Math exam, nationwide, Baccalaureate june 2002, "real" track (science-oriented high schools)

All problems are required. Exam time is 3 hours. You have 10 points from the start - for taking the exam- (total: 100 pnts)

Problem I (30 pnts)

1. Consider the funciton $f : \mathbf{C} \to \mathbf{C},\ f(x) = x^2 + 2x$.

a) (4 pnts) Verify that $f(x) = (x+1)^2 - 1,\ \forall x \in .$

b) (4 pnts) Solve in $\mathbf{R}$ the equation $(f \circ f)(x) = 0$.

c) (2 pnts) Using the method of mathematical induction, show that

$$\text{n times}: \quad (f \circ f \circ ... \circ f)(x) = (x+1)^{2^n} - 1,\ \forall x \in \mathbf{C} \tag{1}$$

2. Consider the function $f : \mathbf{R} \to \mathbf{R},\ f(x) = \ln(x^2 + 1)$.

a) (4 pnts) Calculate $f'(x),\ x \in \mathbf{R}$.

b) (3 pnts) Calculate

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} \tag{2}$$

c) (3 pnts) Calculate $\int_0^1 f(x)dx$.

3. In the cartesian system of coordinates xOy, consider the points $A(n, -n),\ \forall n \in \mathbf{N}$.

a) (3 pnts) Write down the equation of the straight line A_0A_1.

b) (4 pnts) Show that the length of the segment A_nA_{n+1} is independent of $n,\ \forall n \in \mathbf{N}$.

c) (3 pnts) Show that the point A_n is situated on the line $A_0A_1,\ \forall n \in \mathbf{N}$.

Problem II (20 pnts)

1. Consider the system

$$\begin{aligned} x - y + z &= 0 \\ x - 2y + 3z &= 0 \\ x - 3y + 5z &= 0 \end{aligned} \tag{3}$$

where $(x, y, z) \in \mathbf{R^3}$. Denote by A the matrix of the system.

a) (5 pnts) Calculate the determinant and the rank of the matrix A.

b) (3 pnts) Solve the system.

c) Find a solution (x_0, y_0, z_0) of the system for which $x_0 + 2y_0 + 3z_0 = 8$.

2. Consider $(a_n)_{n\geq 1}$ defined by

$$a_n = \frac{1}{1^2} + \frac{1}{2^2} + ... \frac{1}{n^2},\ \forall n \in \mathbf{N}^* \tag{4}$$

Assume it is known that $\lim_{n\to\infty} a_n = \pi^2/6$ and consider $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ defined by $b_n = a_n + 1/n,\ c_n = a_n + 1/(n+1),\ \forall n \in \mathbf{N}$.

a) (3 pnts) Show that $(b_n)_{n\geq 1}$ is strictly decreasing.

b) (3 pnts) Show that $(c_n)_{n\geq 1}$ is strictly increasing.

c) (2 pnts) Show that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \pi^2/6$.

d) (2 pnts) Show that $\lim_{n\to\infty} n(a_n - \pi^2/6) = -1$.

Problem III (20 pnts)

For any nonzero natural number n, consider the set of rational numbers $H_n = \{k/n! | k \in \mathbf{Z}\}$.

a) (4 pnts) Show that, if $x, y \in H_n$, then $x + y \in H_n$.

b) (4 pnts) Verify that, if $x \in H_n$, then $-x \in H_n$.

c) (4 pnts) Show that, if $n < p \in \mathbf{N}^*$, then $H_n \in H_p$.

d) (2 pnts) Show that for any rational number r, there is an $n \in \mathbf{N}^*$ such that $r \in H_n$.

e) (4 pnts) Show that if $(G, +)$ is a subgroup of the group $(\mathbf{Q}, +)$ and $1/n! \in G,\ \forall n \in \mathbf{N}^*$, then $H_n \in G$.

f) (2 pnts) Show that, if $G_1, ..., G_{2002}$ are subgroups of the group $(\mathbf{Q}, +)$ and $\mathbf{Q} = G_1 \cup G_2 \cup ... \cup G_{2002}$, then there is an $i \in \{1, 2, ..., 2002\}$ such that $G_i = \mathbf{Q}$.

Problem IV (20 pnts)

Consider the real numbers $a_1, a_2, ..., a_n$ and the functions $f, F : \mathbf{R} \to \mathbf{R}$,

$$\begin{aligned} f(x) &= a_1 \sin x + a_2 \sin 2x + ... + a_n \sin nx \\ F(x) &= -a_1 \cos x - \frac{a_2}{2} \cos 2x - ... - \frac{a_n}{n} \cos nx,\ \forall n \in bfN^*,\ n \geq 2 \end{aligned} \tag{5}$$

a) (4 pnts) Show that the function F is a primitive (integral) of the function f on $\mathbf{R}$.

b) (4 pnts) Verify that $F(x + 2k\pi) = F(x),\ \forall k \in \mathbf{Z},\ \forall x \in \mathbf{R}$.

c) (2 pnts) Using the result: "If a function $g : \mathbf{R} \to \mathbf{R}$ is periodic and monotonous, then it is a constant", show that if $f(x) \geq 0,\ \forall x \in \mathbf{R}$, then the function F is constant.

d) (4 pnts) Show that if the function F is constant, then $f(x) = 0,\ \forall x \in \mathbf{R}$.

e) (4 pnts) Denote

$$S(p, q) = \int_0^{2\pi} \sin px\ \sin qx\ dx,\ \forall p, q \in \mathbf{N}^* \tag{6}$$

Using the formula

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b),\ \forall a, b \in \mathbf{R} \tag{7}$$

show that

$$S(p, q) = (0,\ \text{if } p \neq q \text{ and } = \pi,\ \text{if } p = q),\ \forall n \in \mathbf{N}^* \tag{8}$$

f) (4 pnts) Show that if $f(x) \geq 0,\ \forall x \in \mathbf{R}$, then $a_1 = a_2 = ... = a_n = 0$.