Math exam, nationwide, Baccalaureate june 2001, "real" track (science-oriented high schools)

All problems are required. Exam time is 3 hours. You have 10 points from the start - for taking the exam- (total: 100 pnts)

Problem I (30 pnts)

1. Consider the polynomial $f = x^3 - x + 3$ with roots $x_1, x_2, x_3 \in \mathbf{C}$ and the polynomial $g = x^2 + x + 1$ with roots $y_1, y_2 \in \mathbf{C}$.

a) (4 pnts) Determine the result and the remainder of the division of the polynomial f by the polynomial g.

b) (2 pnts) Verify that $y_1^3 = y_2^3 = 1$.

c) (2 pnts) Show that the number $a = f(y_1) + f(y_2)$ is natural.

c) (2 pnts) Calculate $b = g(x_1) + g(x_2) + g(x_3)$.

2. Consider the function $f : \mathbf{R} \to \mathbf{R}, f(x) = \frac{2x}{x^2+1}$.

a) (4 pnts) Calculate $f'(x), x \in \mathbf{R}$.

b) (2 pnts) Determine the asymptotes of the graph of the function f.

c) (4 pnts) Calculate $\int_0^1 f(x)dx$.

3. In the cartesian coordinate system xOy consider the points $A(3,0), B(0,4)$ and $C(3,4)$.

a) (3 pnts) Determine the area of the triagle ABC.

b) (4 pnts) Calculate the perimeter of the triangle ABC.

c) (3 pnts) Show that the triangle ABC is right angled.

Problem II (20 pnts)

1. Consider the bynomial $a = (\sqrt{2} + \sqrt{3})^{100}$.

a) (3 pnts) Determine the number of rational terms in the expansion of the bynomial. Denote by S the sum of the rational terms and by T the sum of the irational terms of the bynomial.

b) (1 pnt) Show that $S - T = (\sqrt{2} - \sqrt{3})^{100}$.

c) (3 pnts) Show that $S > T$.

d) (3 pnts) Show that $S - T < 1/3^{100}$.

2. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \int_0^x e^{t^2} dt$.

a) (4 pnts) Show that $f(-x) = -f(x), \forall x \in \mathbf{R}$.

b) (4 pnts) Calculate $f'(x), x \in \mathbf{R}$.

c) (2 pnts) Calculate $\lim_{x\to\infty} f(x)$.

Problem III (20 pnts)

Consider the matrices:

$$X = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad Y = (1 \ \ 2 \ \ -1 \ \ -2), \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and} \tag{1}$$

$$A = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 1 & 2 & -1 & -2 \\ 1 & 2 & -1 & -2 \\ 1 & 2 & -1 & -2 \end{pmatrix} \tag{2}$$

Define $B = aA + I_4, a \in bfR$.
a) (4 pnts) Calculate the matrix $A - XY$.
b) (4 pnts) Calculate the determinant and the rank of the matrix A.
c) (3 pnts) Calculate A^2.
d) (3 pnts) Verify that $2B - B^2 = I_4$.
e) (3 pnts) Show that B is invertible, $\forall a \in \mathbf{R}$ and calculate its inverse.
f) (3 pnts) Prove, using the method of mathematical induction, that $B^n = I_4 + naA, \forall n \in \mathbf{N}^*, \forall a \in \mathbf{R}$.

Problem IV (20 pnts)

Consider the real number $a \in (0;1)$ and $(x_n)_{n\geq 1}, (I_n)_{n\geq 1}$ with general terms:

$$x_n = a - \frac{a^3}{3} + \frac{a^5}{5} + ... + (-1)^{n-1}\frac{a^{2n-1}}{2n-1},\ I_n = \int_0^a \frac{x^{2n}}{1+x^2}dx,\ \forall n \in \mathbf{N}^* \tag{3}$$

a) (4 pnts) Prove the identity:

$$1 - x^2 + x^4 + ... + (-1)^{n-1}x^{2n-2} = \frac{1 + (-1)^{n-1}x^{2n}}{1+x^2},\ \forall n \in \mathbf{N}^*,\ \forall x \in \mathbf{R} \tag{4}$$

b) (4 pnts) Integrating the equality at point a), show that:

$$x_n = arctan\ a + (-1)^{n-1}I_n,\ \forall n \in \mathbf{N}^*,\ \forall a \in (0;1] \tag{5}$$

c) (4 pnts) Show that

$$0 \leq \frac{x^{2n}}{1+x^2} \leq x^{2n},\ \forall x \in \mathbf{R};\ \forall n \in \mathbf{N}^* \tag{6}$$

d) (4 pnts) Show that $\lim_{n\to\infty} I_n = 0$ and $\lim_{n\to\infty} x_n = arctan\ a$.
e) (4 pnts) Calculate

$$\lim_{n\to\infty}(4 - \frac{4}{3} + \frac{4}{5} + ... + \frac{(-1)^n \cdot 4}{2n+1}) \tag{7}$$