Math exam, nationwide, Baccalaureate june 2000, "real" track (science-oriented high schools)

All problems are required. Exam time is 3 hours. You have 10 points from the start -for taking the exam- (total: 100 pnts)

Problem I

1. Consider the polynomial with real coefficients $f = x^3 - 2x^2 - 5x + 6$.

a) (3 pnts) Calculate $f(1)$;

b) (3 pnts) Determine the result and remainder of the division of f by $x - 1$;

c) (4 pnts) Solve the equation $f(x) = 0$.

2. (5 pnts) Solve the equation:

$$(\ln x)^3 - 2(\ln x)^2 - 5\ln x + 6 = 0, x > 0 \tag{1}$$

3. (10 pnts) Determine $x \in \mathbf{R}$ such that

$$\int_0^x (3t^2 - 4t - 5)dt + 6 = 0 \tag{2}$$

4. In the cartesian coordinate system xOy consider the straight lines defined by the equations

$$d_1 : 3x - 2y = 0,\ d_2 : x + 3y - 11 = 0,\ d_3 : 2x - 3y + 5 = 0 \tag{3}$$

a) (3 pnts) Determine the intersection point of the lines d_1 and d_2;

b) (4 pnts) Show that the lines d_1, d_2, d_3 intersect in the same point;

c) (3 pnts) Write down the equation of the circle of center $O(0, 0)$ which passes through the intersection point of the 3 lines.

Problem II

1. Consider the polynomial $f = x^3 - x^2 + aX - 1, a \in \mathbf{R}$. For $n \in \mathbf{N}^*$ denote $S_n = x_1^n + x_2^n + x_3^n$, where $x_1, x_2, x_3 \in \mathbf{C}$ are the roots of the polynomial f.

a) (6 pnts) Show that $S_3 - S_2 + aS_1 - 3 = 0$.

b) (6 pnts) Determine $a \in \mathbf{R}$ such that $S_3 = 1$.

2. For any $x \in [0, 1)$ define the sum:

$$S_n(x) = \sqrt{1-x} + x\sqrt{1-x} + ... + x^{n-1}\sqrt{1-x}, \forall n \in \mathbf{N}^* \tag{4}$$

a) (4 pnts) Show that for any $n \in \mathbf{N}^*$, we have:

$$S_n(x) = \sqrt{1-x} \cdot \frac{1-x^n}{1-x}, x \in [0, 1) \tag{5}$$

b) (4 pnts) Calculate $\lim_{n\to\infty} S_n(x)$.

c) (5 pnts) Calculate $\int_0^1 \sqrt{1-x}dx$.

Problem III

In $M_2(\mathbf{R})$, the set of square matrices of order 2 over $\mathbf{R}$, consider the matrix:

$$X(a) = \begin{pmatrix} 1+5a & 10a \\ -2a & 1-4a \end{pmatrix} \tag{6}$$

a) (3 pnts) Calculate the determinant of the matrix $X(a)$;

b) (3 pnts) Show thar for any $a, b \in \mathbf{R}$, we have:

$$X(a) \cdot X(b) = X(ab + a + b) \tag{7}$$

Consider the set $G = \{X(a) | a \in (-1, \infty)\}$

c) (3 pnts) Show that G is a stable part of $M_2(\mathbf{R})$ with respect to the operation of matrix multiplication;

d) (2 pnts) Determine $(X(1))^2$.

e) (4 pnts) Prove, using the method of mathematical induction, that for any $n \in \mathbf{N}^*$, we have $(X(1))^n = X(2^n - 1)$.

Problem IV

Consider the function $f : \mathbf{R} \to \mathbf{R}, f(x) = \cos x - 1 + x^2/2$.

a) (3 pnts) Calculate $\lim_{x \to 0} f(x)/x^4$.

b) (6 pnts) Determine f' and f''.

c) (2 pnts) Show that $f'(x) > 0, \forall x \in (0, \infty)$ and $f'(x) < 0, \forall x \in (-\infty, 0)$.

d) (2 pnts) Show that for any $x \in \mathbf{R}, f(x) \geq 0$.

e) (2 pnts) For the function $g : \mathbf{R} \to \mathbf{R}, g(x) = \cos x$, show that the area of the plane surface delimited by the graph of the function, the Ox axis and the straight lines of equations $x = 0, x = 1$, is larger than $5/6$.