

# 1 Hamiltonian quantization and BRST -survival guide; notes by Horatiu Nastase

## 1.1 Dirac- first class and second class constraints, quantization

Classical Hamiltonian

Primary constraints:

$$\phi_m(p, q) = 0 \quad (1)$$

imposed from the start. The equations of motion on a quantity  $g(q, p)$  are

$$\dot{g} = [g, H]_{P.B.} \quad (2)$$

Define  $\phi_m \sim 0$  (weak equality), meaning use the constraint only at the end of the calculation, then for consistency  $\dot{\phi}_m \sim 0$ , implying

$$[\phi_m, H]_{P.B.} \sim 0 \quad (3)$$

The l.h.s. will be however in general a linearly independent function (of  $\phi_m$ ). If it is, we can take its time derivative and repeat the process. In the end, we find a complete set of new constrains from the time evolution, called **secondary constraints**. Together, they form the set of constraints,  $\{\phi_j\}, j = 1, J$ .

A quantity  $R(q, p)$  is called **first class** if

$$[R, \phi_j]_{P.B.} \sim 0 \quad (4)$$

for all  $j=1, J$ . If not, it is called **second class**. Correspondingly, constraints are also first class and second class, independent of being primary or secondary.

To the Hamiltonian we can always add a term linear in the constraints, generating the **total Hamiltonian**

$$H_T = H + u_m \phi_m \quad (5)$$

where  $u_m$  are functions of  $q$  and  $p$ . The secondary constraint equations are

$$[\phi_m, H_T] \sim [\phi_m, H] + u_n [\phi_m, \phi_n] \sim 0 \quad (6)$$

where in the first line we used that  $[u_n, \phi_m]\phi_n \sim 0$ . The general solution for  $u_m$  is

$$u_m = U_m + v_a V_{am} \quad (7)$$

with  $U_m$  a particular solution and  $V_{am}$  a solution to  $V_m[\phi_j, \phi_m] =$  and  $v_a$  arbitrary functions of time only. Thus we can *choose* a total hamiltonian that splits into a *first class part*  $H'$

$$H' = H + U_m \phi_m \quad (8)$$

(we can prove that it is so) and a linear combination of first class primary constraints (we can also prove that),

$$H_T = H' + v_a \phi_a; \quad \phi_a \equiv V_{am} \phi_m \quad (9)$$

Then however we find that the first class secondary constraints generate time variations that do not change the state, hence can also be added to the hamiltonian, thus obtaining the **extended hamiltonian**

$$H_E = H_T + v_{a'} \phi_{a'} \quad (10)$$

where  $\phi_{a'}$  are the first class, secondary constraints. In conclusion, we can add all the first class constraints to the hamiltonian, first the primary ones as above, so that we get a first class hamiltonian  $H'$ , and then the secondary ones with linear coefficients, to get  $H_E$ .

### Interpretation

Thus one can think of first class constraints as generating a motion tangent to the constraint hypersurface, whereas second class constraints don't: the algebra of constraints doesn't close. First class constraints are like gauge constraints: the generator of gauge transformations leaves physical quantities unchanged (gauge invariance). Second class constraints are gauge-fixing conditions of a larger, equivalent system.

### Quantization

We take the independent second class constraints  $\chi_s$ , and define

$$c_{ss'}[\chi_{s'}, \chi_{s''}] = \delta_{ss''} \quad (11)$$

and define the Dirac brackets

$$[f, g]_{D.B.} = [f, g] - [f, \chi_s] c_{ss'}[\chi_{s'}, g] \quad (12)$$

such that

$$[f, \chi_s]_{D.B.} = 0 \quad (13)$$

thus we can take  $\chi_s = 0$  as an operator equation and replace the new (Dirac) brackets with the commutator for quantization. Thus in the quantum theory the relevant distinction is between the first class and second class constraints, not between primary and secondary.

Thus in the quantum theory there are fewer degrees of freedom (one can take the example of secondary constraints  $q \sim 0, p \sim 0$ , which means we just drop  $q$  and  $p$  in the Poisson bracket to get Dirac). We have to be careful in the quantum theory, whenever we have relations like  $[\phi_j, \phi_{j'}] \sim 0$ , thus  $[\phi_j, \phi_{j'}] = c_{jj'} \phi_{j''}$  that the  $c$ 's appear on the r.h.s. not on the l.h.s.

**Example: Electromagnetic field**

Action

$$S = -\frac{1}{4} \int F_{\mu\nu}^2 \quad (14)$$

Momenta

$$P^\mu = \frac{\delta L}{\delta A_{\mu,0}} = F^{\mu 0} \quad (15)$$

thus we have the primary constraint  $P^0 = 0$ . Then the Hamiltonian is

$$\begin{aligned} H &= \int d^3x P^\mu A_{\mu,0} - L = \dots = \int d^3x \left[ \frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} F_{i0} F^{i0} + F^{i0} A_{i,0} \right] \\ &= \int d^3x \left[ \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} P^r P^r - A_0 P'_{,r} \right] \end{aligned} \quad (16)$$

The secondary constraint comes from  $[P^0, H] = 0$ , giving  $P'_{,r} \sim 0$ . The next level gives an identity, so the secondary constraint is just  $P'_{,r} = 0$ . Both primary and secondary constraints are first class. Moreover, we can check that  $H$  is first class also, so we can take it as  $H'$ . The total Hamiltonian is

$$H_T = \int d^3x \left[ \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} P^r P^r - A_0 P'_{,r} \right] + \int d^3x v(x) P^0(x) \quad (17)$$

The extended Hamiltonian is

$$H_E = H_T + \int d^3x u(x) B'_{,r}(x) \quad (18)$$

and now we can reabsorb  $A_0(x)$  into  $u'(x) = u(x) - A_0(x)$  in  $H_E$ .

## 1.2 BRST and quantization: Lagrangian BRST

For a general YM theory, usually one adds a gauge fixing term in the action

$$S \rightarrow S - \int \frac{\mathcal{F}[A]^2}{2\alpha} \quad (19)$$

to gauge fix  $\mathcal{F}^a[A] = c^a$ . Then, Fadeev-Popev show that putting a  $\delta(\mathcal{F}^a - c^a)$  in the path integral is equivalent to introducing a  $\det(M(A))$  in it, where

$$M^{ab}(A) = \frac{\partial \mathcal{F}^a}{\partial A_\mu^c(x)} D_\mu^{cb}(x, A) \delta(x - y) \quad (20)$$

which in turn can be exponentiated with ghosts (anticommuting) as an extra term in the action

$$S_{eff} = S - \int \frac{\mathcal{F}[A]^2}{2\alpha} + \int B^a M^{ab} C^b = \int \frac{\mathcal{F}[A]^2}{2\alpha} + \int B^a \delta_{gauge,b} \mathcal{F}^a C^b \quad (21)$$

(the gauge variation is  $\delta_{\xi^b} \mathcal{F}^a = \partial \mathcal{F}^a / \partial A_\mu^c D_\mu^{cb}$ ). For the usual  $\mathcal{F}^a = \partial^\mu A_\mu^a$  then

$$S_{eff} = \int \left[ -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2\alpha} (\partial^\mu A_\mu^a)^2 + B^a \partial_\mu D_\mu^{ab} C^b \right] \quad (22)$$

Then the BRST invariance is a residual invariance of the gauge fixed+ ghost action, that acts on the classical fields (Yang-Mills) as a gauge transformation with parameter  $C^a \Lambda$ , with  $\Lambda$  a noncommuting constant parameter. In total, the transformation is

$$\begin{aligned} \delta A_\mu^a &= D_\mu^{ab} C^b \Lambda \\ \delta C^a &= +\frac{1}{2} g f_{bc}^a C^c \Lambda C^b \\ \delta B^a &= -\frac{1}{\alpha} \partial^\mu A_\mu^a \Lambda \end{aligned} \quad (23)$$

One rewrites the gauge fixing term introducing an auxiliary field (the Lautrup - Nakanishi field)  $d$ , such that the BRST charge (generator of BRST transformations) satisfies  $Q^2 = 0$ . Before we do that, we will generalize a little bit the formalism, for later use. Instead of  $\mathcal{F}^a$  we write  $\mathcal{F}^\alpha$ , the structure constants for the gauge algebra are  $f_{\beta\gamma}^\alpha = -g f_{bc}^a$ , the ghost action is  $B_\alpha F_{,\beta}^\alpha C^\beta$ , and the metric for raising and lowering indices is  $\gamma^{\alpha\beta}$  with  $\gamma^{ab} = -\alpha \delta^{ab}$ . The

gauge transformation on the classical fields  $\phi^I$  is  $\delta_\xi \phi^I = R^I_\alpha \xi^\alpha$ . Then the gauge fixing + ghost action is

$$-\frac{1}{2}d^\alpha d^\beta \gamma^{\alpha\beta} + d_\alpha F^\alpha + B_\alpha F^\alpha_\beta C^\beta \quad (24)$$

Here  $d_\alpha$  is the auxiliary field. And the BRST transformation is

$$\begin{aligned} \delta\phi^I &= R^I_\alpha C^\alpha \Lambda \\ \delta C^\alpha &= -\frac{1}{2}f^\alpha_{\beta\gamma} C^\beta C^\gamma \Lambda \\ \delta B_\alpha &= d_\alpha \Lambda \\ \delta d_\alpha &= 0 \end{aligned} \quad (25)$$

and the BRST charge  $Q$  is nilpotent, i.e. satisfies  $Q^2 = 0$  ( $\delta_{BRST}^2 = 0$ ).

Now one has a quantum action with a BRST invariance. If the invariance remains after regularization (in susy theories, regularization schemes that respect both susy and BRST invariance are not known!), one can renormalize correctly the quantum BRST action.

The Lagrangean formalism above is manifestly covariant (if the gauge fixing term is Lorentz covariant), but one gauge fixes from the start. One can however treat the BRST quantization in the Hamiltonian formalism, where the approach breaks manifest Lorentz invariance, but is gauge invariant until the end.

### 1.3 Hamiltonian BRST (Batalin-Fradkin-Vilkovisky)

One follows Dirac's formalism in the context of BRST quantization. As described by Dirac, from the hamiltonian  $H$ , now denoted  $H_0$  and the first class constraints, with the Poisson brackets replaced by Dirac brackets to deal with the second class constraints, one finds an algebra, with structure functions (depending on  $p$  and  $q$ )  ${}^{(n)}U_{b_1..b_{n+1}}{}^{a_1..a_n}$  (the index  $a$  is defined below, as ghost-antighost and their momenta index), a BRST generalization of the  $c_{jj'j''}$  of Dirac. Then one adds ghost, antighost, Lagrange multiplier fields and the conjugate momenta for *all* of them, creating an extended phase space.

Then one constructs a nilpotent BRST charge  $Q_H$  (i.e.  $Q_H^2 = 0$ ), using the canonical variables of the extended phase space and the structure functions. In addition one constructs a BRST invariant Hamiltonian

$$\{H, Q_H\} = 0 \quad (26)$$

In fact, given any classical observable (gauge invariant, bosonic, its bracket with the first class constraints and  $H_0$  vanishes weakly) one can construct a BRST invariant extension depending on the extended phase space and the structure functions.

The final quantum action is of the type

$$S^{qu} = \int [p^i \dot{q}_i + \pi_\mu \dot{\lambda}^\mu + \dot{\eta}^a P_a - H + \{\psi, Q_H\}] dt \quad (27)$$

where  $\lambda^\mu$  are Lagrange multipliers,  $\pi_\mu$  their conjugate momenta,  $\eta^a$  are the ghosts  $C^\alpha$  and the antighost-momenta  $P(B)_\alpha$ ,  $P_a$  are the antighosts  $B_\alpha$  and ghost-momenta  $P(C)^\alpha$ , and  $\psi$  is a “gauge fixing fermion”, an arbitrary function of the variables of extended phase space, corresponding to a gauge choice, and then the path integral over the quantum action is independent on  $\psi$ . To reproduce the gauge choice  $\xi^\mu$  made in the Lagrangean formulation, we put  $\psi = B_\mu \xi^\mu + \dots$ . The “gauge fixing+ghost” terms from the Lagrangean formulation are then encoded in the  $\psi$  term.

### Structure functions

The structure functions will have “ghost symmetry” (as we can check by counting lower and upper indices). The first (zeroth order) structure functions are the constraints themselves (together with the conjugate momenta to the Lagrange multipliers - “anticonstraint momenta”-)

$${}^{(0)}U_a = G_a \equiv \{\pi_\mu(\lambda), \phi_\alpha(q, p)\} \quad (28)$$

The next order structure functions are the structure constants of the bracket algebra of the constraints

$${}^{(1)}U_{b_1 b_2}{}^{a_1} = -\frac{1}{2}(-)^{b_2} f_{b_1 b_2}{}^{a_1} \quad (29)$$

where  $f$  are the  $c_{jj'j''}$  of before, i.e.  $\{\phi_\alpha, \phi_\beta\} = f_{\alpha\beta}{}^\gamma \phi_\gamma$ .

At the next order, one works out the Jacobi identities for brackets of  $\epsilon^a G_a$ , where  $\epsilon^a$  is a gauge parameter, replaced with the BRST expression on classical fields,  $\Lambda C^a$ . The Jacobi identity can be rewritten as

$$\Lambda_a \Lambda_2 \Lambda_3 C^{b_1} C^{b_2} C^{b_3} ({}^{(1)}D_{b_1 b_2 b_3}{}^{a_1})_A G_{a_1} (-1)^{a_1} = 0 \quad (30)$$

where the symbol  $\Lambda$  denotes ghost-symmetrization. The solution to this equation is

$$({}^{(1)}D_{b_1 b_2 b_3}{}^{a_1})_A = 2({}^{(2)}U_{b_1 b_2 b_3}{}^{a_1 a_2}) G_{a_2} \quad (31)$$

and iteratively if

$${}^{(1)}D_{b_1 \dots b_{n+2}}{}^{a_1 \dots a_n}{}_A = (n+1){}^{(n+1)}U_{b_1 \dots b_{n+2}}{}^{a_1 \dots a_{n+1}}G_{a_{n+1}} \quad (32)$$

we take its bracket *before* ghost-symmetrization with another  $G_a$  getting a Jacobi identity defining  ${}^{(n+1)}D$ , etc.

**Hamiltonian BRST charge  $Q_H$**

Corresponding to

$$G_a = \{\pi_\mu(\lambda), \phi_\alpha\} \quad (33)$$

we have “ghosts”

$$\eta^a = \{P^\mu(B), C^\alpha\} \quad (34)$$

and “antighosts”

$$P_a = \{B_\mu, P_\alpha(C)\} \quad (35)$$

and we have “ghost number” +1 for  $\eta^a$  and -1 for  $P_a$ . Then the nilpotent BRST charge is

$$Q_H = \eta^a G_a + \sum_{n \geq 1} \eta^{b_{n+1}} \dots \eta^{b_1}{}^{(n)}U_{b_1 \dots b_{n+1}}{}^{a_1 \dots a_n} P_{a_n} \dots P_{a_1} \quad (36)$$

We can directly find  $Q_H$ , by putting  $Q_H = c^a G_a + \dots$  and requiring  $\{Q_H, Q_H\} = 0$  and find the result order by order in the ghosts. Assuming that  $G_a$  are real and defining  $\eta^a$  to be real, then we find that  $Q_H$  is real.

**Examples:** U(1) gauge theory:  $Q_H = \eta^a G_a$ . Gauge theory with closed gauge algebra with field-independent structure constants

$$Q_H = \eta G_a - \frac{1}{2} \eta^{b_2} \eta^{b_1} f_{b_1 b_2}{}^a P_a(-)^{b_2} \quad (37)$$

Thus YM gives

$$Q_H = \int d^3x [c^a D_i E_i^a + P(B)^a \pi_0(\lambda) - \frac{1}{2} g C^b C^c f_{cb}{}^a P_a(C)] \quad (38)$$

while the bosonic string has

$$Q_H = \int d\sigma [C^+ \Psi_+ + C^- \Psi_- + P(B)^\mu \pi_\mu(\lambda) - \partial_\sigma C^+ C^+ P_+(C) + \partial_\sigma C^- C^- P_-(C)] \quad (39)$$

**BRST invariant Hamiltonian and observables**

An observable is defined as a real, bosonic, first class quantity  $A_0$ , that can be extended to an BRST invariant quantity  $A$ .

In other words, for a bosonic function  $A$  satisfying

$$\{A_0, G_a\} = W_a{}^b G_b \quad (40)$$

we can extend it to a BRST invariant  $A$ ,  $\{A, Q_H\} = 0$  by

$$A = A_0 + \sum_{n \geq 1} \eta^{b_n} \dots \eta_{b_1} A_{b_1 \dots b_n}{}^{a_1 \dots a_n} P_{a_n} \dots P_{a_1} \quad (41)$$

and the first term is

$$A = A_0 + \eta^a A_a{}^b P_b + \dots \quad (42)$$

By introducing a new real anticommuting ghost  $c_0$  and ghost number 1 and its conjugate momentum  $P_a$ , and also considering  $A_0$  as a new  $G_a$ , we can write an extended charge

$$S = Q_H + c_a A \quad (43)$$

for which we apply the usual  $Q_H$  construction and thus find  $Q_H$  and  $A$  together. For the Hamiltonian, since

$$\{H, \phi_\alpha\} = V_\alpha{}^\beta \phi_\beta \quad (44)$$

we find

$$H_{BRST} = H_0 + \eta^a V_a{}^b P_b + \dots \quad (45)$$

but for a general observable  $A$  there is no general construction.

### Comments

In the Hamiltonian formulation all transformation rules follow from

$$\delta(\text{field}) = \{\Lambda Q_H, \text{field}\} = \{\text{field}, Q_H\} \Lambda \quad (46)$$

Here  $Q_H = \eta^a G_a + \dots$  and the higher orders in ghosts can be found by the *Noether method*, i.e.

The gauge fixing term is manifestly separately gauge invariant. To obtain the usual gauge choice (in the Lagrangean formalism), one takes  $\psi + B_\mu \xi^\mu$  where  $\xi^\mu$  is the usual gauge choice.

One can check that  $Q_H$  is the Noether charge of the rigid BRST symmetry of the quantum action,  $Q_H = \int d^3x j^0$ , with  $j^\mu$  the Noether current.

### Quantum theory



As stated before, the quantum theory is obtained by promoting the constraint algebra to an operatorial algebra, on a physical space. Observables are BRST invariant, thus on physical states

$$[\hat{A}, \hat{Q}_H] = 0; \quad \hat{A} \sim \hat{A}' = \hat{A} + [\hat{K}, \hat{Q}_H] \quad (47)$$

that is, I can add a BRST variation to an observable without affecting the physics. This defines a *cohomology* on the observables ( $Q_H^2 = 0$ , and the equivalence classes are  $Q_H$ -closed modulo  $Q_H$ -exact observables). This gives also a cohomology on the physical states:

**Physical states** are defined by

$$Q_H|\psi\rangle = 0 \quad (48)$$

This is obtained from the fact that  $Q_H^2 = 0$ ,  $Q_H$  is hermitian and thus we obtain  $\|Q_H|\psi\rangle\|^2 = 0$ . For boundary condition (physical states at the extremes of integration,  $t = t_0$  and  $t = t_1$ ), in the path integral formalism, this implies that

$$Q_H(t_0) = 0; \quad Q_H(t_1) = 0 \quad (49)$$

Note that this definition gives also a cohomology on physical states, as physical states are  $Q_H$ -closed, and the equivalence class is defined by  $Q_H$ -exact states (we drop  $Q_H|\chi\rangle$  terms from  $|\psi\rangle$ ).

As noted at the begining of this section, we have the **Fradkin-Vilkovisky theorem** stating that the path integral over the quantum action in the presence of a gauge fermion  $\psi$  is independent of  $\psi$ . But we can't put  $\psi = 0$ , since then the path integral is not well defined! In other words, we MUST fix the gauge, but the result is gauge invariant.

### Physical states and the Kugo-Ojima quartet mechanism

In the BRST formalism, gauge invariance is replaced by BRST invariance. One has the BRST charge  $Q_H$  and the ghost charge  $Q_c$ . The BRST charge has ghost number one, thus  $[iQ_c, Q_H] = Q_H$ . Zero norm states (BRST exact,  $|\psi\rangle = Q_H|\chi\rangle$ ) form a representation of the algebra  $V_0$ , and the physical Hilbert space is closed modulo exact states, i.e.

$$\mathcal{H}_{phys} = V_{phys}/V_0 \quad (50)$$

Physical states are BRST singlets and (because of  $[iQ_c, Q_H] = Q_H$ ), also ghost number zero sector, i.e.

$$Q_H|k, 0\rangle = 0; \quad Q_c|k, 0\rangle = 0 \quad (51)$$

Unphysical states come naturally in BRST doublet representation. Given a state  $|k, N\rangle$  of ghost number  $N$ , i.e.

$$Q_H|k, N\rangle \neq 0; \quad iQ_c|k, N\rangle = N|k, N\rangle \quad (52)$$

we define the zero norm state

$$|k, N+1\rangle \equiv Q_H|k, N\rangle; \quad \langle N+1, k|k, N+1\rangle = 0 \quad (53)$$

and further action with  $Q_H$  yields nothing. However, that is not the whole story. Using  $[iQ_c, Q_H] = Q_H$  in between  $|k, N\rangle$  and a  $\langle k, -(N+1)|$  we can convince ourselves that the latter exists, such that

$$\langle k, -(N+1)|k, N+1\rangle = 1 \quad (54)$$

Then we also define

$$Q_H|k, -(N+1)\rangle \equiv |k, -N\rangle \quad (55)$$

Thus, the BRST doublets come in pairs, forming the BRST quartets or Kugo-Ojima quartets (irreps). We can define the creation operators

$$\begin{aligned} |k, N\rangle &\equiv \chi_k^+|0\rangle; & -|k, -N\rangle &\equiv \beta_k^+|0\rangle \\ |k, N+1\rangle &\equiv \gamma_l^+|0\rangle; & -|k, -(N+1)\rangle &\equiv \bar{\gamma}_k^+|0\rangle \end{aligned} \quad (56)$$

which imply commutation relations

$$[Q_H, \chi_k] = -i\gamma_k; \quad \{Q_H, \bar{\gamma}_k\} = \beta_k; \quad [Q_H, \beta_k] = \{Q_H, \gamma_k\} = 0 \quad (57)$$

Then the projector onto  $n$  unphysical particles is defined recursively as

$$\begin{aligned} P^{(n)} &= \frac{1}{n}[-\beta_k^+ P^{(n-1)} \chi_k - \chi_k P^{(n-1)} \beta_k - w_{kl} \chi_k^+ \\ &P^{(n-1)} \chi_l^+ + i\gamma_k^+ P^{(n-1)} \bar{\gamma}_k - i\bar{\gamma}_k^+ P^{(n-1)} \gamma_k] \end{aligned} \quad (58)$$

whereas the projector onto physical states=singlets is

$$P^{(0)} = \sum_n \frac{1}{n!} \phi_{\alpha_1}^+ \dots \phi_{\alpha_n}^+ |0\rangle \langle 0| \phi_{\alpha_n} \dots \phi_{\alpha_1} \quad (59)$$

And then any representative of the cohomology (state in  $V_{phys}$ , BRST closed, or invariant) can be written as

$$|f\rangle = P^{(0)}|f\rangle + \sum_{n \geq 1} P^{(n)}|f\rangle \quad (60)$$

as a sum of a physical state (in  $\mathcal{H}_{phys}$ ) and a BRST-exact state (in  $V_0$ ), since

$$\sum_{n \geq 0} P^{(n)} = 1 \quad (61)$$

### Final comments: Interpretation and classical BRST

It is important to note that one introduces the ghost variables in the system in order to do BRST quantization, however **BRST** symmetry is actually a **classical symmetry**! One introduces the ghosts and a Grassman-odd BRST charge for any first-class system, defining a classical BRST cohomology. One speaks of functions instead of operators and states for the classical cohomology. Also note that this can only be done in the Hamiltonian case, since otherwise the constraint algebra doesn't close off-shell (or  $Q$  is not nilpotent off-shell).

The path integral interpretation of ghosts doesn't even work (at least not immediately) in the general case (see later, the case of ghosts-for-ghosts). Quantization means promoting the constraint algebras (with Poisson or Dirac brackets) to operatorial (commutation) algebras and impose them on a Hilbert space in order to define a physical subspace. Also note that in the Lagrangean case one obtains the FP terms from the path integral, but now these terms come only from the  $\psi$  term. Without it, one can still speak about BRST symmetry!

The classical cohomology of zero ghost number is isomorphic to the classes of gauge invariant classical "observables", whereas the higher ghost number classical cohomologies are the same as the cohomologies of the operator  $d$  (exterior derivatives), acting on the gauge orbits generated by  $G_a$ , i.e. instead of  $dx^a$  one uses gauge orbits  $\omega^a$ , and then  $d^2 \sim 0$  and the cohomology is defined only weakly,  $dA \sim 0$  is closed and  $A \sim dB$  is exact.

The BRST construction then possesses topological information about gauge orbits and how they fill the constraint surface.

### Example: BRST quantization of the bosonic string

The action

$$S = -\frac{1}{2} \int d^2\xi \sqrt{g(\xi)} g^{ab} \partial_a X^\mu(\xi) \partial_b X_\mu(\xi) \quad (62)$$

is invariant under world sheet reparametrizations  $\xi \rightarrow f(\xi)$  and Weyl transformations  $g^{\alpha\beta} \rightarrow \lambda(\xi)g^{\alpha\beta}(\xi)$ . We fix reparametrization invariance choosing a conformal gauge  $g_{\alpha\beta} = \rho(\xi)\delta_{\alpha\beta}$ . Then the action plus gauge fixing terms and FP ghosts gives

$$S = -\frac{1}{2} \int d^2z \partial_z X^\mu \partial_{\bar{z}} X_\mu - \frac{1}{2} \int d^2z b(z) \partial_{\bar{z}} c(z) - \frac{1}{2} \int d^2z \bar{b}(z) \partial_z \bar{c}(z) \quad (63)$$

The gauge fixed reparametrization generators,  $T_x(z), \bar{T}_x(\bar{z})$  generate conformal transformations

$$L_n = \oint T_x(z) z^{n+1} dz; \quad \bar{L}_n = \oint \bar{T}_x(\bar{z}) \bar{z}^{n+1} d\bar{z} \quad (64)$$

satisfying the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12} n(n^2-1) \delta_{n+m,0} \quad (65)$$

The BRST charge is

$$Q_{BRST} = c^i G_i + \frac{1}{2} c^i c^j f_{ij}^k b_k; \quad [G-i, G_j] = f_{ij}^k G_k \quad (66)$$

and here

$$\begin{aligned} Q_{BRST} &= \oint dz j_{BRST}; \quad T_{gh}(z) = c \partial_z b + 2 \partial_z c \cdot b \\ j_{BRST} &= c(T_x(z) + \frac{1}{2} T_{gh}(z)) + \frac{3}{2} \partial_z^2 c \end{aligned} \quad (67)$$

Then the nilpotency of the BRST charge is obtained from the OPE:

$$\begin{aligned} \{Q_{BRST}, Q_{BRST}\} &= \\ &= \oint j_{BRST}(z) \oint_{|z|>|w|} j_{BRST}(w) + \oint j_{BRST}(w) \oint_{|w|>|z|} j_{BRST}(z) \\ &= \oint dw \oint dz : c(z) [T_x(z) + \frac{1}{2} T_{gh}(z)] :: c(w) [T_x(w) + \frac{1}{2} T_{gh}(w)] : \\ &= -\frac{1}{12} \oint dw \partial_w^3 c(w) c(w) (D-26) \end{aligned} \quad (68)$$

thus nilpotency of  $Q_{BRST}$  implies  $D=26$ . Observables transform as

$$\delta_B \mathcal{O}(z) = [Q, \mathcal{O}(z)] = \oint \frac{dw}{2\pi i} j_{BRST}(w) \mathcal{O}(z) \quad (69)$$

## 1.4 Batalin-Vilkovisky (BV) or field-antifield formalism

It deals with actions, thus is manifestly Lorentz covariant, but fixes the gauge at the very end, thus keeps the beauty of both Lagrangean and Hamiltonian BRST quantization formalisms. One does so by adding to each field an “antifield”, a type of *Lorentz covariant* conjugate momentum with opposite statistics. Now the *action is equal to the BRST charge*.

Thus we want

$$\delta_{BRST}\phi^A = (\phi^A, S\Lambda) \quad (70)$$

which at first sight is a contradiction, since the action is commuting, whereas the BRST charge is anticommuting. But all one really knows is that the *combination* of the BRST charge and the bracket is anticommuting. In the Hamiltonian case one takes the BRST charge to be anticommuting, and the bracket (Poisson or Dirac) to be anticommuting, but now we take *the charge (=action) to be commuting, and the bracket to be anticommuting*. It is called the “**antibracket**”.

For each field  $\phi^A$  we introduce an antifield  $\phi_A^*$  with opposite statistics, such that

$$(\phi^A, \phi_B^*) = \delta_B^A, \quad (\phi^A, \phi^B) = (\phi_A^*, \phi_B^*) = 0 \quad (71)$$

where the delta contains also a 4d delta function. The antibracket of functions of fields and antifields is

$$(f, g) = \partial f / \partial \phi^A \frac{\partial}{\partial \phi_A^*} g - \partial f / \partial \phi_A^* \frac{\partial}{\partial \phi^A} g \quad (72)$$

As usual, we require that BRST transformations are nilpotent, thus

$$(S, S) = 0 \quad (73)$$

This is called the “**master equation**”. The action is composed of a minimal part  $S^{min}$  depending only on  $\phi^i, C^\alpha$  and ghosts-for-ghosts if present, as well as their antifields, but not antighosts or antighost-antifields, and a nonminimal action  $S^{nonmin}$ . The **minimal action** appears as

$$S^{min} = S_{cl}(\phi^i) + \phi_A^*(\delta_{BRST}\phi^A)/\Lambda + \dots \quad (74)$$

where the terms with ... indicate two and more antifields, and  $\phi^i$  are classical fields. The “classical correspondence limit”

$$S_{cl}(\phi^i) = S^{min}(\phi_A^* = 0, \phi^A) \quad (75)$$

is automatic, since  $S$  has ghost number zero, and all ghosts have positive ghost number, and by definition there are no antighosts in  $S^{min}$ , thus at  $\phi_A^* = 0$  there can be no ghosts left in the action.

Note that the action is BRST invariant before gauge fixing, as in the Hamiltonian formalism, thus BRST is a symmetry of the classical action, even before gauge fixing!

Expanding the action in antifields  $\phi_A^*$  we get

$$S = S_0 + S_1 + S_2 + S_3 \dots \quad (76)$$

we get a number of equations from the master equation:

$$(S_0, S_0) = 0; \quad (S_0, S_1) = 0; \quad (S_1, S_1) + 2(S_0, S_2) = 0; \dots \quad (77)$$

Here the first equation is automatically satisfied (no  $\phi_A^*$ 's, so zero bracket), the second expresses the gauge invariance of the classical action (it becomes  $\partial S_0 / \partial \phi^A \times \delta_{BRST} \phi^A = 0$ ). The third equation says that the BRST algebra (commutator of two transformations) is proportional to the field equations. Indeed,  $(S_1, S_1)$  is proportional to  $(\delta_{BRST}, \delta_{BRST})$ , and the second starts with  $\delta S_0 / \delta \phi^A$ . For theories with a closed gauge algebra, the BRST transformations are nilpotent, thus then we must have  $S_2 = 0$ . Otherwise, as in some sugra theories, two BRST transformations yield another one plus field equations, and then one has

$$\delta_{BRST}(C^\alpha \Lambda_1) \delta_{BRST}(C^\alpha \Lambda_2) = \Lambda_1 \Lambda_2 \Delta^{ij} \frac{\partial}{\partial \phi^j} S_0 \quad (78)$$

and  $\Delta^{ij}$  corresponds to  $(S_0, S_2)$ , thus  $S_2$  is a ‘‘nonclosure term’’.

**Ghosts-for-ghosts** appear when the ghost action obtained by Fadeev-Popov gauge quantization has still some gauge invariance. For instance, for the antisymmetric tensor  $A_{\mu\nu}$  after FP quantization we get a ghost action  $B_\nu \partial_\mu [\partial^\mu C^\nu - \partial^{\nu\mu} C^\mu]$ , which has still the gauge invariance  $\partial C^\nu = \partial^\nu \Lambda_1$ . Thus one further makes the gauge choice  $\partial_\nu C^\nu = 0$  and gets ghosts-for-ghosts  $\bar{D}$  and  $D$ , with action  $\bar{D} \partial_\nu \partial^\nu D$ . But also the antighosts have a gauge invariance  $\delta B_\mu = \partial_\mu \Lambda_2$ , and fixing it by  $\partial^\nu B_\nu = 0$ , we get ghosts-for-ghosts  $\bar{E}$  and  $E$ , with action  $\bar{E} \partial_\nu \partial^\nu E$ . Here the anti- is suppressed in terminology. This is partly because one can't really justify this repeated procedure from FP quantization!

In fact, the correct number of ghosts-for-ghosts is 3, not 4, and is correctly obtained in the BV formalism.

Finally, one needs to **add the nonminimal action**

$$S^{nonmin} = \pi_\alpha B^{*\alpha} \quad (79)$$

where  $B^{*\alpha}$  are the antighosts and  $\pi_\alpha$  correspond to the conjugate momenta of the Lagrange multipliers in the Hamiltonian BRST formalism. Then the full action is

$$S = S^{min} + S^{nonmin} \quad (80)$$

and is still nilpotent  $(S, S) = 0$ . But this action is still gauge invariant!

So we must **fix the gauge**, as we did in the Hamiltonian BRST formalism. This is done by acting with a fermionic generator onto the fields and antifields and then setting the (old) antifields to zero. Thus the **final gauge fixed quantum action** is given by

$$S_{qu}(\phi) = S(\phi', \phi^{*'})|_{\phi^*=0};$$

$$\phi'^A = e^\psi \phi^A = \phi^A + (\psi, \phi^A) + \frac{1}{2}(\psi, (\psi, \phi^A)) + \dots; \quad \phi_A^{*'} = e^\psi \phi_A^* = \dots \quad (81)$$

Here the fermionic generator  $\psi$  is completely arbitrary, but one usually makes it independent on antifields, and then one gets

$$S_{qu}(\phi^A) = S(\phi^A, \phi_A^* = \partial\psi/\partial\phi^A) \quad (82)$$

Thus the classical action is obtained by putting in the (minimal plus nonminimal) action the antifields to zero, whereas the quantum action is obtained by first rotating the fields with  $\psi$  and then putting the antifields to zero.

An often used choice is

$$\psi = B_\alpha F^\alpha \quad (83)$$

where  $F^\alpha$  is a gauge choice. Then one gets

$$B^{*\alpha} = F^\alpha; \quad \pi^{*\alpha} = 0, \quad \phi_A^* = B_\alpha \delta F^\alpha / \delta \phi^A \quad (84)$$

and the quantum action is

$$S_{qu} = S_{cl} + B_\alpha \partial F^\alpha / \partial \phi^A (\delta_{BRST} \phi^A) / \Lambda + \dots + \pi_\alpha F^\alpha \quad (85)$$

Here the second term is clearly the FP term and the dots come from possible  $S_2$  and higher  $S_n$  in  $S^{min}$ .

**From Hamiltonian-BRST to BV-BRST**

BV has antifields, whereas Hamiltonian BRST has momenta. Thus we need to extend the BV action in a Hamiltonian direction, by including the momenta among the fields, and then we relate to the Hamiltonian formalism.

We will start with the set  $z^a = \{p^i, q_i, C^\alpha, P(C)_\alpha\}$ . The Lagrange multipliers appear as antifields of the momenta conjugate to the ghosts, i.e.  $\lambda^\alpha = P(C)^{* \alpha}$ .

In the Hamiltonian formalism, we have

$$\begin{aligned} Q_H &= \pi_\alpha P(B)^\alpha + Q_H^1(p^i, q_i, C^\alpha) P(C)_\alpha \\ H_{BRST} &= H_0(p^i, q_i) + C^\alpha V_\alpha^\beta P(C)_\beta + \dots = H(p^i, q_i, C^\alpha, P(C)_\alpha) \end{aligned} \quad (86)$$

Then  $z^a$  have the Poisson (or Dirac) brackets and transformation rules

$$\{z^a, z^a\} = \Omega^{ab}; \quad \delta_{BRST} z^a = \{z^a, Q_H^1 \Lambda\} \quad (87)$$

(note that we also have the BV brackets  $(z^a, z_b^*) = \delta_b^a$ ) and one has the brackets

$$\{Q_H^1, Q_H^1\} = \{Q_H^1, H\} = 0 \quad (88)$$

Then the BV action in Hamiltonian form is

$$\begin{aligned} S_H &= \frac{1}{2} \Omega_{ba} \dot{z}^b \dot{z}^a - H - \{\psi, Q_H^1\} + z_a^* \{z^a, Q_H^1\} \\ &\frac{1}{2} \Omega_{ba} \dot{z}^b \dot{z}^a - H - \{\psi, Q_H^1\} - \frac{1}{2} \Omega^{ab} (z_a^* z_b^*, Q_H^1) \end{aligned} \quad (89)$$

and is BV-bracket-nilpotent, i.e.  $(S_H, S_H) = \{S_H, Q_H\} = 0$ .

For usual Yang-Mills, we don't get anything new from BV.

### Construction of ghosts-for-ghosts

Example: coupling of Yang-Mills fields  $A_\mu^a$  to several antisymmetric tensors  $B_{\mu\nu}^a$  in the adjoint. Then

$$S_{cl} = \int d^4x \left[ -\frac{1}{2} B_{\mu\nu}^a F_a^{\mu\nu} + \frac{1}{2} A_\mu^a A_a^\mu \right] \quad (90)$$

with gauge invariance  $\delta B_{\mu\nu}^a = \epsilon_{\mu\nu\rho\sigma} D^\rho \xi^{\sigma,a}$ ,  $\delta A_\mu^a = 0$ . The FP ghost action for  $D^\mu B_{\mu\nu} = 0$  is

$$\mathcal{L}_{FP} = B_a^\nu (D^\mu \epsilon_{\mu\nu\rho\sigma} D^\rho C^{\sigma,a}) \quad (91)$$

with gauge invariances

$$\delta B_\mu^a = \partial_\mu \lambda^a; \quad \delta C_\mu^a = \partial_\mu \bar{\lambda}^a \quad (92)$$



but at the nonabelian level they are only true on-shell (for  $F_{\mu\nu}^a = 0$ ). One needs to introduce ghosts-for-ghosts  $C^a$  to fix the extra invariance. Naively we would also need a  $B^a$ , but that's not the case.

To find the correct (BRST invariant) quantum action we use BV. Introduce the fields  $\phi^A = \{B_{\mu\nu}^a, A_\nu^a, C_\nu^a, C^a\}$  and their antifields. Then

$$S^{min} = S_{cl} + \int d^4x [B_a^{*\mu\nu} \epsilon_{\mu\nu\rho\sigma} D^\rho C^{\sigma,a} + C_a^{*\mu} D_\mu C^a + \text{"more"}] \quad (93)$$

and requiring nilpotence  $(S, S) = 0$  we find that

$$S_2^{min} = \text{"more"} = -\frac{1}{2} B_a^{*\mu\nu} B_b^{*\rho\sigma} \epsilon_{\mu\nu\rho\sigma} f^{abc} C_c \quad (94)$$

(write down all possible terms in  $S$  with correct ghost number and dimension, and require the master equation  $(S_1, S_1) + 2(S_0, S_2) = 0$ . The nonminimal terms to be added are

$$S^{nonmin} = \pi_a^\mu b_\mu^{*a} + \pi_a b^{*a} + \pi_a^a d^{*a} \quad (95)$$

Here as advocated,  $b_a^\mu$  and  $b_a$  are the antighosts for the ghosts  $c_\mu^a$  and  $C^a$ , and  $\pi_\mu^a, \pi_a$  their auxiliary fields. One finds then the extra ghost  $d^a$ .

Finally, one can add a particular gauge fermion, eliminate the antifields by  $\phi_A^* = \partial\psi/\partial\phi^A$  and plug back in the action to get the BRST quantum action.