μ,

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}A^\nu \tag{2.14}$$

whereas a covariant tensor B_μ transforms as $\partial/\partial x^\mu$, i.e.

$$B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}B_\nu \tag{2.15}$$

and a general tensor transforms as the product of the transformations of the indices. The metric $g_{\mu\nu}$, the Riemann $R^\mu{}_{\nu\rho\sigma}$ and Ricci $R_{\mu\nu}$ and R are tensors, but the Christoffel symbol $\Gamma^\mu{}_{\nu\rho}$ is not.

To describe physics in curved space, we replace the Lorentz metric $\eta_{\mu\nu}$ by the general metric $g_{\mu\nu}$, and Lorentz tensors with general tensors. One important observation is that ∂_μ is not a tensor! The tensor that replaces it is the curved space covariant derivative, D_μ,

$$D_\mu T^\rho_\nu \equiv \partial_\mu T^\rho_\nu + \Gamma^\rho{}_{\mu\sigma} T^\sigma_\nu - \Gamma^\sigma{}_{\mu\nu} T^\rho_\sigma \tag{2.16}$$

We are now ready to describe the dynamics of gravity, in the form of Einstein's equation. It is obtained by postulating an action for gravity. The invariant volume of integration over space is not d^dx anymore as in Minkowski or Euclidean space, but $d^dx\sqrt{-g} \equiv d^dx\sqrt{-\det(g_{\mu\nu})}$ (where the $-$ sign comes from the Minkowski signature of the metric). The Lagrangian has to be invariant under general coordinate transformations, thus it must be a scalar (tensor with no indices). There would be several possible choices for such a scalar, but the simplest possible one, the Ricci scalar, turns out to be correct (i.e. compatible with experiment). Thus, one postulates the Einstein-Hilbert action for gravity

$$S_{gravity} = -\frac{1}{16\pi G}\int d^dx\sqrt{-g}R \tag{2.17}$$

The equations of motion of this action are

$$\frac{\delta S_{grav}}{\delta g^{\mu\nu}} = 0: \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{2.18}$$

and as we mentioned, this action is not fixed, it just happens to agree well with experiments. In fact, in quantum gravity/string theory, S_g could have quantum corrections of different functional form.

The next step is to put matter in curved space, since one of the physical principles was that matter sources gravity. This follows the above mentioned rules. For instance, the kinetic term for a scalar field in Minkowski space was

$$S_\phi = \frac{1}{2}\int d^4x(\partial_\mu\phi)(\partial_\nu\phi)\eta^{\mu\nu} \tag{2.19}$$

and it becomes now

$$\frac{1}{2}\int d^4x\sqrt{-g}(D_\mu\phi)(D_\nu\phi)g^{\mu\nu} = \frac{1}{2}\int d^4x\sqrt{-g}(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu} \tag{2.20}$$

where the last equality, of the partial derivative with the covariant derivative, is only valid for a scalar field. In general, we will have covariant derivatives in the action.

The variation of the matter action gives the energy-momentum tensor (known from electromagnetism). By definition, we have

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \tag{2.21}$$

Then the sum of the gravity and matter action give the equation of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{2.22}$$

known as the Einstein's equation. For a scalar field, we have

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\rho\phi)^2 \tag{2.23}$$

Global Structure: Penrose diagrams

Spaces of interest are infinite in extent, but have complicated topological and causal structure. To make sense of them, we use the Penrose diagrams. These are diagrams that preserve the causal and topological structure of space, and have infinity at a finite distance on the diagram.

To construct a Penrose diagram, we note that light propagates along $ds^2 = 0$, thus an overall factor ("conformal factor") in ds^2 is irrelevant. So we make coordinate transformations that bring infinity to a finite distance, and drop the conformal factors. For convenience, we usually get some type of flat space at the end of the calculation. Then, in the diagram, light rays are at 45 degrees ($\delta x = \delta t$ for light, in the final coordinates).

As an example, we draw the Penrose diagram of 2 dimensional Minkowski space,

$$ds^2 = -dt^2 + dx^2 \tag{2.24}$$

where $-\infty < t, x < +\infty$. We first make a transformation to "lightcone coordinates"

$$u_\pm = t \pm x \Rightarrow ds^2 = -du_+du_- \tag{2.25}$$

followed by a transformation of the lightcone coordinates that makes them finite,

$$u_\pm = \tan\tilde{u}_\pm; \quad \tilde{u}_\pm = \frac{\tau\pm\theta}{2} \tag{2.26}$$

where the last transformation goes back to space-like and time-like coordinates θ and τ. Now the metric is

$$ds^2 = \frac{1}{4\cos^2\tilde{u}_+\cos^2\tilde{u}_-}(-d\tau^2 + d\theta^2) \tag{2.27}$$

and by dropping the overall (conformal) factor we get back a flat two dimensional space, but now of finite extent. Indeed, we have that $|\tilde{u}_\pm| \leq \pi/2$, thus $|\tau\pm\theta| \leq \pi$, so the Penrose diagram is a diamond (a rotated square), as in Fig.4a)

For 3 dimensional Minkowski space the metric is again

$$ds^2 = -dt^2 + dr^2(+r^2d\theta^2) \tag{2.28}$$

and by dropping the angular dependence we get the same metric with as before, just that $r > 0$ now. So everything follows in the same way, just that $\theta > 0$ in the final form. Thus for 3d (and higher) Minkowski space, the Penrose diagram is a triangle (the $\tau > 0$ half of the 2d Penrose diagram), as in Fig.4b.

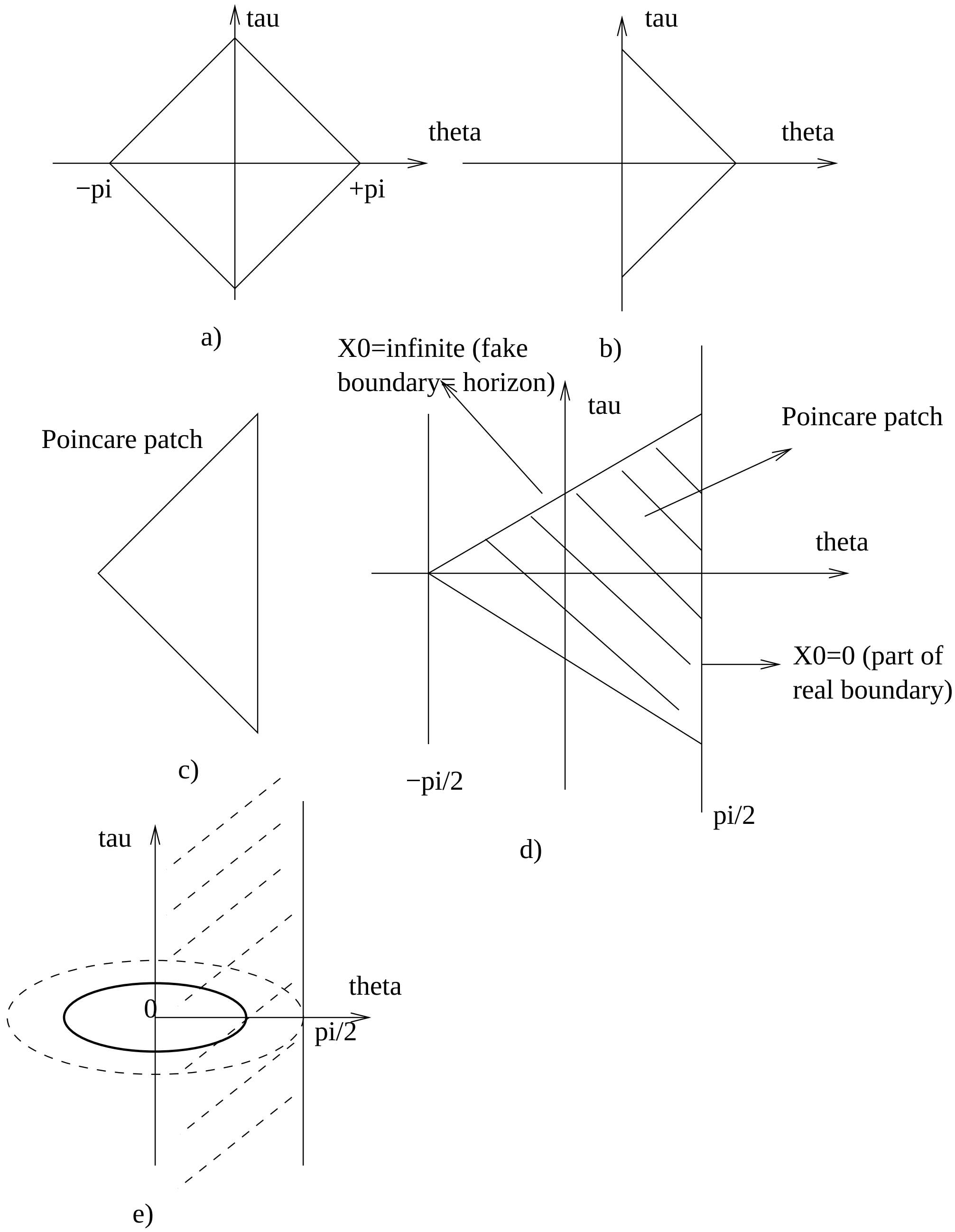

Figure 4: Penrose diagrams. a) Penrose diagram of 2 dimensional Minkowski space. b) Penrose diagram of 3 dimensional Minkowski space. c) Penrose diagram of the Poincare patch of Anti de Sitter space. d) Penrose diagram of global AdS_2 (2 dimensional Anti de Sitter), with the Poincare patch emphasized; $x_0 = 0$ is part of the boundary, but $x_0 = \infty$ is a fake boundary (horizon). e) Penrose diagram of global AdS_d for $d \geq 2$. It is half the Penrose diagram of AdS_2 rotated around the $\theta = 0$ axis.

Anti de Sitter space

Anti de Sitter space is a space of Lorentzian signature $(-++...+)$, but of constant *negative* curvature. Thus is an analog of the Lobachevski space, which was a space of Euclidean signature and of constant negative curvature.

The anti in Anti de Sitter is because de Sitter space is defined as the space of Lorentzian signature and of constant positive curvature, thus an analog of the sphere (the sphere is the space of Euclidean signature and constant positive curvature).

In d dimensions, de Sitter space is defined by a sphere-like embedding in d+1 dimensions

$$ds^2 = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 + dx_{d+1}^2$$
$$-x_0^2 + \sum_{i=1}^{d-1} x_i^2 + x_{d+1}^2 = R^2 \tag{2.29}$$

thus as mentioned, this is the Lorentzian version of the sphere, and it is clearly invariant under the group SO(1,d) (the d dimensional sphere would be invariant under SO(d+1) rotating the d+1 embedding coordinates).

Similarly, in d dimensions, Anti de Sitter space is defined by a Lobachevski-like embedding in d+1 dimensions

$$ds^2 = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 - dx_{d+1}^2$$
$$-x_0^2 + \sum_{i=1}^{d-1} x_i^2 - x_{d+1}^2 = -R^2 \tag{2.30}$$

and is therefore the Lorentzian version of Lobachevski space. It is invariant under the group SO(2,d-1) that rotates the coordinates $x_\mu = (x_0, x_{d+1}, x_1, ..., x_{d-1})$ by $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

The metric of this space can be written in different forms, corresponding to different coordinate systems. In Poincare coordinates,

$$ds^2 = \frac{R^2}{x_0^2}(-dt^2 + \sum_{i=1}^{d-2} dx_i^2 + dx_0^2) \tag{2.31}$$

where $-\infty < t, x_i < +\infty$, but $0 < x_0 < +\infty$. Up to a conformal factor therefore, this is just like (flat) 3d Minkowski space, thus its Penrose diagram is the same, a triangle, as in Fig.4c. However, one now discovers that one does not cover all of the space! In the finite coordinates τ, θ, one finds that one can now analytically continue past the diagonal boundaries (there is no obstruction to doing so).

In these Poincare coordinates, we can understand Anti de Sitter space as a d-1 dimensional Minkowski space in $(t, x_1, ...x_{d-2})$ coordinates, with a "warp factor" (gravitational potential) that depends only on the additional coordinate x_0.

A coordinate system that does cover the whole space is called the global coordinates, and it gives the metric

$$ds_d^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\vec{\Omega}_{d-2}^2) \tag{2.32}$$

where $d\vec{\Omega}^2$ is the metric on the unit sphere. This metric is written in a suggestive form, since the metric on the d-dimensional sphere can be written in a similar way,

$$ds_d^2 = R^2(\cos^2\rho dw^2 + d\rho^2 + \sin^2\rho d\vec{\Omega}_{d-2}^2) \tag{2.33}$$

The change of coordinates $\tan\theta = \sinh\rho$ gives the metric

$$ds_d^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\vec{\Omega}_{d-2}^2) \tag{2.34}$$

where $0 \leq \theta \leq \pi/2$ in all dimensions except 2, (where $-\pi/2 \leq \theta \leq \pi/2$), and τ is arbitrary, and from it we infer the Penrose diagram of global AdS_2 space (Anti de Sitter space in 2 dimensions) which is an infinite strip between $\theta = -\pi/2$ and $\theta = +\pi/2$. The "Poincare patch" covered by the Poincare coordinates, is a triangle region of it, with its vertical boundary being a segment of the infinite vertical boundary of the global Penrose diagram, as in Fig.4d.

The Penrose diagram of AdS_d is similar, but it is a cylinder obtained by the revolution of the infinite strip between $\theta = 0$ and $\theta = \pi/2$ around the $\theta = 0$ axis, as in Fig.4e. The "circle" of the revolution represents in fact a d-2 dimensional sphere. Thefore the boundary of AdS_d (d dimensional Anti de Sitter space) is $\mathbf{R}_\tau \times S_{d-2}$, the infinite vertical line of time times a d-2 dimensional sphere. This will be important in defining AdS-CFT correctly.

Finally, let us mention that Anti de Sitter space is a solution of the Einstein equation with a constant energy-momentum tensor, known as a *cosmological constant*, thus $T_{\mu\nu} = 2\Lambda g_{\mu\nu}$, coming from a constant term in the action, $\int d^4x\sqrt{-g}\Lambda$, so the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}g_\mu R = 16\pi G\Lambda g_{\mu\nu} \tag{2.35}$$

Important concepts to remember

- In general relativity, space is intrinsically curved
- In general relativity, physics is invariant under general coordinate transformations
- Gravity is the same as curvature of space, or gravity = local acceleration.
- The Christoffel symbol acts like a gauge field of gravity, giving the covariant derivative
- Its field strength is the Riemann tensor, whose scalar contraction, the Ricci scalar, is an invariant measure of curvature
- One postulates the action for gravity as $(1-/(16\pi G))\int\sqrt{-g}R$, giving Einstein's equations

- To understand the causal and topological structure of curved spaces, we draw Penrose diagrams, which bring infinity to a finite distance in a controlled way.
- de Sitter space is the Lorentzian signature version of the sphere; Anti de Sitter space is the Lorentzian version of Lobachevski space, a space of negative curvature.
- Anti de Sitter space in d dimensions has $SO(2, d-1)$ invariance.
- The Poincare coordinates only cover part of Anti de Sitter space, despite having maximum possible range (over the whole real line).
- Anti de Sitter space has a cosmological constant.

Exercises, section 2

1) Parallel the derivation in the text to find the metric on the 2-sphere in its usual form,

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.36}$$

from the 3d Euclidean metric.

2) Show that on-shell, the graviton has degrees of freedom corresponding to a transverse (d-2 indices) symmetric traceless tensor.

3) Show that the metric $g_{\mu\nu}$ is covariantly constant ($D_\mu g_{\nu\rho} = 0$) by substituting the Christoffel symbols.

4) The Christoffel symbol $\Gamma^\mu_{\nu\rho}$ is not a tensor, and can be put to zero at any point by a choice of coordinates (Riemann normal coordinates, for instance), but $\delta\Gamma^\mu_{\nu\rho}$ is a tensor. Show that the variation of the Ricci scalar can be written as

$$\delta R = \delta^\rho_\mu g^{\nu\sigma}(\partial_\rho \delta\Gamma^\mu_{\nu\sigma} - \partial_\sigma \delta\Gamma^\mu_{\nu\rho}) + R_{\nu\sigma}\delta g^{\nu\sigma} \tag{2.37}$$

5) Parallel the calculation in 2d to show that the Penrose diagram of 3d Minkwoski space, with an angle ($0 \leq \phi \leq 2\pi$) supressed, is a triangle.

6) Substitute the coordinate transformation

$$X_0 = R\cosh\rho\cos\tau; \quad X_i = R\sinh\rho\Omega_i; \quad X_{d+1} = R\cosh\rho\sin\tau \tag{2.38}$$

to find the global metric of AdS space from the embedding (2,d-1) signature flat space.

3 Basics of supersymmetry

In the 1960's people were asking what kind of symmetries are possible in particle physics?

We know the Poincare symmetry defined by the Lorentz generators J_{ab} of the SO(1,3) Lorentz group and the generators of 3+1 dimensional translation symmetries, P_a.

We also know there are possible internal symmetries T_r of particle physics, such as the local U(1) of electromagnetism, the local SU(3) of QCD or the global SU(2) of isospin. These generators will form a Lie algebra

$$[T_r, T_s] = f_{rs}{}^t T_t \tag{3.1}$$

So the question arose: can they be combined, i.e. $[T_s, P_a] \neq 0, [T_s, J_{ab}] \neq 0$, such that maybe we could embed the SU(2) of isospin together with the SU(2) of spin into a larger group?

The answer turned out to be NO, in the form of the Coleman-Mandula theorem, which says that if the Poincare and internal symmetries were to combine, the S matrices for all processes would be zero.

But like all theorems, it was only as strong as its assumptions, and one of them was that the final algebra is a Lie algebra.

But people realized that one can generalize the notion of Lie algebra to a *graded Lie algebra* and thus evade the theorem. A graded Lie algebra is an algebra that has some generators Q^i_α that satisfy not a commuting law, but an anticommuting law

$$\{Q^i_\alpha, Q^j_\beta\} = \text{other generators} \tag{3.2}$$

Then the generators P_a, J_{ab} and T_r are called "even generators" and the Q^i_α are called "odd" generators. The graded Lie algebra then is of the type

$$[\text{even}, \text{even}] = \text{even}; \quad \{\text{odd}, \text{odd}\} = \text{even}; \quad [\text{even}, \text{odd}] = \text{odd} \tag{3.3}$$

So such a graded Lie algebra generalization of the Poincare + internal symmetries is possible. But what kind of symmetry would a Q^i_α generator describe?

$$[Q^i_\alpha, J_{ab}] = (...)J_{cd} \tag{3.4}$$

means that Q^i_α must be in a representation of J_{ab} (the Lorentz group). Because of the anticommuting nature of Q^i_α we choose the spinor representation. But a spinor field times a boson field gives a spinor field. Therefore when acting with Q^i_α (spinor) on a boson field, we will get a spinor field.

Therefore Q^i_α *gives a symmetry between bosons and fermions, called* **supersymmetry**!

$$\delta\, boson = fermion; \quad \delta\, fermion = boson \tag{3.5}$$

$\{Q_\alpha, Q_\beta\}$ is called the supersymmetry algebra, and the above graded Lie algebra is called the superalgebra.

Here Q^i_α is a spinor, with α a spinor index and i a label, thus the parameter of the transformation law, ϵ^i_α is a spinor also.

But what kind of spinor? In particle physics, Weyl spinors are used, that satisfy $\gamma_5\psi = \pm\psi$, but in supersymmetry one uses Majorana spinors, that satisfy the reality condition

$$\chi^C \equiv \chi^T C = \bar{\chi} \equiv \chi^\dagger i\gamma^0 \tag{3.6}$$

where C is the "charge conjugation matrix", that relates γ_m with γ_m^T. In 4 dimensions, it satisfies

$$C^T = -C; \quad C\gamma^m C^{-1} = -(\gamma^m)^T \tag{3.7}$$

And C is used to raise and lower indices, but since it is antisymmetric, one must define a convention for contraction of indices (the order matters).

The reason we use Majorana spinors is convenience, since it is easier to prove various supersymmetry identities, and then in the Lagrangian we can always go from a Majorana to a Weyl spinor and viceversa.

2 dimensional Wess Zumino model

We will exemplify supersymmetry with the simplest possible models, which occur in 2 dimensions.

A general (Dirac) fermion in d dimensions has $2^{[d/2]}$ complex components, therefore in 2 dimensions it has 2 complex dimensions, and thus a Majorana fermion will have 2 real components. An on-shell Majorana fermion (that satisfies the Dirac equation, or equation of motion) will then have a single component.

Since we have a symmetry between bosons and fermions, the number of degrees of freedom of the bosons must match the number of degrees of freedom of the fermions (the symmetry will map a degree of freedom to another degree of freedom). This matching can be

- on-shell, in which case we have *on-shell supersymmetry* OR
- off-shell, in which case we have *off-shell supersymmetry*

Thus, in 2d, the simplest possible model has 1 Majorana fermion ψ (which has one degree of freedom on-shell), and 1 real scalar ϕ. We can then obtain **on-shell supersymmetry** and get the Wess-Zumino model in 2 dimensions.

The action of a free boson and a free fermion in two dimensions is

$$S = -\frac{1}{2}\int d^2x[(\partial_\mu\phi)^2 + \bar{\psi}\partial\!\!\!/\psi] \tag{3.8}$$

and this is actually the action of the free Wess-Zumino model. From the action, the mass dimension of the scalar is $[\phi] = 0$, and of the fermion is $[\psi] = 1/2$ (the mass dimension of $\int d^2x$ is -2 and of ∂_μ is $+1$, and the action is dimensionless).

To write down the supersymmetry transformation between the boson and the fermion, we start by varying the boson into fermion times ϵ, i.e

$$\delta\phi = \bar{\epsilon}\psi = \bar{\epsilon}_\alpha\psi^\alpha = \epsilon^\beta C_{\beta\alpha}\psi^\alpha \tag{3.9}$$

From this we infer that the mass dimension of ϵ is $[\epsilon] = -1/2$. By dimensional reasons, for the reverse transformation we must add an object of mass dimension 1 with no free vector indices, and the only one such object available to us is $\partial\!\!\!/$, thus

$$\delta\psi = \partial\!\!\!/\phi\epsilon \tag{3.10}$$

We can check that the above free action is indeed invariant on-shell under this symmetry. For this, we must use the Majorana spinor identities, valid both in 2d and 4d

$$\begin{aligned} &1)\ \bar{\epsilon}\chi = +\bar{\chi}\epsilon; \quad 2)\ \bar{\epsilon}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\epsilon \\ &3)\ \bar{\epsilon}\gamma_5\chi = +\bar{\chi}\gamma_5\epsilon \quad 4)\ \bar{\epsilon}\gamma_\mu\gamma_5\chi = +\bar{\chi}\gamma_\mu\gamma_5\epsilon \end{aligned} \tag{3.11}$$

To prove, for instance, the first identity, we write $\bar{\epsilon}\chi = \epsilon^\alpha C_{\alpha\beta}\chi^\beta$, but $C_{\alpha\beta}$ is antisymmetric and ϵ and χ anticommute, being spinors, thus we get $-\chi^\beta C_{\alpha\beta}\epsilon^\alpha = +\chi^\beta C_{\beta\alpha}\epsilon^\alpha$. Then the variation of the action gives

$$\delta S = -\int d^2x[-\phi\Box\delta\phi + \frac{1}{2}\delta\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\bar{\psi}\partial\!\!\!/\delta\psi] = -\int d^2x[-\phi\Box\delta\phi + \bar{\psi}\partial\!\!\!/\delta\psi] \tag{3.12}$$

where in the second equality we have used partial integration together with identity 2) above. Then substituting the transformation law we get

$$\delta S = \int d^2x[-\phi\Box\bar{\epsilon}\psi + \bar{\psi}\partial\!\!\!/\partial\!\!\!/\epsilon] \tag{3.13}$$

But we have

$$\partial\!\!\!/\partial\!\!\!/ = \partial_\mu\partial_\nu\gamma^\mu\gamma^\nu = \partial_\mu\partial_\nu\frac{1}{2}\{\gamma_\mu,\gamma_\nu\} = \partial_\mu\partial_\nu g^{\mu\nu} = \Box \tag{3.14}$$

and by using this identity, together with two partial integrations, we obtain that $\delta S = 0$. So the action is invariant without the need for the equations of motion, so it would seem that this is an off-shell supersymmetry. However, the invariance of the action is not enough, since we have not proven that the above transformation law closes on the fields, i.e. that by acting twice on every field and forming the Lie algebra of the symmetry, we get back to the same field, or that we have a *representation of the Lie algebra* on the fields. The graded Lie algebra of supersymmetry is generically of the type

$$\{Q^i_\alpha, Q^j_\beta\} = 2(C\gamma^\mu)_{\alpha\beta}P_\mu\delta^{ij} + ... \tag{3.15}$$

In the case of a single supersymmetry, for the 2d Wess-Zumino model we don't have any +..., the above algebra is complete. In order to realize it on the fields, we need that (since P_μ is represented by the translation ∂_μ and Q_α is represented by δ_{ϵ_α})

$$[\delta_{\epsilon_{1,\alpha}}, \delta_{\epsilon_{2\beta}}]\begin{pmatrix}\phi \\ \psi\end{pmatrix} = 2\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu\begin{pmatrix}\phi \\ \psi\end{pmatrix} \tag{3.16}$$

We get that

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi = 2\bar{\epsilon}_2\gamma^\rho\epsilon_1\partial_\rho\phi \tag{3.17}$$

as expected, but

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi = 2(\bar{\epsilon}_2\gamma^\rho\epsilon_1)\partial_\rho\psi - (\bar{\epsilon}_2\gamma^\rho\epsilon_1)\gamma_\rho\partial\!\!\!/\psi \tag{3.18}$$

thus we have an extra term that vanishes on-shell ($\partial\!\!\!/\psi = 0$). So on-shell the algebra is satisfied and we have on-shell supersymmetry.

It is left as an exercise to prove these relations. One must use the previous spinor identities together with new ones, called 2 dimensional "Fierz identities" (or "Fierz recoupling"),

$$M\chi(\bar{\psi}N\phi) = -\sum_j \frac{1}{2} MO_j N\phi(\bar{\psi}O_j\chi) \tag{3.19}$$

where M and N are arbitrary matrices, and the set of matrices $\{O_j\}$ is $= \{1, \gamma_\mu, \gamma_5\}$ (in 2 Minkowski dimensions, $\gamma_\mu = (i\tau_1, \tau_2)$ and $\gamma_5 = \tau_3$, where τ_i are Dirac matrices).

Off-shell supersymmetry

In 2 dimensions, an off-shell Majorana fermion has 2 degrees of freedom, but a scalar has only one. Thus to close the algebra of the Wess-Zumino model off-shell, we need one extra scalar field F. But on-shell, we must get back the previous model, thus the extra scalar F needs to be auxiliary (non-dynamical, with no propagating degree of freedom). That means that its action is $\int F^2/2$, thus

$$S = -\frac{1}{2}\int d^2x[(\partial_\mu\phi)^2 + \bar{\psi}\partial\!\!\!/\psi - F^2] \tag{3.20}$$

From the action we see that F has mass dimension $[F] = 1$, and the equation of motion of F is $F = 0$. The off-shell Wess-Zumino model algebra does not close on ψ, thus we need to add to $\delta\psi$ a term proportional to the equation of motion of F. By dimensional analysis, $F\epsilon$ has the right dimension. Since F itself is a (bosonic) equation of motion, its variation δF should be the fermionic equation of motion, and by dimensional analysis $\bar{\epsilon}\partial\!\!\!/\psi$ is OK. Thus the transformations laws are

$$\delta\phi = \bar{\epsilon}\psi; \quad \delta\psi = \partial\!\!\!/\phi\epsilon + F\epsilon; \quad \delta F = \bar{\epsilon}\partial\!\!\!/\psi \tag{3.21}$$

We can similarly check that these transformations leave the action invariant again, and moreover now we have

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\begin{pmatrix}\phi \\ \psi \\ F\end{pmatrix} = 2\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu\begin{pmatrix}\phi \\ \psi \\ F\end{pmatrix} \tag{3.22}$$

so the algebra closes off-shell, i.e. we have an off-shell representation of $\{Q_\alpha, Q_\beta\} = 2(C\gamma^\mu)_{\alpha\beta}P_\mu$.

4 dimensions

Similarly, in 4 dimensions the on-shell Wess-Zumino model has one Majorana fermion, which however now has 2 real on-shell degrees of freedom, thus needs 2 real scalars, A and B. The action is then

$$S_0 = -\frac{1}{2}\int d^4x[(\partial_\mu A)^2 + (\partial_\mu B)^2 + \bar{\psi}\partial\!\!\!/\psi] \tag{3.23}$$

and the transformation laws are as in 2 dimensions, except now B aquires a γ_5 to distinguish it from A, thus

$$\delta A = \bar{\epsilon}\psi; \quad \delta B = \bar{\epsilon}i\gamma_5\psi; \quad \delta\psi = \partial\!\!\!/(A + i\gamma_5 B)\epsilon \tag{3.24}$$

And again, off-shell the Majorana fermion has 4 degrees of freedom, so one needs to introduce one auxiliary scalar for each propagating scalar, and the action is

$$S = S_0 + \int d^4x [\frac{F^2}{2} + \frac{G^2}{2}] \tag{3.25}$$

with the transformation rules

$$\begin{aligned}
&\delta A = \bar{\epsilon}\psi; \quad \delta B = \bar{\epsilon} i\gamma_5 \psi; \quad \delta\psi = \partial\!\!\!/ (A + i\gamma_5 B)\epsilon + (F + i\gamma_5 G)\epsilon \\
&\delta F = \bar{\epsilon}\partial\!\!\!/\psi; \quad \delta G = \bar{\epsilon} i\gamma_5 \partial\!\!\!/\psi
\end{aligned} \tag{3.26}$$

One can form a complex field $\phi = A + iB$ and one complex auxiliary field $M = F + iG$, thus the Wess-Zumino multiplet in 4 dimensions is (ϕ, ψ, M).

We have written the free Wess-Zumino model in 2d and 4d, but one can write down interactions between them as well, that preserve the supersymmetry.

These were examples of $\mathcal{N} = 1$ supersymmetry that is, there was only one supersymmetry generator Q_α. The possible on-shell multiplets of $\mathcal{N} = 1$ supersymmetry that have spins ≤ 1 are

- The Wess-Zumino or chiral multiplet that we discussed, (ϕ, ψ).
- The vector multiplet (λ^A, A^A_μ), where A is an adjoint index. The vector A_μ in 4 dimensions has 2 on-shell degrees of freedom: it has 4 components, minus one gauge invariance symmetry parametrized by an arbitrary ϵ^a, $\delta A^A_\mu = \partial_\mu \epsilon^A$ giving 3 off-shell components. In the covariant gauge $\partial^\mu A_\mu = 0$ the equation of motion $k^2 = 0$ is supplemented with the constraint $k^\mu \epsilon^a_\mu(k) = 0$ ($\epsilon^a_\mu(k)$ =polarization), which has only 2 independent solutions. The two degrees of freedom of the gauge field match the 2 degrees of freedom of the on-shell fermion.

For $\mathcal{N} \geq 2$ supersymmetry, we have Q^i_α with $i = 1, ..., \mathcal{N}$. For $\mathcal{N} = 2$, the possible multiplets of spins ≤ 1 are

- The $\mathcal{N} = 2$ vector multiplet, made of one $\mathcal{N} = 1$ vector multiplet (A_μ, λ) and one $\mathcal{N} = 1$ chiral (Wess-Zumino) multiplet (ψ, ϕ).
- The $\mathcal{N} = 2$ hypermultiplet, made of two $\mathcal{N} = 1$ chiral multiplets (ψ_1, ϕ_1) and (ψ_2, ϕ_2).

For $\mathcal{N} = 4$ supersymmetry, there is a single multiplet of spins ≤ 1, the $\mathcal{N} = 4$ vector multiplet, made of an $\mathcal{N} = 2$ vector multiplet and a $\mathcal{N} = 2$ hypermultiplet, or one $\mathcal{N} = 1$ vector multiplet (A_μ, ψ_4) and 3 $\mathcal{N} = 1$ chiral multiplets $(\psi_i, \phi_1), i = 1, 2, 3$. They can be rearranged into $(A^a_\mu, \psi^{ai}, \phi_{[ij]})$, where $i = 1, .., 4$ is an $SU(4) = SO(6)$ index, $[ij]$ is the 6 dimensional antisymmetric representation of SU(4) or the fundamental representation of SO(6), and i is the fundamental representation of SU(4) or the spinor representation of SO(6). The field $\phi_{[ij]}$ has complex entries but satisfies a reality condition,

$$\phi^\dagger_{ij} = \phi^{ij} \equiv \epsilon^{ijkl}\phi_{kl} \tag{3.27}$$

The action of the $\mathcal{N}=1$ vector multiplet is

$$S_{\mathcal{N}=1SYM} = \int d^4x tr[-\frac{1}{4}F^2_{\mu\nu} - \bar{\lambda}\not{D}\lambda(+\frac{D^2}{2})] \tag{3.28}$$

where $\not{D} = \gamma^\mu D_\mu$ and D is an auxiliary field for the off-shell action. It is just the action of a gauge field, a spinor minimally coupled to it, and an auxiliary field. The transformation rules are

$$\begin{aligned} \delta A^a_\mu &= \bar{\epsilon}\gamma_\mu\psi^a \\ \delta\psi^a &= (-\frac{1}{2}\gamma^{\mu\nu}F^a_{\mu\nu} + i\gamma_5 D^a)\epsilon \\ \delta D^a &= i\bar{\epsilon}\gamma_5\not{D}\psi^a \end{aligned} \tag{3.29}$$

They are similar to the rules of the Wess-Zumino multiplet, except for the gamma matrix factors introduced in order to match the index structure, and for replacing $\partial_\mu\phi$ with $F_{\mu\nu}$.

The action of the $\mathcal{N}=4$ Super Yang-Mills multiplet is

$$\begin{aligned} S_{\mathcal{N}=4SYM} = \int d^4x tr[&-\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}\bar{\psi}_i\not{D}\psi^i - \frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij} \\ &-\frac{1}{2}\bar{\psi}^i[\phi_{ij},\psi^j] + \frac{1}{4}[\phi_{ij},\phi_{kl}][\phi^{ij},\phi^{kl}]] \end{aligned} \tag{3.30}$$

where $D_\mu = \partial_\mu + g[A_\mu, \quad]$. This action however has no (covariant and un-constrained auxiliary fields) off-shell formulation.

The supersymmetry rules are

$$\begin{aligned} \delta A^a_\mu &= \bar{\epsilon}_i\gamma_\mu\lambda^{ai} \\ \delta\phi^{ij}_a &= \bar{\epsilon}^{(i}\lambda^{j)}_a \\ \delta\lambda^{ai} &= -\frac{\gamma^{\mu\nu}}{2}F^a_{\mu\nu}\epsilon^i - \frac{1}{2}(\gamma^{mn})^i{}_j\partial_{(m}\phi^{a,kl}(\gamma_{n)})_{kl}\epsilon^j \end{aligned} \tag{3.31}$$

Important concepts to remember

- A graded Lie algebra can contain the Poincare algebra, internal algebra and supersymmetry.
- The supersymmetry Q_α relates bosons and fermions.
- If the on-shell number of degrees of freedom of bosons and fermions match we have on-shell supersymmetry, if the off-shell number matches we have off-shell supersymmetry.
- For off-shell supersymmetry, the supersymmetry algebra must be realized on the fields.
- The prototype for all (linear) supersymmetry is the 2 dimensional Wess-Zumino model, with $\delta\phi = \bar{\epsilon}\psi, \delta\psi = \not{\partial}\phi\epsilon$.

- The Wess-Zumino model in 4 dimensions has a fermion and a complex scalar on-shell. Off-shell there is also an auxiliary complex scalar.
- The on-shell vector multiplet has a gauge field and a fermion
- The $\mathcal{N} = 4$ supersymmetric vector multiple ($\mathcal{N} = 4$ SYM) has one gauge field, 4 fermions and 6 scalars, all in the adjoint of the gauge field.

Exercises, section 3

1) Prove that the matrix

$$C_{AB} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}; \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.32}$$

is a representation of the 4d C matrix, i.e. $C^T = -C, C\gamma^\mu C^{-1} = -(\gamma^\mu)^T$, if γ^μ is represented by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}; \quad (\sigma^\mu)_{\alpha\dot{\alpha}} = (1, \vec{\sigma})_{\alpha\dot{\alpha}}; \quad (\bar{\sigma}^\mu)^{\alpha\dot{\alpha}} = (1, -\vec{\sigma})^{\alpha\dot{\alpha}} \tag{3.33}$$

2) Prove that if ϵ, χ are 4d Majorana spinors, we have

$$\bar{\epsilon}\gamma_\mu\gamma_5\chi = +\bar{\chi}\gamma_\mu\gamma_5\epsilon \tag{3.34}$$

3) Prove that, for

$$S = -\frac{1}{2}\int d^4x[(\partial_\mu\phi)^2 + \bar{\psi}\partial\!\!\!/\psi] \tag{3.35}$$

we have

$$\begin{aligned} [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi &= 2\bar{\epsilon}_2\partial\!\!\!/\epsilon_1\phi \\ [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi &= 2(\bar{\epsilon}_2\gamma^\rho\epsilon_1)\partial_\rho\psi - (\bar{\epsilon}_2\gamma^\rho\epsilon_1)\gamma_\rho\partial\!\!\!/\psi \end{aligned} \tag{3.36}$$

4) Show that the susy variation of the 4d Wess-Zumino model is zero, paralleling the 2d WZ model.

5) Check the invariance of the N=1 off-shell SYM action

$$S = \int d^4x[-\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2}\bar{\psi}^a D\!\!\!\!/\,\psi_a + \frac{1}{2}D^2_a] \tag{3.37}$$

under the susy transformations

$$\delta A^a_\mu = \bar{\epsilon}\gamma_\mu\psi^a; \quad \delta\psi^a = (-\frac{1}{2}\sigma^{\mu\nu}F^a_{\mu\nu} + i\gamma_5 D^a)\epsilon; \quad \delta D^a = i\bar{\epsilon}\gamma_5 D\!\!\!\!/\,\psi^a \tag{3.38}$$

6)Calculate the number of off-shell degrees of freedom of the on-shell N=4 SYM action. Propose a set of bosonic+fermionic auxiliary fields that could make the number of degrees of freedom match. Are they likely to give an off-shell formulation, and why?

4 Basics of supergravity

Vielbeins and spin connections

We saw that gravity is defined by the metric $g_{\mu\nu}$, which in turn defines the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}(g)$, which is like a gauge field of gravity, with the Riemann tensor $R^\mu{}_{\nu\rho\sigma}(\Gamma)$ playing the role of its field strength.

But there is a formulation that makes the gauge theory analogy more manifest, namely in terms of the "vielbein" e_μ^a and the "spin connection" ω_μ^{ab}. The word "vielbein" comes from the german viel= many and bein=leg. It was introduced in 4 dimensions, where it is known as "vierbein", since vier=four. In various dimensions one uses einbein, zweibein, dreibein,... (1,2,3= ein, zwei, drei), or generically vielbein, as we will do here.

Any curved space is locally flat, if we look at a scale much smaller than the scale of the curvature. That means that locally, we have the Lorentz invariance of special relativity. The vielbein is an object that makes that local Lorentz invariance manifest. It is a sort of square root of the metric, i.e.

$$g_{\mu\nu}(x) = e_\mu^a(x)e_\nu^b(x)\eta_{ab} \tag{4.1}$$

so in $e_\mu^a(x)$, μ is a "curved" index, acted upon by a general coordinate transformation (so that e_μ^a is a covariant vector of general coordinate transformations, like a gauge field), and a is a newly introduced "flat" index, acted upon by a local Lorentz gauge invariance. That is, around each point we define a small flat neighbourhood ("tangent space") and a is a tensor index living in that local Minkowski space, acted upon by Lorentz transformations.

We can check that an infinitesimal general coordinate transformation ("Einstein" transformation) $\delta x^\mu = \xi^\mu$ acting on the metric gives

$$\delta_\xi g_{\mu\nu}(x) = (\xi^\rho\partial_\rho)g_{\mu\nu} + (\partial_\mu\xi^\rho)g_{\rho\nu} + (\partial_\nu\xi^\rho)g_{\rho\nu} \tag{4.2}$$

where the first term corresponds to a translation, but there are extra terms. Thus the general coordinate transformations are the general relativity analog of P_μ translations in special relativity.

On the vielbein e_μ^a, the infinitesimal coordinate transformation gives

$$\delta_\xi e_\mu^a(x) = (\xi^\rho\partial_\rho)e_\mu^a + (\partial_\mu\xi^\rho)e_\rho^a \tag{4.3}$$

thus it acts only on the curved index μ. On the other hand, the local Lorentz transformation

$$\delta_{l.L.}e_\mu^a(x) = \lambda^a{}_b(x)e_\mu^b(x) \tag{4.4}$$

is as usual.

Thus the vielbein is like a sort of gauge field, with one covariant vector index and a gauge group index. But there is one more "gauge field" ω_μ^{ab}, the "spin connection", which is defined as the "connection" ($\equiv$ gauge field) for the action of the Lorentz group on spinors.

Namely, the curved space covariant derivative acting on spinors acts similarly to the gauge field covariant derivative on a spinor, by

$$D_\mu\psi = \partial_\mu\psi + \frac{1}{4}\omega_\mu^{ab}\Gamma^{ab}\psi \tag{4.5}$$

This definition means that $D_\mu\psi$ is the object that transforms as a tensor under general coordinate transformations. It implies that ω_μ^{ab} acts as a gauge field on any local Lorentz index.

If there are no *dynamical* fermions (i.e. fermions that have a kinetic term in the action) then $\omega_\mu^{ab} = \omega_\mu^{ab}(e)$ is a fixed function, defined through the "vielbein postulate"

$$T^a_{[\mu\nu]} = D_{[\mu}e^a_{\nu]} = \partial_{[\mu}e^a_{\nu]} + \omega^{ab}_{[\mu}e^b_{\nu]} = 0 \tag{4.6}$$

Note that we can also start with

$$D_\mu e^a_\nu \equiv \partial_\mu e^a_\nu + \omega^{ab}_\mu e^b_\nu - \Gamma^\rho{}_{\mu\nu}e^a_\rho = 0 \tag{4.7}$$

ant antisymmetrize, since $\Gamma^\rho{}_{\mu\nu}$ is symmetric. This is also sometimes called the vielbein postulate.

Here T^a is called the "torsion", and as we can see it is a sort of field strength of e^a_μ, and the vielbein postulate says that the torsion (field strength of vielbein) is zero.

But we can also construct an object that is a field strength of ω_μ^{ab},

$$R^{ab}_{\mu\nu}(\omega) = \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \omega^{ab}_\mu\omega^{bc}_\nu - \omega^{ac}_\nu\omega^{cb}_\mu \tag{4.8}$$

and this time the definition is exactly the definition of the field strength of a gauge field of the Lorentz group (though there still are subtleties in trying to make the identification of ω_μ^{ab} with a gauge field of the Lorentz group).

This curvature is in fact the analog of the Riemann tensor, i.e. we have

$$R^{ab}_{\rho\sigma}(\omega(e)) = e^a_\mu e^{-1,\nu b}R^\mu{}_{\nu\rho\sigma}(\Gamma(e)) \tag{4.9}$$

The Einstein-Hilbert action is then

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x(\det e)R^{ab}_{\mu\nu}(\omega(e))e_a^{-1,\mu}e^{-1,\nu}|_b \tag{4.10}$$

since $\sqrt{detg} = \det e$.

The formulation just described of gravity in terms of e and ω is the *second order formulation*, so called because ω is not independent, but is a function of e.

But notice that if we make ω an independent variable in the above Einstein-Hilbert action, the ω equation of motion gives exactly $T^a_{\mu\nu} = 0$, i.e. the vielbein postulate that we needed to postulate before. Thus we might as well make ω independent without changing the classical theory (only possibly the quantum version). This is then the *first order formulation* of gravity (Palatini formalism), in terms of $(e^a_\mu, \omega^{ab}_\mu)$.

Supergravity

Supergravity can be defined in two independent ways that give the same result. It is a supersymmetric theory of gravity; and it is also a theory of local supersymmetry. Thus we could either take Einstein gravity and supersymmetrize it, or we can take a supersymmetric model and make the supersymmetry local. In practice we use a combination of the two.

We want a theory of local supersymmetry, which means that we need to make the rigid ϵ^α transformation local. We know from gauge theory that if we want to make a global symmetry local we need to introduce a gauge field for the symmetry. The gauge field would be "A_μ^α" (since the supersymmetry acts on the index α), which we denote in fact by $\psi_{\mu\alpha}$ and call the gravitino.

Here μ is a curved space index ("curved") and α is a local Lorentz spinor index ("flat"). In flat space, $\psi_{\mu\alpha}$ would have the same kind of indices and we can then show that it forms a spin 3/2 field, therefore the same is true in curved space.

The fact that we have a supersymmetric theory of gravity means that gravitino must be transformed by supersymmetry into some gravity variable, thus $\psi_\alpha = Q_\alpha(gravity)$. But the index structure tells us that the gravity variable cannot be the metric, but something with only one curved index, namely the vielbein.

Therefore we see that supergravity needs the vielbein-spin connection formulation of gravity. To write down the supersymmetry transformations, we start with the vielbein. In analogy with the Wess-Zumino model where $\delta\phi = \bar\epsilon\phi$ or the vector multiplet where the gauge field variation is $\delta A_\mu^a = \bar\epsilon\gamma_\mu\psi^a$, it is easy to see that the vielbein variation has to be

$$\delta e_\mu^a = \frac{k}{2}\bar\epsilon\gamma^a\psi_\mu \tag{4.11}$$

where k is the Newton constant. Since ψ is like a gauge field of local supersymmetry, we expect something like $\delta A_\mu = D_\mu\epsilon$. Therefore we must have

$$\delta\psi_\mu = \frac{1}{k}D_\mu\epsilon; \quad D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon \tag{4.12}$$

plus maybe more terms.

The action for a free spin 3/2 field in flat space is the Rarita-Schwinger action which is

$$S_{RS} = -\frac{1}{2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\bar\psi_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma = -\frac{1}{2}\int d^dx\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho \tag{4.13}$$

where the first form is only valid in 4 dimensions and the second is valid in all dimensions ($\epsilon^{\mu\nu\rho\sigma}\gamma_4\gamma_\nu = \gamma^{\mu\rho\sigma}$ in 4 dimensions). In curved space, this becomes

$$S_{RS} = -\frac{1}{2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\bar\psi_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma = -\frac{1}{2}\int d^dx(dete)\bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho \tag{4.14}$$

$\mathcal{N}=1$ (on-shell) supergravity in 4 dimensions

We are now ready to write down $\mathcal{N}=1$ on-shell supergravity in 4 dimensions. Its action is just the sum of the Einstein-Hilbert action and the Rarita-Schwinger action

$$S_{\mathcal{N}=1} = S_{EH}(\omega, e) + S_{RS}(\psi_\mu) \tag{4.15}$$

and the supersymmetry transformations rules are just the ones defined previously,

$$\delta e_\mu^a = \frac{k}{2}\bar\epsilon\gamma^a\psi_\mu; \quad \delta\psi_\mu = \frac{1}{k}D_\mu\epsilon \tag{4.16}$$

However, this is not yet enough to specify the theory. We must specify the formalism and various quantities:

- second order formalism: The independent fields are $e_\mu 6a, \psi_\mu$. ω is not an independent field. But now there is a dynamical fermion (ψ_μ), so the torsion $T^a_{\mu\nu}$ is not zero anymore, thus $\omega \neq \omega(e)$! In fact,

$$\omega_\mu^{ab} = \omega_\mu^{ab}(e,\psi) = \omega_\mu^{ab}(e) + \psi\psi \ \text{ terms} \tag{4.17}$$

is found by varying the action with respect to ω, as in the $\psi = 0$ case:

$$\frac{\delta S_{\mathcal{N}=1}}{\delta \omega_\mu^{ab}} = 0 \Rightarrow \omega_\mu^{ab}(e,\psi) \tag{4.18}$$

- first order formalism: All fields, ψ, e, ω are independent. But now we must supplement the action with a transformation law for ω. It is

$$\begin{aligned}\delta\omega_\mu^{ab}(\text{first order}) &= -\frac{1}{4}\bar\epsilon\gamma_5\gamma_\mu\tilde\psi^{ab} + \frac{1}{8}\bar\epsilon\gamma_5(\gamma^\lambda\tilde\psi^b_\lambda e^a_\mu - \gamma^\lambda\tilde\psi^a_\lambda e^b_\mu)\\ \tilde\psi^{ab} = \epsilon^{abcd}\psi_{cd}; &\quad \psi_{ab} = e_a^{-1\mu}e_b^{-1\nu}(D_\mu\psi_\nu - D_\nu\psi_\mu)\end{aligned} \tag{4.19}$$

General features of supergravity theories
4 dimensions

The $\mathcal{N} = 1$ supergravity multiplet is $(e^a_\mu, \psi_{\mu\alpha})$ as we saw, and has spins (2,3/2).

It can also couple with other $\mathcal{N} = 1$ supersymmetric multiplets of lower spin: the chiral multiplet of spins (1/2,0) and the gauge multiplet of spins (1,1/2) that have been described, as well as the so called gravitino multiplet, composed of a gravitino and a vector, thus spins (3/2,1).

By adding appropriate numbers of such multiplets we obtain the $\mathcal{N} = 2, 3, 4, 8$ supergravity multiplets. Here $\mathcal{N}$ is the number of supersymmetries, and since it acts on the graviton, there should be exactly $\mathcal{N}$ gravitini in the multiplet, so that each supersymetry maps the graviton to a different gravitino.

$\mathcal{N} = 8$ supergravity is the maximal supersymmetric multiplet that has spins ≤ 2 (i.e., an $\mathcal{N} > 8$ multiplet will contains spins > 2, which are not very well defined), so we consider only $\mathcal{N} \leq 8$.

Coupling to supergravity of a supersymmetric multiplet is a generalization of coupling to gravity, which means putting fields in curved space. Now we put fields in curved space and introduce also a few more couplings.

We will denote the $\mathcal{N} = 1$ supersymmetry multiplets by brackets, e.g. (1,1/2), (1/2,0), etc. The supergravity multiplets are compose of the following fields:

$\mathcal{N} = 3$ supergravity: Supergravity multiplet (2,3/2) + 2 gravitino multiplets (3/2,1) + one vector multiplet (1,1/2). The fields are then $\{e^a_\mu, \psi^i_\mu, A^i_\mu, \lambda\}$, i=1,2,3.

$\mathcal{N} = 4$ supergravity: Supergravity multiplet (2,3/2) + 3 gravitino multiplets (3/2,1) + 3 vector multiplets (1,1/2) + one chiral multiplet (1/2,0). The fields are $\{e^a_\mu, \psi^i_\mu, A^k_\mu, B^k_\mu, \lambda^i, \phi, B\}$, where i=1,2,3,4; k=1,2,3, A is a vector, B is an axial vector, ϕ is a scalar and B is a pseudoscalar.

$\mathcal{N} = 8$ supergravity: Supergravity multiplet (2,3/2) + 7 gravitino multiplets (3/2,1) + 21 vector multiplets (1,1/2) + 35 chiral multiplets (1/2,0). The fields are $\{e^a_\mu, \psi^i_\mu, A^{IJ}_\mu, \chi_{ijk}, \nu\}$

which are: one graviton, 8 gravitinos ψ_μ^i, 28 photons A_μ^{IJ}, 56 spin 1/2 fermions χ_{ijk} and 70 scalars in the matrix ν.

In these models, the photons are not coupled to the fermions, i.e. the gauge coupling $g = 0$, thus they are "ungauged" models. But these models have *global* symmetries, e.g. the $\mathcal{N} = 8$ model has SO(8) global symmetry.

One can couple the gauge fields to the fermions, thus "gauging" (making local) some global symmetry (e.g. SO(8)). Thus abelian fields become nonabelian (Yang-Mills), i.e. self-coupled. Another way to obtain the gauged models is by adding a cosmological constant and requiring invariance

$$\delta\psi_\mu^i = D_\mu(\omega(e,\psi))\epsilon^i + g\gamma_\mu\epsilon^i + gA_\mu\epsilon^i \tag{4.20}$$

where g is related to the cosmological constant, i.e. $\Lambda \propto g$. Because of the cosmological constant, it means that gauged supergravities have Anti de Sitter (AdS) backgrounds.

Higher dimensions

In $D > 4$, it is possible to have also antisymmetric tensor fields $A_{\mu_1,\dots,\mu_n}$, which are just an extension of abelian vector fields, with field strength

$$F_{\mu_1,\dots,\mu_{n+1}} = \partial_{[\mu_1}A_{\mu_2,\dots,\mu_{n+1}]} \tag{4.21}$$

and gauge invariance

$$\delta A_{\mu_1,\dots,\mu_n} = \partial_{[\mu_1}\Lambda_{\mu_2,..,\mu_n]} \tag{4.22}$$

and action

$$\int d^dx(\det e)F^2_{\mu_1,\dots,\mu_{n+1}} \tag{4.23}$$

The maximal model possible that makes sense as a 4 dimensional theory is the $\mathcal{N} = 1$ supergravity model in 11 dimensions, made up of a graviton e_μ^a, a gravitino $\psi_{\mu\alpha}$ and a 3 index antisymmetric tensor $A_{\mu\nu\rho}$.

But how do we make sense of a higher dimensional theory? The answer is the so called **Kaluza-Klein (KK) dimensional reduction**. The idea is that the extra dimensions $(d-4)$ are curled up in a small space, like a small sphere or a small $d-4$-torus.

For this to happen, we consider a background solution of the theory that looks like, e.g. (in the simplest case) as a product space,

$$g_{\Lambda\Sigma} = \begin{pmatrix} g^{(0)}_{\mu\nu}(x) & 0 \\ 0 & g^{(0)}_{mn}(y) \end{pmatrix} \tag{4.24}$$

where $g^{(0)}_{\mu\nu}(x)$ is the metric on our 4 dimensional space and $g^{(0)}_{mn}(y)$ is the metric on the extra dimensional space.

We then expand the fields of the higher dimensional theory around this background solution in Fourier-like modes, called spherical harmonics. E.g., $g_{\mu\nu}(x,y) = g^{(0)}_{\mu\nu}(x) + \sum_n g^{(n)}_{\mu\nu}(x)Y_n(y)$, with $Y_n(y)$ being the spherical harmonic (like e^{ikx} for Fourier modes).

Finally, dimensional reduction means dropping the higher modes, and keeping only the lowest Fourier mode, the constant one, e.g.

$$g_{\Lambda\Sigma} = \begin{pmatrix} g^{(0)}_{\mu\nu}(x) + h_{\mu\nu}(x) & h_{\mu m}(x) \\ h_{m\nu}(x) & g^{(0)}_{mn}(y) + h_{mn}(x) \end{pmatrix} \tag{4.25}$$

Important concepts to remember

- Vielbeins are defined by $g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x)\eta_{ab}$, by introducing a Minkowski space in the neighbourhood of a point x, giving local Lorentz invariance.
- The spin connection is the gauge field needed to define covariant derivatives acting on spinors. In the absence of dynamical fermions, it is determined as $\omega = \omega(e)$ by the vielbein postulate: the torsion is zero.
- The field strength of this gauge field is related to the Riemann tensor.
- In the first order formulation (Palatini), the spin connection is independent, and is determined from its equation of motion.
- Supergravity is a supersymmetric theory of gravity and a theory of local supersymmetry.
- The gauge field of local supersymmetry and superpartner of the vielbein (graviton) is the gravitino ψ_μ.
- Supergravity (local supersymmetry) is of the type $\delta e^a_\mu = (k/2)\bar{\epsilon}\gamma^a\psi_\mu + ...$, $\delta\psi_\mu = (D_\mu\epsilon)/k + ...$
- For each supersymmetry we have a gravitino. The maximal supersymmetry in d=4 is $\mathcal{N} = 8$.
- Supergravity theories in higher dimensions can contain antisymmetric tensor fields.
- The maximal dimension for a supergravity theory is d=11, with a unique model composed of $e^a_\mu, \psi_\mu, A_{\mu\nu\rho}$.
- A higher dimensional theory can be dimensionally reduced: expand in generalized Fourier modes (spherical harmonics) around a vacuum solution that contains a compact space for the extra dimensions (like a sphere or torus), and keep only the lowest modes.

Exercises, section 4

1) Prove that the general coordinate transformation on $g_{\mu\nu}$,

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x)\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} \tag{4.26}$$

reduces for infinitesimal tranformations to

$$\partial_\xi g_{\mu\nu}(x) = (\xi^\rho\partial_\rho)g_{\mu\nu} + (\partial_\mu\xi^\rho)g_{\rho\sigma} + (\partial_\nu\xi^\rho)g_{\rho\mu} \tag{4.27}$$

2) Check that

$$\omega_\mu^{ab}(e) = \frac{1}{2}e^{a\nu}(\partial_\mu e_\nu^b - \partial_\nu e_\mu^b) - \frac{1}{2}e^{b\nu}(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) - \frac{1}{2}e^{a\rho}e^{b\sigma}(\partial_\rho e_{c\sigma} - \partial_\sigma e_{c\rho})e_\mu^c \tag{4.28}$$

satisfies the no-torsion (vielbein) constraint, $T^a_{\mu\nu} = D_{[\mu}e^a_{\nu]} = 0$.

3) Check that the equation of motion for ω_μ^{ab} in the first order formulation of gravity (Palatini formalism) gives $T^a_{\mu\nu} = 0$.

4) Write down the free gravitino equation of motion in curved space.

5) Find $\omega_\mu^{ab}(e,\psi) - \omega_\mu^{ab}(e)$ in the second order formalism for N=1 supergravity.

6) Calculate the number of off-shell bosonic and fermionic degrees of freedom of N=8 on-shell supergravity.

7) Consider the Kaluza Klein dimensional reduction ansatz from 5d to 4d

$$g_{\Lambda\Pi} = \phi^{-1/3}\begin{pmatrix} \eta_{\mu\nu} + h_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix} \tag{4.29}$$

Show that the action for the linearized perturbation $h_{\mu\nu}$ contains no factors of ϕ. (Hint: first show that for small $h_{\mu\nu}$, where $g_{\mu\nu} = f(\eta_{\mu\nu} + h_{\mu\nu})$, $R_{\mu\nu}(g)$ is independent of f).

5 Black holes and p-branes

The Schwarzschild solution (1916)

The Schwarzschild solution is a solution to the Einstein's equation without matter ($T_{\mu\nu} = 0$), namely

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{5.1}$$

It is in fact the most general solution of Einstein's equation with $T_{\mu\nu} = 0$ and spherical symmetry (Birkhoff's theorem, 1923). That means that by general coordinate transformations we can always bring the metric to this form.

The 4 dimensional solution is

$$ds^2 = -(1 - \frac{2MG}{r})dt^2 + \frac{dr^2}{1 - \frac{2MG}{r}} + R^2 d\Omega_2^2 \tag{5.2}$$

It is remarkable that Schwarzschild derived this solution while fighting in World War I, (literally, in the trenches: in fact, he even got ill there and died shortly after the end of WWI).

The Newtonian approximation of general relativity is one of weak fields, i.e. $g_{\mu\nu} - \eta_{\mu\nu} \equiv h_{\mu\nu} \ll 1$ and nonrelativistic, i.e. $v \ll 1$. In this limit, one can prove that the metric can be written in the general form

$$ds^2 \simeq -(1 + 2U)dt^2 + (1 - 2U)d\vec{x}^2 = -(1 + 2U)dt^2 + (1 - 2U)(dr^2 + r^2 d\Omega_2^2) \tag{5.3}$$

where U =Newtonian potential for gravity. In this way we recover Newton's gravity theory. We can check that, with a $O(\epsilon)$ redefinition of r, the Newtonian approximation metric matches the Schwarzschild metric if

$$U(r) = -\frac{2MG}{r} \tag{5.4}$$

without any additional coordinate transformations, so at least its Newtonian limit is correct.

Observation: Of course, this metric has a source at $r = 0$, which we can verify in the Newtonian approximation: the solution is given by a point mass situated at $r = 0$. But the point is that if the space is empty at $r \geq r_0$, with r_0 some arbitrary value, and is spherically symmetric, Birkhoff's theorem says that we should obtain the Schwarzschild metric for $r \geq r_0$ (and maybe a modified solution at $r \leq r_0$).

But the solution becomes apparently singular at $r_H = 2MG > 0$, so it would seem that it cannot reach its source at $r = 0$? This would be a paradoxical situation, since then what would be the role of the source? It would seem as if we don't really need a point mass to create this metric.

If the Schwarzschild solution is valid all the way to $r = r_H$ (not just to some $r_0 > r_H$ which is the case for, let's say, the gravitational field of the Earth, in which case r_0 is the Earth's radius), then we call that solution a **Schwarzschild black hole**.

So what does happen at $r_H = 2MG$? We will try to understand it in the following.

First, let's investigate the propagation of light (the fastest possible signal). If light propagates radially ($d\theta = d\phi = 0$), $ds^2 = 0$ (light propagation) implies

$$dt = \frac{dr}{1 - \frac{2MG}{r}} \tag{5.5}$$

That means that near r_H we have

$$dt \simeq 2MG \frac{dr}{r - 2MG} \Rightarrow t \simeq 2MG \ln(r - 2MG) \to \infty \tag{5.6}$$

In other words, from the point of view of an asymptotic observer, that measures coordinates r, t (since at large r $ds^2 \simeq -dt^2 + dr^2 + r^2 d\Omega_2^2$), it takes an infinite time for light to reach r_H. And reversely, it takes an infinite time for a light signal from $r = r_H$ to reach the observer at large r. That means that $r = r_H$ is cut-off from causal communication with $r = r_H$. For this reason, $r = r_H$ is called an "event horizon". Nothing can reach, nor escape from the event horizon.

Observation: However, quantum mechanically, Hawking proved that black holes radiate thermally, thus thermal radiation does escape the event horizon of the black hole.

But is the event horizon of the black hole singular or not?

The answer is actually NO. In gravity, the metric is not gauge invariant, it changes under coordinate transformations. The appropriate gauge invariant (general coordinate transformations invariant) quantity that measures the curvature of space is the Ricci scalar R. One can calculate it for the Schwarzschild solution and one obtains that at the event horizon

$$R \sim \frac{1}{r_H^2} = \frac{1}{(2MG)^2} = \text{finite!} \tag{5.7}$$

Since the curvature of space at the horizon is finite, an observer falling into a black hole doesn't feel anything special at $r = r_H$, other than a finite curvature of space creating some tidal force.

So for an observer at large r, the event horizon looks singular, but for an observer falling into the black hole it doesn't seem remarkable at all. This shows that in general relativity, more than in special relativity, different observers see apparently different events: For instance, in special relativity, synchronicity of two events is relative.

An observer at fixed r close to the horizon sees an apparently singular behaviour: If $dr = 0, d\Omega = 0$, then

$$ds^2 = -\frac{dt^2}{1 - \frac{2MG}{r}} = -d\tau^2 \Rightarrow d\tau = \sqrt{-g_{00}}dt = \frac{dt}{\sqrt{1 - \frac{2MG}{r}}} \tag{5.8}$$

thus the time measured by that observer becomes infinite as $r \to r_H$, and we get an infinite time dilation: an observer fixed at the horizon is "frozen in time" from the point of view of the observer at infinity.

Since there is no singularity at the event horizon, it means that there must exist coordinates that continue inside the horizon, and there are indeed. The first such coordinates were found by Eddington (around 1924!) and Finkelstein (in 1958! He rediscovered it, whithout being aware of Eddington's work, which shows that the subject of black holes was not so popular back then...). The Eddington-Finkelstein coordinates however don't cover all the geometry.

The first set of coordinates that cover all the geometry was found by Kruskal and Szekeres in 1960, and they give maximum insight into the physics, so we will describe them here.

One first introduces the "tortoise" coordinates r_* by imposing

$$\frac{dr}{1-\frac{2MG}{r}} = dr_* \Rightarrow r_* = r + 2mG\ln(\frac{r}{2MG}-1) \tag{5.9}$$

which gives the metric

$$ds^2 = (1-\frac{2MG}{r})(-dt^2+dr_*^2) + r^2(r_*)d\Omega_2^2 \tag{5.10}$$

Next one introduces null (lightcone) coordinates

$$u = t - r_*; \quad v = t + r_* \tag{5.11}$$

such that light ($ds^2 = 0$) travels at u=constant or v=constant. Finally, one introduces Kruskal coordinates,

$$\bar{u} = -4MGe^{-\frac{u}{4MG}}; \quad \bar{v} = +4MGe^{\frac{v}{4MG}} \tag{5.12}$$

Then the region $r \geq 2MG$ becomes $-\infty < r_* < +\infty$, thus $-\infty < \bar{u} \leq 0, 0 \leq \bar{v} < +\infty$. But the metric in Kruskal coordinates is

$$ds^2 = -\frac{2MG}{r}e^{-\frac{r}{2MG}}d\bar{u}d\bar{v} + r^2 d\Omega_2^2 \tag{5.13}$$

where r stands for the implicit $r(\bar{u},\bar{v})$. This metric is non-singular at the horizon $r = 2MG$, thus can be analytically continued for general values of $\bar{u},\bar{v}$, covering all the real line, having 4 quadrants instead of one!

The resulting *Kruskal diagram* (diagram in Kruskal coordinate) in given in Fig.5a and the Penrose diagram (which can be obtained from (5.13) as a subset of the flat 2 dimensional space $ds^2 = d\bar{u}d\bar{v}$ Penrose diagram) is given in Fig.5b. The Penrose diagram of a physical black hole, obtained from a collapsing star, is given in Fig.5c.

Solutions with charge

The Reissner-Nordstrom black hole is obtained by adding a charge Q at $r = 0$, giving the solution

$$ds^2 = -(1-\frac{2MG}{r^2}+\frac{Q^2G}{r^2})dt^2 + \frac{dr^2}{1-\frac{2MG}{r}+\frac{Q^2G}{r^2}} + r^2 d\Omega_2^2 \tag{5.14}$$

together with the electric field given by

$$F_{rt} = \frac{Q}{r^2} \Rightarrow A_t = -\frac{Q}{r} \tag{5.15}$$

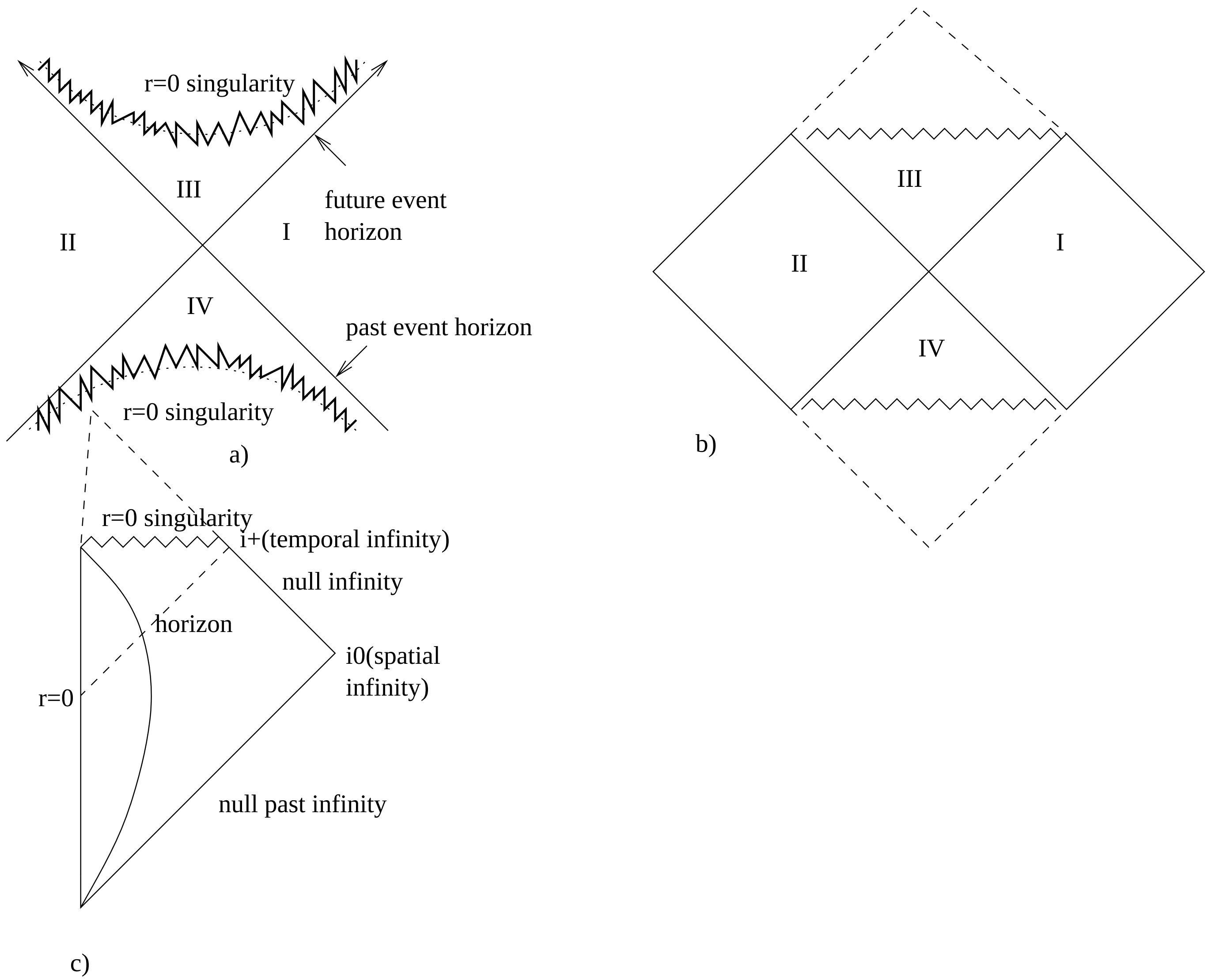

Figure 5: a) Kruskal diagram of the Schwarzschild black hole. b) Penrose diagram of the eternal Schwarzschild black hole (time independent solution). The dotted line gives the completion to the Penrose diagram of flat 2 dimensional (Minkowski) space. c) Penrose diagram of a physical black hole, obtained from a collapsing star (the curved line). The dotted line gives the completion to the Penrose diagram of flat $d > 2$ dimensional (Minkowski) space.

that is, the electric field of a point charge. The event horizon is now where $1 - 2MG/r + Q^2G/r^2 = 0$. In the following we will put $G = 1$ for simplicity, and G can be reintroduced by dimensional analysis. The event horizon is at

$$r = r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{5.16}$$

thus we have now 2 horizons, instead of one, and the metric can be rewritten as

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_2^2; \quad \Delta = (1 - \frac{r_+}{r})(1 - \frac{r_-}{r}) \tag{5.17}$$

However, if $M < Q$, there is no horizon at all, just a "naked singularity" at $r = 0$ (not covered by a horizon), which is believed to be excluded on physics grounds: there are a number of theorems saying that naked singularities should not occur under certain very reasonable assumptions. Therefore we must have $M \geq Q$.

The case $M = Q$ is special and is called the "extremal black hole". Its metric is

$$ds^2 = -(1 - \frac{M}{r})^2 dt^2 + \left(\frac{dr}{1 - \frac{M}{r}}\right)^2 + r^2 d\Omega_2^2 \tag{5.18}$$

and by a change of coordinates $r = M + \bar{r}$ we get

$$ds^2 = -\frac{1}{(1 + \frac{M}{\bar{r}})^2} dt^2 + (1 + \frac{M}{\bar{r}})^2 (d\bar{r}^2 + \bar{r}^2 d\Omega_2^2) \tag{5.19}$$

Here

$$H = 1 + \frac{M}{\bar{r}} \tag{5.20}$$

is a harmonic function, i.e. it satisfies

$$\Delta_{(3)} H = (a) M \delta^3(r) \tag{5.21}$$

So we see that the extremal solutions are defined by a harmonic function in 3 dimensions.

One can put this Reissner-Nordstrom black hole inside an Anti de Sitter space as well as follows. The Anti de Sitter metric can be written (by a coordinate transformation) as

$$ds^2 = -(1 - \frac{\Lambda r^2}{3}) dt^2 + \frac{dr^2}{1 - \frac{\Lambda r^2}{3}} + r^2 d\Omega_2^2 \tag{5.22}$$

Then the Anti de Sitter charged black hole metric is

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_2^2; \quad \Delta \equiv 1 - \frac{2M}{r} + \frac{Q}{r^2} - \frac{\Lambda r^2}{3} \tag{5.23}$$

The only other parameter one can add to a black hole is the angular momentum J, in which case however the metric is quite complicated. There are so called "no hair theorems" stating that black holes are characterized only by Q,M and J (any other charge or parameter would be called hair of the black hole).

P-branes

Black holes that extend in p spatial dimensions are called p-branes (the terminology comes from the word mem-brane which is now called a 2-brane, that is, extends in 2 spatial dimensions).

In 4 dimensions, the only localized p-branes are the black holes. An extended object can be either a cosmic string (one spatial extension) or a domain wall (two spatial extensions). However, we will shortly see that the p-branes are defined by harmonic functions in D-p-1 dimensions (the black hole, with p=0, in d=4 is defined by a harmonic function in 3 dimensions). Thus for a cosmic string, the harmonic function would be in 2 dimensions, which is $H = \ln |z|$ ($z = x_1 + ix_2$), whereas for a domain wall, the harmonic function would be in one dimension, which is $H = 1 + a|x|$. In both cases, the harmonic function increases away from its source, so both the cosmic string and the domain wall p-brane solutions would affect the whole space. They are therefore quite unlike black holes, and not quite physical.

But in dimensions higher than 4, we can have black-hole like objects extended in p spatial dimensions that are localized in space (don't grow at infinity). These are the "black p-branes", and have complicated metrics.

We will focus on the case of D=10, which is the case relevant for string theory, as we will see in the next section. We will also focus on extremal objects (with Q=M), which are very special: in fact, they are very relevant for string theory. The solution of the D=10 supergravity theory that approximates string theory at moderate energies is of the general type

$$
\begin{aligned}
&ds^2_{string} = H_p^{-1/2}(-dt^2 + d\vec{x}_p^2) + H_p^{1/2}(dr^2 + r^2 d\Omega^2_{8-p} = H_p^{-1/2}(-dt^2 + d\vec{x}_p^2) + H_p^{1/2} d\vec{x}^2_{9-p})\\
&e^{-2\phi} = H_p^{\frac{p-3}{2}}\\
&A_{01\dots p} = -\frac{1}{2}(H_p^{-1} - 1) \qquad (5.24)
\end{aligned}
$$

where H_p is a harmonic function of $\vec{x}_{9-p}$, i.e.

$$
\Delta_{(9-p)} H_p = (a) Q \delta^{(9-p)}(x^i); \quad \Rightarrow H_p = 1 + \frac{(\dots)Q}{r^{7-p}} \qquad (5.25)
$$

Here ds^2_{string} is known as the "string metric" and is related to the usual "Einstein metric" defined until now by

$$
ds^2_{Einstein} = e^{-\phi/2} ds^2_{string} \qquad (5.26)
$$

and $A_{01\dots p}$ is some antisymmetric tensor ("gauge") field present in the 10 dimensional supergravity theory (there are several), and ϕ is the "dilaton" field, which is a scalar field that is related to the string theory coupling constant by $g_s = e^{-\phi}$.

We noted that a black hole carries electric (or magnetic!) charge Q, with respect to the electromagnetic potential A_μ. Specifically, for a static electric charge, only A_0 is nonzero. That means that there is a source coupling in the action, of the type

$$
\int d^4x j^\mu A_\mu = \int j^0 A_0 \qquad (5.27)
$$

and if j^0 is taken to be the current of a static charge, $j^0 = Q\delta^3(x)$, the source term gives rise by the A_μ equation of motion to the solution of nonzero A_0.

Similarly, we find that a p-brane carries electric charge Q with respect to the p+1-form field $A_{\mu_1 \dots \mu_{p+1}}$. By analogy with the above means that there should be a source coupling

$$\int d^d x j^{\mu_1 \dots \mu_{p+1}} A_{\mu_1 \dots \mu_{p+1}} \to \int j^{01 \dots p} A_{01 \dots p} \tag{5.28}$$

It therefore follows that a source for the $A_{01 \dots p}$ field will be of the type $j^{01 \dots p} = Q\delta^{(d-p-1)}(x)$, which is therefore an object extended in p spatial dimensions plus time. The solution of the source coupling is an object with nonzero $A_{01..p}$, and indeed the p-brane has such a nonzero field.

Important concepts to remember

- The Schwarzschild solution is the most general solution with spherical symmetry and no sources. Its source is localed behind the event horizon.
- If the solution is valid down to the horizon, it is called a black hole.
- Light takes an infinite time to reach the horizon, from the point of view of the far away observer, and one has an infinite time dilation at the horizon ("frozen in time").
- Classically, nothing escaper the horizon. (quantum mechanically, Hawking radiation)
- The horizon is not singular, and one can analytically continue inside it via the Kruskal coordinates.
- Black hole solutions with charge have $Q \geq M$. The $Q = M$ solutions (extremal) are defined by a harmonic function and have a collapsed horizon.
- P-brane solutions are (extremal) black hole solutions that extend in p spatial dimensions. They also carry charge under an antisymmetric tensor field $A_{\mu_1 \dots \mu_{p+1}}$, and are determined by a harmonic function.

Exercises, section 5

1) Check the transformation from Schwarzschild coordinates to Kruskal coordinates.

2) Find the equation for the r=0 singularity in Kruskal coordinates (the singularity curve on the Kruskal diagram). Hint: calculate the equation at $r = r_0$=arbitrary and then extrapolate the final result to $r = 0$.

3) Check that the transformation of coordinates $r/R = \sinh\rho$ takes the AdS metric between the global coordinates

$$ds^2 = R^2(-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega^2) \tag{5.29}$$

and the coordinates (here $R = \sqrt{-\Lambda/3}$)

$$ds^2 = -(1 - \frac{\Lambda}{3}r^2)dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2} + r^2 d\Omega^2 \tag{5.30}$$

4) Check that $H = 1 + a/r^{7-p}$ is a good harmonic function for a p-brane. Check that r=0 is an event horizon (it traps light).

5) The electric current of a point charge is $j^\mu = Q\frac{dx^\mu}{d\tau}\delta^{d-1}(x^\mu(\tau))$. Write an expression for the p+1-form current of a p-brane, $j^{\mu_1\cdots\mu_{p+1}}$.

6 String theory actions and spectra

The Nambu-Goto action

String theory is the theory of relativistic strings. That is, not strings like the violin strings, but strings that move at the speed of light. They don't have a compression mode (the energy density along a string is not a Lorentz invariant, so cannot appear as a physical variable in a relativistic theory). They only have a vibration mode, unlike, e.g. a massive cosmic string or a violin string.

However, they can have tension, which resists against pulling the string apart (energy per unit length). The point is that if one stretches the string the energy density stays the same, just the length increases, thus *energy = tension × length*.

Because they have tension, the only possible action for a string is the one that minimizes the area traversed by the string, i.e. the "worldsheet". The coordinates for the position of the string are $X^\mu(\sigma,\tau)$, where σ =worldsheet length and τ = worldsheet time, i.e. (σ,τ) are intrinsic coordinates on the surfacem drawn by the moving string (worldsheet), as in Fig.6a. The string action, due to Nambu and Goto, is

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{\det(h_{ab})} \tag{6.1}$$

where $1/(2\pi\alpha') = T$ is the string tension. The metric h_{ab} is the metric induced on the worldsheet by the motion through spacetime, or "pull-back" of the spacetime metric,

$$h_{ab}(\sigma,\tau) = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) \tag{6.2}$$

and $d\tau d\sigma\sqrt{\det h}$ is the "volume element" (infinitesimal area) on the worldsheet. This is similar to the case of the metric on a 2-sphere in 3 dimensional Euclidean space given as an example at the begining of the General Relativity section. As there, the metric is obtained by the fact that the metric on the worldsheet is expressed in two ways

$$ds^2|_{on\ \mathcal{M}} = d\xi^a d\xi^b h_{ab}(\xi) = g_{\mu\nu}(X)dX^\mu dX^\nu; \quad \xi^a = (\sigma,\tau) \tag{6.3}$$

What does the Nambu-Goto action calculate though? It calculates $X^\mu(\sigma,\tau)$, the string trajectory through spacetime.

So, an analog of the string action is the particle action in flat space

$$S = -m\int ds = -m\int d\tau\sqrt{-\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu}} = -m\int dt\sqrt{1-\vec{v}^2} \tag{6.4}$$

By varying it with respect to X^μ we get the equation of motion

$$-\frac{m}{2}\frac{d}{d\tau}[\frac{\dot{X}^\nu\eta_{\mu\nu}}{\sqrt{(\dot{X}^\mu)^2}}] = -\frac{m}{2}\frac{d}{d\tau}[\dot{X}^\nu\eta_{\mu\nu}] = 0 \Rightarrow \frac{d^2}{d\tau}\dot{X}_\mu = 0 \tag{6.5}$$

Here $\dot{X} = dX/d\tau$ and we have used $(\dot{X}^\mu)^2 = ds^2/d\tau^2 \equiv 1$.

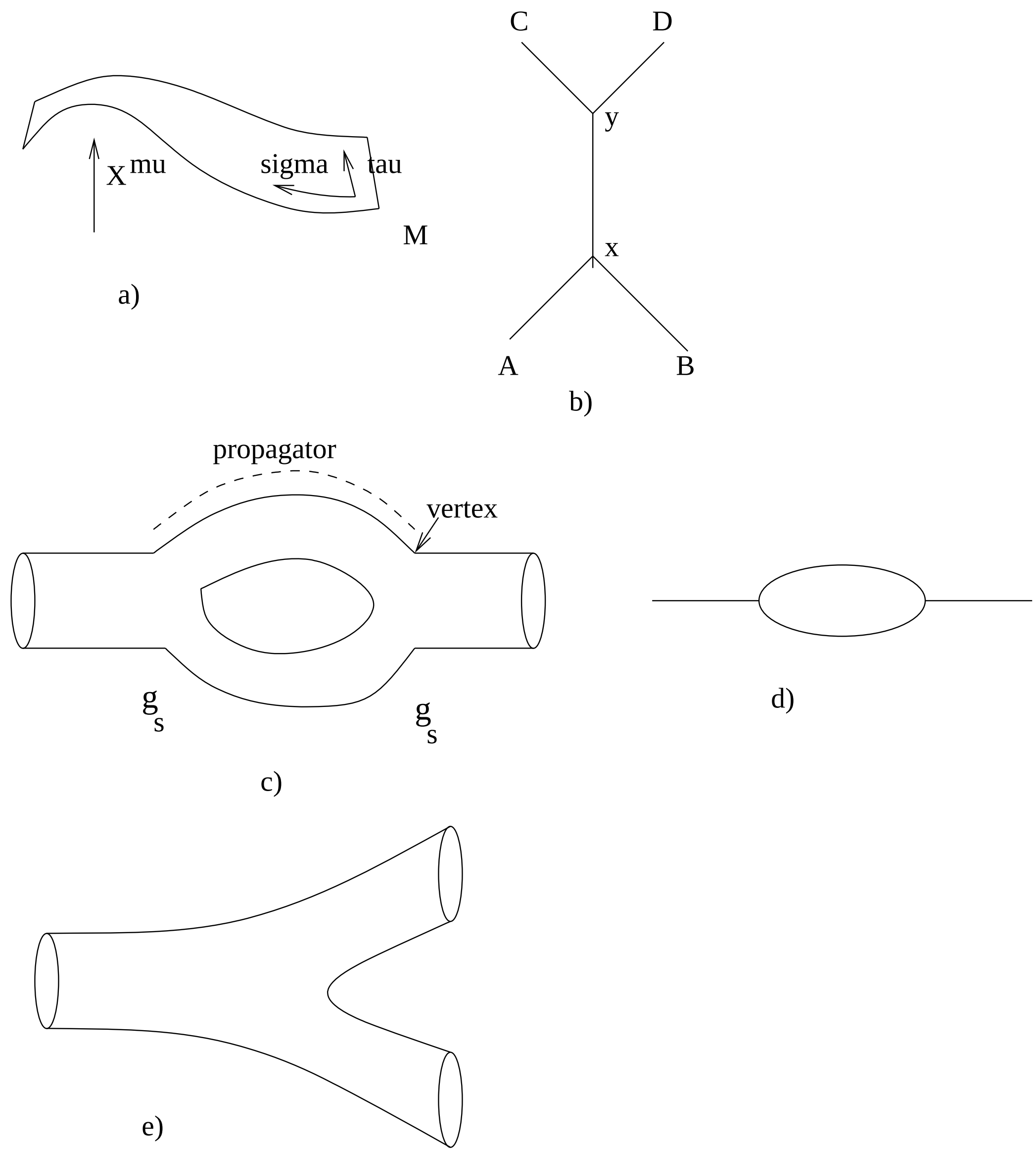

Figure 6: a)String moving in spacetime parametrized by X^μ spans a worldsheet $\mathcal{M}$ spanned by σ (coordinate along the string) and τ (worldsheet time). b)Feynman diagram in x space: from x to y we have the particle propagator. c)String loop diagram. The vertices are not pointlike, but are spread out, and have a coupling g_s. d)By comparison, a particle loop diagram. e) Basic string interaction: "pair of pants"= vertex for a string to split in two strings.

This of course looks a little trivial, we obtain just the free motion in a straight line. However, if the write the same action in curved space instead, replacing $\eta_{\mu\nu} \to g_{\mu\nu}$, we will get the free motion along a geodesic in spacetime. The geodesic equation is then nontrivial, and can be understood as the interaction of the particle with the gravitational field. In more general terms, we can say that background fields (like the metric) appearing in the particle or string actions will give interaction effects.

But what is the usefulness of the particle action for quantum field theory?

Let us suppose that we don't know how to do quantum field theory and/or the precise theory we have. We can then still *construct Feynman diagrams*, considered as describing actual particles propagating in spacetime, for instance as in Fig.6b.

To construct such a Feynman diagram, we need

- the propagator from x to y
- the vertex factor at x and y: this contains the coupling g, thus it defines a particular theory.
- rules about how to integrate (in this case, $\int d^4x \int d^4y$). For particles, this is obvious, but for strings, we need to carefully define a path integral construction. There are subtleties due to the possibility of overcounting if we use naive integration.

The propagator from x to y for a massless particle is (here, $\Box$= kinetic operator)

$$< x|\Box^{-1}|y >= \int_0^\infty d\tau < y|e^{-\tau\Box}|x > \tag{6.6}$$

But now we can use a trick: A massive nonrelativistic particle has the Hamiltonian $H = p^2/(2m) = \Box/(2m)$ (in Euclidean x space). Using $m = 1/2$ we get $H = \Box$ and therefore we can use quantum mechanics to write a path integral representation of the transition amplitude

$$< y|e^{-\tau H}|x >= \int_x^y \mathcal{D}x(t) e^{-\frac{1}{4}\int_0^\tau dt\dot{x}^2} \tag{6.7}$$

Since $H = \Box$, we use this representation to express the propagator of a massless relativistic particle in (6.6) as

$$< x|\Box^{-1}|y >= \int_0^\tau \int_x^y \mathcal{D}x(t) e^{-\frac{1}{4}S_p} \tag{6.8}$$

where $S_p = \int_0^\tau dt\dot{x}^2$ is the massless particle action (in fact, we have not quite seen that yet, we just looked at the massive particle action, but we will see soon that it is as we said).

So the particle action defines the propagator, and to complete the perturbative definition of the quantum field theory by Feynman diagrams we need to add the vertex rules (specifying the interactions of the theory: for instance, in the $V = \lambda\phi^4$ example in section 1 we had a vertex $-\lambda$), as well as the integration rules (trivial, in the case of the particle).

We will do the same for string theory: we will define perturbative string theory by defining Feynman diagrams. We will write a worldsheet action that will give the propagator, and then interaction rules and integration rules.

Before that however, we need to understand better the **possible particle actions**. Specifically, we can write down a first order action for the massive particle that is more fundamental than the one we wrote. First order means that we introduce an independent "worldline metric" field, $\gamma_{\tau\tau}(\tau)$, not defined by embedding in the spacetime metric. Rather, we will use the vielbein, or rather einbein in this case, $e(\tau) = \sqrt{-\gamma_{\tau\tau}(\tau)}$.

Then we can write the first order particle action

$$\tilde{S}_p = \frac{1}{2}\int d\tau (e^{-1}(\tau)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu} - em^2) \tag{6.9}$$

Then the $e(\tau)$ equation of motion gives

$$-\frac{1}{e^2}\dot{X}^2 - m^2 = 0 \Rightarrow e^2(\tau) = -\frac{\dot{X}^\mu \dot{X}_\mu}{m^2} \tag{6.10}$$

Substituting in $\tilde{S}_p$ we get

$$\tilde{S}_p = \frac{1}{2}\int d\tau \left[\frac{m}{\sqrt{-\dot{X}^2}}\dot{X}^2 - \frac{\sqrt{-\dot{X}^2}}{m}m^2\right] = -m\int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} = S_p \tag{6.11}$$

so we do indeed get the previous (second order) action by solving the $e(\tau)$ equation of motion and substituting.

Note that now we can take the $m \to 0$ limit of the first order action $\tilde{S}_p$, unlike the second order action S_p which is proportional to m. The first order action has a gauge invariance, which is the reparametrization invariance, $\tau \to \tau'(\tau)$ that gives $e \to e d\tau/d\tau'$. Therefore by a reparametrization $\tau'(\tau)$ I can set e to whatever value. In particular it is convenient to choose the gauge $e = 1$. Then the massless particle action in this gauge is

$$\tilde{S}_{m=0,e=1} = \int d\tau \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu} \tag{6.12}$$

which is the result we used above, in the calculation of the massless particle propagator.

Note that now the equation of motion for $X^\mu(\tau)$ is

$$\frac{d}{d\tau}(\frac{dX^\mu}{d\tau}) = 0 \tag{6.13}$$

Note also that, since we work in the gauge $e = 1$, we must impose the $e(\tau)$ equation of motion as a constraint on the solutions. It gives

$$\frac{ds^2}{d\tau^2} = \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu} = 0 \tag{6.14}$$

which is just the statement that the particle is massless.

We now go back to strings and mimic what we did for particles, to write down a first order action. It is called the **Polyakov action**. In flat spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$), it is

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \tag{6.15}$$

Here γ^{ab} is an independent metric on the worldsheet. Its equation of motion gives

$$h_{ab} - \frac{1}{2}\gamma_{ab}(\gamma^{cd}h_{cd}) = 0 \tag{6.16}$$

where

$$h_{ab} = \partial_a^\mu \partial_b X^\nu \eta_{\mu\nu} \tag{6.17}$$

as before. We then obtain

$$\frac{h_{ab}}{\sqrt{-h}} = \frac{\gamma_{ab}}{\sqrt{-\gamma}} \Rightarrow S_P = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-h} = S_{NG} \tag{6.18}$$

thus indeed, the Polyakov action is the first order form of the Nambu-Goto action.

The Polyakov action has the following invariances:

- Spacetime Poincare invariance
- Worldsheet diffeomorphism invariance, defined by two transformations $(\sigma'(\sigma,\tau), \tau'(\sigma,\tau))$, that give $X'^\mu(\sigma',\tau') = X^\mu(\sigma,\tau)$
- Worldsheet Weyl invariance: for any $\omega(\sigma,\tau)$, we have

$$X'^\mu(\sigma,\tau) = X^\mu(\sigma,\tau); \quad \gamma'_{ab}(\sigma,\tau) = e^{2\omega(\sigma,\tau)}\gamma_{ab}(\sigma,\tau) \tag{6.19}$$

The Weyl invariance is very important in the following, and is not present in the Nambu-Goto action. Therefore the Polyakov form is more fundamental. Classically, the two actions are equivalent, as we saw. But quantum mechanically, they are not.

Strings have spatial extension, but that means we also need boundary conditions for them. They can be open, in which case the endpoints of the string are different (and can have either Neumann or Dirichlet boundary conditions) or closed. We will study closed strings in the following.

Closed string spectrum.

The Polyakov action has 3 worldsheet invariances (defined by arbitrary functions): 2 diffeomorphisms ($\sigma'(\sigma,\tau)$ and $\tau'(\sigma,\tau)$) and one Weyl invariance ($\omega(\sigma,\tau)$). That means that we can choose the 3 independent elements of the symmetric matrix $h_{\alpha\beta}(\sigma,\tau)$ (the worldsheet metric) to be anything we want. We will choose the gauge

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.20}$$

called the conformal gauge. Then, the Polyakov action becomes

$$S = -\frac{T}{2}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{6.21}$$

The X^μ equation of motion gives the 2 dimensional wave equation

$$\Box X^\mu = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = -\partial_+\partial_- X^\mu = 0 \tag{6.22}$$

We define

$$\sigma^{\pm} = \tau \pm \sigma; \quad \partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{6.23}$$

Then the general solution of the 2 dimensional wave equation is

$$X^\mu(\sigma, \tau) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+) \tag{6.24}$$

For a closed string, that is everything, since we don't need to impose a boundary condition, and we can expand this general solution in Fourier modes:

$$\begin{aligned} X_R^\mu &= \frac{1}{2}x^\mu + \frac{l^2}{2}p^\mu(\tau - \sigma) + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)} \\ X_L^\mu &= \frac{1}{2}x^\mu + \frac{l^2}{2}p^\mu(\tau + \sigma) + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \end{aligned} \tag{6.25}$$

Note that the zero mode has been written in a particular way: The zero mode is $X^\mu = x^\mu + l^2 p^\mu \tau$ but has been split into a X_L^μ part and a X_R^μ part.

As in the case of the particle action, because we work in a gauge for 2 dimensional invariance, we need to impose the equation of motion of the worldsheet metric γ^{ab} as a constraint

$$\frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta \gamma^{\alpha\beta}} \equiv T_{\alpha\beta} = 0 \tag{6.26}$$

So the constraint is that the worldsheet energy-momentum tensor must be equal to zero. We expand also this constraint in Fourier modes and define

$$\begin{aligned} L_m &= \frac{T}{2}\int_0^\pi e^{-2im\sigma} T_{--} d\sigma \\ \tilde{L}_m &= \frac{T}{2}\int_0^\pi e^{2im\sigma} T_{++} d\sigma \end{aligned} \tag{6.27}$$

The zero modes of the constraints give

$$L_0 + \tilde{L}_0 = - \Rightarrow p^\mu p_\mu \equiv M^2 = \frac{2}{\alpha'}\sum_{n\geq 1}(\alpha_{-n}^\mu \alpha_n^\mu + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\mu) \tag{6.28}$$

We are still left with $L_0 - \tilde{L}_0$ to impose.

But there is in fact a quantum correction, that one can calculate, giving in fact $L_0 + \tilde{L}_0 = 2$, and modifying the mass relation (see below). One quantizes these oscillation modes (as is familiar from, let's say, the phonon quantization or the quantization of sounds modes in a cavity) by setting, for $m > 0$,

$$\alpha_m^\mu = \sqrt{m} a_m^\mu; \quad \alpha_{-m}^\mu = \sqrt{m} a_m^{\dagger\ \mu}; \quad [a_m^\mu, a_n^{\dagger\ \nu}] = \delta_{mn}\delta^{\mu\nu} \tag{6.29}$$

Then one obtains the mass spectrum

$$\alpha' M^2 = -4 + 2\sum_{m,\mu} m(N_m^\mu + \tilde{N}_m^\mu); \quad N_m^\mu = a_m^{\dagger\ \mu} a_m^\mu = \text{number operator} \tag{6.30}$$

where the constant actually depends on dimension. Quantum consistency requires D=26, and then the constant is -4 as above.

So this string theory makes sense at the quantum level only if it is defined in 26 dimensions, so in order to get a 4 dimensional theory we must use the Kaluza-Klein idea of dimensional reduction.

There is now an extra condition for physical states. It can be understood in two ways. We can say that is the invariance of translations along the closed string, $\sigma \to \sigma + s$, which means that P_σ, the translation generator along σ, should act trivially on states, and it turns out that we must impose

$$P_\sigma = -\frac{2\pi}{l}\sum_{m,\mu} m(N_m^\mu - \tilde{N}_m^\mu) = 0 \tag{6.31}$$

Another way of saying this is that P_σ is proportional to $L_0 - \tilde{L}_0$, which was still left to be imposed on states. In any case, that means that

$$N \equiv \sum_{m,\mu} m N_m^\mu = \sum_{m,\mu} m\tilde{N}_m^\mu \equiv \tilde{N} \tag{6.32}$$

Then, the closed string spectrum starts with a tachyon of $\alpha' M^2 = -4$. It is denoted by $|0,0;k>$, that is, vacuum for α_m oscillators, vacuum for $\tilde{\alpha}_m$ oscillators, and with zero-mode momentum $p = k$. At the next level, we have 1 excitation on the level m=1. But then $N_m^\mu = \tilde{N}_m^\mu = 1$ and we must have both a α_{-1} and a $\tilde{\alpha}_{-1}$ excitation. We get that this excitation has $\alpha' M^2 = -4 + 2\cdot 2 = 0$, so these are massless states. These states will be of the type

$$\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0,0;k> \tag{6.33}$$

This will then be a tensor state $A^{\mu\nu}$ (with two spacetime indices). It decomposes into a symmetric traceless tensor part $g_{\mu\nu}$, an antisymmetric tensor part $B_{\mu\nu}$ and a trace part ϕ. These massless modes of the string correspond to the graviton $g_{\mu\nu}$, a field called the antisymmetric tensor field (or B field) $B_{\mu\nu}$ and the dilaton field ϕ.

This was for the simplest string action, the bosonic string, and we saw that the ground state is tachyonic, thus unstable ($M^2 < 0$ means that we are perturbing a potential $V(\Phi)$, where Φ is the tachyon field, around a maximum, $V(\Phi) \simeq V_0 + M^2(\delta\Phi)^2;\ \ M^2 < 0$) instead of a minimum. It then means that this vacuum will decay to the true vacuum. The bosonic string thus is not very well undestood.

Instead, one defines the superstring, which is a supersymmetric string. One extends the Polyakov action to a supersymmetric action. Then the spectrum of the supersymmetric closed string is in part obtained by projecting out some of the bosonic string states. The tachyon ground state is projected out, but the massless states remain.

So the superstring has a ground state composed of the massless states $(g_{\mu\nu}, B_{\mu\nu}, \phi)$, together with some supersymmetric partners. Quantum consistency of the superstring now requires D=10. Thus we still need to use the Kaluza-Klein idea of dimensional reduction in order to get to a 4 dimensional theory.

Above this ground state, the string modes have increasing mass. Each string mode corresponds to a spacetime field of a given mass. But since the mass scale of the modes is set by $1/\alpha'$, in the limit of $\alpha' \to 0$ only the massless fields remain. The massless fields then aquire VEVs that correspond to classical backgrounds (with quantum corrections).

By self-consistency, we write down the propagation of the string in backgrounds generated by the massless string modes, i.e. $g_{\mu\nu}, B_{\mu\nu}, \phi$. The action is

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma[\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X^\rho) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X^\rho) - \alpha'\sqrt{h}\mathcal{R}^{(2)}\Phi(X^\rho)] \tag{6.34}$$

where $\mathcal{R}^{(2)}$ is the 2 dimensional Ricci scalar and the quantity

$$\frac{1}{4\pi}\int d^2\sigma\sqrt{h}\mathcal{R}^{(2)} = \chi \tag{6.35}$$

is a topological invariant, i.e. a negative integer that counts the number of holes the topology of the 2 dimensional surface has (times -2, specifically, $\chi = 2(1-g)$). But e^{-S} contains then $e^{-\chi\Phi} \sim (e^{\Phi})^{2g}$. Therefore, the addition of a hole to a whorldsheet, which is interpreted as an extra loop in the quantum interaction of a string, as in Fig.6c, gives a factor of $e^{2\Phi}$, prompting the identification of e^{Φ} with the string coupling constant, g_s.

This procedure, of putting the string in a background ("condensate") of its own ground state modes, needs a self-consistency condition: the procedure must preserve the original invariances of the action, specifically Weyl invariance (or conformal invariance, see next section). Imposing Weyl invariance of the action in fact turns out to give the equations of motion for $g_{\mu\nu}, B_{\mu\nu}, \phi$.

When using this self-consistency on the full superstring, the background will be a supersymmetric theory of gravity = supergravity! It will contain the fields $g_{\mu\nu}, B_{\mu\nu}, \phi$ among others. That means that the $\alpha' \to 0$ (low energy limit) of string theory, which is the theory of the massless backgrounds of string theory, will be supergravity in 10 dimensions.

Now, to construct string theory perturbatively, as for the particle case, we construct S matrices through Feynman diagrams, as in Fig.6c. The basic interaction that gives the Feynman digrams is the "pants diagram" in Fig.6e. The Polyakov action will define the propagator, the vertices are defined such that we reproduce supergravity vertices in the low energy limit $\alpha' \to 0$, and as noted, one needs to define integration carefully at each loop order, since as we can see the vertex is "smoothed out".

Important concepts to remember

- String theory is the theory of relativistic strings, with tension = energy/ length.
- The string action is the area spanned by the moving string, and its minimization is due to its tension
- For the Feynman diagram construction of quantum field theory, we need the particle action to define the propagator, the vertex factors to define the theory, and integration rules.

- The first order particle action is more fundamental: it contains the massless case. The Polyakov string action is also more fundamental: it has more symmetries.
- By fixing a gauge, the closed string action reduces to free 2 dimensional bosons, which contain left and right moving wave modes.
- By quantizing these modes, we get the particle spectrum. The massless particles are the graviton, an antisymmetric tensor and a scalar.
- The bosonic string is unstable. The superstring is stable and lives in 10 dimensions, thus we need to use Kaluza-Klein dimensional reduction.
- Self-consistent backgrounds for the string are given by the theory of the massless modes of the superstring, namely supergravity.
- Thus the low energy limit ($\alpha' \to 0$) of string theory is supergravity.
- One knows how to construct string theory S matrices from Feynman diagrams by defining the propagator, vertices and integration rules.

Exercises, section 6

1) Write down the worldline reparametrization invariance for the particle, both the finite and infinitesimal versions.

2) Calculate $L_m, \tilde{L}_m$ and $L_0 + \tilde{L}_0$.

3) Derive P_σ.

4) Write down the states of the first massive closed string level.

5) Show that the coupling to $B_{\mu\nu}$ is of the type of p-brane sources, thus a string is a 1-brane source for the field $B_{\mu\nu}$.

7 Elements of conformal field theory; D-branes

Conformal transformations and the conformal group

Consider flat space (either Euclidean or Minkowski), and quantum field theory in it. Conformal transformations are then generalizations of the scale transformations

$$x'^{\mu} = \alpha x^{\mu} \Rightarrow ds^2 = d\vec{x}'^2 = \alpha^2 d\vec{x}^2 \tag{7.1}$$

Before we define conformal transformations, let's understand scale transformations in field theory.

The procedure of **renormalization** involves a cut-off ϵ and bare coupling λ_0 and mass m_0. For example, dimensional regularization of scalar field theory for $V(\phi) = m^2\phi^2/2 + \lambda\phi^4$ gives

$$\lambda_0 = \mu^{\epsilon}(\lambda + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k}); \quad m_0^2 = m^2(1 + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\epsilon^k}) \tag{7.2}$$

where μ is the renormalization scale, out of which we extract the renormalized coupling $\lambda = \lambda(\mu, \epsilon; \lambda_0, m_0)$, which in general depends on scale.

This *running of the coupling constant* with the scale is characterized by the β function,

$$\beta(\lambda, \epsilon) = \mu \frac{d\lambda}{d\mu}|_{m_0, \lambda_0, \epsilon} \tag{7.3}$$

A scale invariant theory (i.e. a theory independent of α in (7.1)) must then be μ-independent, thus have a zero β function. There are two ways in which this can happen:

- $\beta = 0$ everywhere, which means a cancellation of Feynman diagrams that gives no infinities. OR
- a nontrivial interacting theory: the β function is nontrivial, but has a zero (fixed point) away from $\lambda = 0$, at which a nontrivial (nonperturbative) theory emerges: a conformal field theory. For the case in Fig.7c, Λ_F is called an IR stable point. Indeed, if $\lambda > \lambda_F$, $\beta(\lambda) > 0$, thus λ decreases if μ decreases (thus in the IR). And f $\lambda < \lambda_F$, $\beta(\lambda) < 0$, thus λ increases if μ again decreases (in the IR). That means that if we go to the IR, wherever we start, we are driven to $\lambda = \lambda_F$, that has $\beta(\lambda_F) = 0$.

If we have a theory with classical scale invariance, it must be respected in the quantum theory. But a priori there could be a quantum anomaly (that is, there are Feynman diagrams that could potentially break scale invariance of the quantum averaged theory). So one must require as a consistency of the theory the absence of quantum anomalies to Weyl (scale) invariance, which will give constraints on the theory.

Most theories that are quantum mechanically scale invariant (thus have $\beta = 0$), have a larger invariance, called **conformal invariance**.

In flat d dimensions, i.e. on $R^{1,d-1}$, conformal transformations are defined by $x_\mu \to x'_\mu(x)$ such that

$$dx'_\mu dx'_\mu = [\Omega(x)]^{-2} dx_\mu dx_\mu \tag{7.4}$$

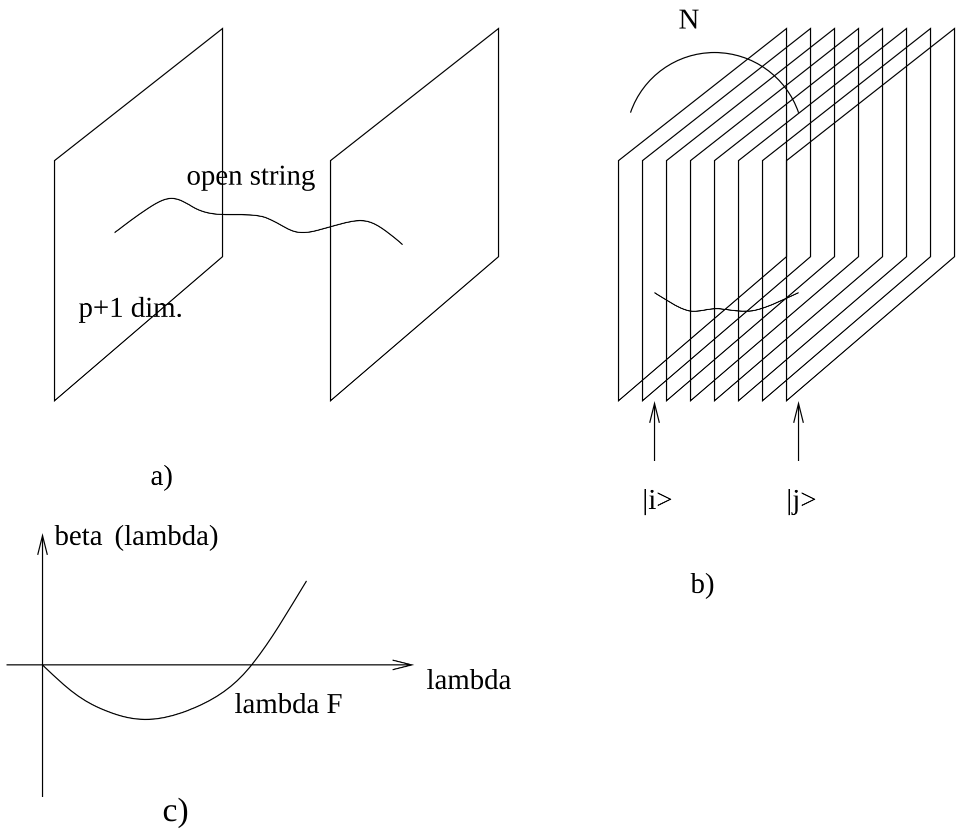

Figure 7: a)Open string between two D-p-branes (p+1 dimensional "walls"). b)The end-points of the open string are labelled by the D-brane they end on (out of N D-branes), here $|i>$ and $|j>$. c)$\beta(\lambda)$ for the case of an IR stable point.

Note that conformal invariance is NOT the same as general coordinate invariance, since the metric is modified, from flat $ds^2 = dx'_\mu dx'_\mu$ to "conformally flat" $ds^2 = [\Omega(x)]^{-2} dx_\mu dx_\mu$. This is a statement of the fact that conformal transformations are generalizations of scale transformations (7.1) that change the distance between points.

The infinitesimal conformal transformation is then

$$\begin{aligned} x'_\mu &= x_\mu + v_\mu(x); \quad \Omega(x) = 1 - \sigma_v(x) \\ &\Rightarrow \partial_\mu v_\nu + \partial_\nu v_\mu = 2\sigma_v \delta_{\mu\nu} \Rightarrow \sigma_v = \frac{1}{d}\partial \cdot v \end{aligned} \tag{7.5}$$

D=2 is special, and will be analyzed separately. But except for d=2, the most general solution to this equation is

$$v_\mu(x) = a_\mu + \omega_{\mu\nu} x_\nu + \lambda x_\mu + b_\mu x^2 \tag{7.6}$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ (antisymmetric) and $\sigma_v(x) = \lambda - 2b \cdot x$. Thus the parameters of conformal transformations are $\lambda, a_\mu, b_\mu, \omega_{\mu\nu}$, corresponding respectively to scale transformations, translations, a new type of transformations, and rotations. The new type of transformations parametrized by b_μ is called "special conformal transformations". Together there are $1 + d + d + d(d-1)/2 = (d+1)(d+2)/2$ components for the parameters of conformal transformatios.

These transformations form together a symmetry group. Its generators are: P_μ for a_μ and $J_{\mu\nu}$ for $\omega_{\mu\nu}$ forming together the Poincare group, as expected. For them, we have the particular case of $\Omega(x) = 1$. The new generators are K_μ for the special conformal transformations b_μ and dilatation generator D for λ. Counting shows that we can assemble these generators in a group defined by an antisymmetric $(d+2) \times (d+2)$ matrix,

$$\bar{J}_{MN} = \begin{pmatrix} J_{\mu\nu} & \bar{J}_{\mu,d+1} & \bar{J}_{\mu,d+2} \\ -\bar{J}_{\nu,d+1} & 0 & D \\ -\bar{J}_{\nu,d+2} & -D & 0 \end{pmatrix} \tag{7.7}$$

where

$$\bar{J}_{\mu,d+1} = \frac{K_\mu - P_\mu}{2}; \quad \bar{J}_{\mu,d+2} = \frac{K_\mu + P_\mu}{2}; \quad \bar{J}_{d+1,d+2} = D \tag{7.8}$$

By looking at the Lie algebra of $\bar{J}_{MN}$ we find that the metric in the $d+2$ direction is negative, thus the symmetry group is $SO(2,d)$. So conformal invariance in flat $(1, d-1)$ dimensions ($d > 2$) corresponds to the symmetry group $SO(2,d)$, the same as the symmetry group of $d+1$-dimensional Anti de Sitter space, AdS_{d+1}.

This is in fact the first hint of a relation between d-dimensional conformal field theory, i.e. a field theory on d-dimensional Minkwoski space that is invariant under the conformal group, and a gravity theory in d+1 dimensional Anti de Sitter space. The precise relation between the two will be AdS-CFT, defined in the next section.

A comment is in order here. Strictly speaking, SO(2,d) is a group that only contains elements continously connected to the identity, however the conformal group is an extension that also contains the *inversion*

$$I : x'_\mu = \frac{x_\mu}{x^2} \Rightarrow \Omega(x) = x^2 \tag{7.9}$$

In fact, all conformal transformations can be generated by combining the inversion with the rotations and translations. The finite version of the special conformal transformation is

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2} \tag{7.10}$$

and the finite version of the scale transformation is $x^\mu \to \lambda x^\mu$.

Since we will be defining AdS-CFT in Euclidean space, we should note that the conformal group on R^d (Euclidean space) is $SO(1, d+1)$.

Conformal fields in 2 dimensions

As noted, d=2 is special. In d=2, the conformal group is much larger: in fact, it has an infinite set of generators.

To describe conformal fields in Euclidean d=2, we will use complex coordinates $(z, \bar{z})$,

$$ds^2 = dz d\bar{z} \tag{7.11}$$

It is easy then to see that the most general solution of the conformal transformation condition (7.5) is a general *holomorphic* transformation, i.e. $z' = f(z)$ (but not a function of $\bar{z}$). Then,

$$ds'^2 = dz' d\bar{z}' = \frac{\partial z'}{\partial z}\frac{\partial \bar{z}'}{\partial \bar{z}} dz d\bar{z} = \Omega^{-2}(z, \bar{z}) dz d\bar{z} \tag{7.12}$$

The simplest example of a euclidean d=2 conformal field theory is just a set of free scalar fields, with action

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma [\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu] \tag{7.13}$$

As we can see, this is nothing but the Polyakov string action in conformal gauge. In fact, the choice of conformal gauge was actually related to Weyl (scale) invariance, which is a part of conformal invariance. We can check in fact that the string action before imposing conformal gauge is conformally invariant.

Using complex coordinates

$$z = \sigma^1 + i\sigma^2; \quad \bar{z} = \sigma^1 - i\sigma^2; \quad \partial_z = \frac{\partial_1 - i\partial_2}{2}; \quad \partial_{\bar{z}} = \frac{\partial_1 + i\partial_2}{2} \tag{7.14}$$

we get the action

$$S = \frac{1}{2\pi\alpha'}\int d^2 z \partial X^\mu \bar{\partial} X_\mu \tag{7.15}$$

giving the equation of motion

$$\partial\bar{\partial} X^\mu(z, \bar{z}) = 0 \tag{7.16}$$

with the general solution

$$X^\mu = X^\mu(z) + X^\mu(\bar{z}) \tag{7.17}$$

The continuation to Minkowski space is done by $\sigma^2 = i\sigma^0 = i\tau$, and under it a holomorphic function (function of z only) becomes a function of $-(\tau - \sigma)$, i.e. left-moving, and an anti-holomorphic function (function of $\bar{z}$ only) becomes a function of $\bar{z} = \tau + \sigma$, i.e. right-moving. We thus recover the Minkwoski space treatment of the string in the previous section.

There we had defined

$$L_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma}T_{--}d\sigma$$
$$\tilde{L}_m = \frac{T}{2}\int_0^\pi e^{2im\sigma}T_{++}d\sigma \tag{7.18}$$

In complex coordinates, L_m and $\tilde{L}_m$ are defined (equivalently) as Laurent coefficients of T_{zz} and $\tilde{T}_{\bar{z}\bar{z}}$, namely

$$T_{zz}(z) = \sum_{m\in Z}\frac{L_m}{z^{m+2}}; \quad \tilde{T}_{\bar{z}\bar{z}}(\bar{z}) = \sum_{m\in Z}\frac{\tilde{L}_m}{\bar{z}^{m+2}} \tag{7.19}$$

By commuting the L_m's one finds the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n} \tag{7.20}$$

and similarly for the $\tilde{L}_m$'s. The algebra at $c = 0$ is the classical part, and the term with c is a quantum correction. Here $c =$ "central charge" is a parameter of the theory in general. In string theory it can be fixed.

The Virasoro algebra defines the "conformal group" in 2 dimensions, which means L_m's are conserved charges, corresponding to symmetry operators. But it is not really a usual group, since it has an infinite number of generators and more importantly the algebra contains a constant term (proportional to c), therefore the algebra does not close in the usual sense. However, L_0, L_1 and L_{-1} for a closed algebra without central charge:

$$[L_1, L_{-1} = 2L_0; \quad [L_0, L_1] = -L_1; \quad [L_0, L_{-1}] = L_{-1} \tag{7.21}$$

which is the algebra of the group $Sl(2,C)$, whose finite transformations act on z as

$$z \to \frac{az+b}{cz+d} \tag{7.22}$$

This is then a subalgebra of the Virasoro algebra that sometimes is called (by an abuse of notation) the conformal algebra in 2 dimensions.

In 2 dimensions, we define tensors of general relativity as objects that under a general coordinate transformation $(z_1, z_2) \to (z_1', z_2')$ transform as

$$T_{i_1\ldots i_n}(z_1, z_2) = T'_{j_1\ldots j_n}\frac{\partial z'_{j_1}}{\partial z_{i_1}}\ldots\frac{\partial'_{j_n}}{\partial z_{i_n}} \tag{7.23}$$

Under a conformal transformation, in $z, \bar{z}$ notation, i.e. $z' = z'(z), \bar{z}' = \bar{z}'(\bar{z})$, we obtain

$$T_{z\ldots z\bar{z}\ldots\bar{z}}(z,\bar{z}) = T'_{z\ldots z\bar{z}\ldots\bar{z}}(z'\bar{z}')(\frac{dz'}{dz})^h(\frac{d\bar{z}'}{d\bar{z}})^{\bar{h}} \tag{7.24}$$

where there are h indices of type z and $\bar{h}$ indices of type $\bar{z}$.

In two dimensions there are no distinctions between fields and composite operators as there are in 4 dimensions (where the two have different properties). Then either a field $\phi(z,\bar{z})$ or an operator $\mathcal{O}(z,\bar{z})$ is called a tensor operator or a *primary field* of dimensions $(h,\bar{h})$ id it transforms as $T_{z...z\bar{z}...\bar{z}}$ above under a *conformal* transformation. But unlike in the example above that used GR tensors, in general h and $\bar{h}$ need not be integers!

Under a scale transformation $z \to \lambda z$, $\bar{z} \to \lambda \bar{z}$, $T_{z...z\bar{z}...\bar{z}}$ transforms as

$$T_{z...z\bar{z}...\bar{z}} \to T_{z...z\bar{z}...\bar{z}}(\lambda)^{h+\bar{h}}, \tag{7.25}$$

so $\Delta = h + \bar{h}$ is called the scaling dimension.

Back to $d > 2$

We now define primary operators in $d > 2$ also, but in a slightly different manner.

Representations of the conformal group are defined by eigenfunctions of the scaling operator D with eigenvalue $-i\Delta$, where Δ is the scaling dimension, i.e. under $x \to \lambda x$, we get

$$\phi(x) \to \phi'(x) = \lambda^{\Delta}\phi(\lambda x) \tag{7.26}$$

Then Δ is increased by P_μ, since

$$[D, P_\mu] = -iP_\mu \Rightarrow D(P_\mu \phi) = P_\mu(D\phi) - iP_\mu\phi = -i(\Delta+1)(P_\mu\phi) \tag{7.27}$$

and decreased by K_μ, since

$$[D, K_\mu] = iK_\mu \tag{7.28}$$

thus we can think of K_μ as an annihilation operator a and P_μ as a creation operator $a^\dagger$. Since P_μ and K_μ are symmetry operators, by succesive action of them we get other states in the theory. The representation then is built as if using creation/ annihilation operators.

There will be an operator of lowest dimension, Φ_0, in the representation of the conformal group. Then, it follows that $K_\mu \Phi_0 = 0$, and Φ_0 is called the primary operator. The representation is obtained from Φ_0 and operators obtained by acting succesively with P_μ ($\sim a^\dagger$) on Φ_0 ($\sim |0>$).

In **d=4**, $\mathcal{N} = 4$ Super Yang-Mills theory is such a representation of the conformal group. $\mathcal{N} = 4$ Super Yang-Mills theory with SU(N) gauge group has the fields $\{A^a_\mu, \psi^{ai}_\alpha, \phi^a_{(ij)}\}$. Here we have used SU(4) notation ($i \in SU(4)$) and $a \in SU(N)$. Indeed, one can calculate the β function of the theory and obtain that it is zero, thus the theory has quantum scale invariance. It is in fact, quantum mechanically invariant under the full conformal group.

Observation: The quantum conformal dimension (scaling dimension) Δ need not be the same as the free (at coupling $g = 0$) scaling dimension for an operator in $\mathcal{N} = 4$ Super Yang-Mills, since $\beta = 0$ just means that there are no infinities, but there still can be finite renormalizations giving nontrivial quantum effects (so that $\Delta = \Delta_0 + o(g)$).

Classically, the fundamental fields have dimensions $[A^a_\mu] = 1$, $[\psi^{ai}_\alpha] = 3/2$, $[\phi^a_{(ij)}] = 1$, and we form operators out of them, for instance $trF^2_{\mu\nu}$ which will have classical dimension 4. For some of these, the classical dimension will be exact, for some it will get quantum corrections.

D-branes

Closed strings are free to move arbitrarily through space. Open strings however need to have boundary conditions defined on the endpoints. By varying the Polyakov string action, we get the an extra boundary term for an open string,

$$\delta S_{P,boundary} = -\frac{1}{2\pi\alpha'}\int d\tau\sqrt{-\gamma}\delta X^\mu \times \partial^\sigma X_\mu|_{\sigma=0}^{\sigma=l} \tag{7.29}$$

which must vanish independently. This means that the possible boundary conditions are

- Neumann boundary condition: $\partial^\sigma X_\mu = 0$ at $\sigma = 0$ and l. It implies that the endpoints must move at the speed of light.
- Dirichlet boundary condition: $\delta X^\mu = 0$ at $\sigma = 0$ and l, thus $X^\mu =$ constant at $\sigma = 0$ and l. Thus in this case the endpoints of the string are fixed in space.

But, we can choose $p + 1$ Neumann boundary conditions for p spatial dimensions and time, and $d - p - 1$ Dirichlet boundary conditions. This means that the endpoints of the string are constrained to live on a $p + 1$-dimensional wall in spacetime. But different string endpoints could be on a different wall, as in Fig.7a.

Dai, Leigh and Polchinski, in 1989, proved that in fact this wall is dynamical, i.e. it can fluctuate and respond to external interactions, and that it has degrees of freedom living on it.

The wall was then called a D-brane, from Dirichlet-brane (as in Dirichlet boundary conditions). For p=2, we would have a Dirichlet mem-brane. By extension, we have a Dirichlet p-brane, or D p-brane.

The endpoints of strings can have a label $|i>$, called "Chan-Patton factor", that corresponds to a label of the D-brane on which the string ends, as in Fig.7b.

An open string state then will have labels of the type $|i>|j>\lambda^a_{ij}$, which means they are $N \times N$ matrices if there are N D-branes. One can prove it is a $U(N)$ matrix, and the open string state lives in the adjoint of $U(N)$. So we have a theory of open strings in the adjoint of $U(N)$ living on the D-branes. One can prove that in fact, the low energy limit of this theory is a SU(N) Yang-Mills theory. Since it also has $\mathcal{N} = 4$ supersymmetry in 4 dimensions, the theory on the 4 dimensional world-volume of N D3-branes (D-branes for p=3) is $\mathcal{N} = 4$ Super Yang-Mills theory with gauge group $SU(N)$ (the $U(1) = U(N)/SU(N)$ corresponds to the "center of mass" D-brane and decouples for most questions).

Important concepts to remember

- Conformal transformations act on flat space and give a space-dependent scale factor $[\Omega(x)]^{-2}$, thus conformal invariance is not part of general coordinate invariance.
- A scale invariant theory (with zero beta function) is generally conformal invariant. The absence of anomalies requires consistency conditions on the theory.

- In $d > 2$ Minkowski dimensions, the conformal group is $SO(d,2)$, the same as the invariance group of AdS_{d+1}.
- In 2 dimensions, conformal invariance is an infinite algebra, the Virasoro algebra, of a more general type (with a constant term). A normal subgroup is $Sl(2,C)$.
- Primary fields of dimensions $(h,\bar{h})$ in 2 dimensions scale under $z \to \lambda z, \bar{z} \to \lambda \bar{z}$ as $\phi \to (\lambda)^{h+\bar{h}}$, and in 4 dimensions primary fields of dimension Δ scale as $\phi \to \phi(\lambda)^{\Delta}$.
- In d=4, a representation of the conformal algebra is obtained by acting with P_μ on the primary field.
- D-branes are $(p+1)-$dimensional endpoints of strings, that act as dynamical walls.
- N coincident D-branes give an U(N) gauge group, and the theory on the D3-branes (in 4 dimensions) is $\mathcal{N}=4$ Super Yang-Mills.

Exercises, section 7

1) Check that

$$v_\mu = a_\mu + \omega_{\mu\nu} x_\nu + \lambda x_\mu + b_\mu x^2 - 2x_\mu b \cdot x$$
$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\sigma_v \delta_{\mu\nu}; \quad \sigma_v = \frac{1}{d}\partial \cdot v \tag{7.30}$$

and that if $x'_\mu = x_\mu + v_\mu$, then the conformal factor is $\Omega(x) = 1 - \sigma_v(x)$.

2) Derive the conformal algebra in terms of $P_\mu, J_{\mu\nu}, K_\mu, D$ from the SO(d,2) algebra, given that $J_{\mu,d+1} = (K_\mu - P_\mu)/2$, $J_{\mu,d+2} = (K_\mu + P_\mu)/2$, $J_{d+1,d+2} = D$.

3) Prove that the special conformal transformation

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2} \tag{7.31}$$

can be obtained by an inversion, followed by a translation, and another inversion.

4) Prove that a circle $(x_\mu - c_\mu)^2 = R^2$ remains a circle after a general finite conformal transformation.

5) The action for the U(1) gauge field on a D-brane is

$$S = T_p \int d^{p+1}\xi \sqrt{\det(g_{\mu\nu} + \alpha' F_{\mu\nu})} \tag{7.32}$$

Show that as $\alpha' \to 0$, the action becomes the action for electromagnetism.

8 The AdS-CFT correspondence: motivation, definition and spectra

The AdS-CFT correspondence is a relation between a conformal field theory (CFT) in d dimensions and a gravity theory in $d+1$-dimensional Anti de Sitter space (AdS). We already saw the first hint that this should be possible: Both such theories will have the same symmetry group, $SO(2,d)$. Specifically, the case of interest for us in the following will be d=4, in which case the CFT will be $\mathcal{N}=4$ Supersymmetric Yang-Mills theory with gauge group $SU(N)$ and the gravitational theory will be string theory.

D-branes =p-branes

The first step towards finding such an equivalence is to prove that D-branes are the same as p-branes. A D-brane is a dynamical wall on which strings can end. Then a string state will contain a factor $\lambda_{ij}^a|i>\otimes|j>$ from the D-branes i and j on which the two endpoints lie. Although we have not proved this here, the massless state of an open string can be shown to be a vector, thus the massless open string state will be

$$|\mu>\otimes|i>\otimes|j> \tag{8.1}$$

which is a gauge field A_μ^a in an $SU(N)$ gauge group (if there are N branes, i.e. $i,j=1,...,N$). Here λ_{ij}^a are generators of the adjoint representation. In the $\alpha'\rightarrow 0$ only the massless string states remain, therefore the low energy theory living on the N D-branes is Supersymmetric Yang-Mills with SU(N) gauge group. But string theory has 32 supercharges (32 components Q_α^i), which form a 11-dimensional spinor or 8 4-dimensional spinors, thus $\mathcal{N}=8$ supersymmetry in d=4. But a D-brane background breaks 1/2 of the supersymmetry, thus for p=3 the 4 dimensional worldvolume of the D3-branes will contain $\mathcal{N}=4$ Supersymmetric Yang-Mills with SU(N) gauge group, a conformal field theory.

On the other hand, extremal p-branes are solutions of supergravity, which is the low energy limit ($\alpha'\rightarrow 0$) of string theory. Therefore the extremal p-branes discussed in section 6 are solutions of string string theory, of a solitonic-like character (although not quite solitonic). The extremal p-branes, as we saw, have $Q=M$, saturating the bound $|Q|\leq M$, which bound can be derived in two ways

- In gravity, it comes from the fact that singularities must be hidden behind a horizon, as we saw. As mentioned, there are "no naked singularity" theorems, and for $Q>M$, we would obtain a naked singularity.
- On the other hand, in a supersymmetric theory, this bound comes from the supersymmetry algebra and is known as the "BPS bound". When the bound is saturated, the solution preserves the maximum amount of supersymmetry, which is 1/2, i.e. the solution is left invariant by a half of the supersymmetry generators.

Therefore the extremal p-branes also have $\mathcal{N}=4$ supersymmetry in d=4, and are solutions of supergravity with horizons at $r=0$ (singularity=horizon).

Polchinski, in 1995, has proven that in fact D-branes and extremal p=branes are one and the same, thus the dynamical endpoints of open strings correspond to extremal solutions of supergravity. The proof involves computing p-brane charges and tensions of the endpoints of open strings, and matching with the supergravity solutions.

Thus D-branes curve space and N D3-branes (p=3) correspond to the supergravity solution (here wedge $\wedge$ means antisymmetrization and $F_5 = F_{\mu_1 \dots \mu_5} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_5}$)

$$\begin{aligned} ds^2 &= H^{-1/2}(r) d\vec{x}_{||}^2 + H^{1/2}(r)(dr^2 + r^2 d\Omega_5^2) \\ F_5 &= (1 + *) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge (df^{-1}) \\ H(r) &= 1 + \frac{R^4}{r^4}; \quad R = 4\pi g_s N \alpha'^2; \quad Q = g_s N \end{aligned} \tag{8.2}$$

But if we go a bit from the extremal limit $Q = M$ by adding a small mass δM, this solution will develop an event horizon at a small $r_0 > 0$, and like the Schwarzschild black hole, it will emit "Hawking radiation" (thermal radiation produced by the event horizon).

But if this supergravity solution represents a D-brane also, one can derive this Hawking radiation in the D-brane picture from a unitary quantum process: Two open strings living on a D-brane collide to form a closed string, which then is not bound to the D-brane anymore and can peel off the D-brane and move away as Hawking radiation, as in Fig.8a.

This then also means that there should be a relation between the theory of open strings living on the D3-brane, i.e. $\mathcal{N} = 4$ Super Yang-Mills, and the gravity theory of fields living in the space curved by the D3-brane (8.2) (the "Hawking radiation").

Motivation

We will now motivate (heuristically derive) this relation by studying string theory in the presence of D3-branes from two points of view.

Point of view nr.1

Consider the D-branes viewed as endpoints of open strings. Then string theory with D3-branes has 3 ingredients

- the open strings living on the D3-branes, giving a theory that reduces to $\mathcal{N} = 4$ Super Yang-Mills in the low energy limit.
- the closed strings living in the bulk (the whole) of spacetime, giving a theory that is supergravity coupled to the massive modes of the string. In the low energy limit, only supergravity remains.
- the interactions between the two, giving for instance Hawking radiation through the process I just described.

Thus the action of these strings will be something like

$$S = S_{bulk} + S_{brane} + S_{interactions} \tag{8.3}$$

In the low energy limit $\alpha' \to 0$, the massive string modes drop out, and $S_{bulk} \to S_{supergravity}$, also $S_{brane} \to S_{\mathcal{N}=4SYM}$. Moreover, since

$$S_{int} \propto k \sim g_s \alpha'^2 \tag{8.4}$$

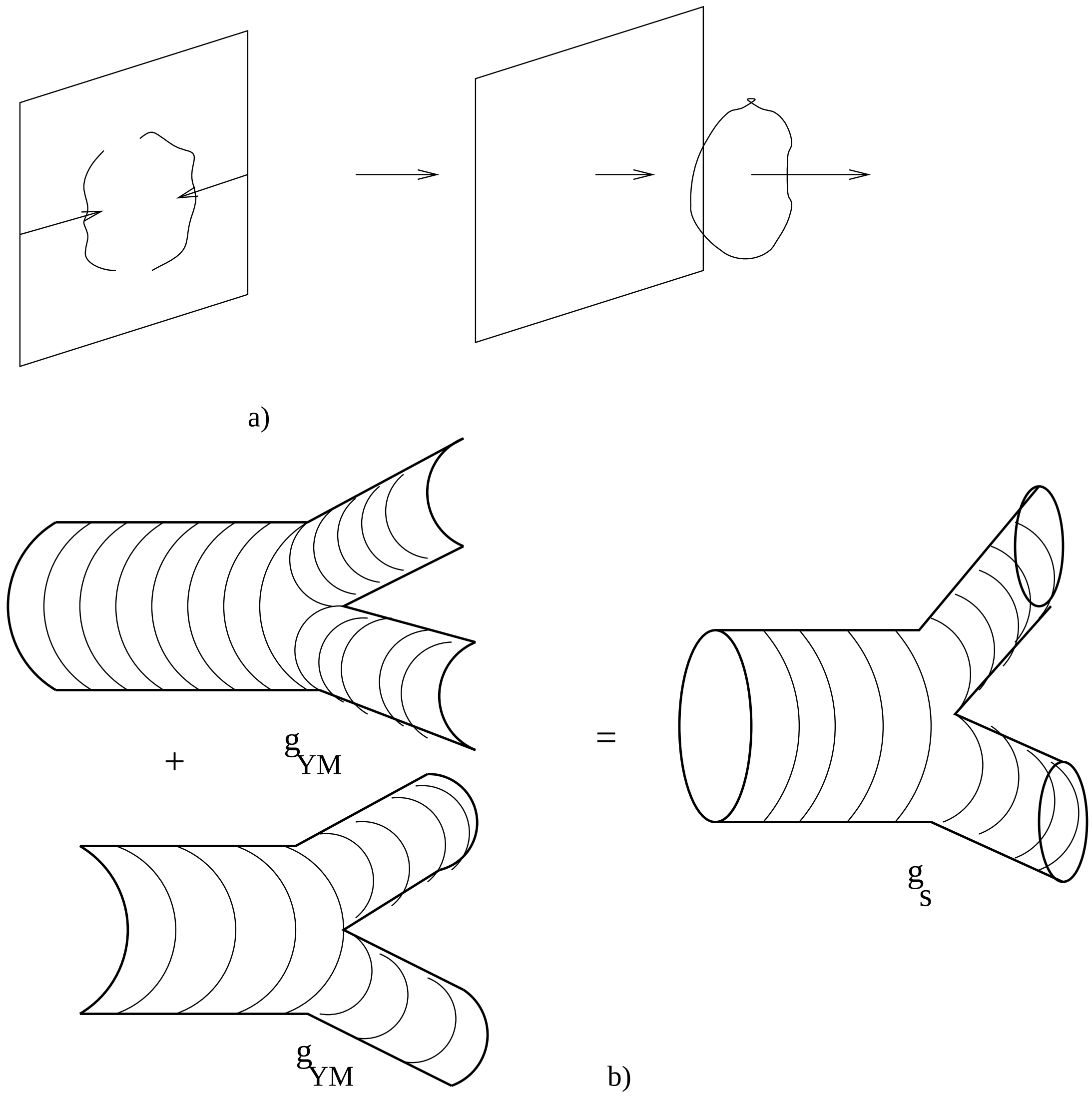

Figure 8: a)Two open strings living on a D-brane collide and form a closed string, that can then peel off and go away from the brane. b) Two open string splitting interactions can be glued on the edges to give a closed string interaction ("pair of pants"), therefore $g_{YM}^2 = g_s$.

where $k = \sqrt{Newton\ constant}$ and $\alpha' \to 0$, whereas g_s is the string coupling and stays fixed. Then we see that $S_{int} \to 0$ and moreover, since the Newton constant $k^2 \to 0$, gravity (thus supergravity also) becomes free. Thus in this limit we get two decoupled systems (non-interacting)

- free gravity in the bulk of spacetime
- 4 dimensional $\mathcal{N} = 4$ gauge theory on the D3-branes.

Point of view nr.2.

We now replace the D3 brane by the supergravity solution (p-brane).

Then the energy E_p measured at a point r and the energy E measured at infinity are related by

$$E_p \sim \frac{d}{d\tau} = \frac{1}{\sqrt{-g_{00}}}\frac{d}{dt} \sim \frac{1}{\sqrt{-g_{00}}}E \Rightarrow E = H^{-1/4}E_p \sim rE_p \tag{8.5}$$

Therefore for fixed E_p, as $r \to 0$, the energy observed at infinity, E, goes to zero, i.e. we are in the low energy regime.

Thus from this point of view, we also have two decoupled low energy systems of excitations

- At large distances ($\delta r \to \infty$), therefore at low energies (energy $\sim$ 1/length), gravity becomes free (the gravitational coupling has dimensions, therefore the effective dimensionless coupling is $GE^2 \to 0$ as $E \to 0$). Thus again we have free gravity at large distances.
- At small distances $r \to 0$, we have also low energy excitations, as we saw.

The fact that these two systems are decoupled can be seen in a couple of ways. One can calculate that waves of large r have vanishing absorbtion cross section on D-branes. One can also show that reversely, the waves at $r = 0$ can't climb out of the gravitational potential and escape at infinity.

Thus in the second point of view we again have two decoupled low energy systems, one of which is free gravity at large distances. Therefore, we can identify the other low energy system in the two point of view and obtain that

The 4 dimensional gauge theory on the D3-branes, i.e.
$\mathcal{N} = 4$ Super Yang-Mills with gauge group SU(N), at large N is =
= gravity theory at $r \to 0$ in the D-brane background, if we take $\alpha' \to 0$.

This is called AdS-CFT, but at this moment it is just a vague statement.

Definition: limit, state map, validity

Let us therefore define better what we mean. If we take $r \to 0$, then the harmonic function $H \simeq R^4/r^4$, and we obtain the supergravity background solution

$$ds^2 \simeq \frac{r^2}{R^2}(-dt^2 + d\vec{x}_3^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 \tag{8.6}$$

By changing the coordinates $r/R \equiv R/x_0$, we get

$$ds^2 = R^2 \frac{-dt^2 + d\vec{x}_3^2 + dx_0^2}{x_0^2} + R^2 d\Omega_5^2 \tag{8.7}$$

which is the metric of $AdS_5 \times S_5$, i.e. 5 dimensional Anti de Sitter space times a 5-sphere of the same radius R, where AdS_5 is in Poincare coordinates.

From the point of view of the supergravity background solution, the gauge theory lives in the original metric (before taking the $r \to 0$ limit). Therefore in the new $AdS_5 \times S_5$ space we can say that the gauge theory lives at $r \to \infty$, or $x_0 \to 0$, which as we have proven when analyzing AdS space, is part of the real boundary of global AdS space, and in Poincare coordinates $x_0 \to 0$ is a Minkowski space.

Therefore the gravity theory lives in $AdS_5 \times S_5$, whereas the Super Yang-Mills theory lives on the 4 dimensional Minkowski boundary of AdS_5.

We still need to understand the $\alpha' \to 0$ limit. We want to keep arbitrary excited string states at position r as we take $r \to 0$ to find the low energy limit. Therefore the energy at point p in string units, $E_p\sqrt{\alpha'}$ needs to be fixed. Since $H \simeq R^4/r^4 \propto \alpha'^2/r^4$, the energy measured at infinity is

$$E = E_p H^{-1/4} \propto E_p r/\sqrt{\alpha'} \tag{8.8}$$

But at infinity we have the gauge theory, therefore the energy measured at infinity (in the gauge theory) must also stay fixed. Then since $E_p\sqrt{\alpha'} \sim E\alpha'/r$ must be fixed, it follows that

$$U \equiv \frac{r}{\alpha'} \tag{8.9}$$

is fixed as $\alpha' \to 0$ and $r \to 0$ and can be thought of as an energy scale in the gauge theory (since we said that E/U was fixed). The metric is then ($R^4 = \alpha'^2 4\pi g_s N$)

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}}(-dt^2 + d\vec{x}_3^2) + \sqrt{4\pi g_s N}(\frac{dU^2}{U^2} + d\Omega_5^2) \right] \tag{8.10}$$

where $\alpha' \to 0$ but everything inside the brackets is finite.

Here in the gravity theory N is the number of D3-branes and g_s is the string coupling. In the Super Yang-Mills gauge theory, N is the rank of the $SU(N)$ gauge group, (which is the low energy gauge group on the N D3-branes). And g_s is related to the Yang-Mills coupling by

$$g_s = g_{YM}^2 \tag{8.11}$$

since g_{YM} is the coupling of the gauge field A_μ^a, which we argued that is the massless mode of the open string living on the D3-branes. But out of two open strings we can make a closed string, therefore out of two open string splitting interactions, governed by the g_{YM} open string coupling, we can make one closed string splitting interaction, governed by the g_s coupling, as in Fig.8b.

The last observation that one needs to make is that in the limit $\alpha' \to 0$, string theory becomes its low energy limit, supergravity.

Therefore AdS-CFT relates string theory, in its supergravity limit, in the background (8.10), with $\mathcal{N} = 4$ Super Yang-Mills with gauge group SU(N) living in d=4, at the boundary of AdS_5.

Now it remains to define the **limits of validity** of this identification.

As it was stated, the supergravity approximation of string theory means that

- the curvature of the background (8.10) must be large compared to the string length, i.e. $R = \sqrt{\alpha'}(g_s N)^{1/4} \gg \sqrt{\alpha'} = l_s$. That means that we are in the limit $g_s N \gg 1$, or $g_{YM}^2 N \gg 1$.
- string corrections, governed by g_s, are small, thus $g_s \to 0$.

Therefore, for supergravity to be valid, we need to have $g_s \to 0, N \to \infty$, but $\lambda = g_s N = g_{YM}^2 N$ must be fixed and large ($\gg 1$).

On the other hand, in an $SU(N)$ gauge theory, as 't Hooft showed, at large N, the effective coupling is the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$, therefore if perturbation theory is valid, $\lambda \ll 1$, which is the opposite case of the supergravity approximation of AdS-CFT.

That is the reason why AdS-CFT is called a *duality*, since the two descriptions (gauge theory perturbation theory and supergravity in $AdS_5 \times S_5$) are valid in opposite regimes ($\lambda \ll 1$ and $\lambda \gg 1$, respectively). That means that such a duality will be hard to check, since in one regime we can use a description to calculate, but not the other.

So finally, we have come to the definition of AdS-CFT as a duality between supergravity on $AdS_5 \times S_5$ as in (8.10) and 4 dimensional $\mathcal{N} = 4$ Super Yang-Mills with SU(N) gauge group, living at the AdS_5 boundary, with $g_s \to 0, N \to \infty$ and $g_s N$ fixed and large.

But AdS-CFT can have then several possible versions:

- The weakest version is the one that was just described: AdS-CFT is valid only at large $g_s N$, when we have just the supergravity approximation of string theory in the background (8.10). If we go to the full string theory (away from large $g_s N$), we might find dissagreements.
- A stronger version would be that the AdS-CFT duality is valid at any finite $g_s N$, but only if $N \to \infty$ and $g_s \to 0$, which means that α' corrections, given by $\alpha'/R^2 = 1/\sqrt{g_s N}$ agree, but g_s corrections might not.
- The strongest version would be that the duality is valid at any g_s and N, even if we can only make calculations in certain limits. This is what is believed to be true, since many examples were found of α' and g_s corrections that agree between AdS and CFT theories.

Next we will turn to the relation between various observables in the two theories.

State map

Let us take an operator $\mathcal{O}$ in the $\mathcal{N} = 4$ Super Yang-Mills CFT. It will be characterized by a certain conformal dimensions Δ (since we are in a conformal field theory) and a representation index I_n for the $SO(6) = SU(4)$ symmetry.

In the gravity theory (string theory) in $AdS_5 \times S_5$ it will correspond to a field.

In this discussion we will restrict to the supergravity limit. Then we have supergravity on $AdS_5 \times S_5$, where S_5 is a compact space, thus we can apply the Kaluza-Klein procedure of compactification: We expand the supergravity fields in "spherical harmonics" (Fourier-like modes) on the sphere. For instance, a scalar field would be expanded as

$$\phi(x,y) = \sum_n \sum_{I_n} \phi^{I_n}_{(n)}(x) Y^{I_n}_{(n)}(y) \tag{8.12}$$

where n is the level, the analog of the n in $e^{inx/R}$ for a Fourier mode around a circle of radius R. I_n is an index in a representation of the symmetry group, x is a coordinate on AdS_5 and y a coordinate on S_5, and the spherical harmonic $Y^{I_n}_{(n)}(y)$ is the analog of $e^{inx/R}$ for a Fourier mode.

Then the field $\phi^{I_n}_{(n)}$ living in AdS_5, of mass m, corresponds to an operator $\mathcal{O}^{I_n}_{(n)}$ in 4 dimensional $\mathcal{N}=4$ Super Yang-Mills, of dimension Δ. The relation between m and Δ is

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2} \tag{8.13}$$

The dimensional reduction on S_5, i.e. keeping only the lowest mode in the Fourier-like expansion, should give a supergravity theory in AdS_5. But as we mentioned in section 4, supergravity theories that admit Anti de Sitter backgrounds (with a cosmological constant) are actually gauged supergravity theories, so we obtain maximal 5 dimensional gauged supergravity.

The symmetry group of the reduction, under which $\phi_{(n)}$ has representation index I_n, is $SO(2,4) \times SO(6)$, with $SO(2,4)$ being the symmetry group of AdS_5 and the conformal group of the $\mathcal{N}=4$ Super Yang-Mills, and $SO(6)$ being the symmetry group of S_5 and the "R-symmetry group" of $\mathcal{N}=4$ Super Yang-Mills, the global symmetry rotating the SYM fields.

The level n indicates fields $\phi_{(n)}$ of increasing mass m, and by the above relation, SYM fields of increasing conformal dimension Δ.

"Experimental evidence"

One can now analyze the set of fields $\phi_{(n)}$ obtained by the spherical harmonic expansion of 10 dimensional supergravity around the background solution $AdS_5 \times S_5$ and match against the set of operators in the conformal field theory that belong to definite representations of the symmetry groups. One then matches I_n's and Δ's versus m's.

However, this is not as simple as it sounds, since we mentioned that even though $\mathcal{N}=4$ Super Yang-Mills has zero beta function, there are still quantum corrections to the conformal dimensions Δ of operators. Since we are working in the deeply nonperturbative gauge theory regime, od effective coupling $\lambda \gg 1$, it would seem that we have no control over the result for the quantum value of Δ of a given operator.

But we are saved by the large amount of symmetry available. Supersymmetry together with the conformal group $SO(2,4)$ give the superconformal group $SU(2,2|4)$.

Representations of the conformal group are given as we said by a primary operator $\mathcal{O}$ and their "descendants," obtained by acting with P_μ on them like a creation operator on the vacuum $(P_{\mu_1}...P_{\mu_n}\mathcal{O})$.

Representations of the superconformal group are correspondingly larger (there are more symmetries, which must relate more fields), so they will include many primary operators of the conformal group (there are 2^{16} primaries for a generic representation of $\mathcal{N}=4$ in d=4, since there are 16 supercharges).

However, there are special, *short* representations of the superconformal group, that are generated by *chiral primary operators*, which are primary operators that are annihilated by some combination of Q's (thus they preserve some supersymmetry by themselves), i.e. $[Q\ comb.]\mathcal{O}_{ch.pr}=0$. The conformal dimension Δ of chiral primary operators is uniquely determined by the R-symmetry, thus it does not receive quantum corrections, i.e. the $\lambda \gg 1$ value is the same as the $\lambda = 0$ value, and we can check it using AdS-CFT!

The representations I_n of the symmetry groups are in fact such small representations for supergravity fields (non-supergravity string fields will in general belong to large representations), thus Kaluza-Klein supergravity modes in AdS_5 correspond to chiral primary fields in Super Yang-Mills, with dimensions protected against quantum corrections.

KK scalar fields in AdS_5 belong to 5 families, and correspondingly we find 5 families of chiral primary representations. For simplicity, we analyze three of these 5, which are

- $tr(\phi^{(I_1}...\phi^{I_n)})$ (in the symmetric representation), plus its fermionic partners, which therefore has dimension $\Delta = n$ (there are n fields of dimension 1), and by the above we expect to correspond to a KK field of mass $m^2 = n(n-4), n \geq 2$

- $tr(\epsilon^{\alpha\beta}\lambda_{\alpha A}\lambda^{\beta B}\phi^{I_1}...\phi^{I_{n-1}})$ of dimension $\Delta = n+2$ (λ has dimension 3/2), therefore corresponding to a KK field of mass $m^2 = (n+2)(n-2), n \geq 0$.

- $tr(F_{\mu\nu}F^{\mu\nu}\phi^n)$ where ϕ is a complex scalar, of dimension $\Delta = n+4$ (A_μ has dimension 1), corresponding to $m^2 = n(n+4)$.

We find that indeed 3 of the KK families have such masses, therefore we have "experimental evidence" for AdS-CFT.

Global AdS-CFT

We obtained AdS-CFT in the Poincare patch, but AdS space is larger, therefore AdS-CFT must relate global AdS_5 space to the $\mathcal{N}=4$ Super Yang-Mills theory in 4 dimensions. But the twist is that then one must make a conformal transformation in Euclidean space.

String theory in the Poincare patch of AdS space is related to $\mathcal{N}=4$ Super Yang-Mills living the 4d dimensional Minkowski space at the boundary. The boundary of global AdS_5 space is, as we saw, an $R_t \times S_3$, therefore string theory in global AdS_5 is related to gauge theory on the $R_t \times S_3$ space at its boundary.

The metric of global AdS_5 (times S_5) is

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_3^2)(+R^2 d\Omega_5^2) \tag{8.14}$$

If we put $\theta = \pi/2$ in it, we obtain the boundary

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\Omega_3^2) \tag{8.15}$$

where $1/\cos^2\theta \to \infty$, whereas in Poincare coordinates

$$ds^2 = R^2 \frac{d\vec{x}^2 + dx_0^2 + x_0^2 d\Omega_5^2}{x_0^2} = R^2 \frac{d\vec{x}^2}{x_0^2} \tag{8.16}$$

with $x_0 \to 0$.

Then indeed, the Euclidean versions of the two metrics, $d\vec{x}^2$ and $d\tau^2 + d\Omega_3^2$ are related by a conformal transformation.

$$ds^2 = d\vec{x}^2 = dx^2 + x^2 d\Omega_3^2 = x^2((d\ln x)^2 + d\Omega_3^2) = x^2(d\tau^2 + d\Omega_3^2) \tag{8.17}$$

Thus string theory in global $AdS_5 \times S_5$ is related to $\mathcal{N} = 4$ Super Yang-Mills on $R_t \times S_3$ (conformally related to R^4).

Important concepts to remember

- D-branes are the same as (extremal) p-branes, and we have $\mathcal{N} = 4$ Super Yang-Mills with gauge group $SU(N)$ on the worldvolume of N D3-branes.
- AdS-CFT states that the $\mathcal{N} = 4$ Super Yang-Mills with gauge group $SU(N)$ at large N equals string theory in the $\alpha' \to$ limit, in the $r \to 0$ of the D3-brane metric, which is $AdS_5 \times S_5$.
- The most conservative statement of AdS-CFT relates supergravity in $AdS_5 \times S_5$ with $\mathcal{N} = 4$ Super Yang-Mills with gauge group $SU(N)$ and $g_{YM}^2 = g_s$ at $g_s \to 0$, $N \to \infty$ and $\lambda = g_s N$ fixed and large ($\gg 1$).
- The strongest version of AdS-CFT is believed to hold: string theory in $AdS_5 \times S_5$ is related to $\mathcal{N} = 4$ Super Yang-Mills with gauge group $SU(N)$ at any $g_{YM}^2 = g_s$ and N., but away from the above limit it is hard to calculate anything
- AdS-CFT is a duality, since weak coupling calculations in string theory $\alpha' \to 0, g_s \to 0$ are strong coupling (large $\lambda = g_{YM}^2 N$) in $\mathcal{N} = 4$ Super Yang-Mills, and vice versa.
- Supergravity fields in $AdS_5 \times S_5$, Kaluza-Klein dimensionally reduced on S_5, correspond to operators in $\mathcal{N} = 4$ Super Yang-Mills, and the conformal dimension of operators is related to the mass of supergravity fields.
- Chiral primary operators are primary operators that preserve some supersymmetry, and belong to special (short) representations of the superconformal group. The dimension of chiral primary operators matches with what is expected from the mass of the corresponding AdS_5 fields.
- AdS-CFT is actually defined in global AdS space, which has a $S^3 \times R_t$ boundary. The $\mathcal{N} = 4$ Super Yang-Mills theory lives at this boundary, which is conformally related to R^4.

Exercises, section 8

1) The metric for an "M2 brane" solution of d=11 supergravity (and of so called "M theory," related to string theory, by extension) is given by

$$ds^2 = H^{-2/3}(d\vec{x}_3)^2 + H^{+1/3}(dr^2 + r^2 d\Omega_7^2); \quad H = 1 + \frac{2^5\pi^2 l_P^6}{r^6} \tag{8.18}$$

Check that the same limit taken for D3 branes gives M theory on $AdS_4 \times S_7$ if $l_P \to 0$, $U \equiv r^2/l_P^3$ fixed.

2) Let Y^A be 6 cartesian coordinates for the 5-sphere S^5. Then Y^A are vector spherical harmonics and $Y^{A_1...A_n} = Y^{(A_1}...Y^{A_n)}-$traces is a totally symmetric traceless spherical harmonic (i.e. $Y^{A_1...A_n}\delta_{A_m A_p} = 0$, $\forall\ 1 \leq m, p \leq n$). Check that, as polynomials in 6d, $Y^{A_1...A_n}$ satisfy $\Box_{6d} Y^{A_1...A_n} = 0$. Expressing $\Box_{6d}$ in terms of $\Box_{S^5}$ amd ∂_r (where $Y^A Y^A \equiv r^2$), check that $Y^{A_1...A_n}$ are eigenfunctions with eigenvalues $-k(k+6-1)/r^2$.

3) Check that the $r \to 0$ limit of the Dp-brane metric gives $AdS_{p+2} \times S_{8-p}$ only for p=3.

4) String corrections to the gravity action come about as g_s corrections to terms already present and α' corrections appear generally as $(\alpha'\mathcal{R})^n$, with $\mathcal{R}$ the Ricci scalar, or some particular contraction of Riemann tensors. What then do α' and g_s string corrections correspond to in SYM via AdS-CFT (in the $N \to \infty$, $\lambda = g_{YM}^2 N$ fixed and large limit)?

5) Show that the time it takes a light ray to travel from a finite point in AdS to the real boundary of space and back is finite, but the times it takes to reach the center of AdS ($x_0 = \infty$, or $r = 0$, or $\rho = 0$) is infinite. Try this in both Poincare and global coordinates.

9 Witten prescription and 3-point function calculation

Witten prescription

A precise correspondence between the fundamental observables, the correlators of the CFT and the correlators of Supergravity, was proposed by Witten. This prescription relates the Euclidean version of AdS_5 (Lobachevski space) with the CFT on Euclidean R^4. The physical case of Minkowski space is harder. One needs to analytically continue the Euclidean space final results to Minkowski space.

An operator $\mathcal{O}$ in $\mathcal{N}=4$ SYM of dimension Δ is related to a field ϕ of mass m in $AdS_5 \times S_5$ supergravity where the relation between Δ and m is (8.13). A massless field $m=0$ corresponds to a field ϕ of $\Delta = d$ living at the boundary of space.

The boundary of $AdS_5 \times S_5$, S_5 shrinks to zero size, and the boundary is either $R_t \times S_3$ in global coordinates, or the conformally equivalent R^4 in the Poincare patch, and is identified with the space where $\mathcal{N}=4$ SYM lives. But the massless field ϕ will have a value ϕ_0 on the boundary of AdS_5, which therefore should have a corresponding meaning in the gauge theory.

The natural interpretation is that ϕ_0 is a source for $\mathcal{O}$, i.e. that it couples to it. Since ϕ_0 has no gauge indices (there is no "gauge group" in gravity), $\mathcal{O}$ has none, so it must be a gauge invariant operator, therefore composite (since fundamental fields have gauge indices).

One is then led to consider the partition function with sources for the composite operator $\mathcal{O}$, $Z_{\mathcal{O}}[\phi_0]$, which is a generating functional of correlation functions of $\mathcal{O}$, as we discussed in section 1.

In Euclidean space, we have

$$\begin{aligned} Z_{\mathcal{O}}[\phi_0] &= \int \mathcal{D}[SYM\ fields] \exp(-S_{\mathcal{N}=4\ SYM} + \int d^4x \mathcal{O}(x)\phi_0(x)) \\ \Rightarrow &< \mathcal{O}(x_1)...\mathcal{O}(x_n) >= \frac{\delta^n}{\delta\phi_0(x_1)...\delta\phi_0(x_n)} Z_{\mathcal{O}}[\phi_0] \end{aligned} \tag{9.1}$$

We now need to understand how to compute $Z_{\mathcal{O}}[\phi_0]$ in AdS_5. It should be a partition function of string theory in AdS_5 for the field ϕ, with the source ϕ_0 on its boundary, i.e. the field ϕ approaches ϕ_0 on its boundary.

But if we are in the supergravity limit, $g_2 \to 0, \alpha' \to 0$, $R^4/\alpha'^2 = g_s N \gg 1$, we have no quantum corrections, therefore the classical supergravity is a good approximation. Then the partition function $Z[\phi_0]$ of the field ϕ in classical supergravity, for $\phi \to \phi_0$ on the boundary, becomes

$$Z[\phi_0] = \exp[-S_{sugra}[\phi[\phi_0]]] \tag{9.2}$$

i.e., one finds the classical solution $\phi[\phi_0]$ and replaces it in S_{sugra}.

Therefore, Witten's prescription for the correlation functions of massless fields in AdS-CFT is

$$Z_{\mathcal{O}}[\phi_0]_{CFT} = \int \mathcal{D}[] e^{-S+\int d^4x \mathcal{O}(x)\phi_0(x)} = Z_{class}[\phi_0]_{AdS} = e^{-S_{sugra}[\phi[\phi_0]]} \tag{9.3}$$

One can define a classical AdS_5 Green's function (for instance, in the Poincare patch).

The bulk-to boundary propagator K_B is defined by

$$"\Box_{\vec{x},x_0}" K_B(\vec{x}, x_0; \vec{x}') = \delta^4(\vec{x} - \vec{x}') \tag{9.4}$$

where "$\Box_{\vec{x},x_0}$" is the kinetic operator and the delta function is a source on the flat 4 dimensional boundary of AdS_5. Then the field ϕ is written as

$$\phi(\vec{x}, x_0) = \int d^4\vec{x}' K_B(\vec{x}, x_0; \vec{x}')\phi_0(\vec{x}') \tag{9.5}$$

and one replaces this in $S_{sugra}[\phi]$.

The simplest **example** is a CFT 2-point function of the operator $\mathcal{O}$ corresponding to ϕ,

$$< \mathcal{O}(x_1)\mathcal{O}(x_2) >= \frac{\delta^2}{\delta\phi_0[x_1]\delta\phi_0[x_2]} e^{-S_{sugra}[\phi[\phi_0]]}|_{\phi_0=0} \tag{9.6}$$

But since

$$S_{sugra}[\phi[\phi_0]] \sim \int (d^5x\sqrt{g}) \int d^4\vec{x}' \int d^4\vec{y}' "\partial_\mu" K_B(\vec{x}, x_0; \vec{x}')\phi_0(\vec{x}') "\partial^\mu" K_B(\vec{x}, x_0; \vec{y}')\phi_0(\vec{y}') + O(\phi_0^3) \tag{9.7}$$

where "$\partial_\mu \cdot \partial^\mu$" = "$\Box$" = kinetic operator, we get that

$$S_{sugra}[\phi[\phi_0]]|_{\phi_0=0} = 0; \quad \frac{\delta S}{\delta\phi_0}[\phi[\phi_0]]|_{\phi_0=0} = 0 \tag{9.8}$$

and only second derivatives and higher give a nonzero result. Then

$$\begin{aligned} < \mathcal{O}(x_1)\mathcal{O}(x_2) > &= \frac{\delta}{\delta\phi_0[x_1]}\left(-\frac{\delta S_{sugra}}{\delta\phi_0[x_2]} e^{-S_{sugra}}\right)|_{\phi_0=0} = -\frac{\delta^2 S_{sugra}[\phi[\phi_0]]}{\delta\phi_0(x_1)\delta\phi_0(x_2)}|_{\phi_0=0} \\ &= -\frac{\delta^2}{\delta\phi_0[x_1]\delta\phi_0[x_2]} \int d^5x\sqrt{g} \int d^4\vec{x}' \int d^4\vec{y}' "\partial_{\mu_{\vec{x},x_0}}" K_B(\vec{x}, x_0; \vec{x}')\phi_0(\vec{x}') \times \\ &\times "\partial^\mu_{\vec{x},x_0}" K_B(\vec{x}, x_0; \vec{y}')\phi_0(\vec{y}') = \int d^5x\sqrt{g} "\partial_{\mu_{\vec{x},x_0}}" K_B(\vec{x}, x_0; \vec{x}') "\partial^\mu_{\vec{x},x_0}" K_B(\vec{x}, x_0; \vec{y}') \end{aligned} \tag{9.9}$$

This is the general approach one can use for any n-point function, but in the particular case of the 2-point function the problem simplifies, and the integral that needs to be done is simpler. We are working in Euclidean AdS_{d+1} (Lobachevski space) in the Poincare patch, with metric

$$ds^2 = R^2 \frac{d\vec{x}^2 + dx_0^2}{x_0^2} \tag{9.10}$$

As we just saw, because we take two ϕ_0 derivatives and afterwards put ϕ_0 to zero, the interacting terms in the supergravity action can be neglected for the calculation of the two point function. Therefore we are considering only a free scalar field, satisfying $\Box\phi = 0$, and with action

$$S = \int d^5x\sqrt{g}(\partial_\mu\phi)\partial^\mu\phi = -\int d^5x\sqrt{g}\phi\Box\phi + \int d^5x\sqrt{g}\partial_\mu(\phi\partial^\mu\phi) = \int_{boundary} d^4x\sqrt{h}(\phi\vec{n}\cdot\vec{\nabla}\phi) \tag{9.11}$$

where h is the metric on the boundary. From the bulk-to-boundary propagator equation (9.4) one finds that

$$K_B(\vec{x}, x_0; \vec{x}') = \frac{C x_0^d}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^d} \tag{9.12}$$

and we have $\sqrt{h} = x_0^{-d}$; $\vec{n} \cdot \vec{\nabla} = x_0 \partial / \partial x_0$. We also have $\phi(\vec{x}, x_0) \to \phi_0(\vec{x})$ as $x_0 \to 0$ and

$$x_0 \frac{\partial}{\partial x_0} \phi(\vec{x}, x_0) = x_0 \frac{\partial}{\partial x_0} \int d^d \vec{x}' K_B(\vec{x}, x_0; \vec{x}') \phi_0(\vec{x}') \to c d x_0^d \int d^d \vec{x} \frac{\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \tag{9.13}$$

thus we obtain

$$S_{sugra}[\phi] = \lim_{x_0 \to 0} \int d^d \vec{x} x_0^{-d} \phi(\vec{x}, x_0) x_0 \frac{\partial}{\partial x_0} \phi(\vec{x}, x_0) = \frac{Cd}{2} \int d^d \vec{x} d^d \vec{x}' \frac{\phi_0(\vec{x}) \phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2d}} \tag{9.14}$$

and therefore

$$< \mathcal{O}(x_1) \mathcal{O}(x_2) > = -\frac{Cd/2}{|\vec{x} - \vec{x}'|^{2d}} \tag{9.15}$$

which is the correct behaviour for a field of conformal dimension $\Delta = d$. As we said, the massless scalar field should indeed correspond to an operator of protected dimension $\Delta = d$, so we have our first check of AdS-CFT!

But a real test comes at the level of interactions. The two-point function behaviour is kinematically fixed (by conformal invariance), and the numerical factor $-Cd/2$ is only a normalization constant. Not so for a 3-point function. Even though the functional form will be dictated by conformal invariance (plus the conformal dimensions, tested in the two-point functions), the actual numerical coefficient could provide a test of the dynamics. But, as for the conformal dimension, the numerical coefficient of a 3-point function will in general receive quantum corrections. So we need to find quantitites that are not renormalized.

R current anomaly

Luckily, the first such example is easy to find. There is an $SU(4) = SO(6)$ R-symmetry in the $\mathcal{N} = 4$ SYM theory, which in principle can be broken by quantum anomalies, as described in section 1. And as we said there, these anomalies appear only at 1-loop, so they can be calculated exactly. Also, in 4 dimensions, the only one-loop diagram that gives a quantum anomaly is the triangle graph, which has 3 external points, therefore contributes to the 3-point function.

The SU(4) R-symmetry currents J_μ^a are gauge invariant, composite operators of the type of $\mathcal{O}$, which by Witten's AdS-CFT prescription couple to fields A_μ^a in AdS_5, that have boundary values a_μ^a. Here a is a SU(4)=SO(6) index, and in gravity SO(6) is the symmetry of the sphere S_5. From this one can infer that the fields A_μ^a are the gauge fields of the gauged supergravity that is the Kaluza-Klein dimensional reduction of 10 dimensional supergravity compactified on the S_5. The Witten prescription gives

$$Z = \int \mathcal{D}[fields] e^{-S + \int d^4 x J_\mu^a a_\mu^a} = e^{-S_{sugra}[A[a]]} \tag{9.16}$$

The R symmetry currents are obtained as follows. The $\mathcal{N}=4$ SYM action in Euclidean space is

$$S_{\mathcal{N}=4\ SYM}=Tr\int d^4x[\frac{1}{4}F_{\mu\nu}^2-\frac{i}{2}\bar{\psi}_i \not{D}\psi^i-\frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij}$$
$$-\frac{i}{2}\bar{\psi}_i[\phi^{ij},\psi_j]+\frac{1}{4}[\phi_{iij},\phi_{kl}][\phi^{ij},\phi^{kl}]] \tag{9.17}$$

and the SU(4) R symmetry transformations are

$$\delta\psi^i=\epsilon^a(T_a)^i{}_j\frac{1+\gamma_5}{2}\psi^j;\quad \delta\phi_{ij}=\epsilon^a(T_a)_{ij}{}^{kl}\phi_{kl} \tag{9.18}$$

and then the Noether current (1.41) is

$$J_a^\mu(x)=\frac{1}{2}\phi(x)T_a^\phi(\overset{\leftrightarrow\mu}{\partial}+2A^\mu(x))\phi(x)-\frac{i}{2}\bar{\psi}(x)T_a^\psi\gamma^\mu\frac{1+\gamma_5}{2}\psi(x) \tag{9.19}$$

The conventions we use here are that

$$T_aT_b=\frac{1}{2}(f_{ab}{}^c-id_{ab}{}^c)T_c\Rightarrow[T_a,T_b]=f_{ab}{}^cT_c;\quad \{T_a,T_b\}=-id_{ab}{}^cT_c \tag{9.20}$$

and that

$$Tr_R(T_aT_b)=-C_R\delta_{AB} \tag{9.21}$$

where C_R is the Casimir in the corresponding representation ($C_f=1/2$). We note that the R-symmetry is carried in particular by the chiral fermions $\frac{1}{2}(1+\gamma_5)\psi$. The d=4 anomaly is given by a triangle diagram as in Fig.9a), where the loop (triangle) is formed by chiral fermions. Anomaly means that

$$\frac{\partial}{\partial x_\mu}<J_\mu^a(x)J_\nu^b(y)J_\rho^c(z)>\neq 0 \tag{9.22}$$

The quantum anomaly has in general the properties

- is one-loop exact, so we expect to find the same result from AdS-CFT
- is proportional to $d_{abc}=Tr(T_a\{T_b,T_c\})$, which is totally symmetric under the interchange of a,b,c indices, therefore the anomaly (9.22) will be totally symmetric under the interchange of a,b,c.
- is antisymmetric in the indices μ,ν,ρ.

In a similar manner to the 2-point function calculation in (9.9) we get

$$<J_\mu^a(x)J_\nu^b(y)J_\rho^c(z)>=\frac{\delta^3e^{-S_{sugra}[A_\mu^a[a_\nu^b]]}}{\delta a_\mu^a(x)\delta a_\nu^b(y)\delta a_\rho^c(z)}|_{a=0}=-\frac{\delta^3S_{sugra}[A_\mu^a[a_\nu^b]]}{\delta a_\mu^a(x)\delta a_\nu^b(y)\delta a_\rho^c(z)}|_{a=0} \tag{9.23}$$

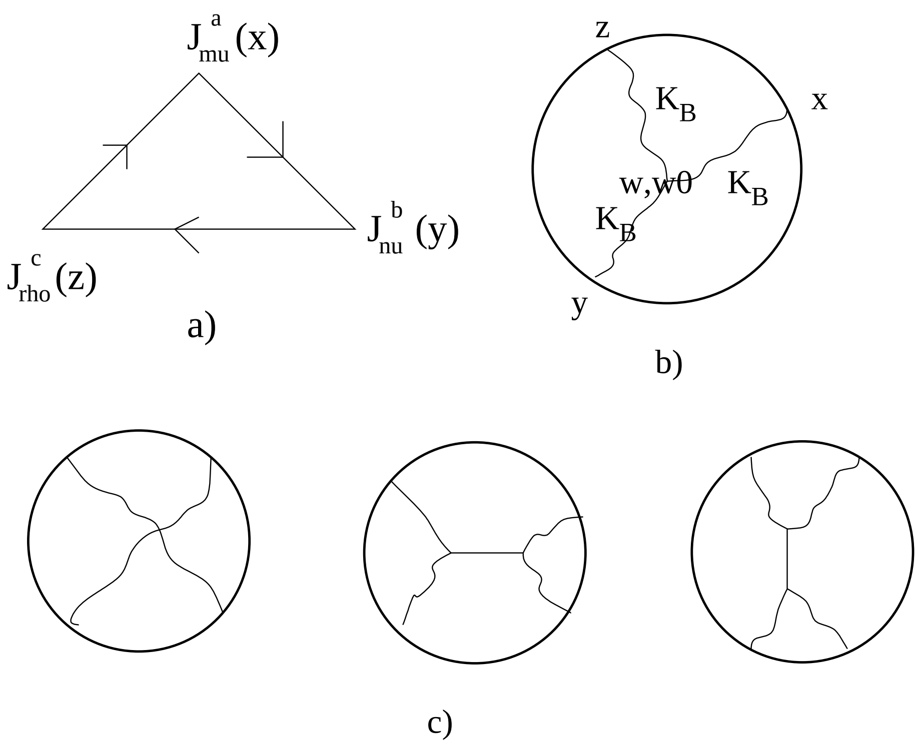

Figure 9: a)Triangle diagram contributing to the $< J_\mu^a(x) J_\nu^b(y) J_\rho^c(z) >$ correlator. Chiral fermions run in the loop. b) Tree level "Witten diagram" for the 3-point function in AdS space. c)Tree level Witten diagrams for the 4-point function in AdS space.

Since $A^a_\mu \propto a^a_\mu$, to get a nonzero result we look for the term with 3 A^a_μ's in S_{sugra}. Moreover, since we are interested in the anomaly, which is antisymmetric in μ, ν, ρ, we look for a term in the 5 dimensional gauged supergravity (since A^a_μ belongs to it) that is antisymmetric in μ, ν, ρ. This is the so-called Chern-Simons term. It can be written as

$$S_{CS}(A) = \frac{iN^2}{16\pi^2} Tr \int_{B_5=\partial M_6} \epsilon^{\mu\nu\rho\sigma\tau}(A_\mu(\partial_\nu A_\rho)\partial_\sigma A_\tau + A^4 \text{ terms} + A^5 \text{ terms})$$
$$= \frac{iN^2}{16\pi^2} Tr \int_{M_6} \epsilon^{\mu\nu\rho\sigma\tau} F_{\mu\nu} F_{\rho\sigma} F_{\tau\epsilon} \tag{9.24}$$

and we can see that it is symmetric under the interchange of the 3 F's. The term is 5-dimensional, but when written in 5 dimensions it looks complicated, an A^3 term that we are interested in for the calculation of the 3-point function and A^4, A^5 terms. But it looks simple when written in 6 dimensions as a boundary term, since

$$\epsilon^{\epsilon\mu\nu\rho\sigma\tau}\partial_\epsilon(A_\mu(\partial_\nu A_\rho)\partial_\sigma A_\tau + A^4 \text{ terms} + A^5 \text{ terms}) = \epsilon^{\epsilon\mu\nu\rho\sigma\tau} F_{\epsilon\mu} F_{\nu\rho} F_{\sigma\tau} \tag{9.25}$$

The Chern-Simons term is easily seen to be proportional (upon performing the trace) to $d_{abc} = Tr(T_a\{T_b, T_c\})$, thus it indeed gives a contribution to the quantum anomaly.

The supergravity action for A^a_μ is of the type

$$S[A] = \int (A_\mu)^2 \text{ term } + \int (A^a_\mu A^b_\nu A^c_\rho \text{ term}) + ... \tag{9.26}$$

and the quadratic term gives a propagator, whereas the cubic term gives a 3-point vertex, out of which we construct so-called "Witten diagrams" (a particular type of Feynman diagrams, really) as in Fig.9b,c. In Fig.9b) we have the unique tree diagram contributing to the sought after 3-point function, a 3-point vertex in the middle of AdS space, with 3 bulk-to-boundary propagators connecting it to 3 points on the boundary. In Fig.9c) we similarly have the only tree diagrams contributing to the 4-point function: a 4-point vertex united with the 4 boundary points, and two diagrams with two internal 3-vertices each, connected to each other and to the boundary. We can draw similar tree diagrams for any n-point function. Since we use the classical supergravity action (we are in the classical supergravity limit), we will only get tree diagrams. Loop diagrams would correspond to quantum corrections, therefore will only appear in the full string theory, and are suppressed in this limit.

Coming back to our case, the Chern-Simons term contains the only d_{abc} 3-point vertex, therefore gives the only anomalous contribution to the 3-point correlator:

$$< J^{\mu a}(x) J^{\nu b}(y) J^{\rho c}(z) >_{CFT,\, d_{abc}\, part} = -\frac{\delta^3 S^{3-pnt\ vertex}_{CS,sugra}[A^a_\mu[a^d_\sigma]]}{\delta a^a_\mu(x)\delta a^b_\nu(y)\delta a^c_\rho(z)} \tag{9.27}$$

We could continue by substituting $A^a_\mu[a^d_\sigma]$ and doing the intergrals and differentiations, but there is a simpler way in the case of the anomaly.

The gauge variation

$$\delta A^a_\mu = (D_\mu \Lambda)^a = \partial_\mu \lambda^a + g f^a{}_{bc} A^b_\mu \lambda^c \tag{9.28}$$

of the Chern-Simons term gives

$$\begin{aligned}
\delta_\Lambda S_{CS} &= \frac{iN^2}{16\pi^2} Tr \int_{B_5} d^5x \epsilon^{\mu\nu\rho\sigma\tau}(\delta A_\mu F_{\nu\rho} F_{\sigma\tau}) \\
&= \frac{iN^2}{16\pi^2} d_{abc} \int_{B_5} d^5x \epsilon^{\mu\nu\rho\sigma\tau}(D_\mu \Lambda)^a F^b_{\nu\rho} F^c_{\sigma\tau} \\
&= -\frac{iN^2}{16\pi^2} d_{abc} \int_{B_5} d^5x \epsilon^{\mu\nu\rho\sigma\tau}\partial_\tau(\Lambda^a \partial_\mu(A^b_\nu \partial_\rho A^c_\sigma + \frac{1}{4} f^c{}_{de} A^b_\nu A^d_\rho A^e_\sigma)) \\
&= -\frac{iN^2}{16\pi^2} d_{abc} \int_{boundary} d^4x \epsilon^{\mu\nu\rho\sigma} \Lambda^a \partial_\mu (A^b_\nu \partial_\rho A^c_\sigma + \frac{1}{4} f^c{}_{de} A^b_\nu A^d_\rho A^e_\sigma)
\end{aligned} \tag{9.29}$$

where in the third line we have used partial integration and $D_{[\mu}F_{\nu\rho]} = 0$ and in the last expression we can substitute A^a_μ's with their boundary values a^a_μ.

But the AdS-CFT prescription (9.16) implies that

$$\delta_\Lambda S_{class}[a^a_\mu] = \delta_\Lambda(-\ln Z[a^a_\mu]) = \int d^4x \delta a^{\mu a}(x) J^a_\mu(x) = \int d^4x (D^\mu \Lambda)^a J^a_\mu(x) = -\int d^4x \Lambda^a [D^\mu J_\mu]^a \tag{9.30}$$

Substituting $\delta_\Lambda S_{CS}$ we get (at leading order in N)

$$(D^\mu J_\mu)^a(x) \equiv \frac{\partial}{\partial x^\mu} J^a_\mu + f^a{}_{bc} a^{\mu b} J^c_\mu = \frac{iN^2}{16\pi^2} d_{abc} \epsilon^{\mu\nu\rho\sigma} \partial_\mu (a^b_\nu \partial_\rho a^c_\sigma + \frac{1}{4} f^c{}_{de} a^b_\nu a^d_\rho a^e_\sigma) \tag{9.31}$$

which is exactly the operator equation for the R-current anomaly in the CFT (coming from the 1-loop CFT computation). At a=0, the 1-loop result for the anomaly of the 3-point function is

$$\frac{\partial}{\partial z^\rho} < J^a_\mu(x) J^b_\nu(y) J^c_\rho(z) >_{CFT, d_{abc}} = -\frac{(N^2-1) i d_{abc}}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \frac{\partial}{\partial x_\rho} \frac{\partial}{\partial y_\sigma} \delta(x-y)\delta(y-z) \tag{9.32}$$

which indeed matches with the above at leading order in N (and a careful analysis matches also at subleading order).

We want now to calculate the full 3-point function, not only the anomalous part. Since the d_{abc} part is anomalous, the other group invariant that appears in the 3-point vertex is f_{abc}, which will thus give the non-anomalous part of the 3-point function. This calculation could in principle be done in x space and in p space. The p space calculation is more familiar in field theory, but in gravity is somewhat more involved, so we will describe the x-space calculation.

In x-space we can use conformal invariance to simplify the calculations. It dictates that the 3-point function of currents should have the general form

$$< J^a_\mu(x) J^b_\nu(y) J^c_\rho(z) >_{f_{abc}} = f_{abc}(k_1 D^{sym}_{\mu\nu\rho}(x,y,z) + k_2 C^{sym}_{\mu\nu\rho}(x,y,z)) \tag{9.33}$$

where k_1, k_2 are arbitrary coefficients and $C^{sym}_{\mu\nu\rho}$ and $D^{sym}_{\mu\nu\rho}$ stand for the symmetrized version of the objects

$$\begin{aligned}
D_{\mu\nu\rho}(x,y,z) &= \frac{1}{(x-y)^2(z-y)^2(x-z)^2} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \log(x-y)^2 \frac{\partial}{\partial z^\rho} \log\left(\frac{(x-z)^2}{(y-z)^2}\right) \\
C_{\mu\nu\rho}(x,y,z) &= \frac{1}{(x-z)^4} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial z^\sigma} \log(x-z)^2 \frac{\partial}{\partial y^\nu} \frac{\partial}{\partial z_\sigma} \log(y-z)^2 \frac{\partial}{\partial z^\rho} \log\left(\frac{(x-z)^2}{(y-z)^2}\right)
\end{aligned} \tag{9.34}$$

By conformal invariance we can fix one point, e.g. $z=0$, and another, e.g. $y\rightarrow\infty$. Then the form of the two structures becomes

$$
\begin{aligned}
&D_{\mu\nu\rho}(x,y,0)\stackrel{y\rightarrow\infty}{\rightarrow}\frac{4}{y^6x^4}I_{\nu\sigma}(y)\{\delta_{\mu\rho}x_\sigma-\delta_{\mu\sigma}x_\rho-\delta_{\sigma\rho}x_\mu-2\frac{x_\mu x_\rho x_\sigma}{x^2}\}\\
&C_{\mu\nu\rho}(x,y,0)\stackrel{y\rightarrow\infty}{\rightarrow}\frac{8}{y^6x^4}I_{\nu\sigma}(y)\{\delta_{\mu\rho}x_\sigma-\delta_{\mu\sigma}x_\rho-\delta_{\sigma\rho}x_\mu+4\frac{x_\mu x_\rho x_\sigma}{x^2}\}\\
&I_{\mu\nu}(x)\equiv\delta_{\mu\nu}-2\frac{x_\mu x_\nu}{x^2}
\end{aligned}
\tag{9.35}
$$

On the other hand, the AdS calculation comes from the 3-point vertex proportional to f_{abc}, which is

$$
\frac{1}{2g_{SG}^2}\int\frac{d^dwdw_0}{w_0^{d+1}}if_{abc}\partial_{[\mu}A^a_{\nu]}w_0^4A^b_\mu(w)A^c_\nu(w)
\tag{9.36}
$$

The bulk to boundary propagator now has a vector index and depends on the gauge for A in which we work

$$
A^a_\mu(z)=\int d^4\vec{x}G_{\mu\alpha}(z,\vec{x})a^a_\alpha(\vec{x})
\tag{9.37}
$$

where α and $\vec{x}$ denote boundary values. The gauge symmetry of A implies the gauge transformation of the bulk to boundary propagator

$$
G_{\mu\alpha}(z,\vec{x})\rightarrow G_{\mu\alpha}(z,\vec{x})+\frac{\partial}{\partial z_\mu}\Lambda_\alpha(z,\vec{x})
\tag{9.38}
$$

We can choose the propagator that is conformally invariant on the boundary, in order to be able to take advantage of the conformal invariance properties. In principle we can do the calculation with any other propagator, but it will be longer. The conformally invariant propagator is

$$
G_{\mu\alpha}(z,\vec{x})=C^d\left(\frac{z_0}{(z-x)^2}\right)^{d-2}\partial_\mu\left(\frac{(z-\vec{x})_\alpha}{(z-\vec{x})^2}\right)
\tag{9.39}
$$

where C^d is a constant. Then one finds

$$
\begin{aligned}
&<J^a_\alpha(x)J^b_\beta(y)J^c_\gamma(z)>_{f_{abc}}=-\frac{if_{abc}}{2g_{SG}^2}2F^{symm}_{\alpha\beta\gamma}(\vec{x},\vec{y},\vec{z})\\
&F_{\alpha\beta\gamma}(\vec{x},\vec{y},\vec{z})\equiv\int\frac{d^dwdw_0}{w_0^{d+1}}\partial_{[\mu}G_{\nu]\alpha}(w,\vec{x})w_0^4G_{\mu\beta}(w,\vec{y})G_{\nu\gamma}(w,\vec{z})
\end{aligned}
\tag{9.40}
$$

After some algebra, one finds

$$
F_{\alpha\beta\gamma}(\vec{x},\vec{y},\vec{z})=-\tilde{C}^d\frac{J_{\beta\delta}(\vec{y}-\vec{x})}{|\vec{y}-\vec{x}|^{2(d-1)}}\frac{J_{\gamma\epsilon}(\vec{z}-\vec{x})}{|\vec{z}-\vec{x}|^{2(d-1)}}\frac{1}{|\vec{t}|^d}(\delta_{\delta\epsilon}t_\alpha+(d-1)\delta_{\alpha\delta}t_\epsilon+(d-1)\delta_{\alpha\epsilon}t_\delta-d\frac{t_\alpha t_\epsilon t_\delta}{|\vec{t}|^2})
\tag{9.41}
$$

where

$$
\vec{t}\equiv(\vec{y}-\vec{x})'-(\vec{z}-\vec{x})' \text{ and } (\vec{w})'\equiv\frac{\vec{w}}{\vec{w}^2}
\tag{9.42}
$$

We can now put $\vec{z} = 0$ and $|\vec{y}| \to \infty$ in this result and compare with the CFT result (9.33) and (9.35) and we can then find that

$$F^{symm}_{\alpha\beta\gamma}(\vec{x},\vec{y},\vec{z}) = \frac{1}{\pi^4}(D^{sym}_{\alpha\beta\gamma}(\vec{x},\vec{y},\vec{z}) - \frac{C^{sym}_{\alpha\beta\gamma}(\vec{x},\vec{y},\vec{z})}{8}) \tag{9.43}$$

One can in fact check that this matches the *1-loop* result of CFT, even though we are at strong coupling ($\lambda \equiv g^2 N \gg 1$). That implies that there should exist some nonrenormalization theorem at work, similar to the one for the quantum anomaly. In fact, such a theorem was subsequently proved for 3-point functions, using superconformal symmetry. Thus in fact, in $\mathcal{N} = 4$ SYM the 3-point functions of currents are 1-loop exact and match with the AdS space calculation!

Important concepts to remember

- The Witten prescription states that the exponential of (minus) the supergravity action for fields ϕ with boundary values ϕ_0 is the partition function for operators $\mathcal{O}$ corresponding to ϕ, and with sources ϕ_0.
- The bulk to boundary propagator, together with the AdS supergravity (gauged supergravity) vertices, define "Witten diagrams" from which we calculate the boundary (2-, 3-, 4-,...-point) correlators.
- The 2-point functions match, but they are kinematic. Dynamics is encoded in 3-point functions and higher
- To compare both sides of the duality, we need correlators that do not get renormalized. The R-current anomaly is such an object
- The R-current anomaly in field theory is given by a one-loop triangle Feynman diagram contribution to the 3-point function of R-currents, and comes from the AdS (gauged) supergravity Chern-Simons term. It matches.
- Even the full 3-point function of R-currents matches with the AdS space calculation of gauge field 3-point function. It was later understood to come from non-renormalization theorems.

Exercises, section 9

1) Knowing that parts of the gauge terms $tr F_{\mu\nu}^2$ and S_{CS} used for the AdS-CFT calculation of the 3-point function of R-currents come from the 10d Einstein term $\sim \frac{1}{g_s^2}\int d^{10}x\sqrt{G^{(10)}}\mathcal{R}$ (here $\mathcal{R}$= 10d Ricci scalar), prove that the overall factor in $S_{sugra}[A_\mu(a_\rho)]$, and thus in the 3-point function of R-currents, is N^2 (no g_{YM} factors). Use that $R_{AdS_5} = R_{S^5} = \sqrt{\alpha'}(g_s N)^{1/4}$.

2) Consider the equation $(\Box - m^2)\phi = 0$ in the Poincare patch of AdS_{d+1}. Check that near the boundary $x_0 = 0$, the two independent solutions go like $x_0^{2h_\pm}$, with

$$2h_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2R^2} \tag{9.44}$$

(so that $2h_+ = \Delta$, the conformal dimension of the operator dual to ϕ).

3) Check that near $x_0 = 0$, the massless scalar field $\phi = \int K_B\phi_0$, with

$$K_B(\vec{x}, x_0; \vec{x}') = c\left(\frac{x_0}{x_0^2 + |\vec{x} - \vec{x}'|^2}\right)^d \tag{9.45}$$

goes to a constant, ϕ_0. Then check that for the massive scalar case, replacing in K_B the power d by $2h_+$, we have $\phi \to x_0^{2h_-}\phi_0$ near the boundary.

4) Check that the (1-loop) anomaly of R-currents is proportional to N^2 at leading order, by doing the trace over indices in the diagram.

5) Write down the classical equations of motion for the 5d Chern-Simons action for A_μ^a.

6) Consider a scalar field ϕ in AdS_5 supergravity, with action

$$S = \int \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \lambda\frac{\phi^3}{3} \tag{9.46}$$

Is the 4-point function of operators $\mathcal{O}$ sourced by ϕ, $< \mathcal{O}(x_1)...\mathcal{O}(x_4) >$, zero or nonzero, and why?

10 Quarks and the Wilson loop

External quarks in QCD

Quarks in QCD can be introduced as

- fundamental: light quarks, appearing in the action
- external probes: (infinitely) heavy quarks, external (not in the action).

QCD is confining, which means light quarks are not free in the vacuum, they appear in pairs with an antiquark. Thus even if we put external quarks (not in the theory), we don't expect to be able to put a single quark in the vacuum, we need at least two: a quark and an antiquark.

Since the external quarks are very heavy, they will stay fixed, i.e. the distance between q and $\bar{q}$ will stay fixed in time, as in Fig.10a. The question is then how do we measure the interaction potential between two such quarks, $V_{q\bar{q}}(L)$? We need to define physical observables that can measure it. One such physical, gauge invariant object is called the Wilson loop.

We first define the path ordered exponential

$$\Phi(y,x;P) = P\ \exp\{i\int_x^y A_\mu(\xi)d\xi^\mu\} \equiv \lim_{n\to\infty}\prod_n e^{iA_\mu(\xi_n^\mu-\xi_{n-1}^\mu)} \tag{10.1}$$

where $A_\mu \equiv A_\mu^a T_a$.

Consider first an $U(1)$ **gauge field** A_μ. Under a gauge transformation $\delta A_\mu = \partial_\mu\chi$

$$e^{iA_\mu d\xi^\mu} \to e^{iA_\mu d\xi^\mu + i\partial_\mu\chi d\xi^\mu} = e^{iA_\mu d\xi^\mu}e^{i\chi(x+dx)-i\chi(x)} \tag{10.2}$$

which implies

$$\begin{aligned}\Phi(y,x;P) &= \prod e^{iA_\mu d\xi^\mu} \to \prod(e^{iA_\mu d\xi^\mu}e^{i\chi(x+dx)-i\chi(x)})\\ &= e^{i\chi(y)}(\prod e^{iA_\mu d\xi^\mu})e^{-i\chi(x)} = e^{i\chi(y)}\Phi(y,x;P)e^{-i\chi(x)}\end{aligned} \tag{10.3}$$

If we have a complex field ϕ charged under this U(1), i.e. transforming as

$$\phi(x) \to e^{i\chi(x)}\phi(x) \tag{10.4}$$

then the multiplication by $\Phi(y,x;P)$ gives

$$\Phi(y,x;P)\phi(x) \to e^{i\chi(y)}\Phi(y,x;P)e^{-i\chi(x)}e^{i\chi(x)}\phi(x) = e^{i\chi(y)}(\Phi(y,x;P)\phi(x)) \tag{10.5}$$

thus it defines parallel transport, i.e. the field $\phi(x)$ was parallel transported to the point y.

On the other hand, for a closed curve, i.e. for y=x, we have

$$\Phi(x,x;P) \to e^{i\chi(x)}\Phi(x,x;P)e^{-i\chi(x)} = \Phi(x,x;P) \tag{10.6}$$

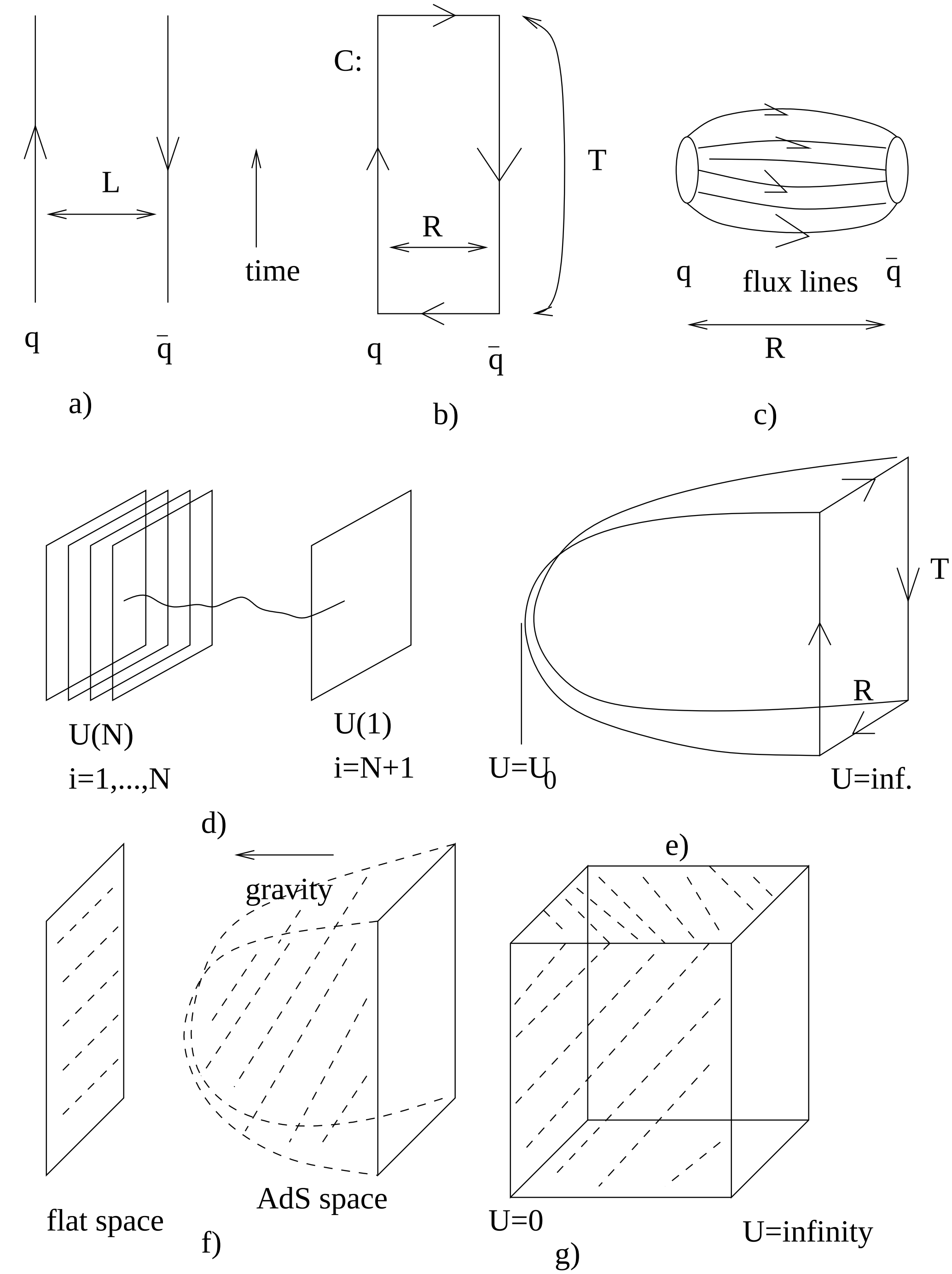

Figure 10: a)Heavy quark and antiquark staying at fixed distance L. b)Wilson loop contour C for the calculation of the quark-antiquark potential. c) Between a quark and an antiquark in QCD, flux lines are confined: they live in a flux tube. d)One D-brane separated from the rest (N) D-branes acts as a probe on which the Wilson loop is located. e) The Wilson loop contour C is located at $U = \infty$ and the string worldsheet ends on it and stretches down to $U = U_0$. f) In flat space, the string worldsheet would form a flat surface ending on C, but in AdS space 5 dimensional gravity pulls the string inside AdS. g) The free "W bosons" are strings that would stretch in all of the AdS space, from $U = \infty$ to $U = 0$, straight down, forming an area proportional to the perimeter of the contour C.

i.e. it is a gauge invariant object.

For a **nonabelian gauge field**, the gauge transformation is

$$A_\mu \to \Omega(x) A_\mu \Omega^{-1}(x) - i(\partial_\mu \Omega)\Omega^{-1} \tag{10.7}$$

An infinitesimal transformation $\Omega(x) = e^{i\chi(x)}$ for small $\chi(x) = \chi^a T_a$ gives

$$\delta A_\mu = D_\mu \chi = \partial_\mu \chi - i[A_\mu, \chi] \tag{10.8}$$

which implies

$$\begin{aligned} e^{iA_\mu d\xi^\mu} &\simeq (1 + iA_\mu d\xi^\mu) \to 1 + \Omega(A_\mu d\xi^\mu)\Omega^{-1} + d\xi^\mu(\partial_\mu \Omega)\Omega^{-1} \\ &= [e^{i\chi(x)}(1 + iA_\mu d\xi^\mu) + d\xi^\mu \partial_\mu e^{i\chi(x)}]e^{-i\chi(x)} \\ &\simeq e^{i\chi(x+dx)}(1 + iA_\mu d\xi^\mu)e^{-i\chi(x)} \simeq e^{i\chi(x+dx)}e^{iA_\mu d\xi^\mu}e^{-i\chi(x)} \end{aligned} \tag{10.9}$$

where we have neglected terms of order $o(dx^2)$.

By taking products, we get again

$$\Phi(y, x; P) \to e^{i\chi(y)}\Phi(y, x; P)e^{-i\chi(x)} \tag{10.10}$$

but unlike for the U(1) gauge field, the order of the terms matters now. So again, $\Phi(y, x; P)$ defines parallel transport, for the same reason.

However, now for a closed path (y=x), Φ is not gauge invariant anymore, but rather gauge covariant:

$$\Phi(y, x; P) \to e^{i\chi(x)}\Phi(x, x; P)e^{-i\chi(x)} \neq \Phi(x, x; P) \tag{10.11}$$

But now the trace of this object is gauge invariant (since it is cyclic). Thus we define the Wilson loop

$$W(C) = tr\Phi(x, x; C) \tag{10.12}$$

which is gauge invariant and independent of the particular point x on the closed curve C, since

$$tr[e^{i\chi(x)}\Phi e^{-i\chi(x)}] = Tr[\Phi] \tag{10.13}$$

In the abelian case, for x=y we can use the Stokes theorem to put Φ in an explicitly gauge invariant form

$$\Phi_C = e^{i\int_{C=\partial A} A_\mu d\xi^\mu} = e^{i\int_A F_{\mu\nu} d\sigma^{\mu\nu}} \tag{10.14}$$

In the nonabelian case, we can do something similar, but we have corrections. If we take a small square of side a in the plane defined by directions μ and ν, we get

$$\Phi_{\Box_{\mu\nu}} = e^{ia^2 F_{\mu\nu}} + o(a^4) \tag{10.15}$$

Since $F_{\mu\nu}$ transforms covariantly:

$$F_{\mu\nu} \to \Omega(x) F_{\mu\nu} \Omega^{-1}(x) \tag{10.16}$$

then the Wilson loop, defined for convenience with a $1/N$ since there are N terms in the trace for a $SU(N)$ gauge field, becomes

$$W_{\Box_{\mu\nu}} = \frac{1}{N} tr\{\Phi_{\Box_{\mu\nu}}\} = 1 - \frac{a^4}{2N} Tr\{F_{\mu\nu}F_{\mu\nu}\} + O(a^6) \tag{10.17}$$

where we don't have a sum over the indices μ, ν. Here $Tr\{F_{\mu\nu}F_{\mu\nu}\}$ is a gauge invariant operator (even if it is not summed over μ, ν, thus to first nontrivial order this is explicitly gauge invariant, and moreover we obtain the kinetic term in the action.

The object of interest is therefore

$$W[C] = tr P \exp[\int iA_\mu d\xi^\mu] \tag{10.18}$$

and for the calculation of the static quark-antiquark potential we are interested in a loop as in Fig.10b, a rectangle with length T in the time direction and R in the spatial direction, with $T \gg R$.

The statement of confinement is that there is a constant force that resists when pulling the quark and the antiquark away, therefore that

$$V_{q\bar{q}}(R) \sim \sigma R \tag{10.19}$$

i.e. a linear potential, with σ called the (QCD) string tension. The "QCD string" is a confined flux tube for the QCD color electric flux, as in Fig.10c. It is not a fundamental object, but an effective description due to the confinement which forces the flux lines to stay (be confined) in a tube.

On the other hand, for QED with infinitely massive (external) quarks, we have the Coulomb static potential

$$V_{q\bar{q}}(R) \sim \frac{\alpha}{R} \tag{10.20}$$

and this model is in fact conformal, since it is scale invariant. This is the kind of potential we therefore expect in a conformally invariant theory.

One can prove that the VEV of the Wilson loop in Fig.9b behaves as

$$< W(C) >_0 \propto e^{-V_{q\bar{q}}(R)T} \tag{10.21}$$

if $T \to \infty$.

Therefore in a confining theory like QCD we get

$$< W(C) >_0 \propto e^{-\sigma T \cdot R} = e^{-\sigma A} \tag{10.22}$$

where $A = area$, thus this behaviour is known as the area law. In fact, since

$$W(C_1 \cup C_2) = W(C_1)W(C_2) \tag{10.23}$$

we can extend the area law to any smooth curve C, not just to the infinitely thin rectangle analyzed here, since we can approximate any area by such infinitely thin rectangles, as in Fig.11.

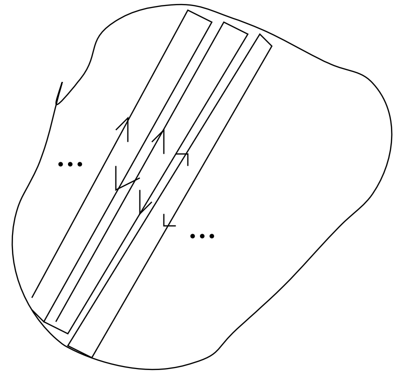

Figure 11: Approximation of a curve C by infinitely thin rectangles.

Therefore, confinement means that for any smooth curve C,

$$< W(C) >_0 \propto e^{-\sigma A(C)} \tag{10.24}$$

On the other hand, in conformally invariant cases like QED with external quarks we find the scale invariant result for the infintely thin curve

$$< W(C) >_0 \propto e^{-\alpha \frac{T}{R}} \tag{10.25}$$

and for more complicated curves we don't have an answer, but we just know that the answer must be scale invariant (independent on the overall size of the curve).

Finally, although here we have only shown how to extract the quark antiquark potential from Wilson loop VEVs, they are actually very important objects. We can in principle extract all the dynamics of the theory if we know the (complete operator) Wilson loop.

Defining the (VEV of the) $\mathcal{N}=4$ SYM Wilson loop via AdS-CFT

AdS-CFT obtains a $U(N)$ gauge group from a large number ($N \to \infty$) of D-branes situated at the same point. Strings with two ends on different branes are massless, since there is no physical separation between the D-branes, and correspond to gauge fields, $A^a_\mu = (\lambda^a)_{ij}|i> \otimes |j> \otimes |\mu>$.

If we consider $N+1$ D-branes, giving a $U(N+1)$ gauge group, and take one of the D-branes and separate it from the rest, as in Fig.10d), it means that we are breaking the gauge group, via a Higgs-like mechanism, to $U(N) \times U(1)$ (where $U(N)$ corresponds to the N D-branes that are still at the same point).

The strings that have one end on one of the N D-branes and one end on the extra D-brane will be massive, with *mass = string tension × D-brane separation*. These strings have a state

$$|i_0> \otimes |i> = |N+1> \otimes |i> \tag{10.26}$$

which is therefore in the fundamental representation of the remaining $U(N)$ (i is a fundamental index). Its mass is

$$M = \frac{1}{2\pi\alpha'} r = \frac{U}{2\pi} \tag{10.27}$$

This string behaves as a "W boson," since as the Standard Model particle, it is a vector field (gauge field) made massive by a Higgs mechanism, that in our case breaks $U(N+1) \rightarrow U(N) \times U(1)$. The string state (or rather, its $|i>$ endpoint) acts in the $U(N)$ gauge theory as a source for the $U(N)$ gauge fields, or as a quark, and as a quark, is in the fundamental representation of $U(N)$.

From (10.27), to get an infinite mass we need to take $U \rightarrow \infty$. Therefore the introduction of infinitely massive external quark is obtained by having a string stretched in AdS space, in the metric (8.10), between infinity in U and a finite point.

Since infinity in (8.10) is also where the $\mathcal{N}=4$ SYM gauge theory lives, we put the Wilson loop contour C at infinity, as a boundary condition for the string. So the string worldsheet stretches between the contour C at infinity down to a finite point in AdS, forming a smooth surface, as in Fig.10e.

But there is a subtlety. Strings must also extend on the S_5, parametrized by coordinates θ^I (since the dual of $\mathcal{N}=4$ SYM is $AdS_5 \times S_5$, not just AdS_5). And θ^I correspond to the scalars X^I of $\mathcal{N}=4$ SYM, which transform in the $SO(6)$ symmetry group (R symmetry of $\mathcal{N}=4SYM$ and invariance symmetry of S_5). Because of that, one finds that supersymmetry dictates that the strng worldsheet described above is not a source for the usual Wilson loop, but for the supersymmetric generalized Wilson loop

$$W[C] = \frac{1}{N} Tr\ P \exp[\oint (iA_\mu \dot{x}^\mu + \theta^I X^I(x^\mu)\sqrt{\dot{x}^2})d\tau] + \text{fermions} \tag{10.28}$$

where $x^\mu(\tau)$ parametrizes the loop and θ^I is a unit vector that gives the position on S_5 where the string is sitting. The fermions in that expression give quantum fluctuations that are suppressed in the supergravity limit, so we will not be bothered by them. We will also consider the case of θ^I =constant.

Then the prescription for calculating $< W[c] >$ is as a partition function for the string with boundary on C. In the supergravity limit ($g_s \rightarrow 0, g_s N$ fixed and large) we obtain

$$< W[C] >= Z_{string}[C] = e^{-S_{string}[C]} \tag{10.29}$$

where S_{string}=string worldsheet action= $1/(2\pi\alpha')\times$ area of worldsheet (area in $AdS_5 \times S_5$, not area of 4 dimensional projection!). That however doesn't necessarily give the (4 dimensional) area law for C, since the worldsheet has an area bigger than the 4 dimensional area enclosed by C.

The string has tension, and it wants to have a minimum area. In flat space, that would mean that it would span just the flat surface enclosed by C, giving the area law (see Fig.10f). However, in AdS space, we have a gravitational field

$$ds^2 = \alpha' \frac{U^2}{R^2}(-dt^2 + d\vec{x}^2) + ... \tag{10.30}$$

To understand the physics, we compare with the Newtonian approximation (though it is not a good approximation now, but we do get the correct qualitative picture)

$$ds^2 = (1+2V)(-dt^2 + ...) \tag{10.31}$$

where V is the Newton potential. Newtonian gravity means that the string would go to the minimum V. In our case, that would mean the minimum U. Therefore the string worldsheet with boundary at $U=\infty$ "drops" down to $U=U_0$ as in Fig.10f) and is stopped (held back) by its tension.

But the prescription is not done, since the area of the worldsheet stretching from $U=\infty$ to $U=U_0$ is divergent, so we would get $<W[C]>=0$. In fact, we must remember that we said the string stretched between the $|i>$ and $|N+1>$ D-branes, and therefore between $U=\infty$ and $U=U_0$ also represents an infinitely massive "W boson," whose mass ϕ we must now subtract. The "free W boson" would stretch along all of AdS_5, thus from $U=\infty$ to $U=0$, in a straight line, parallel with C, as in Fig.10g. Thus the action that we must subtract is ϕl, where $l=$ length of loop C and $\phi=$ free W boson (free string) mass, $U/(2\pi)$. Then

$$<W[C]>=e^{-(S_\phi-l\phi)} \tag{10.32}$$

Calculation of the quark-antiquark potential

We take the contour C to be the infinitely thin rectangle, with $T\rightarrow\infty$, and a quark q at $x=-L/2$ and an antiquark $\bar{q}$ at $x=+L/2$. The metric is

$$ds^2=\alpha'[\frac{U^2}{R^2}(dt^2+d\vec{x}^2)+R^2\frac{dU^2}{U^2}+R^2 d\Omega_5^2];\quad R^2=\sqrt{4\pi g_s N} \tag{10.33}$$

and the Nambu-Goto action for the string is

$$S_{string}=\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{\det G_{MN}\partial_\alpha X^M\partial_\beta X^N} \tag{10.34}$$

We choose a gauge where the worldsheet coordinates equal 2 spacetime coordinates, specifically $\tau=t$ and $\sigma=x$. This choice is known as a static gauge, and it is consistent to take it since we are looking for a static solution. Then we approximate the worldhseet to be translationally invariant in the time direction, which is only a good approximation if $T/L\rightarrow\infty$ (otherwise the curvature of the worldsheet near the corners becomes important). Since we also are looking at a static configuration, we have a single variable for the worldsheet, $U=U(\sigma)$ which becomes $U=U(x)$.

We calculate $h_{\alpha\beta}=G_{MN}\partial_\alpha X^M\partial_\beta X^N$ and obtain

$$h_{11}=\alpha'\frac{U^2}{R^2}(\frac{dt}{d\tau})^2=\alpha'\frac{U^2}{R^2};\quad h_{22}=\alpha'\frac{U^2}{R^2}(\frac{dx}{d\sigma})^2+\alpha'\frac{R^2}{U^2}(\frac{dU}{d\sigma})^2=\alpha'(\frac{U^2}{R^2}+\frac{R^2}{U^2}U'^2);\quad h_{12}=0 \tag{10.35}$$

thus

$$S_{string}=\frac{1}{2\pi}T\int dx\sqrt{(\partial_x U)^2+\frac{U^4}{R^4}} \tag{10.36}$$

and we have reduced the problem to a 1 dimensional mechanics problem.

We define U_0 as the minimum of $U(x)$ and $y=U/U_0$. Then we can check that the solution is defined by

$$x=\frac{R^2}{U_0}\int_1^{U/U_0}\frac{dy}{y^2\sqrt{y^4-1}} \tag{10.37}$$

which gives $x(U, U_0)$ and inverted gives $U(x, U_0)$. To find U_0 we note that at $U = \infty$ we have $x = L/2$, therefore

$$\frac{L}{2} = \frac{R^2}{U_0}\int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} = \frac{R^2}{U_0}\frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2} \tag{10.38}$$

Then from the Wilson loop prescription

$$S_\phi - l\phi = TV_{q\bar{q}}(L) \tag{10.39}$$

and we regularize this formula by integrating only up to U_{max}. Then $l \simeq 2T$ and the mass of the string,

$$\phi = \frac{U_{max} - U_0}{2\pi} + \frac{U_0}{2\pi} = \frac{U_0}{2\pi}\int_1^{y_{max}} dy + \frac{U_0}{2\pi} \tag{10.40}$$

therefore (there is a factor of 2 since we integrate from U_{max} to U_0 and then from U_0 to U_{max})

$$TV_{q\bar{q}}(L) = T\frac{2U_0}{2\pi}[\int_1^\infty dy(\frac{y^2}{\sqrt{y^4-1}} - 1) - 1] \tag{10.41}$$

Finally, by substituting U_0 and R^2, we get

$$V_{q\bar{q}}(L) = -\frac{4\pi^2}{\Gamma(1/4)^4}\frac{\sqrt{2g_{YM}^2 N}}{L} \tag{10.42}$$

So we do get $V_{q\bar{q}}(L) \propto 1/L$ as expected for a conformally invariant theory (therefore no area law). However, we also get that $V_{q\bar{q}}(L) \propto \sqrt{g_{YM}^2 N}$ which is a nonpolynomial, therefore nonperturbative result. That means that this cannot be obtained by a finite loop order calculation. For example, the 1-loop result would be proportional to $g_{YM}^2 N$.

Important concepts to remember

- Introducing external quarks in the theory, we can measure the quark-antiquark potential between heavy sources.
- The Wilson loop, $W[C] = tr\ P \exp \int iA_\mu dx^\mu$ is gauge invariant.
- By choosing the contour C as a rectangle with 2 sides in the time direction, of length T, and two sides in a space direction, of length $R \ll T$, we have a contour C from which we can extract $V_{q\bar{q}}(R)$ by $< W(C) >_0 = \exp(-V_{q\bar{q}}(R))$.
- In a confining theory like QCD, $V_{q\bar{q}}(R) \sim \sigma R$, thus we have the area law: $< W(C) >_0 \propto \exp(-\sigma A(C))$ for any smooth curve C, and reversely, if we find the area law the theory is confining.
- In a conformally invariant theory like QED with external quarks, $V_{q\bar{q}}(R) = \alpha/R$ and the Wilson loop is conformally (scale) invariant. For the above C, $< W(C) >_0 \propto \exp(-\alpha T/R)$.

- In AdS-CFT, the Wilson loop one finds has also coupling to scalars (and fermions), and is defined by $< W(C) >_0 = \exp(-S_{string}(C))$, where the string worldsheet ends at $U = \infty$ on the curve C and drops inside AdS space. One needs to subtract the mass of the free strings extending straight down over the whole space.
- The result or the calculation is nonperturbative (proportional to $\sqrt{\lambda}$), but has the expected conformal (Coulomb) behaviour.

Exercises, section 10

1) Check that in the nonabelian case, for a closed square contour of side a, in a plane defined by $\mu\nu$, we have

$$\Phi_{\Box_{\mu\nu}} = e^{ia^2 F_{\mu\nu}} + o(a^4) \qquad (10.43)$$

2) Check that if a free relativistic string in 4 flat dimensions is stretched between q and $\bar{q}$ and we use the AdS-CFT prescription for the Wilson loop, $W[C] = e^{-S_{string}[C]}$, we get the area law.

3) Consider a circular Wilson loop C, of radius R. Give an argument to show that W[C] in N=4 SYM, obtained from AdS-CFT as in the rectangular case, is also conformally invariant, i.e. independent of R.

4) Check that if AdS_5 terminates at a fixed $U = U_1$ and strings are allowed to reach U_1 and get stuck there, then we get the area law for $< W[C] >$. (This is similar to what happens in the case of finite temperature AdS-CFT).

5) Finish the steps left out in the calculation of the quark antiquark potential to get the final result for $V_{q\bar{q}}(L)$.

11 Finite temperature and scattering processes

We now turn to phenomena that seem to have relevance for QCD, at least in terms of qualitative behaviour. We will examine two important examples, finite temperature field theory and scattering processes.

Finite temperature in field theory; periodic time

In quantum mechanics, we write down a transition amplitude between points q, t and q', t' as

$$\begin{aligned} < q', t'|q, t > &= < q'|e^{-i\hat{H}(t'-t)}|q >= \sum_{nm} < q'|n >< n|e^{-i\hat{H}(t'-t)}|m >< m|q > \\ &= \sum_n \psi_n(q')\psi_n^*(q)e^{-iE_n(t'-t)} \end{aligned} \tag{11.1}$$

On the other hand it can also be written as a path integral

$$< q', t'|q, t >= \int \mathcal{D}q(t)e^{iS[q(t)]} \tag{11.2}$$

If we perform a Wick rotation to Euclidean space by $t \to -it_E$, $t' - t \to -i\beta$, $iS \to -S_E$ and look at closed paths $q' = q(t_E + \beta) = q = q(t_E)$, we obtain

$$< q, t'|q, t >= Tr(e^{-\beta\hat{H}}) = \int_{q(t_E+\beta)=q(t_E)} \mathcal{D}q(t_E)e^{-S_E[q(t_E)]} \tag{11.3}$$

That means that the Euclidean path integral on a closed path equals the statistical mechanics partition function at temperature $T = 1/\beta$ (Boltzmann constant $k = 1$).

Similarly in field theory we obtain for the euclidean partition function

$$Z_E[\beta] = \int_{\phi(\vec{x},t_E+\beta)=\phi(\vec{x},t_E)} \mathcal{D}\phi e^{-S_E[\phi]} = Tr(e^{-\beta\hat{H}}) \tag{11.4}$$

Therefore the partition function at finite temperature T is expressed again as a euclidean path integral over periodic euclidean time paths. One can then extend this formula by adding sources and calculating propagators and correlators, exactly as for zero temperature field theory.

Thus the finite temperature field theory, for static quantities only (time-independent!), is obtained by considering periodic imaginary time, with period $\beta = 1/T$.

Black hole temperature

We can use this approach to deduce that black holes radiate thermally at a given temperature T, a process known as Hawking radiation.

We want to describe quantum field theory in the black hole background. As always, it is best described by performing a Wick rotation to Euclidean time. The Wick-rotated Schwarzschild black hole is

$$ds^2 = +(1 - \frac{2M}{r})d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2 \tag{11.5}$$

Having now Euclidean signature, it doesn't make sense to go inside the horizon, at $r < 2M$, since then the signature will not be euclidean anymore (unlike for Lorentz signature, when the only thing that happens is that the time t and radial space r change roles), but will be $(- - ++)$.

Therefore, if the Wick rotated Schwarzschild solution represents a Schwarzschild black hole, the horizon must not be singular, yet there must not be a continuation inside it, i.e. it must be smoothed out somehow. This is possible since in Euclidean signature one can have a conical singularity if

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 \tag{11.6}$$

but $\theta \in [0, 2\pi - \Delta]$. If $\Delta \neq 0$, then $\rho = 0$ is a singular point, and the metric describes a cone, as in Fig.12b, therefore $\rho = 0$ is known as a conical singularity. However, if $\Delta = 0$ we don't have a cone, thus no singularity, and we have a (smooth) euclidean space.

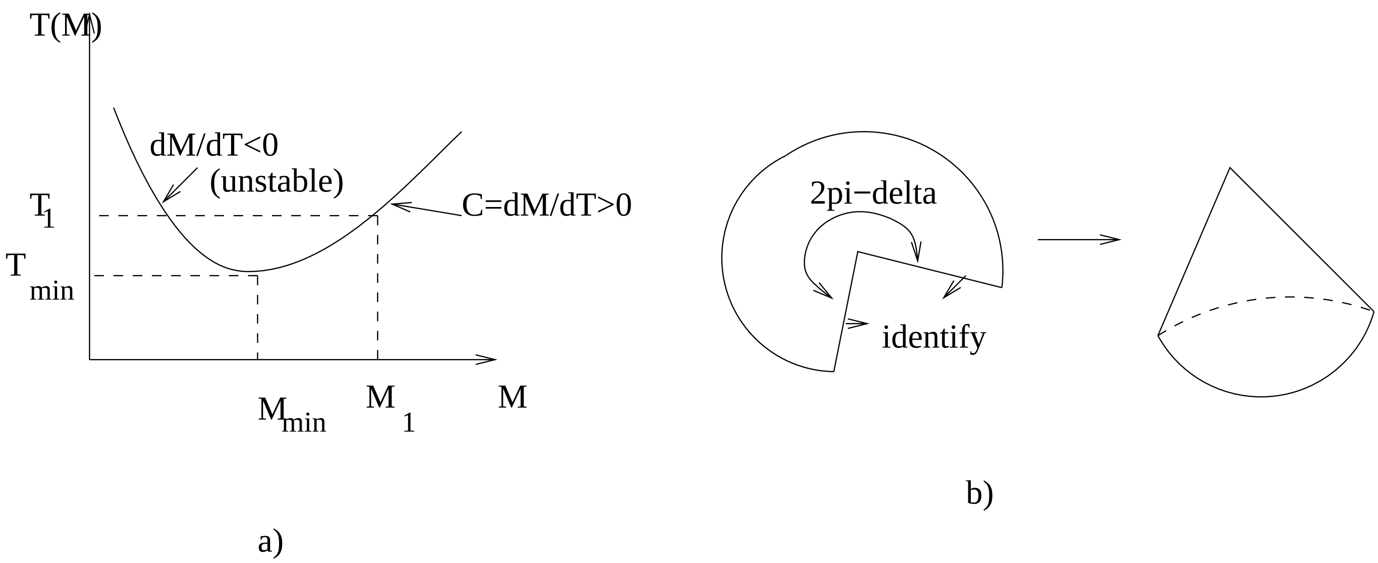

Figure 12: a) $T(M)$ for the AdS black hole. The lower M branch is unstable, having $\partial M/\partial T < 0$. The higher M branch has $\partial M/\partial T > 0$, and above T_1 it is stable. b) A flat cone is obtained by cutting out an angle from flat space, so that $\theta \in [0, 2\pi - \Delta]$ and identifying the cut.

A similar situation applies to the Wick rotated Schwarzschild black hole. Near $r = 2M$, we have

$$ds^2 \simeq \frac{\tilde{r}}{2M} d\tau^2 + 2M \frac{d\tilde{r}^2}{\tilde{r}} + (2M)^2 d\Omega_2^2 \tag{11.7}$$

where $r - 2M \equiv \tilde{r}$. By defining $\rho \equiv \sqrt{\tilde{r}}$ we get

$$ds^2 \simeq 8M(d\rho^2 + \frac{\rho^2 d\tau^2}{(4M)^2}) + (2M)^2 d\Omega_2^2 \tag{11.8}$$

so near the horizon the metric looks like a cone. If τ has no restrictions, the metric doesn't make much sense. It must be periodic for it to make sense, but for a general period we get a cone, with $\rho = 0 (r = 2M)$ a singularity. Only if $\tau/(4M)$ has period 2π we avoid the conical singularity and we have a smooth space, that cannot be continued inside $r = 2M$.

Therefore we have periodic euclidean time, with period $\beta_\tau = 8\pi M$. By the previous analysis, this corresponds to finite temperature quantum field theory at temperature

$$T_{BH} = \frac{1}{\beta_\tau} = \frac{1}{8\pi M} \tag{11.9}$$

We can therefore say that quantum field theory in the presence of a black hole has a temperature T_{BH} or that black holes radiate thermally at temperature T_{BH}.

Does that mean that we can put a quantum field theory at finite temperature by adding a black hole? Not quite, since the specific heat of the black hole is

$$C = \frac{\partial M}{\partial T} = -\frac{1}{8\pi T^2} < 0 \tag{11.10}$$

therefore the black hole is thermodynamically unstable, and it does not represent an equilibrium situation.

But we will see that in Anti de Sitter space we have a different situation. Adding a black hole does provide a thermodynamically stable system, which therefore does represent an equilibrium situation.

Before anlyzing that however, we will try to understand better the Schwarzschild solution in Euclidean space.

At $r \to \infty$, the solution is $R^3 \times S^1$ (since τ is periodic, but the metric is flat), which is a Kaluza-Klein vacuum. That is, it is a background solution around which we can expand the fields in Fourier modes (in general, we have spherical harmonics, but for compactification on a circle we have actual Fourier modes) and perform a dimensional reduction by keeping only the lowest modes.

In particular, fermions can in principle acquire a phase $e^{i\alpha}$ when going around the S^1_τ circle at infinity:

$$\psi \to e^{i\alpha}\psi \tag{11.11}$$

These are known as "spin structures", and $\alpha = 0$ and $\alpha = \pi$ are always OK, since the Lagrangian has always terms with an even number of fermions, thus such a phase would still leave it invariant. If $\mathcal{L}$ has additional symmetries, there could be other values of α allowed.

At $r \to 2M$ (the horizon), the solution is $R^2 \times S^2$, with R^2 being $d\rho^2 + \rho^2 d\theta^2$ $(\theta = \tau/(4M))$ and S^2 from $d\Omega_2^2$. But $R^2 \times S^2$ is simply connected, which means that there are no nontrivial cycles, or that any loop on $R^2 \times S^2$ can be smoothly shrunk to zero. That means that there cannot be nontrivial fermion phases as you go in around any loop on $R^2 \times S^2$, or that there is a unique spin structure.

We must therefore find to what does this unique spin structure correspond at infinity? The relevant loop at infinity is $\tau \to \tau + \beta_\tau$, which near the horizon is $\theta \to \theta + 2\pi$, i.e a rotation in the 2d plane R^2. Under such a rotation a fermion picks up a minus sign.

Indeed, a fermion can be defined as an object that gives a minus sign under a complete spatial rotation, i.e. an object that is periodic under 4π rotations instead of 2π. In 4d, the way to see that is that the spatial rotation $\psi \to S\psi$ around the axis defined by $\vec{\nu}$ is given by

$$S(\vec{\nu},0) = \cos\frac{\theta}{2} I + i\vec{\nu}\cdot\vec{\Sigma}\sin\frac{\theta}{2}; \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{11.12}$$

where $\vec{\sigma}$ are Pauli matrices. We can see that under a 2π rotation, $S = -1$.

Therefore the unique spin strucure in the Euclidean Schwarzschild black hole background is one that makes the fermions antiperiodic at infinity, around the Euclidean time direction. That can only happen if they have some Euclidean time dependence, $\psi = \psi(\theta)$. That in turns means that the fermions at infinity get a nontrivial mass under dimensional reductions, since the 4 dimensional free flat space equation (valid at $r \to \infty$) gives ($\Box = \partial\!\!\!/^2$)

$$0 = \Box_{4d} = (\Box_{3d} + \frac{\partial^2}{\partial\theta^2})\psi = (\Box_{3d} + m^2)\psi \tag{11.13}$$

where $m^2 \neq 0$ is a 3 dimensional spinor mass squared.

Bosons on the other hand have no such restrictions on them, and we can have bosons that are periodic at infinity under $\theta \to \theta + 2\pi$, thus also the simplest case of bosons that are independent of θ. Then at infinity

$$0 = \Box_{4d}\phi = (\Box_{3d} + \frac{\partial^2}{\partial\theta^2})\phi = \Box_{3d}\phi \tag{11.14}$$

and therefore they can be massless in 3 dimensions.

But if one would have supersymmetry in flat 3 dimensional euclidean space, we would need that $m_{scalar} = m_{fermion}$. That is not the case in the presence of the black hole, since we can have $m_{fermion} \neq 0$, but $m_{boson} = 0$, therefore the presence of the black hole breaks supersymmetry.

In fact, one can prove that finite temperature always breaks supersymmetry, in any field theory.

Therefore, one of the ways to break the unwelcome $\mathcal{N} = 4$ supersymmetry in AdS-CFT and get to more realistic field theories is by having finite temperature, specifically by putting a black hole in AdS space. We will discuss this prescrition in the following. Of course, in the way shown above we obtain a non-supersymmetric 3 dimensional field theory, but there are ways (that we will not explain) to obtain a 4 dimensional nonsupersymmetric theory in a similar manner.

Witten prescription

Witten gave a prescription about how to put AdS-CFT at finite temperature by introducing a black hole in AdS_5.

As we have seen, the metric of global Anti de Sitter space can be written as

$$ds^2 = -(\frac{r^2}{R^2} + 1)dt^2 + \frac{dr^2}{\frac{r^2}{R^2} + 1} + r^2 d\Omega^2 \tag{11.15}$$

where in relation to the previous form in (5.22) (which was for AdS_4) we have renamed $1/R^2 \equiv -\Lambda/3$ (the cosmological constant Λ is < 0.

Then the black hole in (n+1)-dimensional Anti de Sitter space is

$$ds^2 = -(\frac{r^2}{R^2} + 1 - \frac{w_n M}{r^{n-2}})dt^2 + \frac{dr^2}{\frac{r^2}{R^2} + 1 - \frac{w_n M}{r^{n-2}}} + r^2 d\Omega_{n-2}^2 \quad (11.16)$$

which solves Einstein's equation with a cosmological constant

$$R_{\mu\nu} = -\frac{n}{R^2} g_{\mu\nu} \equiv \Lambda g_{\mu\nu} \quad (11.17)$$

Here

$$w_n = \frac{16\pi G_N}{(n-1)\Omega_{n-1}} \quad (11.18)$$

and Ω_{n-1} is the volume of the unit sphere in $n-1$ dimensions. For n=3 (AdS_4), $\Omega_2 = 4\pi$ and $w_3 = 2G_N$.

Repeating the above analysis for the horizon of the Wick rotated AdS black hole, we find that the temperature of the black hole is

$$T = \frac{nr_+^2 + (n-1)R^2}{4\pi R^2 r_+} \quad (11.19)$$

where r_+ is the largest solution of

$$\frac{r^2}{R^2} + 1 - \frac{w_n M}{r^{n-2}} = 0 \quad (11.20)$$

which is called the outer horizon. Then T(M) looks like in Fig.12a, having a minimum of

$$T_{min} = \frac{\sqrt{n(n-2)}}{2\pi R} \quad (11.21)$$

The low M brach has $C = \partial M \partial T < 0$ therefore is thermodynamically unstable, like the Schwarzschild black hole in flat space (it is in fact a small perturbation of that solution, since the black hole is small compared to the radius of curvature of AdS space).

The high M branch however has $C = \partial M / \partial T > 0$, thus is thermodynamically stable. We also need to check the free energy of the black hole solution, F_{BH}, is smaller than the free energy of pure AdS space, F_{AdS}.

The free energy is defined as

$$Z = \sum e^{-\beta F} \quad (11.22)$$

where $\beta = 1/T$ (if $k = 1$). But in a gravitational theory,

$$Z_{grav} = e^{-S} \quad (11.23)$$

where S = euclidean action. We have seen this for example when defining correlators in AdS-CFT. Then we have that

$$S(euclidean\ action) = \frac{F}{T} \quad (11.24)$$

and therefore we need to compare

$$F_{BH} - F_{AdS} = T(S_{BH} - S_{AdS}) \tag{11.25}$$

and an explicit calculation shows that it is < 0 if

$$T > T_1 = \frac{n-1}{2\pi R} > T_{min} \tag{11.26}$$

There is one more problem. At $r \to \infty$, the metric is

$$ds^2 \simeq (\frac{r}{R}dt)^2 + (\frac{R}{r}dr)^2 + r^2 d\Omega_{n-1}^2 \tag{11.27}$$

therefore the Euclidean time direction is a circle of radius $(r/R) \times (1/T)$, and the transverse $n-1$ dimensional sphere has radius r. Thus both are proportional to $r \to \infty$, however the $\mathcal{N} = 4$ SYM gauge theory that lives at $r \to \infty$ has conformal invariance, therefore only relative scales are relevant for it, so we can drop the overall r. Then the topology at infinity, where $\mathcal{N} = 4$ SYM lives, is $S^{n-1} \times S^1$, but we want to have a theory defined on $R^{n-1} \times S^1$ instead, namely n-dimensional flat space at finite temperature (with periodic Euclidean time).

That means that we need to scale the ratio of sizes to infinity

$$\frac{r}{\frac{r}{R}\frac{1}{R}} = R \cdot T \to \infty \tag{11.28}$$

Therefore we must take $T \to \infty$, only possible if $M \to \infty$, and we must rescale the coordinates to get finite quantities. The rescaling is

$$r = \left(\frac{w_n M}{R^{n-2}}\right)^{1/n} \rho; \quad t = \left(\frac{w_n M}{R^{n-2}}\right)^{-1/n} \tau \tag{11.29}$$

and $M \to \infty$. Under this rescaling, the metric becomes

$$ds^2 = (\frac{\rho^2}{R^2} - \frac{R^{n-2}}{\rho^{n-2}})d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{R^2} - \frac{R^{n-2}}{\rho^{n-2}}} + \rho^2 \sum_{i=1}^{n-1} dx_i^2 \tag{11.30}$$

and the period of τ is

$$\beta_1 = \frac{4\pi R}{n} \tag{11.31}$$

Since for $\rho \to \infty$ we get

$$ds^2_{\rho\to\infty} \simeq \rho^2 (\frac{d\tau^2}{R^2} + d\vec{x}^2) \tag{11.32}$$

considering string theory in the metric (11.30)puts $\mathcal{N} = 4$ SYM at constant finite temperature

$$T = \frac{R}{\beta_1} = \frac{n}{4\pi} \tag{11.33}$$

As we saw before, in this AdS black hole metric, supersymmetry is broken. At ($r =$) infinity, the fermions are antiperiodic around the Euclidean time direction, thus if we dimensionally reduce the $\mathcal{N} = 4$ SYM theory to 3 dimensions (compactify on the Euclidean time) the fermions become massive. The gauge fields are protected by gauge invariance and remain massless under this dimensional reduction. The scalars as we saw remain massless in 3 dimension, at the classical level. At the quantum level, they also get a mass at 1 loop.

Therefore the 3 dimensional theory obtained by dimensionally reducing $\mathcal{N} = 4$ SYM on the compact Euclidean time is pure QCD (only gauge fields A_μ^a and nothing else)! This is the perturbative specrtrum of the theory, and it is defined as pure 3 dimensional QCD. But like the real world 4 dimensional QCD, in 3 dimensions the vacuum structure is very interesting, with nonperturbative states that aquire a mass despite being composed of massless QCD fields. This phenomenon is known as a mass gap (spontaneous appearence of a minimum mass of physical states in a system of massless fields).

Application: mass gap

We would like to understand the mass gap from AdS-CFT. The spontaneous appearence of a mass in for physical states of $\mathcal{N} =$ SYM dimensionally reduced to 3 dimensions translates into having a classical mass for physical states living in the gravitational dual (11.30). We therefore study the fields living in the bulk of the Witten metric (11.30) and we would like to find a nonzero 3 dimensional mass (for n=4). We look for solutions of the free massless field equation of motion, $\Box\phi = 0$ on this space, such that

$$\phi(\rho, \vec{x}, \tau) = f(\rho) e^{i\vec{k}\cdot\vec{x}} \tag{11.34}$$

is independent of τ (dimensionally reduced) and factorizes in ρ and $\vec{x}$ dependence. At the horizon $\rho = b$ we need to impose that the solution is smooth, i.e. $df/d\rho = 0$. On the other hand, at $\rho \to \infty$ we need to impose that the solution is normalizable, which gives

$$f \sim \frac{1}{\rho^4} \tag{11.35}$$

There is also a non-normalizable solution that goes to a constant at infinity. Then one plugs the ansatz and boundary conditions in the Klein-Gordon equation and finds a discrete positive spectrum of values for $\vec{k}^2 \equiv m^2$, which is the value of the 3 dimensional mass squared. That means that the finite temperature AdS space (11.30) behaves like a quantum mechanical box with the zero mode removed. This is exactly the statement of the mass gap.

QCD scattering and the Polchinski-Strassler scenario

We saw that for generic fields ϕ living in AdS space, they take some value ϕ_0 on the boundary, and then ϕ_0 acts as a classical source for ϕ through

$$\phi = \int K_B \phi_0 \tag{11.36}$$

and as sources for composite operators in the CFT that lives at the boundary of AdS space, out of which we can construct correlators. But in QCD we are interested in S matrices that describe scattering of physical asymptotic states, and the LSZ formalism relates the S matrices to correlators. But that assumes the existence of separated asymptotic states.

In a conformal field theory however, there is no notion of scale, therefore there is no notion of infinity, and no asymptotic states, so we cannot construct S matrices from correlators.

Therefore in order to construct S matrices so that we can study scattering of states as in QCD, we need to break the conformal invariance.

It has been understood how to modify $AdS_5 \times S_5$ in order to get something closer to QCD on the field theory side. There are many examples of possible modifications. To obtain something that behaves like real QCD, the "gravity dual" background looks like $AdS_5 \times X_5$ at large ρ (large fifth dimension), which describe the UV (ultraviolet) behaviour of QCD (QCD is conformal in the UV, where any small physical masses are irrelevant). Here X_5 is some compact space. This $AdS_5 \times X_5$ is then modified in some way at small ρ, corresponding to the IR (infrared) behaviour of QCD.

The simplest possible model that captures some of the properties of QCD is then to just cut off $AdS_5 \times X_5$ at a certain value of r, $r_{min} = R^2\Lambda$, where Λ is the QCD scale (the scale of the lowest fundamental excitations).

Fields in $AdS_5 \times X_5$ correspond to 4d composite operators, which correspond to gauge invariant, composite particles. Examples are nucleons and meson or glueballs. An example of glueball operator is $tr\ F_{\mu\nu}F^{\mu\nu}$.

The wavefunction for a glueball state, for instance $e^{ik\cdot x}$, corresponds via AdS-CFT to a wavefunction Φ for the corresponding $AdS_5 \times S_5$ field which equals the glueball wavefunction times a wavefunction in the extra coordinates, e.g.

$$\Phi = e^{ik\cdot x} \times \Psi(\rho, \vec{\Omega}_5) \tag{11.37}$$

In the example of the mass gap, this wavefunction was (11.34).

Then, Polchinski and Strassler made an ansatz for the scattering of gauge invariant states in QCD. The amplitude $\mathcal{A}(p_i)$ in QCD and in the "gravity dual" is related by convolution as

$$\mathcal{A}_{QCD}(p_i) = \int dr d^5\Omega \sqrt{-g} \mathcal{A}_{string}(\tilde{p}_i) \prod_i \Psi_i(r, \vec{\Omega}) \tag{11.38}$$

Since

$$ds^2 = \frac{r^2}{R^2} d\vec{x}^2 + ... \tag{11.39}$$

the momentum $p_\mu = -i\partial/\partial x^\mu$ is rescaled between QCD (p_μ) and string theory ($\tilde{p}_\mu$) by

$$\tilde{p}_\mu = \frac{R}{r} p_\mu \tag{11.40}$$

Important concepts to remember

- Finite temperature field theory is obtained by having a periodic euclidean time, with period $\beta = 1/T$. The partition function for such periodic paths gives the thermal partition function, from which we can extract correlators by adding sources, etc.

- The Wick rotated Schwarzschild black hole has a smooth (non-singular) "horizon" only if the euclidean time is periodic with period $\beta = 1/T_{BH} = 8\pi M$. Thus black holes Hawking radiate.

- Quantum field theory in the presence of a black hole does not have finite temperature though, since the Schwarzschild black hole is thermodynamically unstable ($C = \partial M/\partial T < 0$).

- Fermions in the Wick rotated black hole are antiperiodic around the Euclidean time at infinity, thus they are massive if we dimensionally reduce the theory on the periodic time. Since bosons are massless, the black hole (and finite temperature) breaks supersymmetry.

- By putting a black hole in AdS space, the thermodynamics is stable if we are at high enough black hole mass M.

- The Witten prescription for finite temperature AdS-CFT is to put a black hole of mass $M \to \infty$ inside AdS_5 and to take a certain scaling of coordinates, giving the metric (11.30).

- By dimensionally reducing d=4 $\mathcal{N} = 4$ Super Yang-Mills on the periodic euclidean time, we get pure Yang-Mills in 3 dimensions, which has a mass gap (spontaneous appearence of a lowest nonzero mass state in a massless theory).

- The mass gap is obtained in AdS space from solutions of the wave equation in AdS that have a 3 dimensional mass spectrum like the one of a quantum mechanical box with the ground state removed. Thus the Witten metric is similar in terms of eigenmodes to a finite box.

- Since $\mathcal{N} = 4$ Super Yang-Mills is conformal, it does not have asymptotic states, so no S matrices. To define scattering, one must modify the duality and introduce a fundamental scale (break scale invariance). The simplest model is to cut-off AdS space at an $r_{min} = R^2\Lambda_{QCD}$.

- Gauge invariant scattered states (nucleons, mesons, glueballs) correspond to fields in $AdS_5 \times S_5$.

- Other models ("gravity duals") look like $AdS_5 \times X_5$ in the UV and cut-off in the IR and give theories that better mimic QCD.

- The Polchinski-Strassler scenario for the scattering amplitude of QCD (or the QCD-like model) is a convolution of the amplitude for scattering in the gravity dual.

Exercises, section 11

1) Parallel the calculation of the Schwarzschild black hole to show that the extremal (Q=M) black hole has zero temperature.

2) Derive $T(r_+)$ and $T_{min}(M)$ for the AdS black hole.

3) Check that the rescaling plus the limit given in () gives the Witten background for finite temperature AdS-CFT.

4) Take a near-horizon nonextremal D3-brane metric,

$$ds^2 = \alpha'\{\frac{U^2}{R^2}[-f(U)ds^2 + d\vec{y}^2] + R^2\frac{dU^2}{U^2 f(U)} + R^2 d\Omega_5^2\}$$
$$f(U) = 1 - \frac{U_T^4}{U^4} \quad (11.41)$$

where U_T is fixed, $U_T = TR^2$ (T=temperature). Note that for $f(U) = 1$ we get the near-horizon extremal D3 brane, i.e. $AdS_5 \times S^4$. Check that a light ray travelling between the boundary at $U = \infty$ and the horizon at $U = U_T$ takes a finite time (at $U_T = 0$, it takes an infinite time to reach U=0).

5) Check that the rescaling

$$U = \rho \cdot (TR); \quad t = \frac{\tau}{TR}; \quad \vec{y} = \frac{\vec{x}}{T} \quad (11.42)$$

where $R = AdS$ radius and T=temperature, takes the above near horizon nonextremal D3 brane metric to the Witten finite T AdS-CFT metric.

6) Near the boundary at $r = \infty$, the normalizable solutions (wavefunctions) of the massive AdS laplaceian go like $(x_0^\Delta \sim) r^{-\Delta}$ (where $\Delta = 2h_+ = d/2 + \sqrt{d^2/4 + m^2R^2}$). Substitute in the Polchinski-Strassler formula to obtain the r dependence of the intehral at large r, and using that $r \sim 1/p$, estimate the hard scattering (all momenta of the same order, p) behaviour of QCD amplitudes.

12 The PP wave correspondence and spin chains

The Penrose limit in gravity and pp waves

PP waves are plane fronted gravitational waves that is, solutions of the Einstein equation that correspond to perturbations moving at the speed of light, having a plane wave front.

In a flat background, the pp wave metric is

$$ds^2 = 2dx^+dx^- + (dx^+)^2H(x^+, x^i) + \sum_i dx_i^2 \tag{12.1}$$

For this metric, the only nonzero component of the Ricci tensor is

$$R_{++} = -\frac{1}{2}\partial_i^2 H(X^+, x^i) \tag{12.2}$$

PP waves can be defined in pure Einstein gravity, supergravity, or any theory that includes gravity.

In particular, in the maximal 11 dimensional supergravity, we find a solution that has the above metric, together with

$$F_4 = dx^+ \wedge \phi \tag{12.3}$$

where ϕ is a 3-form that satisfies (Here $\wedge$ denotes antisymmetrization, $\phi \equiv \phi_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho$, $|\phi|^2 \equiv \phi_{\mu\nu\rho}\phi^{\mu\nu\rho}$ and $(*\phi)_{\mu_1\dots\mu_8} \equiv \epsilon_{\mu_1\dots\mu_{11}}\phi^{\mu_9\mu_{10}\mu_{11}}$)

$$d\phi = d*\phi = 0; \quad \partial_i^2 H = \frac{1}{12}|\phi|^2 \tag{12.4}$$

For $\phi = 0$ we have a solution with

$$H = \frac{1}{|x - x_0|^2} \tag{12.5}$$

that corresponds to a D0-brane that is localized in space and time.

On the other hand, if

$$H = \sum_{ij} A_{ij}x^i x^j; \quad 2trA = \frac{1}{12}|\phi|^2 \tag{12.6}$$

we have a solution that is not localized in space and time (the spacetime is not flat at infinity). For $\phi = 0$ we have purely gravitational solutions that obey $trA = 0$. A solution for generic (A, ϕ) preserves 1/2 of the supersymmetry, namely the supersymmetry that satisfies $\Gamma_- \epsilon = 0$ (where ϵ is a generic supersymmetry parameter). There is however a very particular case, that has been found by Kowalski-Glickmann in 1984, that preserves ALL the supersymmetry. It is

$$\begin{aligned} A_{ij}x^i x^j &= -\sum_{i=1,2,3} \frac{\mu^2}{9} x_i^2 - \sum_{i=4}^{9} \frac{\mu^2}{36} x_i^2 \\ \phi &= \mu dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{12.7}$$

The only background solutions that preserve all the supersymmetry of 11 dimensional supergravity are Minkowski space, $AdS_7 \times S_4$, $AdS_4 \times S_7$ and the maximally supersymmetric wave (12.7).

Observation There is one other particular type of pp wave that is relevant, the shockeave of Aichelburg and Sexl. The solution has a delta function source, corresponding to a black hole boosted to the speed of light (while keeping its momentum fixed), by

$$\delta^n(x^i, x^1) \to \delta^{n-1}(x^i)\delta(x^+) \tag{12.8}$$

which implies that the harmonic function H of the pp wave splits as follows

$$H(x^i, x^+) = \delta(x^+)h(x^i) \tag{12.9}$$

Horowitz and Steif (1990) proved that in a pp wave background there are no α' corrections to the equations of motion (all possible R^2 corrections vanish on-shell, i.e. by the use of the Einstein equation), therefore pp waves give exact string solutions!

In particular, 10 dimensional type IIB string theory, the one that has $AdS_5 \times S_5$ as a background solution, contains solutions of the pp wave type, with metric (12.1), together with

$$F_5 = dx^+ \wedge (\omega + *\omega); \quad H = \sum_{ij} A_{ij} x^i x^j \tag{12.10}$$

satisfying

$$d\omega = d * \omega = 0; \ ; \ \partial_i^2 H = -32|\omega|^2 \tag{12.11}$$

As in 11 dimensions, here the general metric preserves 1/2 of the supersymmetry defined by $\Gamma_- \epsilon = 0$. There is also a maximally supersymmetric solution, that has

$$H = \mu^2 \sum_i x_i^2; \quad \omega = \frac{\mu}{2} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \tag{12.12}$$

Penrose limit

There is a theorem due to Penrose, which states that near a null geodesic (the path of a light ray) in any metric, the space becomes a pp wave.

Formally, it says that in the neighbourhood of a null geodesic, we can always put the metric in the form

$$ds^2 = dV(dU + \alpha dV + \sum_i \beta_i dY^i) + \sum_{ij} C_{ij} dY^i dY^j \tag{12.13}$$

and then we can take the limit

$$U = u; \quad V = \frac{v}{R^2}; \quad Y^i = \frac{y^i}{R}; \quad R \to \infty \tag{12.14}$$

and obtain a pp wave metric in u, v, y^i coordinates.

The interpretation of this procedure is: we boost along a direction, e.g. x, while taking the overall scale of the metric to infinity. The boost

$$t' = \cosh\beta \ \ t + \sinh\beta \ \ x; \quad x' = \sinh\beta \ \ t + \cosh\beta \ \ x \tag{12.15}$$

implies

$$x' - t' = e^{-\beta}(x - t); \quad x' + t' = e^{\beta}(x + t) \tag{12.16}$$

so if we scale all coordinates (t, x and the rest, y^i) by $1/R$ and identify $e^{\beta} = R \to \infty$ we obtain (12.14).

We can show that the maximally supersymmetric pp waves are Penrose limits of maximally supersymmetric $AdS_n \times S_m$ spaces. In particular, the maximally supersymmetric IIB solution (12.12) is a Penrose limit of $AdS_5 \times S_5$. This can be seen as follows. We boost along an equator of S_5 and stay in the center of AdS_5, therefore expanding around this null geodesic means expanding around $\theta = 0$ (equator of S_5) and $\rho = 0$ (center of AdS_5), as in Fig.13a, giving

$$\begin{aligned} ds^2 &= R^2(-\cosh^2\rho d\tau_d^2\rho^2 + \sinh^2\rho d\Omega_3^2) + R^2(\cos^2\theta d\psi^2 + d\theta^2 + \sin^2\theta d{\Omega'_3}^2) \\ &\simeq R^2(-(1+\frac{\rho^2}{2})d\tau^2 + d\rho^2 + \rho^2 d\Omega_3^2) + R^2((1-\frac{\theta^2}{2})d\psi^2 + d\theta^2 + \theta^2 d{\Omega'_3}^2) \end{aligned} \tag{12.17}$$

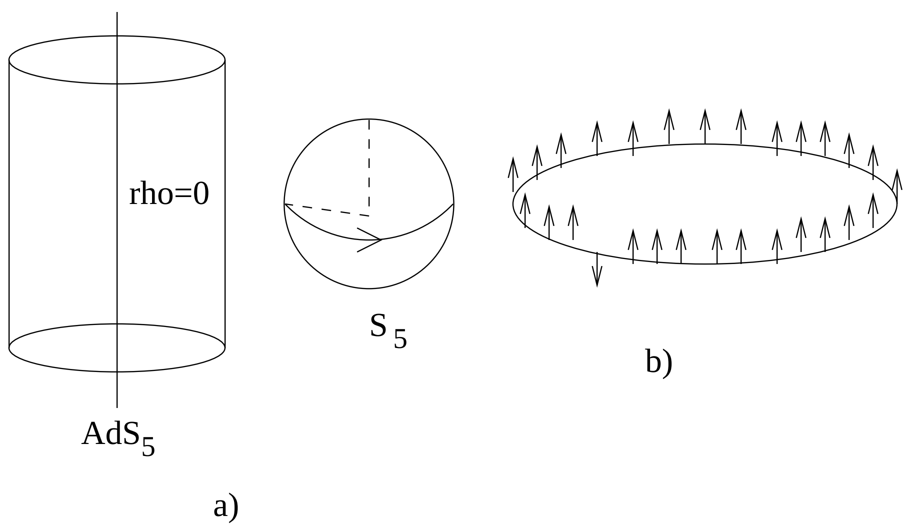

Figure 13: a) Null geodesic in $AdS_5 \times S_5$ for the Penrose limit giving the maximally supersymmetric wave. It is in the center of AdS_5, at $\rho = 0$, and on an equator of S_5, at $\theta = 0$. b)A periodic spin chain of the type that appears in the pp wave string theory. All spins are up, except one excitation has one spin down.

We then define the null coordinates $\tilde{x}^{\pm} = (\tau \pm \psi)/\sqrt{2}$, since ψ parametrizes the equator at $\theta = 0$. And we make the rescaling (12.14), i.e.

$$\tilde{x}^+ = x^+; \quad \tilde{x}^- = \frac{x^-}{R^2}; \quad \rho = \frac{r}{R}; \quad \theta = \frac{y}{R} \tag{12.18}$$

and we get

$$ds^2 = -2dx^+dx^- - (\vec{r}^2 + \vec{y}^2)(dx^+)^2 + d\vec{y}^2 + d\vec{r}^2 \tag{12.19}$$

which is the maximally supersymmetric wave (12.12). The F_5 field also matches.

Penrose limit of AdS-CFT; large R charge

Since the maximally supersymmetric wave is the Penrose limit of $AdS_5 \times S_5$, which defines AdS-CFT, we would like to understand what it means to take the Penrose limit of AdS-CFT.

The energy in AdS space is given by $E = i\partial_\tau$ (the energy is the Noether generator of time translations) and the angular momentum (Noether generator of rotations) for rotations in the plane of two coordinates X^5, X^6 is $J = -i\partial_\psi$, where ψ is the angle between X^5 and X^6.

But by the AdS-CFT dictionary the energy E corresponds to the conformal dimension Δ in $\mathcal{N} = 4$ SYM, whereas the angular momentum J corresponds to an R-charge, specifically a U(1) subgroup of $SU(4) = SO(6)$ that rotates the scalar fields X^5 and X^6.

After taking the Penrose limit, we will have momenta $p^\pm$ in the pp wave background defined as

$$\begin{aligned} p^- &= -p_+ = i\partial_{x^+} = i\partial_{\tilde{x}^+} = i(\partial_\tau + \partial_\psi) = \Delta - J \\ p^+ &= -p_- = i\partial_{x^-} = i\frac{\partial_{\tilde{x}^-}}{R^2} = \frac{i}{R^2}(\partial_t - \partial_\psi) = \frac{\Delta + J}{R^2} \end{aligned} \quad (12.20)$$

We would like to describe string theory on the pp wave, which is the Penrose limit of $AdS_5 \times S_5$. That means that we need to keep the pp wave momenta p^+, p^- (momenta of physical states on the pp wave) finite as we take the Penrose limit. That means that we must take to infinity the radius of AdS space, $R \to \infty$, but keep $\Delta - J$ and $(\Delta + J)/R^2$ of $\mathcal{N} = 4$ SYM operators fixed in the limit. Therefore we must consider only SYM operators that have $\Delta \simeq J \sim R^2 \to \infty$, thus only operators with large R charge!

From the supersymmetry algebra we can obtain that $\Delta \geq |J|$ (in a similar manner to the condition $M \geq |Q|$, the BPS condition), which means that $p^\pm > 0$. Since $R^2/\alpha' = \sqrt{g_s N} = \sqrt{g_{YM}^2 N}$, if we keep g_s fixed, $J \sim R^2$ means that $J/\sqrt{N}$ is fixed, or we look at operators with R-charge $J \sim \sqrt{N}$.

String spectrum from Super Yang-Mills

One can calculate the lightcone Hamiltonian (for time= x^+) for a string moving in the pp wave and obtain

$$H_{l.c.} \equiv p^- = -p_+ = \sum_{n \in Z} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} \quad (12.21)$$

where N_n is the total occupation number,

$$N_n = \sum_i a_n^{i\,\dagger} a_n^i + \sum_\alpha b_n^{\alpha\dagger} b_n^\alpha \quad (12.22)$$

Here $n > 0$ are left-movers and $n < 0$ are right-movers. Note that the formula includes the n=0 mode! We also have the condition that the total momentum along the closed string should be zero, by translational invariance (the same as for the flat space string), giving

$$P = \sum_{n \in Z} n N_n = 0 \quad (12.23)$$

A physical state is then $|\{n_i\}, p^+ >$.

We note that the flat space limit $\mu \to 0$ gives

$$p^+ p^- = \frac{1}{\alpha'} \sum_n n N_n \tag{12.24}$$

which is indeed the flat space spectrum in lightcone parametrization ($M^2 = p^+ p^- + \vec{p}^2$).

If we translate the Hamiltonian in SYM variables, using that $E/\mu = (\Delta - J)$, $p^+ = (\Delta + J)/R^2 \simeq 2J/R^2 = 2J/(\alpha' \sqrt{g_{YM}^2 N}$, we get

$$(\Delta - J)_n = w_n = \sqrt{1 + \frac{g_{YM}^2 N}{J^2}} \tag{12.25}$$

where we are in the limit that

$$\frac{g_{YM}^2 N}{J^2} = \text{fixed} \tag{12.26}$$

Thus we will try to find operators with fixed $\Delta - J$ in the above limit.

The SYM fields

In SYM J rotates X^5 and X^6, therefore the field $Z = \Phi^5 + i\Phi^6$ is charged with charge $+1$ under those rotations, and $\bar{Z}$ is charged with charge -1, whereas the rest of the 6 SYM scalars, $\phi^i, i = 1, ..., 4$ are neutral. The gaugino χ splits under this symmetry into 8 components $\chi^a_{J=+1/2}$ and 8 components $\chi^a_{J=-1/2}$. The 4 gauge field components A_μ complete the SYM multiplet.

These SYM fields are arranged under their $g_{YM} = 0$ value for $\Delta - J$ as follows. At $\Delta - J = 0$ we have a single field, Z ($\Delta = 1$ and $J = 1$). At $\Delta - J = 1$ we have Φ^i ($\Delta = 1$ and $J = 0$), $\chi^a_{J=+1/2}$ ($\Delta = 3/2$ and $J = 1/2$) and A_μ ($\Delta = 1$ and $J = 0$). The rest have $\Delta - J > 1$ ($\bar{Z}$ and $\chi^a_{J=1/2}$ have $\Delta - J = 2$).

The vacuum state of the string then must be represented by an operator of momentum p^+, therefore with charge J, and with zero energy, thus with $\Delta - J = 0$. From the above analysis, the unique such operator is

$$|0, p^+ >= \frac{1}{\sqrt{J} N^{J/2}} Tr[Z^J] \tag{12.27}$$

The string oscillators (creation operators) on the pp wave at the $n = 0$ level are 8 bosons and 8 fermions of $p^- = 1$, therefore should correspond to fields of $\Delta - J = 1$ to be inserted inside the operator corresponding to the vacuum, (12.27). They must be gauge covariant, in order to obtain a gauge invariant operator. It is easy to see then that the unique possibility is the 8 $\chi^a_{J=+1/2}$ for the fermions and the 4 ϕ^i's, together with 4 covariant derivatives $D_\mu Z = \partial_\mu Z + [A_\mu, Z]$ for the bosons. We have replaced the 4 A_μ's with the covariant derivatives $D_\mu Z$ in order to obtain a covariant object.

These fields are to be inserted inside the trace of the vacuum operator (12.27), for example a state with 2 n=0 excitations will be ($a^\dagger_{0,r}$ corresponds to Φ^r and $b^\dagger_{0,b}$ to $\psi^b_{J=1/2}$)

$$a^\dagger_{0,r} b^\dagger_{0,b} |0, p^+ >= \frac{1}{N^{J/2+1}\sqrt{J}} \sum_{l=1}^{J} Tr[\Phi^r Z^l \psi^b_{J=1/2} Z^{J-l}] \tag{12.28}$$

where we have put the ϕ^r field on the first position in the trace by cyclicity of the trace.

The string oscillators at levels $n > 1$ are obtained in a similar manner. But now they correspond to excitations that have a momentum $e^{inx/L}$ around the closed string of length L. Since the closed string is modelled by the vacuum state $Tr[Z^J]$, the appropriate operator corresponding to an $a^\dagger_{n,4}$ insertion is

$$a^\dagger_{n,4}|0,p^+>= \frac{1}{\sqrt{J}}\sum_{l=1}^{J}\frac{1}{\sqrt{J}N^{J/2+1/2}}Tr[Z^l\phi^4 Z^{J-l}]e^{\frac{2\pi inl}{J}} \tag{12.29}$$

Actually, this operator vanishes by cyclicity of the trace, and the corresponding string state doesn't satisfy the equivalent zero momentum constraint (cyclicity). In order to obtain a nonzero state, we must introduce at least two such insertions.

Discretized string action

We can now also derive a Hamiltonian that acts on the above string states, from the SYM interactions. One starts by noting that in SYM, the operator

$$\mathcal{O} = Tr[Z^l\phi Z^{J-l}] \tag{12.30}$$

can be mapped to a state

$$|b_l>\equiv Tr[(a^\dagger)^l b^\dagger (a^\dagger)^{J-l}]|0> \tag{12.31}$$

The reason this mapping can be done is a bit complicated, but one can do it. One then performs a rather involved derivation, which can be understood as follows. One introduces b_l^+ operators for inserting b^+ among a^+'s as above. Then the action of the interacting piece of the Lagrangian, $\mathcal{L}_{int}$ on the 2-point function of operators $\mathcal{O}$, $<\mathcal{O}\mathcal{O}>$ via Feynman diagrams, becomes the action of a Hamiltonian on a state $|b_l>$. One can then find the Hamiltonian

$$H\sim\sum_l b_l^+ b_l + \frac{g_{YM}^2 N}{(2\pi)^2}[(b_l+b_l^+)-(b_{l+1}+b_{l+1}^+)]^2 \tag{12.32}$$

Then the first term is a usual harmonic oscillator, giving a discrete version of $\int dx[\dot{\phi}(x)^2_\phi(x)^2]$. Since a discretized relativistic field is written as $\phi(x)\sim b_l+b_l^+$, the second term is a discretized version of ϕ'^2, giving the continuum version of the Hamiltonian

$$H = \int_0^L d\sigma\frac{1}{2}[\dot{\phi}^2+\phi'^2+\phi^2] \tag{12.33}$$

where

$$L = J\sqrt{\frac{\pi}{g_s N}} \tag{12.34}$$

is the length of the string. This Hamiltonian then is exactly the Hamiltonian of the string on a pp wave (before quantization).

This description of the insertion of ϕ's and their corresponding b_l^+ operators among a loop of Z's and their corresponding a^+ operators reminds one of spin chains. A spin chain is a one dimensional system of length L of spins with only up $|\uparrow>$ or down $|\downarrow>$ degrees of freedom, as in Fig.13b. This equivalence can in fact be made exact.

The Heisenberg XXX spin chain Hamiltonian, H_{XXX}

A spin chain is a model for magnetic interactions in one dimension, where the only relevant degrees of freedom are the electron spins. Heisenberg (1928) wrote a simple model for the rotationally invariant interaction of a system of spin 1/2, called the Heisenberg XXX model, with Hamiltonian

$$H = J\sum_{j=1}^{L} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} \tag{12.35}$$

Here $\vec{\sigma}_j$ are Pauli matrices (spin 1/2 operators) at site j, with periodic boundary conditions, i.e. $\vec{\sigma}_{L+1} \equiv \vec{\sigma}_1$, and J is a coupling constant.

- If $J < 0$ the system is ferromagnetic, and the interaction of spins is minimized if the spins are parallel, therefore the vacuum is $|\uparrow\uparrow ... \uparrow>$.
- If $J > 0$ the system is antiferromagnetic and the interaction is minimized for antiparallel spins, therefore the vacuum is $|\uparrow\downarrow\uparrow\downarrow ... \uparrow\downarrow>$.

The XXX stands for rotationally invariant. The XYZ model is not rotationally invariant and has

$$H = \sum_j (J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z) \tag{12.36}$$

and $J_x \neq J_y \neq J_z$.

The solution of the H_{XXX} model was done by Bethe in 1931, by what is now known as the Bethe ansatz.

Denote by $|x_1, ..., x_N>$ the state with spins up at sites x_i along the chain of spins down, e.g. $|1, 3, 4>_{L=5} = |\uparrow\downarrow\uparrow\uparrow\downarrow>$. Each spin up excitation is called a "magnon".

Then the one-magnon (one pseudoparticle) state is

$$|\psi(p_1)> = \sum_{x=1}^{L} e^{ip_1x}|x> \tag{12.37}$$

which diagonalizes the Hamiltonian

$$H|\psi(p_1)> = 8J\sin^2\frac{p_1}{2}|\psi(p_1)> = E_1|\psi(p_1)> \tag{12.38}$$

The ansatz for the 2-magnon state is

$$|\psi(p_1, p_2)> = \sum_{1\leq x_1<x_2\leq L} \psi(x_1, x_2) > |x_1, x_2> \tag{12.39}$$

where the wavefunction $\psi(x_1, x_2)$ is

$$\psi(x_1, x_2) = e^{i(p_1x_1+p_2x_2)} + S(p_2, p_1)e^{i(p_2x_1+p_1x_2)} \tag{12.40}$$

Plugging this ansatz in the Schrodinger equation we obtain solutions for S and the total energy E. The total energy is just the sum of the 2 magnons' energies, $E = E_1 + E_2$, and

$$S(p_1, p_2) = \frac{\phi(p_1) - \phi(p_2) + i}{\phi(p_1) - \phi(p_2) - i}; \quad \phi(p) \equiv \frac{1}{2} \cot \frac{p}{2} \tag{12.41}$$

One can then also write down ansatze for a multiple magnon state, and the Schrodinger equation will give a set of equations (Bethe equations) for the ansatz, but we will not describe them here.

The SU(2) sector and H_{XXX} from SYM

In the previous analysis we had only few insertions of b_l^+ operators, i.e. we had only few ϕ_i's among mostly Z's inside the operators and no $\bar{Z}$'s.

We can instead consider instead of such operators the "SU(2) sector" constructed out of two complex fields,

$$Z = \phi^1 + i\phi^2; \quad \text{and} \quad W = \phi^3 + i\phi^4 \tag{12.42}$$

and no ϕ^5, ϕ^6, nor $\bar{Z}$ and $\bar{W}$. And consider operators with large, but arbitrary numbers of both Z and W, that is, operators of the type

$$\mathcal{O}_\alpha^{J_1, J_2} = Tr[Z^{J_1} W^{J_2}] + ...(\text{permutations}) \tag{12.43}$$

Then one can similarly calculate the Hamiltonian acting on states corresponding to such operators, and obtain that if we only consider 1-loop interactions and planar diagrams (that can be drawn on a plane without self-intersections; these diagrams are leading in a $1/N$ expansion, for an $SU(N)$ gauge group),

$$H_{1-loop,planar} = \frac{g_{YM}^2 N}{8\pi^2} H_{XXX1/2} \tag{12.44}$$

Important concepts to remember

- pp waves are gravitational waves (gravitational solutions for perturbations moving at the speed of light), having a plane wave front.
- Both the maximal 11 dimensional supergravity and the 10 dimensional supergravity that is the low energy limit of string theory have a pp wave solution that preserves maximal supersymmetry.
- The maximally supersymmetric pp wave solution of 10 dimensional supergravity is the Penrose limit of $AdS_5 \times S_5$: look near a null geodesic at $\rho = \theta = 0$.
- The Penrose limit of AdS-CFT corresponds to large R charge J, $J \simeq \Delta \sim R^2 \sim \sqrt{g_{YM}^2 N}$.
- String energy levels on the pp wave are recovered from AdS-CFT if $g_{YM}^2 N/J^2 = $ fixed. String oscillators correspond to insertion of $\phi^i, D_\mu Z$ and $\chi^a_{J=1/2}$ inside $Tr[Z^J]$, with some momentum $e^{2\pi i n/J}$.

- The discretized string action is obtained from Super Yang-Mills's Feynman diagramatic action. Thus the long operator acts as a discretized closed string.
- The Heisenberg XXX spin chain Hamiltonian is diagonalized by Bethe ansatz, for excitations ("magnons") of spin up propagating in a sea of spin down states.
- One loop planar interactions in the SU(2) sector (Z, W) of large R-charge Super Yang-Mills gives the Heisenberg XXX spin chain Hamiltonian.

Exercises, section 12

1) An Aichelburg-Sexl shockwave is a gravitational solution given by a massless source of momentum p, i.e. $T_{++} = p\delta(x^+)\delta(x^i)$. Find the function $H(x^+, x^i)$ defining the pp wave in D dimensions.

2) If the null geodesic moves on S^5, one can choose the coordinates such that it moves on an equator, thus the Penrose limit gives the maximally supersymmetric pp wave. Show that if instead the null geodesic moves on AdS_5, the Penrose limit gives 10d Minkowski space (choose again $\rho = 0$)

3) Write down the N=4 SYM fields (including derivatives) with $\Delta - J = 2$.

4) Check that, by cyclicity of the trace, the operator with 2 insertions of Φ^1, Φ^2 at levels $+n$ and $-n$ equals (up to normalization)

$$Tr[\Phi^1 Z^l \Phi^2 Z^{J-l}] \tag{12.45}$$

5) Check that the Bethe ansatz for 2 magnons, with

$$E = E_1 + E_2; \quad S(p_1, p_2) = \frac{\phi(p_1) - \phi(p_2) + i}{\phi(p_1) - \phi(p_2) - i}; \quad \phi(p) = \frac{1}{2} cot \frac{p}{2} \tag{12.46}$$

solves the Schrodinger equation for $H_{XXX1/2}$.