1 Duality- Notes by Horatiu Nastase A very basic introduction

1.1 Abelian

Electromagnetism

Action

$$S = \frac{1}{4}\int F_{\mu\nu}^2 = \frac{1}{4}\int F * F \tag{1}$$

Particles: Equations of motion

$$d * F = *J \to \partial_{[M} * F_{NP]} = *J_{MNP} \to \partial_N F^{NP} = J^P \tag{2}$$

and Bianchi identity (postulate)

$$dF = 0 \to F = dA \to \partial_{[M} F_{NP]} = 0 \to \partial_N * F^{NP} = 0 \tag{3}$$

Particles+monopoles: If one adds a source to the Bianchi identity, one gets a monopole:

$$dF = X; \quad X = d\omega \to F = dA + \omega; \quad \partial_{[M} F_{NP]} = X_{MNP} \to \partial_M * F^{MN} = *X^P \tag{4}$$

An electron will have have the current (source)

$$(*J)_{123} = e\delta^3(y) \tag{5}$$

which implies that the electric charge of a field configuration (Gauss law) is

$$e = \int_{S^2} *F = \int_{M^3} *J \tag{6}$$

A single Dirac (abelian) monopole will have the source

$$(X)_{123} = g\delta^3(y) \tag{7}$$

which implies that the magnetic charge (Gauss law) is

$$g = \int_{S^2} F = \int_{M^3} X \tag{8}$$

Here we have used the generalized Gauss formula

$$\int_M dA = \int_{\partial M} A \tag{9}$$

The electric charge of an electron and the magnetic charge of another object are related by the Dirac quantization condition

$$eg = 2\pi n \tag{10}$$

Now we have manifest *duality*:

$$F \rightarrow *F; \quad e \rightarrow g; \quad *J \rightarrow X \tag{11}$$

P-form "electromagnetism" (abelian): Action

$$S = \int \frac{F^2_{(d+1)}}{2(d+1)!} \tag{12}$$

p=d-1-brane+ generalized monopoles: Equations of motion

$$d(*F)_{D-d-1} = (*J)_{D-d} \tag{13}$$

and "Bianchi identities"

$$dF_{d+1} = X_{d+2} = d\omega_{d+1} \rightarrow F_{d+1} = dA_d + \omega_{d+1} \tag{14}$$

The p-brane singular electrically charged object (analog of electron) has

$$(*J)_{1...D-d} = e_d \delta^{D-d}(y) \tag{15}$$

whereas the D-d-3 brane singular magnetically chaged object (analog of the monopole) has

$$X_{1...D+2} = f_{D-d-2}\delta^{d-2}(y) \tag{16}$$

and the charges are defined by the Gauss law

$$e_d = \int_{S^{D-d-1}} (*F)_{D-d-1} = \int_{M^{D-d}} *J_{D-d} \tag{17}$$

and the magnetic Gauss law

$$g_{D-d-2} = \int_{S^{d+1}} F_{d+1} = \int_{M^{d+2}} X_{d+2} \tag{18}$$

obeying the generalized Dirac quantization condition

$$e_d g_{D-d-2} = 4\pi\frac{n}{2} \tag{19}$$

We have the same manifest duality as in the particle case.

Dirac monopole

Analogous to the static electric charge

$$\vec{E} = \frac{e}{4\pi r^2}\hat{r} \tag{20}$$

we have the Dirac monopole configuration

$$\vec{B} = \frac{g}{4\pi r^2}\hat{r} \tag{21}$$

However, since now $F = dA + \omega$, and $F = dA$ only locally (and away from the singularity at x=0), we can't define A globally, only on patches we can write $\vec{B} = \nabla \times \vec{A}$. We can write for instance in the northern and southern hemispheres of 3d space

$$\begin{aligned}\vec{A}_N &= \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\vec{e}_\phi \\ \vec{A}_S &= -\frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\vec{e}_\phi\end{aligned} \tag{22}$$

We see that $\vec{A}_N$ is well defined over the whole space except at the South pole $\theta = \pi$, whereas $\vec{A}_S$ is well defined over the whole space except at the North pole $\theta = 0$.

This is a particular case of the statement that there is always a so-called "Dirac string" singularity, extending from the source to infinity, if we try to write down a globally defined gauge field. For $\vec{A}_N$ it is the South pole of space (for varying r), and for $\vec{A}_S$ it is the North pole of space (for varying r). The Dirac string thus is not physical, and can be moved around by gauge transformations. When matching A's valid on two different patches of space, like $\vec{A}_N$ and $\vec{A}_S$, we get a gauge transformation in between them, here

$$\vec{A}_N - \vec{A}_S = -\nabla\chi; \quad \chi = \frac{g}{4\pi}\phi \tag{23}$$

Then the magnetic charge is

$$g = \int_{S^2}\vec{B}d\vec{S} = \int_{equator}\Delta\vec{A}\cdot d\vec{e} = \chi(0) - \chi(2\pi) \neq 0 \tag{24}$$

where in general, equator can be replaced by any curve on which we match.

We see that we can classify the gauge field configurations by the magnetic charges, which are obtained by taking maps from an equator of S^2_∞ to the $U(1)$ gauge group, formally by the first homotopy group (winding number)

$$\Pi_1(U(1)) = \mathcal{Z} \tag{25}$$

In general, we can classify gauge fields defined over a sphere S_n (at infinity or otherwise) by the (n-1)th homotopy group

$$\Pi_{n-1}(G) \tag{26}$$

by looking at S_{n-1} equators of S_n and mapping them into G.

Equivalently, one classifies by the magnetic charge which is also the first Chern class

$$c_1(F) - \int_{S_2} \frac{F}{2\pi} \tag{27}$$

which is the cohomology class of $F/2\pi$ in the space of 2-forms (equivalence class of $dF = 0$ mod $F = dA$), which is identified with the integral.

Quantization conditions

In the presence of an electrically charged and a magnetically charged object, quantum mechanics dictates allowed values for the two.

The Dirac quantization condition can be obtained from the fact that in quantum mechanics, the gauge transformation χ around the equator would be a priori an operator, thus must be put to 1. The Schrodinger wave function transforms as $\psi \to e^{-ie\chi}\psi$ when going from patch to patch, and operatorially one has also $\vec{A} \to \vec{A} - i/e e^{-e\chi}\nabla e^{-ie\chi}$.

Thus $e^{ie\chi}$ has to be well defined and continous, meaning that $eg = 2\pi n$.

An equivalent way to say this is that an Aharonov-Bohm type experiment, where we encircle the Dirac string, would detect the phase change $\delta\chi$, unless it is quantized as given.

But this is not invariant under the general duality transformation (see below). The invariant relation was derived by Zwanziger and Schwinger, thus the DZS condition, for two dyons of charges (e_1, g_1) and (e_2, g_2), is given by the following. There is a more rigorous proof, but a simple one is to calculate the generalized Lorentz force on a dyon moving in the field of the other. It is

$$m\frac{d\vec{v}_1}{dt} = e_1(\vec{E}_2 + \vec{v}_1 \times \vec{B}_2) + g_1(\vec{B}_2 - \vec{v}_1 \times \vec{E}_2) \tag{28}$$

Expressing $\vec{E}_2$ and $\vec{B}_2$ and doing some vector algebra, we get

$$\frac{d}{dt}[\vec{r}\times m\vec{v}-\frac{e_1g_2-e_2g_1}{4\pi}\frac{\vec{r}}{r}]=0 \tag{29}$$

allowing us to interpret the second term as an angular momentum, thus quantized at the quantum level as

$$\frac{e_1g_2-e_2g_1}{4\pi}=n\frac{\hbar}{2} \tag{30}$$

The case of p-branes (p+1-form abelian fields) was investigated by Nepomechie and Teitelboim and they found that a similar relation occurs, as stated before. Moreover, they proved that there can be no nonabelian charges for p-form gauge fields at the quantum level.

1.2 Nonabelian

't Hooft-Polyakov monopole

The Dirac monopole is singular (has a delta function source) and necessitates an unphysical Dirac string singularity if we want to define a global gauge field. It can be modified to a non-singular and physical (no Dirac string) monopole if we use a theory spontaneously broken to U(1). In the core, we will take advantage of the extra gauge structure to get rid of the singular source, and the embedding also avoids the Dirac string, as the gauge field is now nonabelian.

In particular, the 't Hooft-Polyakov monopole solution to the SO(3) Georgi-Glashow model is such an example. The model has an SO(3) gauge field A^a_μ and a Higgs triplet Φ^a, with action (${f^a}_{bc}=\epsilon^{abc}$)

$$\begin{aligned}
&\mathcal{L}=-\frac{1}{4}(F^a_{\mu\nu})^2+\frac{1}{2}(D_\mu\Phi^a)^2-V(\Phi)\\
&V(\Phi)=\lambda(\Phi^a\Phi^a-v^2)^2\\
&F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+e\epsilon^{abc}A^b_\mu A^c_\nu\\
&D_\mu\Phi^a=\partial_\mu\Phi^a+e\epsilon^{abc}A^b_\mu\Phi^c
\end{aligned} \tag{31}$$

The model has a Higgs vacuum that breaks SO(3) to U(1): $\Phi^a\Phi^a=v^2$. And the moduli space of the Higgs (space of inequivalent vacua) is $\mathcal{M}_H=G/H=SO(3)/SO(2)=S^2$. Since at infinity we are in a Higgs vacuum, the classification of solutions of the model can be done by mapping the sphere S_2 at infinity into G/H, thus by $\Pi_1(G/H)$.

It is a theorem that $\Pi_2(G/H) = \Pi_1(H)_G$, where the subscript G means that we choose paths that can be extended to (homotopically) trivial paths in G. For simply connected G, as is SO(3), $\Pi_1(H)_G = \Pi_1(H)$, and thus the 't Hooft-Polyakov monopole offers a nonsingular and Dirac-string free generalization of the Dirac monopole. Note that for electroweak theory, $G = SU(2) \times U(1)$ and H is a linear combination, therefore $\Pi_1(H)_G = 0$ and any monopole will have a Dirac string. (Thus there are no 't Hooft monopoles -classical solutions- for the electroweak theory, only Dirac monopoles -fundamental particle)

The energy is

$$H = \int d^3 r T_{00} = \frac{1}{2}\int d^3 r[\vec{E}^a \vec{E}^a + \vec{B}^a \vec{B}^a + \Pi^a \Pi^a + \vec{D}\Phi^a \vec{D}\Phi^a] + \int d^3 r V(\Phi) \tag{32}$$

where $\Pi^a = D_0 \Phi^a$, and it can be rewritten as

$$\frac{1}{2}\int d^3 r[\vec{E}^a \vec{E}^a + \Pi^a \Pi^a + (\vec{B}^a - \vec{D}\Phi^a)^2 + 2\vec{B}^a \vec{D}\Phi^a] + \int d^3 r V(\Phi) \tag{33}$$

In the BPS limit $\lambda \to 0$, the last term is zero, and then we have the BPS bound

$$M \geq \int d^3 r \vec{B}^a \vec{D}\Phi^a = \int_{S^2_\infty} \Phi^a \vec{B}^a d\vec{S} \tag{34}$$

With the definition $F^a_{\mu\nu} = \frac{\Phi^a}{v} F_{\mu\nu}$ we see that the bound is topological

$$M \geq v \int_{S^2_\infty} \vec{B} d\vec{S} = vg \tag{35}$$

which is saturated for ($V(\Phi) = 0$ and) $\vec{E}^a = 0, \Pi^a = 0$ and $\vec{B}^a = D\Phi^a$.

Then $F_{\mu\nu}$ is the field strength of the unbroken U(1) gauge field, and satisfies the sourceless equations of motion and Bianchi identities *outside* the core of the monopole

$$d * F = 0; \quad dF = 0 \tag{36}$$

and

$$g = \int_{S^2_\infty} \vec{B} d\vec{S} = \frac{1}{2ev^3}\int_{S^2_\infty} dS^i \epsilon^{ijk} \epsilon^{abc} \Phi^a \partial^j \Phi^b \partial^k \Phi^c = \frac{4\pi}{e} n \tag{37}$$

One can write down an ansatz and solve (numerically) for the monopole.

The ansatz includes

$$A_0^a = 0; \quad A_i^a = -\epsilon^a{}_{ij}\frac{x^i}{er^2}(1 - K(r)) \tag{38}$$

where $K(r)$ is a function that decays exponentially at infinity. One can then calculate that the monopole number n is the same as the winding number

$$n = \frac{e^3}{4\pi^2}\int \epsilon^{abc}\epsilon^{ijk}A_i^a A_j^b A_k^c \tag{39}$$

so one can understand the monopole charge as also being classified by the map of the compactified 3d space ($= S_3$) to the gauge group G.

Thus there are 3 classifications involved: $S_3 \rightarrow G$ by winding number, coming purely from the nonabelian nature of the group, $\Pi_1(H)_G$ (which is also $= \Pi_1(H)/\Pi_1(G)$) coming from the unbroken part (the same as the abelian case), and $\Pi_2(G/H)$, coming from the fact that we have a Higgs model, and all are characterized by the same number, the monopole charge.

Note that for electroweak theory we can have an SU(2) winding number only if SU(2) is unbroken, in the broken case there are no 't Hooft monopoles and no solutions with nontrivial winding number, only unbroken U(1) Dirac monopoles.

More generally, one can write a BPS bound for solutions with electric charge, i.e. Julia-Zee dyons.

Defining also the abelian electric charge

$$q = \int_{S^2_\infty}\vec{E}d\vec{S} = \int d^3r\vec{E}^a\vec{D}\Phi^a = \int_{S^2_\infty}\Phi^a\vec{E}^a d\vec{S} \tag{40}$$

one can rewrite the energy as

$$\begin{aligned}&\frac{1}{2}\int d^3r[(\vec{E}^a - \vec{D}\Phi^a\sin\theta)^2 + \Pi^a\Pi^a + (\vec{B}^a - \vec{D}\Phi^a\cos\theta)^2]\\ &+\int d^3rV(\Phi) + v(q\sin\theta + g\cos\theta)\end{aligned} \tag{41}$$

and thus the BPS bound in the BPS limit is

$$M \geq v\sqrt{q^2 + g^2} \tag{42}$$

Instantons: We have:

$$Tr(F_{\mu\nu} * F^{\mu\nu}) = Tr(F \wedge F) = d\mathcal{L}_{CS} = d(A\wedge dA + 2/3A\wedge A\wedge A) \tag{43}$$

Then

$$\int_{M_4} Tr(F_{\mu\nu} * F^{\mu\nu}) = \int_{\partial M_4} \mathcal{L}_{CS} = \int_{M_3, x_4=\infty} \mathcal{L}_{CS} - \int_{M_3, x_4=0} \mathcal{L}_{CS} = n \quad (44)$$

Where n is defined as follows. Consider a large (with winding number, i.e. cannot be continously deformed to the identity) 3d gauge transformation

$$A_i \to U A_i U^{-1} - \frac{i}{g} \partial_i U U^{-1} \quad (45)$$

that goes to the identity at infinity. In this way, 3d space is effectively compactified and U is a map between S_3 (compactified 3d space) and G. Under it

$$\begin{aligned} S_{CS}(A) = \frac{g^2}{4\pi^2} \int_{M_3} \mathcal{L}_{CS}(A) \to \\ S_{CS}(A) + \frac{1}{4\pi^2} \int d^3x \epsilon^{ijk} Tr(\partial_i U U^{-1} \partial_j U U^{-1} \partial_k U U^{-1}) \end{aligned} \quad (46)$$

But then the expression

$$\frac{1}{4\pi^2} \int d^3x \epsilon^{ijk} Tr(\partial_i U U^{-1} \partial_j U U^{-1} \partial_k U U^{-1}) = \frac{g^3}{4\pi^2} \int d^3x \epsilon^{ijk} Tr(\tilde{A}_i \tilde{A}_j \tilde{A}_k) = n \quad (47)$$

is an integer called winding number, where $\tilde{A}_i$ is a gauge field that is pure (large) gauge.

Going back to 4d, we can use (44) to say that in the A_0 gauge, we can define at given time t the winding number as before, with $\tilde{A}_i$ replaced by A_i, and then the difference in winding numbers is given by a number called instanton number or Pontryagin index:

$$n = \frac{g^2}{16\pi^2} \int d^4x tr F_{\mu\nu} * F^{\mu\nu} \quad (48)$$

We can always write (both in Euclidean and Minkowski space)

$$\frac{1}{4g^2} \int d^4x (F^a_{\mu\nu})^2 = \int d^4x [\frac{1}{4g^2} F_{\mu\nu} * F^{\mu\nu} + \frac{1}{8g^2} (F - *F)^2] \quad (49)$$

but only in Euclidean space can we have a real solution to the equation $F = *F$, called instanton. That means that in Euclidean space, just the

YM action will give the instanton number, and moreover, this number is a minimum of the YM action.

Since $n_{inst} = n_{winding}(\infty) - n_{winding}(0)$, and n_{inst} is a minimum of the YM action in euclidean space, we can have a transition between static configurations with different winding numbers (monopole charges, as we saw before). This is so as the transition probability is given by the euclidean path integral between two given configurations, which by the stationary phase is just the minimum euclidean classical action

$$T(A^{(0)} \rightarrow A^{(n)}) \sim e^{-S_{min}} = e^{-min\int d^4x \frac{1}{4}(F^a_{\mu\nu})^2} = e^{-\frac{16\pi^2}{g^2}n} \tag{50}$$

Note that the instantons have therefore finite action, and are concentrated in a finite region of euclidean space. Therefore the fields go to zero at infinity and then space can be considered to be compactified to S_4, thus the instanton number is a map between S_4 and the group G. The instanton solution for the SU(2) case was written down by 't Hooft.

In the case of electroweak theory, as we mentioned, there are no 't Hooft monopoles (solutions, that can appear in cosmology via the Kibble mechanism), but there are instantons that can induce baryon B assymmetry (and lepton number L) via the anomaly that is proportional to the instanton number

$$\partial^\mu j^B_\mu = \frac{N_{gen}}{16\pi^2} tr(F * F) \tag{51}$$

There could be unbroken U(1) Dirac monopoles, but solutions with winding number only if SU(2) is unbroken, thus there the SU(2) instantons are not relevant there.

However, in SU(5) unification there are 't Hooft monopoles that can appear in cosmology via the Kibble mechanism, but the instanton induced transitions are anyway negligible compared to Kibble production.

In QCD (SU(3) group), there is no Higgs mechanism, so no 't Hooft monopoles, (solutions in a Higgs system) but there could be a different type of monopoles, characterized by the winding number n, that can change via instantons. Note that these monopoles are not of the Dirac type (with singular source) either, but they are solutions of the sourceless field equations with no mechanism for their creation other than instanton transitions. We will call them solitons for lack of a better name.

Note that a 3d (Dirac, 't Hooft or soliton) monopole can also be an instanton for a 2d theory. In 2d, the only topological invariant is abelian, namely

$$\int_{M_2} F \tag{52}$$

(if we would make F nonabelian this would be zero since $TrT^a = 0$) which is nothing other than the abelian magnetic charge. Thus a 3d (euclidean space) monopole can interpolate between two 2d Nielsen-Olesen vortex solutions (monopole solutions of the spontaneously broken abelian Higgs model).

Thus in a 2+1d theory abelian Higgs model (maybe embedded in a higher group), we can have an instanton solution= monopole in euclidean 3d space, (in the case that the model is embedded in the group corresponding to that monopole) that can interpolate between different vortex vacua, thus creating a transition probability.

Similarly, an abelian Nielsen-Olesen vortex can interpolate between a two static kink vacua (topological number doesn't involve gauge fields, as the vortex U(1) is broken at infinity).

Finally, one should note that the 4d instantons of groups bigger than SU(2) can be reduced to instantons of SU(2) subgroups (at least for SU(N)).

Theta vacua and Witten effect

Theta vacua:

Since we can have static configurations with different winding numbers n (which can be obtained by instanton transitions), the true QCD vacuum will be

$$|\theta >= \sum_n e^{-in\theta}|n > \tag{53}$$

which is equivalent to adding a term to the lagrangean,

$$S_\theta = \frac{\theta e^2}{64\pi^2}\int d^4x Tr F_{\mu\nu} * F^{\mu\nu} \tag{54}$$

As we mentioned, in Euclidean space we could put $F = *F$, which would give this term just from the YM action, but in Minkowski space we would need to add it explicitly and it violates P and CP.

In the usual vacuum we would write

$$< \mathcal{O} >= \frac{\int \mathcal{D}\phi e^{iS}\mathcal{O}}{\int \mathcal{D}\phi e^{iS}} \tag{55}$$

But now we write

$$< \theta|\mathcal{O}|\theta >= \sum_{mn} < n|e^{in\theta}\mathcal{O}e^{-im\theta}|m >=$$
$$\sum < n|\mathcal{O}e^{-i(m-n)\theta}|m >= \frac{\sum e^{-i(m-n)\theta}\int_{m-n}\mathcal{D}\phi e^{iS}\mathcal{O}}{\sum e^{-i(m-n)\theta}\int_{m-n}\mathcal{D}\phi e^{iS}} \quad (56)$$

that is, we sum over configurations with 4d Minkowski space "instanton number" (Pontryagin index really, there is no classical instanton solution, but we can integrate over them) p=m-n, with weight $e^{-ip\theta}$, which can be put in the action, as we did before.

So the theta vacuum is equivalent to the inclusion of the theta term in the Minkowski action. Experimentally, θ is very small in real QCD $\sim 10^{-10}$.

Note that the presence of the θ vacuum is unrelated to instanton QCD calculations. Indeed, the θ term is completely neglijible compared to the YM action, and the latter becomes of the same type as the former on the instanton solution, thus YM suffices. One evaluates the Euclidean path integral by assuming some instanton density (as these are classical solutions in Euclidean space), that allows stationary phase evaluation of nonperturbative effects. The instantons have also arbitrary size (there is no scale in the theory), so it is not very clear what kind of instanton density to take.

Witten effect:

In the presence of the θ term, a Dirac monopole configuration acquires electric charge:

$$\vec{E} = \nabla A_0; \quad \vec{B} = \nabla \times \vec{A} + \frac{g\vec{e}_r}{4\pi r^2} \quad (57)$$

implies

$$L_\theta = \frac{\theta e^2}{8\pi^2}\int d^3x \vec{E}\vec{B} = -\frac{\theta e^2 g}{8\pi^2}\int d^3x A_0 \delta^3(\vec{x}) \quad (58)$$

so that (using the Dirac quantization condition)

$$Q_{el} = en_e - \frac{\theta e^2 g}{8\pi^2} = en_e - \frac{e\theta}{2\pi}n_m \quad (59)$$

Alternatively and more generally, namely for the 't Hooft monopole, one uses the Noether charge in quantum theory (the generalization of the usual quantization of charge).

The abelian (unbroken) electric charge operator is, using the Noether theorem and the fact that for the unbroken part

$$\delta A_\mu^a = \frac{\epsilon}{ev}(D_\mu \Phi)^a; \;\; \delta \Phi^a = 0 \tag{60}$$

given by

$$N = \int d^3x \frac{\delta L}{\delta \partial_0 A_\mu^a}\delta A_\mu^a = \int d^3x(-\frac{1}{2}E^{ia} - \frac{\theta e^2}{16\pi^2}B^{ia})\frac{(D_i\Phi)^a}{ev} = \frac{Q_{el}}{e} + \frac{\theta e}{8\pi^2}Q_m \tag{61}$$

and the operator that performs gauge transformations at infinity is

$$e^{2\pi i N} = 1 \rightarrow N = n_e \rightarrow Q_{el} = en_e - \frac{e\theta}{2\pi}n_m \tag{62}$$

1.3 Duality: Towards Seiberg-Witten and S duality

Montonten-Olive conjecture (wrong)

Maxwell duality

Spontaneously broken Georgi-Glashow SO(3) YM-Higgs in BPS limit ($\lambda \rightarrow 0$) has duality

$$\vec{E} \rightarrow \vec{B}, \quad \vec{B} \rightarrow -\vec{E}, \quad e \rightarrow g = \frac{4\pi}{e} \tag{63}$$

Arguments:

-solution is spherically symmetric and it carries unit of magnetic charge: monopole is elementary

-BPS bound also valid for (saturated by) W bosons in classical theory

-intermonopole force for oppositely charged is

$$\frac{2g_0^2}{4\pi r^2} \tag{64}$$

and zero for same charge

Problems:

-dyons can't decay into magnetic monopoles and electrically charged bosons.

-$\beta(g) < 0$ (asymptotic freedom), contradicts duality, since changing the coupling will change the theory. We need $\beta(g) = 0$.

-BPS mass of monopole will get renormalized in a nonsusy theory.

-spin of monopole is zero, but I need something of spin 1 to be the dual photon.

S duality

Definitions (coupling constant, electric and magnetic charges):

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}; \quad Q_e = en_e - \frac{\theta}{2\pi} n_m; \quad Q_m = \frac{4\pi}{e} n_m \tag{65}$$

Then S duality is the generalization of the abelian $e \to 4\pi/e$ to include the θ term. It is

$$S : \tau \to -\frac{1}{\tau} \tag{66}$$

As of now, it is not clear why this should be the correct generalization, but we will see shortly. We also have the "T" duality (not to be confused with the T duality of string theory)

$$T : \tau \to \tau + 1 \tag{67}$$

which shifts the θ angle by 2π, therefore irrelevant ($e^{i\theta n}$ of the θ vacuum is 1). Together, they generate $SL(2,Z)$, as

$$\tau \to \frac{a\tau + b}{c\tau + d} \;; \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{68}$$

Specifically

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{69}$$

The BPS bound for Julia-Zee dyons can then be written as

$$M^2 \geq v^2(Q_{el}^2 + Q_m^2) = 4\pi v^2 \begin{pmatrix} n_e & n_m \end{pmatrix} \frac{1}{Im\tau} \begin{pmatrix} 1 & -Re\tau \\ -Re\tau & |\tau|^2 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} \tag{70}$$

which is manifestly S duality invariant under the τ transformation, coupled with the action on charges

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to M \begin{pmatrix} n_e \\ n_m \end{pmatrix} \tag{71}$$

as we can explicitly check for S and T. We note that in the classical theory, the invariance would be Sl(2,R), and is restricted to Sl(2,Z) in the quantum theory.

Finally, with the definitions

$$a = ve; \quad a_D = \tau a \tag{72}$$

the bound becomes

$$M_{BPS} = |an_e + a_D n_m| = | \begin{pmatrix} n_e & n_m \end{pmatrix} \begin{pmatrix} a \\ a_D \end{pmatrix} | \tag{73}$$

which is now explicitly SL(2,Z) invariant, as we can check that S and T act the same way on the (a, a_D) vector as on the charge vector:

$$\begin{pmatrix} a \\ a_D \end{pmatrix} \rightarrow M \begin{pmatrix} a \\ a_D \end{pmatrix} \tag{74}$$

So the Montonen-Olive conjecture can be extended to S duality (Sl(2,Z)), but it is wrong (doesn't apply for the Georgi-Glashow model).

Witten-Olive (correct)

$\mathcal{N} = 4$ susy YM has S duality. $\mathcal{N} = 2$ susy YM has effective duality.

In a Higgs theory with an adjoint Higgs S^a, define the electric and magentic charges

$$\begin{aligned} Q_e &= \frac{1}{< S >} \int d^3 x \partial_i (S^a F^a_{0i}) \\ Q_m &= \frac{1}{< S >} \int d^3 x \frac{1}{2} \epsilon_{ijk} \partial_i (S^a F^a_{jk}) \end{aligned} \tag{75}$$

Then one computes the susy algebra by the Noether method and finds for the $\mathcal{N} = 2$ theory algebra

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij} \gamma^\mu_{\alpha\beta} P_\mu + \epsilon_{ij} (\delta_{\alpha\beta} U + (\gamma_5)_{\alpha\beta} V) \tag{76}$$

that one has

$$U =< S > A_e; \quad V =< S > Q_m \tag{77}$$

Actually, this is true in general without defining $< S >$ (namely $U = \int d^3 x \partial_i ()...$), but for the susy Georgi-Glashow model, where the gauge group is broken as $SO(3) \rightarrow U(1)$, this is what we get. Then the susy BPS bound (from the algebra) becomes

$$M \geq < S > \sqrt{Q_e^2 + Q_m^2} \tag{78}$$

and as we saw, this bound was an argument for Sl(2,Z) duality for the non-susy GG model in the BPS limit $\lambda \rightarrow 0$. In the susy theory, we get the bound always, from the susy algebra.

1.4 Previous dualities- analogy

Sine-Gordon vs. massive Thirring model

This is the first and the only know example of exact duality (we will see that there is also an $\mathcal{N}=2$ susy abelian self-duality, but that is not so interesting).

The 1+1d sine-Gordon theory is

$$I_{SG} = \int d^2x[(\partial\phi)^2 + \frac{\alpha}{\beta^2}(\cos\beta - 1)] \tag{79}$$

Here the mass is given by $m^2 = \alpha$ and the usual $\lambda = \alpha\beta^2$ (but β is the coupling constant). In $\tilde{\phi} = \beta\phi$ the equations of motion are

$$\Box\tilde{\phi} + \sin\phi = 0 \tag{80}$$

and around the zero energy vacua $\tilde{\phi}_N = 2\pi N$ we have perturbative "mesons" of mass m, while interpolating between two vacua we have the topological soliton solutions (with velocity y)

$$\begin{aligned} \tilde{\phi} &= \pm 4tan^{-1}[exp(\frac{\tilde{x}-\tilde{x}_0-u\tilde{t}}{\sqrt{1-u^2}})] \\ \tilde{x}-\tilde{x}_0 &= \pm\int_{\tilde{\phi}(\tilde{x}_0)}^{\tilde{\phi}(\tilde{x})}\frac{d\phi}{2\sin(\phi/2)} \end{aligned} \tag{81}$$

with nonperturbative mass $M = 8m/\beta^2$, that carry topological charge

$$j^\mu = \epsilon^{\mu\nu}\partial_\nu\phi;\ \ Q = N_1 - N_2 \tag{82}$$

The massive Thirring model is a massive fermion with 4-Fermi interaction,

$$I_{mTh} = \int d^2x[\bar{\psi}i\partial\!\!\!/\psi - m_F\bar{\psi}\psi - \frac{g}{2}(\bar{\psi}\gamma^\mu\psi)^2] \tag{83}$$

The massless model is sensible for $g > \pi$ and exactly solvable. The massive model is not exactly solvable anymore, but the perturbative series is well defined. This model doesn't have solitons, just perturbative fermions, but does have $\psi - \bar{\psi}$ bound states.

Coleman found an exact duality between the two models. At classical level this is the bosonization formula

$$\psi_\pm(x) = exp[\frac{2\pi}{i\beta}\int_{-\infty}^{x}\frac{\partial\phi(x')}{\partial t}dx' \mp \frac{i\beta}{2}\phi(x)] \tag{84}$$

This says a theory of bosons is the same as a theory of fermions, but in 2d there is no way of discerning between bosons and fermions if we measure local fields and the particles are massless (another way of saying this is that there is no little group in 2d).

The correspondence between coupling constants is

$$\frac{\beta}{4\pi^2} = \frac{1}{1+g\pi} \tag{85}$$

thus strong coupling is mapped to weak coupling and viceversa.

At the quantum level the relation is

$$\psi_{\pm}(x) = C_{\pm} : exp[\frac{2\pi}{i\beta}\int_{-\infty}^{x} \frac{\partial\phi(x')}{\partial t}dx' \mp \frac{i\beta}{2}\phi(x)] : \tag{86}$$

Thus renormalization is defined by normal-ordering and one has $m^2/\beta \rightarrow m_0^2/\beta$.

Then the duality proven by Coleman based on this vertex operator construction showed that perturbative Green's functions of ϕ mesons in sine-Gordon are the same as perturbative Green's functions of $\bar{\psi}\psi$ in massive Thirring. One has

$$\frac{m_0^2}{\beta^2}\cos[\beta\phi] = -m_F\bar{\psi}\psi \tag{87}$$

But reversely, a sine-Gordon soliton is the same as a fermion in massive Thirring. One can prove that the topological charge in sine-Gordon is the same as the fermion charge in Thirring

$$-\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi = \bar{\psi}\gamma^\mu\psi \tag{88}$$

In higher dimensions, one needs gauge fields as well for dualities to work. That is because spin makes a difference in higher dimensions, and gauge fields and fundamental particles are in different representations.

The dual superconductor

The mechanism for magnetic confinement was found in the BCS theory for superconductivity (type I superconductor). There one has a composite scalar, a "Higgs"

$$H = \psi^{\uparrow}\psi^{\downarrow} = \epsilon^{ij}\psi^i\psi^j \tag{89}$$

which is to say that electrons condense into Cooper pairs making the photon massive by the Higgs mechanism:

$$\mathcal{H} = \frac{\dot{H}^2}{2} + \frac{\vec{E}^2 + \vec{B}^2}{2} + |(\vec{\nabla} + 2ie\vec{H}|^2 + V(|H|); \quad V(|H|) = \lambda(|H|^2 - F^2)^2 \quad (90)$$

Thus for $T < T_C$ we have condensation of electronic Cooper pairs into an interacting ground state (there is a potential for the Higgs). This is unlike Bose-Einstein condensation which doesn't have interactions.

In a type II superconductor we will have magnetic vortices, which are magnetic flux lines confined to live in a fixed area (Nielsen-Olesen-type vortices), forming tubes of constant energy density. The magnetic flux is quantized $\Phi = n\pi/e$, and the (Abrikosov) vortices are like a dipole (same field lines as a permanent magnet). The Higgs is in the vacuum outside the vortex. If there would be magnetic monopoles, they could break the flux tube and we could have a flux tube in between a monopole and an antimonopole (as opposed to a vortex which extends to infinity in the transverse direction). Then there would be a linear potential between M and $\bar{M}$, proportional to the distance of the flux tube in between them, thus magnetic monpoles are permanently *confined*.

The so called "dual superconductor" idea is a fictitious superconductor at strong coupling, in the presence of monopoles. By abelian electromagnetic duality, there should be a dual description where monopole fields are elementary (not electrons). The perturbative description of the system would be in terms of the weakly coupled monopoles, who would condense in the dual superconductor to form monopole "Cooper pairs", thus a dual Higgs. Then there would be electric flux tubes in between electrons and positrons, thus the electric flux would be *confined*, with a linear potential.

This is the kind of mechanism one expects in QCD, except there the group is nonabelian.

't Hooft's mechanism for confinement in QCD

But what 't Hooft does instead is to look at the maximal abelian subgroup of $SU(3)$ (Cartan subgroup), which is $U(1) \times U(1)$, defined as

$$\Omega = \begin{pmatrix} e^{i\frac{\Lambda_1}{2} + i\frac{\Lambda_2}{3}} & 0 & 0 \\ 0 & e^{-i\frac{\Lambda_1}{2} + i\frac{\Lambda_2}{3}} & 0 \\ 0 & 0 & e^{-2i\frac{\Lambda_2}{3}} \end{pmatrix} \quad (91)$$

and partially fix a gauge such that only this subgroup is left. The procedure is known as "abelian projection".

Then the off-diagonal components of the vector potential matrix correspond to 6 charged photons (with respect to the two U(1)'s). So the YM gluons would be either neutral or with (color) electric charges

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \pm 1/2 \\ \pm 1 \end{pmatrix} \tag{92}$$

and from the Dirac quantization condition we would get that the YM gluons have also (color) magnetic charge spectrum

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 1/2 \end{pmatrix} n_1' + \begin{pmatrix} \pm 1 \\ -1/2 \end{pmatrix} n_2' \tag{93}$$

Thus the gluons are (singular, i.e. Dirac) monopole-like objects, thus we can have a dual superconductor picture.

The quarks also have only an allowed set of color electric and color magnetic charges.

The phases of the QCD vacuum then are

- Higgs phase: color magnetic confinement mode (similar to type II superconductor: condensation of electrically charged bilinears)
- Coulomb phase: magnetic and electric charges coexist- this is the usual vacuum in electromagnetism.
- dual Higgs phase: quark confinement phase. By condensation of monopole field bilinears (that become light). This is the analog of the dual superconductor. The meson is a $q\bar{q}$ system with a color electric flux in between, and a glueball is a loop of color electric flux. Baryons are harder to picture though.

For a general SU(N) model, things are similar, the maximal abelian subgroup being $U(1)^{N-1}$.

The gauge fixing to the abelian projection is done by looking at a gauge function X in the adjoint representation. One can take either

$$X = F_{\mu\nu}F^{\mu\nu}, F_{\mu\nu}D^2F^{\mu\nu}, F_{12} \tag{94}$$

and they all transform covariantly under gauge transformation, that is

$$X \to X' = \Omega X \Omega^{-1} \tag{95}$$

The eigenvalues of X are gauge invariant, so one fixes the gauge by saying that *X is diagonal.* Then

$$X = \begin{pmatrix} \lambda_1 & & 0 \\ & ... & \\ 0 & & \lambda_n \end{pmatrix} \tag{96}$$

That doesn't fix the gauge completely, as we can check, for a gauge transformation in the Cartan subgroup it is left invariant (abelian projection). There are singular points in this construction, when two of the eigenvalues λ_i coincide. These are genuine points (not higher dimension). In fact, the only gauge condition with only point-like singularities is achieved if the largest abelian subgroup is left unbroken, giving the abelian projection.

In the presence of a θ parameter, one can have phase transitions as θ is varied. Indeed, in a $2(N-1)$-dimensional plot of electric charges q_i and magnetic charges h_i, one has a lattice of allowed charges. But when θ varies, charges vary.

In a particular mode, charges that have relative Dirac number zero with the fundamental condensing particle condense also, and lie on a line through the origin of h-q space (these would be electrons in a superconductor). Particles that have relative nonzero number are confined and lie away from the line (these would be monopoles in superconductor).

As θ varies, the condensing line gets tilted (as the electric charge varies), and phase transitions occur whenever we cross another condensing line (that was a line at $\theta = 0$).

- But now there is an unusual fourth type of phase called "oblique confinement". In the SU(2) theory it occurs at $\theta = \pi$. Here different kind of particles can form a bound state that condenses. In the SU(2) case it is two different type of monopoles, in the SU(3) case it is a bound state of 3 monopoles and a gluon. But the reason why this is different is that quarks are not confined in the usual sense now. Quarks can form bound states with monopoles that lie on the condensing line, so they can condense and escape as free particles.

So any gauge theory, including susy theories will have these 4 types of phases, Higgs phase, Coulomb phase, confinement phase and oblique confinement phase(s).

Abelian duality in N=2 SYM

It is the only known example of explicit duality transformation at the quantum level (on the path integral).

Define the dual $\mathcal{N}=1$ chiral superfield by

$$\Phi_D = \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} \tag{97}$$

which implies for the scalar components

$$a_D = \mathcal{F}'(a) \tag{98}$$

In order to change variables we make a Legendre transformation

$$\mathcal{F}_D(\Phi_D) = \mathcal{F}(\Phi) - \Phi\Phi_D \tag{99}$$

which implies

$$\mathcal{F}'_D(\Phi_D) = -\Phi \tag{100}$$

The Bianchi identity for $F_{\mu\nu}$ is in the superspace formulation part of the superspace constraint

$$Im(D_\alpha W^\alpha) = 0; \quad \bar{D}^2 D_\alpha V \equiv W_\alpha \tag{101}$$

The path integral is over an unconstrained gauge superfield V that includes A_μ, and duality would be proven by exchanging $\int \mathcal{D}V$ with $\int \mathcal{D}V_D$.

First, we exchange the path integral over V with path integral over W_α with the constraint, by a Langrange multiplier V_D:

$$\int \mathcal{D}V e^{iS} = \int \mathcal{D}W_\alpha DV_D e^{iS+ct.\frac{i}{2}Im\int d^4xd^2\theta d^2\bar{\theta}V_d D_\alpha W^\alpha} \tag{102}$$

But for an abelian $\mathcal{N}=2$ theory

$$16\pi S_{N=2} = Im\int d^4xd^4\theta\mathcal{F}(\Psi) =$$
$$Im\int d^4x[\mathcal{F}''(\Phi)W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\Phi^+\mathcal{F}(\Phi)] \tag{103}$$

We use

$$Im\int \Phi^+\mathcal{F}'(\Phi) = -Im\int[\mathcal{F}'(\Phi)]^+\Phi = Im\int \Phi_D^+\mathcal{F}'_D(\Phi_D) \tag{104}$$

and the fact that (by partial integration)

$$\int d^4xd^2\theta d^2\bar{\theta}V_D D_\alpha W^\alpha = \int d^4xd^2\theta\bar{D}^2V_D D_\alpha W^\alpha =$$
$$\int d^4xd^2\theta(\bar{D}^2D_\alpha V_D)W^\alpha = -4\int d^4xd^2\theta W_{D,\alpha}W^\alpha \tag{105}$$

Then we get

$$\int \mathcal{D}W exp[\frac{i}{16\pi}Im\int d^4xd^2\theta[\mathcal{F}''(\Phi)W^\alpha W_\alpha - 2\int d^4xd^2\theta(W_D)_\alpha W^\alpha] =$$
$$exp[\frac{i}{16\pi}Im\int d^4xd^2\theta(-\frac{1}{\mathcal{F}''(\Phi)}W_{D\alpha}W_D^\alpha)] \tag{106}$$

which means that we rewrote the theory in dual variables, where

$$\mathcal{F}_D''(\Phi_D) = -\frac{d\Phi}{d\Phi_D} = -\frac{1}{\mathcal{F}''(\Phi)} \Rightarrow \tau_D(a_D) = -\frac{1}{\tau(a)} \tag{107}$$