Sugra-Notes by Horatiu Nastase
A very basic introduction (survival guide)

1 Generalities

Supergravity is the supersymmetric theory of gravity, or equivalently, a theory of local supersymetry. The gauge field of supersymmetry is called a gravitino, and since it gauges the ϵ^α supersymmetry, it is a spin 3/2 field: ψ_μ^α. It is also the partner of the graviton, more precisely of the vielbein e_μ^a.

Writing down any supersymmetric theory, but in particular supergravity, for which there are many different types of fields, starts with a counting of degrees of freedom. Since susy relates bosons and fermions, the number of d.o.f. for bosons has to match the number for fermions. If they only match on-shell, we have on-shell sugra, if they match off-shell we have off-shell sugra.

Off-shell counting:

-vielbein or graviton: e_μ^a has d^2 components, but on it we act with the symmetries: General coordinate invariance, characterized by an arbitrary vector $\xi^\mu(x)$ and local Lorentz invariance characterized by an antisymmetric tensor $\lambda^{mn}(x)$. Thus we have $d^2 - d - d(d-1)/2 = d(d-1)/2$ components. This is the same number as for the metric $g_{\mu\nu}$ (symmetric tensor), if the index μ runs over d-1 instead of d components.

-spinor (i.e. spin 1/2) ψ^α has $n = 2^{[d/2]}$ components. We work with irreducible spinors, usually Majorana (then the components above are real).

-gravitino: ψ_μ^α has nd components. But we have susy acting on it, characterized by the arbitrary spinor $\epsilon^\alpha(x)$. Thus we have $nd - n = n(d-1)$ components. Thus again we have the index μ running over d-1 components.

-scalar fields (propagating or auxiliary): 1 component.

-gauge fields: A_μ have d components, but gauge invariance has an arbitrary scalar parameter $\lambda(x)$, thus we have d-1 components again.

-antisymmetric tensors $A_{\mu_1\ldots\mu_r}$ have $d(d-1)\ldots(d-r+1)/(1\cdot 2\cdot \ldots \cdot r)$. But one has a gauge invariance defined by a parameter $\lambda_{\mu_1\ldots\mu_{r-1}}$, thus the number of components is

$$\frac{d(d-1)...(d-r+1)}{1\cdot 2\cdot ...\cdot r} - \frac{d(d-1)...(d-r+2)}{1\cdot 2\cdot ...\cdot r-1} = \frac{(d-1)...(d-r)}{1\cdot 2\cdot ...\cdot r} \quad (1)$$

Obs: auxiliary fields can be also tensors, not only scalars.

On-shell:

-vielbein or graviton: The equation of motion for the linearized graviton follows from the Fierz-Pauli action

$$\mathcal{L} = \frac{1}{2}h_{\mu\nu,\rho}^2 + h_\mu^2 - h^\mu h_{,\mu} + \frac{1}{2}h_{,\mu}^2; \quad h_\mu \equiv \partial^\nu h_{\nu\mu}; \quad h \equiv h^\mu{}_\mu \quad (2)$$

If we impose the de Donder gauge condition

$$\partial^\nu \bar{h}_{\mu\nu} = 0; \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}\frac{h}{2} \quad (3)$$

we get just $\Box\bar{h}_{\mu\nu} = 0$. Thus imposing the field equation means restriction of the number of polarizations: $k^2 = 0 \Rightarrow k^\mu \epsilon_{\mu\nu}(k) = 0$, which kills d degrees of freedom, thus the number of components is

$$\frac{d(d-1)}{2} - d = \frac{(d-1)(d-2)}{2} - 1 \quad (4)$$

that is, transverse traceless symmetric tensor.

-scalar field: the KG equation doesn't restrict anything, we still have 1 component. However, for an auxiliary field, the equation kills it.

-gauge field: In the covariant gauge $\partial^\mu A_\mu = 0$, the equation of motion $\Box A_\mu = 0$ has only d-2 polarizations: For $k^2 = 0$, $k^\mu \epsilon_\mu^a(\vec{k})$ has d-2 independent solutions.

-spinor: The Dirac equation relates 1/2 of components to the other 1/2, i.e. the independent solutions to $(/p - m)u(p) = 0$ are 1/2 as many as components, so $n/2$.

-antisymmetric tensor: In the covariant gauge $\partial^{\mu_1} A_{\mu_1...\mu_r} = 0$, the equation of motion $\Box A_{\mu_1...\mu_r} = 0$ has only a number of polarizations equaling the components, with d replaced by $d-2$ (transverse): For $k^2 = 0$, $k^{\mu_1}\epsilon^a_{\mu_1....\mu_r}(\vec{k})$ has $(d-2)...(d-1-r)/[1\cdot 2\cdot ....\cdot (r-2)]$ independent solutions.

-gravitino: in the gauge $\gamma^\mu \psi_\mu = 0$ (since only the gamma-traceless part is an irreducible representation of the Lorentz group), the counting is as for a gauge field $\times$ a spinor, i.e. $[(d-2)n - n]/2 = (d-3)n/2$.

Let's try to make s supermultiplet out of the graviton and the gravitino. Applying the rules above, we get:

3d Off-shell: Bosons: e_μ^a: $3 \cdot 2/2 = 3$. Fermions: ψ_μ^α: $2 \cdot 2 = 4$. Thus we need one bosonic auxiliary field, S.

On-shell: e_μ^a: $1 \cdot 2/2 - 1 = 0$ vs ψ_μ^α: $(3-3) \cdot 2/2 = 0$, so nothing on-shell.

4d Off shell: Bosons: e_μ^a: $4 \cdot 3/2 = 6$. Fermions: ψ_μ^α: $4 \cdot 3 = 12$. We need 6 bosonic auxiliary components. There existseveral constructions, but the minimal one is a scalar S, a pseudoscalar P and an axial vector A_μ. The scalar and pseudoscalar can be combined into a complex scalar $M = S + iP$.

On-shell: e_μ^a: $3 \cdot 2/2 - 1 = 2$. vs ψ_μ^α: $1 \cdot 4/2 = 2$, thus they match!

Thus the simplest case is in 3d, but there is no dynamics, and off-shell we need only one auxiliary scalar.

The simplest on-shell case is 4d.

We will start with on-shell susy in 4d and then move to 3d where things are simpler.

2 4d on-shell

Susy rules

Supergravity is a supersymmetric theory, so we expect on general grounds that we have (see the WZ model)

$$\delta e_\mu^a = \frac{k}{2}\bar{\epsilon}\gamma^a\psi_\mu \tag{5}$$

(plus maybe more terms), where the $k/2$ is by dimensional reasons and the γ^a is needed to match indices.

Supergravity is also a local theory of susy, with the gravitino as its gauge field, so we also expect

$$\delta\psi_\mu = \frac{1}{k}D_\mu\epsilon; \quad D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon \tag{6}$$

(plus mabe more terms), where ω_μ^{ab} is the spin connection and defines the covariant derivative acting on spinors in curved space (as we wrote above).

Action

For the gravity action, one takes the Einstein-Hilbert action. One usually writes the Ricci scalar in terms of Christoffel symbols, i.e. $R = R(\Gamma)$. For Γ one usually takes $\Gamma^\lambda_{\mu\nu}(g)$ (symmetric, and expressed in terms of the metric).

This is called the second order formulation. One can also take Γ and g to be independent, and then the field equation $\delta S/\delta\Gamma^\lambda_{\mu\nu}=0$ in the absence of matter fixes $\Gamma=\Gamma(g)$. This is called the first order formulation, or Palatini formalism.

But we saw that spinors need the spin connection and susy needs the vielbein. So we must use instead the formulation with the vielbein and spin connection. The Riemann tensor is just the YM curvature (field strength) for the spin connection,

$$R^{ab}_{\mu\nu}(\omega)=\partial_\mu\omega^{ab}_\nu-\partial_\nu\omega^{ab}_\mu+\omega^{ac}_\mu\omega^{cb}_\nu-\omega^{ac}_\nu\omega^{cb}_\mu;\ \ (R=d\omega+\omega\wedge\omega) \tag{7}$$

For the second order formulation one can use the "vielbein postulate" (the vielbein is covariantly constant)

$$D_\mu e^a_\nu\equiv\partial_\mu e^a_\nu+\omega^{ab}_\mu(e)e^b_\nu-\Gamma^\rho_{\mu\nu}(g)e^a_\rho=0\Rightarrow De^a=0:\ \ \partial_{[\mu}e^a_{\nu]}+\omega^{ab}_{[\mu}(e)e^b_{\nu]}=0 \tag{8}$$

The solution to the "no torsion" constraint (vielbein postulate) is $\omega=\omega(e)$. Torsion is the YM curvature of the vielbein, $T^a=De^a=0$. For the first order formulation (Palatini formalism) one treats the spin connection as independent. Then

$$R^{ab}_{\mu\nu}(\omega)=R^\lambda{}_{\rho\mu\nu}(\Gamma)e^{\rho a}e^b_\lambda \tag{9}$$

and so the action is

$$-\frac{1}{2k^2}eR(e,\omega) \tag{10}$$

The equation of motion for ω^{ab}_μ in the absence of matter is just the vielbein postulate (no torsion constraint), giving $\omega=\omega(e)$. In the presence of matter we get some torsion, i.e. the equation of motion for ω^{ab}_μ get contributions from the fermions (which as we saw have covariant derivatives that involve the spin connection), thus we will have $\omega=\omega(e,\psi)$.

Note: As for any YM theory, the spin connection curvature satisfies

$$R^{rs}_{\mu\nu}(\omega)\gamma_{rs}=[D_\mu(\omega),D_\nu(\omega)] \tag{11}$$

but moreover, by defining flat covariant derivatives $D_a=e^\mu_aD_\mu$ we can check that we have

$$[D_a,D_b]=(e^\mu_ae^\nu_bD_{[\mu}e^c_{\nu]})D_c+(e^\mu_ae^\nu_bR^{rs}_{\mu\nu}(\omega)\gamma_{rs}\equiv T^c_{ab}D_c+R^{rs}_{ab}M_{rs} \tag{12}$$

where M_{rs} are the Lorentz generators. This definition of torsion and curvature will be generalized to superspace and YM theories.

The action for the gravitino is the Rarita-Schwinger action (unique action for a spin 3/2 field), which in curved space and in a general dimension is

$$-\frac{e}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho \tag{13}$$

and in 4 dimensions can also be rewritten as

$$-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho(\omega)\psi_\sigma \tag{14}$$

and the curved space covariantization followed the general rules.

1, 2 and 1.5 order formalisms

So the action must be of the type (10) plus (14), and the invariance of the type (5) and (6).

But we must be more precise. In second order formalism, in the action (10) plus (14) and susy rules (5) and (6) we have ω replaced with $\omega(e,\psi)$ which is the solution of the $\delta S/\delta\omega_\mu^{ab}=0$ equation.

In first order formalism, the action is (10) plus (14), and the susy rules are (5) and (6), but we also need

$$\delta\omega_\mu^{ab}(\text{first order}) = -\frac{1}{4}\bar{\epsilon}\gamma_5\gamma_\mu\tilde{\psi}^{ab} + \frac{1}{8}\bar{\epsilon}\gamma_5(\gamma^\lambda\tilde{\psi}^b_\lambda e^a_\mu - \gamma^\lambda\tilde{\psi}^a_\lambda e^b_\mu);\quad \tilde{\psi}^{ab}=\epsilon^{abcd}\psi_{cd} \tag{15}$$

but on the right hand side we have equations of motion, so we can see that ω is auxiliary.

Finally, there is the so-called 1.5 order formalism, which uses the best of both worlds. We use 2-nd order formalism, but in the action $S(e,\psi,\omega(e,\psi))$ we don't vary $\omega(e,\psi)$ by the chain rule (as if we were in first order formalism and ω was independent), since its variation will always be multiplied by $\delta S/\delta\omega = 0$!

Finally, for completeness, the action of Einstein and local Lorentz transformations on e_μ^a and ω_μ^{ab} is as follows. Einstein:

$$\begin{aligned}\delta_E e_\mu^a &= \xi^\nu\partial_\nu e^a_\mu + \partial_\mu\xi^\nu e^a_\nu\\ \delta_E\omega_\mu^{ab} &= \xi^\nu\partial_\nu\omega_\mu^{ab} + \partial_\mu\xi^\nu\omega_\nu^{ab}\end{aligned} \tag{16}$$

and local Lorentz:

$$\begin{aligned}\delta_{lL}e^a_\mu &= \lambda^{ab}e^b_\mu\\ \delta_{lL}\omega_\mu^{ab} &= -D_\mu\lambda^{ab} \equiv -\partial_\mu\lambda^{ab} - \omega_\mu^{ac}\lambda^{cb} - \omega_\mu^{bc}\lambda^{ca}\end{aligned} \tag{17}$$

In other dimensions and for extended supergravity we can have more fields, and correspondingly more terms in the action and transformation rules, and also we could have more fermions added in $\omega(e,\psi)$, but it is always the solution of the ω equation of motion.

Thus for the rest of features we have to go case by case, and the only universal features of on-shell supergravity have already been explained.

3 3d off-shell

Everything that was said in 4d goes through, but in the Rarita-Schwinger action (13) we have $\gamma^{\mu\nu\rho}=\epsilon^{\mu\nu\rho}$.

We saw that on-shell there is no dynamics, so we need to introduce auxiliary fields to find something nontrivial.

We need to introduce auxiliary fields to close the algebra. Indeed, in global susy, we want to represent the algebra, most notably $[\bar\epsilon_1 Q,\bar\epsilon_2 Q]=1/2\bar\epsilon_2\gamma^\mu\epsilon_1 P_\mu$, on the fields. One finds that the algebra does not close on all fields, i.e. that one has additional equation of motion terms spoiling the algebra. Only after we add the auxiliary fields does the algebra closes.

The same thing happens in local supersymmetry (supergravity), but with one important difference. We might expect that P_μ is represented in global susy by the general coordinate transformations, but not quite. In fact, we find that the local algebra corresponding to the global $[\bar\epsilon_1 Q,\bar\epsilon_2 Q]=1/2\bar\epsilon_2\gamma^\mu\epsilon_1 P_\mu$ is

$$
\begin{aligned}
&[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]=\delta_{gen.coord.}(\xi^\mu)+\delta_{local\ Lorentz}(\xi^\mu\omega_\mu^{ab})+\delta_Q(-\xi^\mu\psi_\mu)\\
&\xi^\mu=\frac{1}{2}\bar\epsilon_2\gamma^\mu\epsilon_1 \qquad (18)
\end{aligned}
$$

which is the local version of the super-Poincare algebra. It cannot be derived from group theory alone (from the global algebra)!!

Both in 3 and 4d the local algebra does not close on the gravitino without auxiliary fields (we get extra equation of motion terms).

After the introduction of auxiliary fields, the algebra closes on all fields, but we need to add auxiliary field terms to the parameters of the general coordinate, local Lorentz and susy transformations on the r.h.s. of (18), and those terms depend on dimension (otherwise the algebra is the same in all dimensions), and on the particular set of auxiliary fields chosen.

In 3d, we saw that we only need one auxiliary scalar, S. It comes with the curved space auxiliary action

$$-\frac{e}{2}S^2 \tag{19}$$

In flat space this would be completely trivial, but now note that S couples to the vielbein! This will be very important later on, when one couples supergravity with matter.

One also needs to add to the susy rules

$$\begin{aligned} &\delta(S)\psi_\mu = cS\gamma_\mu\epsilon \\ &\delta S = -k\bar{\epsilon}\gamma^\mu\psi_\mu S - ce^{-1}\epsilon^{\mu\nu\rho}\bar{\epsilon}\gamma_\mu D_\nu(\omega)\psi_\rho \end{aligned} \tag{20}$$

and the algebra closes with the new local Lorentz parameter on the r.h.s. of (18)

$$\xi^\mu\omega_\mu^{ab} \to \xi^\mu\omega_\mu^{ab} + 4ck\bar{\epsilon}_2\gamma^{ab}\epsilon_a S \tag{21}$$

4 Superspace-general formalism (can be skipped)

4.1 Coset theory

Superspace can be understood as a coset space. To define a coset for a graded (super) Lie algebra $[T_a, T_b\} = f_{ab}{}^c T_c$ for a group G, we need a split $T_a = \{H_i, K_\alpha\}$, such that we have a reductive algebra, i.e.

$$[H_i, H_j] = f_{ij}{}^k H_k; \quad [H_i, K_\alpha\} = f_{i\alpha}{}^\beta K_\beta \tag{22}$$

(splitting into H and G/H) and where

$$[K_\alpha, K_\beta\} = f_{\alpha\beta}{}^i H_i + f_{\alpha\beta}{}^\gamma K_\gamma \tag{23}$$

and if $f_{\alpha\beta}{}^\gamma = 0$ we have a symmetric algebra (but not needed). Then by definition

$$e^{z^\alpha K_\alpha}h, \quad \forall h \tag{24}$$

is a coset element, and for h=1 we have a coset representative $L(z)$. Then z^α are coordinates on the coset space and we can also write $h = exp(y^i H_i)$ in general. A general group element g induces a motion on the coset:

$$ge^{z^\alpha K_\alpha} = e^{z'^\alpha K_\alpha}h(z,g) \tag{25}$$

With the definitions $x \cdot K = x^\mu K_\mu$ (coset motion) and $dx \cdot K = dx^m K_m$ (1-forms), we can act on the right with an infinitesimal (1-form) G/H element

$$e^{x\cdot K} e^{dx\cdot K} = e^{x\cdot K + dx^m e_m^\mu(x) K_\mu} e^{dx^m e_m^\mu(x)\omega_\mu^i(x) H_i} + o(dx)^2 \tag{26}$$

and on the left with an infinitesimal general group element

$$e^{dg^a T_a} e^{x\cdot K} = e^{x\cdot K + dg^a f_a^\mu(x) K_\mu} e^{-dg^a \Omega_a^i(x) H_i} + o(dg^2) \tag{27}$$

thus defining the (inverse) vielbein $e_m^\mu(x)$ on the coset, the H connection $\omega_\mu^i(x)$, the Lie vector f_a^μ and the H compensator Ω_a^i. Here μ is called a curved index and m is called a flat index. Equivalently, we may define the vielbein and H connection as

$$L^{-1}(z)\partial_\mu L(z) = e_\mu^m(x) K_m + \omega_\mu^i(x) H_i \tag{28}$$

One can derive the vielbein postulate ($De^m = 0$) and the flat (spin) connection on the coset manifold

$$\omega_\mu{}^m{}_n(x) = e_\mu^r(x)\omega_\mu{}^m{}_n(0) + \omega_\mu^i(x) f_{in}{}^m \tag{29}$$

One can also define the Lie derivative

$$l = dg^A f_A^\mu(x)\partial_\mu \tag{30}$$

One takes fields $\phi^a(x)$ in representations $D^a{}_b(h)$ of H. Defining

$$\omega_{\mu b}^a(x) = \omega_\mu^i(x)(H_i)^a{}_b \tag{31}$$

(H connection in that representation), we have the covariant derivative

$$D_m\phi^a(x) = e_m^\mu[\partial_\mu\phi^a(x) + \omega_{\mu b}^a\phi^b(x)] \tag{32}$$

and the variation of $\phi^a(x)$ under the infinitesimal group element dg^A is called H-covariant Lie derivative,

$$\delta\phi^a(x) \equiv \mathcal{L}_H\phi^a(x) = l\phi^a(x) + dg^A\Omega_A^i(x)(H_i)^a{}_b\phi^b(x) \tag{33}$$

Here $l\phi^a(x)$ is an orbital part (independent of the H representation of the field), and the rest is a spin part (depends on representation, and is =0 in an H-scalar representation, like for instance for the coset space coordinates).

The H-covariant Lie derivatives commute with the covariant derivatives

$$[D_m, \mathcal{L}_H] = 0 \tag{34}$$

One can also define the group-invariant integration measure on the coset

$$\int_M \mu x f(x) d^n x \tag{35}$$

from the Jacobian of the transformation $x \to x'$ on the coset to be

$$\mu(x) = (dete_\mu^m(x))\mu(0) \tag{36}$$

4.2 Rigid superspace

Superspace is the coset super-Poincare/Lorentz, that is, we write a general super-Poincare group element g as

$$e^{\xi^\mu P_\mu + \epsilon^A Q_A + \epsilon^{\dot{A}} Q_{\dot{A}}} e^{\lambda^{mn} M_{mn}} \tag{37}$$

and by acting with it on the coset representative

$$e^{x^\mu P_\mu + \theta^A Q_A + \bar{\theta}^{\dot{A}} Q_{\dot{A}}} \tag{38}$$

we find the action of super-Poincare on superspace $(x^\mu, \theta^A, \bar{\theta}^{\dot{A}})$. Note that in 3d, there are no dotted indices, and everything else follows.

Since for super-Poincare, $[K, K\}$ doesn't have any H components, we can check that $\omega_\Lambda^i = \Omega_a^i = 0$.

Then Lie derivatives are just the usual supercharges $Q_A, Q_{\dot{A}}$ and momenta P_μ and thus they satisfy the super Poincare algebra (i.e. form a representation).

$$\begin{aligned} l_A &= \partial_A + i\sigma^\mu_{A\dot{B}} \bar{\theta}^{\dot{B}} \partial_\mu \\ l_{\dot{A}} &= \partial_{\dot{A}} + i\sigma^\mu_{B\dot{A}} \bar{\theta}^B \partial_\mu; \quad P_\mu = i\partial_\mu \end{aligned} \tag{39}$$

Lie derivatives $l =$ H-covariant Lie derivatives $\mathcal{L}_H$ (since $\Omega_a^i = 0$), thus commute with covariant derivatives D_m.

Superfields are classified as before after the H representation, in this case their Lorentz group representation (i.e. spin), as scalar, vector superfields, etc.

Components are defined by acting with the covariant derivatives D_m, which as we saw commute with the supercharges= Lie derivatives.

Vielbein, curvature, measure, constraints

From the general coset formalism, we calculate the rigid superspace supervielbein. It is

$$E_M^\Lambda = \begin{pmatrix} \delta_m^\mu & 0 & 0 \\ -i\sigma^\mu_{A\dot{B}}\theta^{\dot{B}} & \delta_A^B & \\ -i\sigma^\mu_{B\dot{A}}\theta^B & & \delta_{\dot{A}}^{\dot{B}} \end{pmatrix} \tag{40}$$

We can then calculate the covariant derivatives $D_M = E_M^\Lambda(\partial_\Lambda + \omega_\Lambda^i T_i)$ and since $\omega_\Lambda^i = 0$ we get $D_M = E_M^\Lambda \partial_\Lambda$, i.e.

$$\begin{aligned} &D_A = \partial_A - i\sigma^\mu_{A\dot{B}}\bar{\theta}^{\dot{B}}\partial_\mu \\ &D_{\dot{A}} = \partial_{\dot{A}} - i\sigma^\mu_{B\dot{A}}\bar{\theta}^B\partial_\mu; \quad D_m = \partial_m \end{aligned} \tag{41}$$

Then we can use the general definition of curvature and torsion (for any space, or any gauge theory)

$$[D_m, D_n] = T^p_{mn}(x)D_p + R^i_{mn}(x)T_i \tag{42}$$

where

$$\begin{aligned} &R^i_{mn} = e^\mu_m e^\nu_n R^i_{\mu\nu} \\ &R^i_{\mu\nu} = \partial_\mu\omega^i_\nu - \partial_\nu\omega^i_\mu + f^i_{jk}\omega^j_\mu\omega^k_\nu \\ &T^o_{mn} = e^\mu_m(D_\mu e^\nu_n)e^p_\nu - (m \leftrightarrow n) \\ &D_\mu e^\nu_n = \partial_\mu e^\nu_n + \omega^i_\mu f^p_{ni} e^\nu_p \end{aligned} \tag{43}$$

When we apply this definition to the rigid superspace, we find that there is no curvature (bosonic or fermionic) and the only torsion is $T^m_{A\dot{B}}$, thus the only nontrivial commutator is

$$[D_A, D_{\dot{B}}] = T^m_{A\dot{B}} D_m \tag{44}$$

For the superspace coordinate transformation

$$\begin{pmatrix} x' \\ \theta' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} = M \begin{pmatrix} x \\ \theta \end{pmatrix} \tag{45}$$

we get the superjacobian

$$J = \frac{det(A - BD^{-1}C)}{det(D)} = sdet(M) \tag{46}$$

and then the superspace invariant integration measure is

$$\mu = sdet E_\Lambda^M \tag{47}$$

and in the case of rigid superspace it is trivial (=1).

One can impose susy-preserving constraints on the general superfields (in the general representations of the Lorentz-H group). As we saw, the covariant derivatives commute with the supercharges (Lie derivatives), thus we can impose constraints in terms of covariant derivatives, called *covariant constraints*.

The simplest example is the chiral constraint $\bar{D}_{\dot{A}}\phi = 0$, defining the chiral superfield and the other simple possibility is ϕ real and $D^A D_A \phi = 0$, giving the linear multiplet. One can analyze systematically all covariant constraints, by combining all possible derivatives and analyzing them in order of dimension (e.g. $[D_A] = 1/2$, $[D^A D_A] = 1$, etc.), thus finding all irreps. The YM multiplet can be defined instead of using constraints, by defining the field strength of a real superfield V by $W_A = \bar{D}^2 D_A V$.

Covariant formulation

We can also mimic what happens for YM theory in superspace. We start with a real superfield with a super-index $A_\Lambda^a(x,\theta)$ (super-connection) and then define super-covariant derivatives

$$\begin{aligned}
&\mathcal{D}_\Lambda = \partial_\Lambda + A_\Lambda^a T_a \\
&\mathcal{D}_M = E_M^\Lambda \mathcal{D}_\Lambda = E_M^\Lambda \mathcal{D}_\Lambda = E_M^\Lambda \partial_\Lambda + E_M^\Lambda A_\Lambda = D_M + A_M
\end{aligned} \tag{48}$$

The define torsion T_{MN}^P, spacetime curvature R_{MN}^{mn} and YM curvature F_{MN}^a by

$$\begin{aligned}
&[\mathcal{D}_M, \mathcal{D}_N\} = T_{MN}^P \mathcal{D}_P + \frac{1}{2} R_{MN}^{mn} M_{mn} + F_{MN}^a T_a \\
&\Rightarrow F_{MN}^a = D_M A_N - (-)^{MN}(M \leftrightarrow N) + [A_M, A_N\} - T_{MN}^P A_P
\end{aligned} \tag{49}$$

As before, the torsion is f_{MN}^P (structure constants), and the curvature is zero. We need to impose constraints on the YM curvature components to obtain usual YM theory.

The constraints that give usual YM are

- "Representation preserving constraints" $F_{\dot{A}\dot{B}} = 0, F_{AB} = 0$ (necessary)
- "Conventional constraints" (optional) $F_{A\dot{B}} = 0$ (analogy: in GR $\omega = \omega(e)$ is either equation of motion or constraint).

Instead of solving the constraints, one can solve the Bianchi identities, following from Jacobi identities on supercovariant derivatives (and using the definition of curvatures and torsion):

$$[\mathcal{D}_M, [\mathcal{D}_N, \mathcal{D}_P\}\} + ... = 0 \to [\mathcal{D}_M, (T_{NP}^S \mathcal{D}_S + F_{NP}^a T_a)\} + \text{supercyclic} = 0 \quad (50)$$

In all cases, we are back to the usual definition of the YM multiplet.

4.3 Local superspace

The definition of local superspace is more tricky, and there are in fact more versions of how to do it.

Coset approach to 3d sugra

We consider rigid superspace and want on top of it to gauge the super-Poincare Lie algebra ("YM theory of super-Poincare on rigid superspace"). Corresponding to the generators $T_I = \{P_\mu, Q_\alpha, M_{rs}\}$ we write "gauge fields" H_A^I where now actually the YM index I corresponds to the curved index $M = \{\mu, \alpha\}$ (α=fermionic) together with the Lorentz index (rs), whereas A is a flat index $\{m, a\}$ (a= fermionic).

Exactly as in the covariant YM formulation, one introduces new kinds of (gauge and super-) covariant derivatives, including the YM fields:

$$\nabla_A = D_A + H_A^I T_I \quad (51)$$

where D_A are the flat index rigid superspace covariant derivatives.

The generators T_I are represented by H-covariant Lie derivatives $\mathcal{L}_H$, which commute with D_M, thus we can find another basis for T_I, namely: $T_I = \{D_M, M_{rs}\}$. When we write this change of basis, we get a linear combination of components of H_A^I:

$$H_A^I T_I = h_A^M D_M + \frac{1}{2}\phi_A^{rs} M_{rs} \quad (52)$$

thus

$$\begin{aligned} \nabla_A &= D_A + h_A^M D_M + \frac{1}{2}\phi_A^{rs} M_{rs} \equiv E_A^M D_M + \frac{1}{2}\phi_A^{rs} M_{rs} \\ E_A^M &= \delta_A^M + h_A^M \end{aligned} \quad (53)$$

We again define torsion and curvature by the commutator of flat covariant derivatives, just that now there is no extra YM curvature, since the gauge group is the supergroup itself, thus

$$\{\nabla_A, \nabla_B\} = T_{AB}^C \nabla_C + \frac{1}{2} R_{AB}^{rs} M_{rs} \tag{54}$$

One then needs to impose constraints on torsion and curvature components in order to get the usual supergravity multiplet (by now there are too many fields). One imposes the *conventional constraints*

$$\begin{aligned} &\{\nabla_a, \nabla_b\} = 2i\nabla_{ab}; \quad (\nabla_{ab} \equiv (\gamma^m)_{ab}\nabla_m) \\ &\leftrightarrow T_{a,b}^{cd} = 2i\delta_a^{(c}\delta_b^{d)}; \;\; T_{a,b}^c = 0; \quad R_{a,b}^{rs} = 0 \end{aligned} \tag{55}$$

and

$$T_{a,bc}^{de} = 0 \tag{56}$$

By solving the first set of constraints and the Bianchi identities (that follow from the super-Jacobi identities for ∇_A's), one expresses everything in terms of E_a^M and ϕ_a^{rs}. By using the last constraint, one can also express ϕ_a^{rs} in terms of $E_a^M(x,\theta)$, which is thus the only independent superfield.

Then, we have superspace invariances: local Lorentz L^{rs}, which can be used to fix $E_a^\alpha = \delta_A^\alpha \psi$; fermonic super-Einstein k^α, used to put $E_a^{\alpha\beta}\delta_\alpha^a = 0$ ($(\alpha\beta)$ is a bosonic index expanded in fermions); bosonic super-Einstein $k^{\alpha\beta}$, which is

$$k^{\alpha\beta}(x,\theta) = \xi^{\alpha\beta}(x) + i\theta^{(\alpha}\epsilon^{\beta)} + i\theta_\gamma \eta^{(\gamma\alpha\beta)} + i\theta^2\xi^{\alpha\beta} \tag{57}$$

where the first component is general coordinate transformation, the second is local susy transformation, and the last two can be used to fix a WZ gauge in which

$$\begin{aligned} &\psi(x,\theta) = e_{m\mu}\delta^{\mu m} + i\theta^\alpha\gamma^\mu\psi_\mu(x) + i\theta^2 S \\ &E^{(a\alpha\beta)}(x,\theta) = \delta_{ab}^{\alpha\beta}(\theta_d h^{(abcd)} + i\theta^2\psi^{(abc)}) \end{aligned} \tag{58}$$

and the supergravity multiplet is: $\psi^{(abc)}$ is the gamma-traceless gravitino, $h^{(abcd)}$ is the symmetric vielbein, S is the auxiliary field, and h= trace part of symmetric vielbein, $\gamma^\mu\psi_\mu$ is the gamma-trace of the gravitino.

The local superspace invariant integration measure is, as one would expect, the same as for rigid superspace,

$$\int d^3x \; d^2\theta \; sdet E_M^A \tag{59}$$

(with the obvious generalization to any dimension)

One can find that the correct action is

$$\frac{1}{k^2}\int d^3xd^2\theta\ sdetE_M^A(R+\Lambda) \tag{60}$$

which gives supergravity with a cosmological constant.

Super-Geometric approach

Instead of using the coset approach for rigid superspace and putting YM fields in the covariant approach for the supergroup, one can mimic GR on superspace (not defined formally, i.e. like in the coset approach, just a space with x,θ coordinates), and write superfields for the vielbein and spin connection, with superindices.

Thus one has $E_\Lambda^M(x,\theta)$ and $\Omega_\Lambda^{MN}(x,\theta)$ and one restricts the number of components using invariances and physical input, and also adding constraints on torsion and curvatures constructed using covariant derivatives for the curved superspace.

$$[D_M, D_N\} = T^P_{MN}D_P + \frac{1}{2}R^{rs}_{MN}M_{rs} \tag{61}$$

I haven't seen this formulation for 3d, although I am sure it exists.

5 Superspace-bottom line

One can represent the rigid super-Poincare algebra in terms of

$$\begin{aligned} l_A &= \partial_A + i\sigma^\mu_{A\dot{B}}\bar{\theta}^{\dot{B}}\partial_\mu \\ l_{\dot{A}} &= \partial_{\dot{A}} + i\sigma^\mu_{B\dot{A}}\bar{\theta}^B\partial_\mu; \quad P_\mu = i\partial_\mu \end{aligned} \tag{62}$$

One can also define flat index covariant derivatives that commute with the above susy generators

$$\begin{aligned} D_A &= \partial_A - i\sigma^\mu_{A\dot{B}}\bar{\theta}^{\dot{B}}\partial_\mu \\ D_{\dot{A}} &= \partial_{\dot{A}} - i\sigma^\mu_{B\dot{A}}\bar{\theta}^B\partial_\mu; \quad D_m = \partial_m \end{aligned} \tag{63}$$

which are related to the curved index covariant derivatives D_Λ by the rigid supervielbein ($D_M = E_M^\Lambda\partial_\Lambda$)

$$E_M^\Lambda = \begin{pmatrix} \delta_m^\mu & 0 & 0 \\ -i\sigma^\mu_{A\dot{B}}\theta^{\dot{B}} & \delta_A^B & \\ -i\sigma^\mu_{B\dot{A}}\theta^B & & \delta_{\dot{A}}^{\dot{B}} \end{pmatrix} \tag{64}$$

One can also define curvature and torsion by the usual

$$[D_m, D_n] = T^p_{mn}(x) D_p + R^i_{mn}(x) T_i \tag{65}$$

and find that we have no curvature and only $T^m_{A\dot{B}}$. The superjacobian is

$$J = \frac{det(A - BD^{-1}C)}{det(D)} = sdet(M) \tag{66}$$

and then the superspace invariant integration measure is

$$\mu = sdet E^M_\Lambda \tag{67}$$

and in this case of rigid superspace it is trivial (=1). For local superspace, it will be the same, except then it will be nontrivial. For a YM theory, one can define YM and super- covariant derivatives

$$\mathcal{D}_\Lambda = \partial_\Lambda + A^a_\Lambda T_a; \quad \mathcal{D}_M = E^\Lambda_M \mathcal{D}_\Lambda = D_M + A_M \tag{68}$$

and imposing constraints $F_{AB} = 0, F_{\dot{A}\dot{B}} = 0, F_{A\dot{B}} = 0$ on their curvatures

$$[\mathcal{D}_M, \mathcal{D}_N\} = T^P_{MN} \mathcal{D}_P + \frac{1}{2} R^{mn}_{MN} M_{mn} + F^a_{MN} T_a \tag{69}$$

one can find the YM supermultiplet.

In the local superspace case, things are more complicated. One choice is one can write down (gauge and super-) covariant derivatives

$$\nabla_A = E^M_A D_M + \frac{1}{2} \phi^{rs}_A M_{rs} \tag{70}$$

where D_M is the curved index rigid super-covariant derivative, E^M_A is the inverse supervielbein. By imposing constraints on the torsion and curvatures derived from it

$$\{\nabla_A, \nabla_B\} = T^C_{AB} \nabla_C + \frac{1}{2} R^{rs}_{AB} M_{rs} \tag{71}$$

together with their Bianchi identities (Jacobi identities), one gets the usual supergravity. We have seen that in 3d, the constraints are $\{\nabla_A, \nabla_B\} = i\nabla_{AB}$ and $T^{de}_{a,bc} = 0$.

Another choice is the super-geometric approach. One defines superspace supervielbein $E^M_\Lambda(x, \theta)$ and superconnection $\Omega^{MN}_\Lambda(x)$ and constrains their components and the torsion and curvature derived from curved superspace covariant derivatives.

The latter procedure can be generalized to any dimension and for any types of fields. For instance, in 11d, one can write down a general superspace superfield 3-form $A_{\Lambda\Pi\Omega}$ also, and the constraints are also in terms of its curvature, $H = dA$.

6 4d off-shell and in superspace

Off-shell

As we saw, in order to match degrees of freedom off-shell and to close the algebra on the fields (specifically, on the gravitino), we need to introduce extra fields: scalar S and pseudoscalar P (together complex scalar $M = S + iP$) and axial vector A_μ.

One needs to add auxiliary terms to the susy rules, which become

$$\begin{aligned}
&(\delta e_\mu^m = \frac{k}{2}\bar{\epsilon}\gamma^m\psi_\mu)\\
&\delta\psi_\mu = (\frac{1}{k}D_\mu\epsilon) + \frac{i}{2}A_\mu\gamma_5\epsilon - \frac{1}{2}\gamma_\mu\eta\epsilon; \quad \eta \equiv -\frac{1}{3}(S - i\gamma_5 P - i\not{A}\gamma_5)\\
&\delta S = \frac{1}{4}\bar{\epsilon}\gamma\cdot R^{cov}\\
&\delta P = -\frac{i}{4}\bar{\epsilon}\gamma_5\gamma\cdot R^{cov}\\
&\delta A_\mu = \frac{3i}{4}\bar{\epsilon}\gamma_5(R_\mu^{cov} - \frac{1}{3}\gamma_\mu\gamma\cdot R^{cov})
\end{aligned} \tag{72}$$

where the symbol $R^{\mu,cov}$ is the gravitino field equation $R^\mu = \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma$, but with *supercovariant derivatives* (i.e. their variation doesn't have $\partial_\mu\epsilon$ terms)

$$R^{\mu,cov} = \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu(D_\rho\psi_\sigma - \frac{i}{2}A_\sigma\gamma_5\psi_\sigma + \frac{1}{2}\gamma_\sigma\eta\psi_\rho) \tag{73}$$

The Lagrangean is

$$\mathcal{L} = -\frac{e}{2}R(e,\omega) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma - \frac{e}{3}(S^2 + P^2 - A_\mu^2) \tag{74}$$

Again, note that if in flat space the auxiliary fields would not couple to anything, now they couple to the vielbein!

The introduction of auxiliary fields closes the algebra, and one just gets extra auxiliary field-dependent terms in the parameters of transformations. Specifically, one has

$$
\begin{aligned}
&[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{gen.coord.}(\xi^\alpha) + \delta_Q(-\xi^\alpha \psi_\alpha) \\
&+\delta_{local\ Lorentz}[\xi^\mu \hat{\omega}_\mu^{mn} + \frac{1}{3}\bar{\epsilon}_2 \sigma^{mn}(S - i\gamma_5 P)\epsilon_1] \\
&\hat{\omega}_{\mu,ab} = \omega_{\mu,ab} - \frac{i}{3}\epsilon_{\mu abc}A^c, \quad \xi^\mu = \bar{\epsilon}_2 \gamma^\mu \epsilon_1
\end{aligned}
\tag{75}
$$

Superspace

Indices: flat: M= m (bosonic), a (fermionic); curved: $\Lambda = \mu$ (bosonic), α (fermionic). Flat fermionic, a= $A, \dot{A}$.

Using as a starting point the super-geometric approach, one has a superpace supervielbein $E_\Lambda^M(x,\theta)$, on which one can act with super-Einstein transformations, defined by $\xi^\Lambda(x,\theta)$ and with super-local Lorentz $\Lambda^{MN}(x,\theta)$, but since we don't want to mix bosons and fermions, we have to restrict it to

$$
\Lambda^{MN} = \begin{pmatrix} \Lambda^{mn} & 0 & 0 \\ 0 & -\frac{1}{4}(\sigma_{mn})_{AB}\Lambda^{mn} & 0 \\ 0 & 0 & +\frac{1}{4}(\sigma_{mn})_{\dot{A}\dot{B}}\Lambda^{mn} \end{pmatrix}
\tag{76}
$$

We also have a super-spin connection $\Omega_\Lambda^{MN}(x,\theta)$, but since it should be a connection for the local Lorentz transformation, it has the same structure as Λ^{MN}, namely

$$
\Omega_\Lambda^{MN} = \begin{pmatrix} \Omega_\Lambda^{mn} & 0 & 0 \\ 0 & -\frac{1}{4}(\sigma_{mn})_{AB}\Omega_\Lambda^{mn} & 0 \\ 0 & 0 & +\frac{1}{4}(\sigma_{mn})_{\dot{A}\dot{B}}\Omega_\Lambda^{mn} \end{pmatrix}
\tag{77}
$$

One can then define (super-GR) covariant derivatives

$$
D_\Lambda = \partial_\Lambda + \frac{1}{2}\Omega_\Lambda^{mn} M_{mn}; \quad D_M = E_M^\Lambda D_\Lambda
\tag{78}
$$

and torsion and curvatures

$$
[D_N, D_M\} = T_{NM}^P D_P + \frac{1}{2}R_{NM}^{mn} M_{mn}
\tag{79}
$$

and correspondingly, Bianchi identities from the Jacobi identities of the covariant derivatives

$$
[D_M, [D_N, D_P\}\} + \text{supercyclic} = 0
\tag{80}
$$

Note that the covariant derivatives have the right limit for rigid superspace, when $E_M^\Lambda \to {E^{(0)}}_M^\Lambda$, $\Omega_\Lambda^{mn} = 0$.

One has to make a gauge choice in which

$$E_\mu^m(x,\theta=0) = e_\mu^m; \quad E_\mu^a(x,\theta=0) = \psi_\mu^a, \quad \Omega_\mu^{mn}(x,\theta=0) = \omega_\mu^{mn} \tag{81}$$

This is really a gauge choice and not a definition since the one has to check that these components transform in the right way.

Note that this is different from what we had in the 3d coset approach, where the independent components had flat fermionic index only (and curved bosonic and fermionic), so there is no general prescription for what components are of interest.

Still, we have much too many fields left, so one has to impose constraints.

The constraints are

$$T_{mn}^p = T_{am}^n = T_{ab}^c = T_{ab}^m + \frac{1}{4}(C\gamma)_{ab}^r = 0 \tag{82}$$

They can be split into: Conventional

$$\begin{aligned} T_{mn}^p &= T_{AB}^C = T_{A\dot{B}}^{\dot{C}} = 0 \\ T_{A\dot{B}}^m &+ \frac{i}{4}(\sigma^m)_{A\dot{B}} = T_{Am}^n(\bar{\sigma}_m^n)_{\dot{B}\dot{C}} = 0 \end{aligned} \tag{83}$$

Representation preserving

$$T_{AB}^{\dot{C}} = T_{AB}^m = 0 \tag{84}$$

and Super-conformal choice

$$T_{Am}^m = 0 \tag{85}$$

After solving the constraints and Bianchi identities we get that supertorsions and supercurvatures can be expressed in terms of 3 superfields,

$$R, \quad G_{A\dot{A}}; \quad W_{ABC} \tag{86}$$

where $G_{A\dot{A}}$ is Hermitean, R and W_{ABC} are chiral superfields ($D_{\dot{A}}R = D_{\dot{A}}W_{BCD} = 0$), W_{ABC} is totally symmetric and we also have

$$D^A G_{A\dot{A}} = \bar{D}_{\dot{A}}R^*; \quad D^A W_{ABC} = {D_B}^{\dot{E}} G_{C\dot{E}} + B \leftrightarrow C \tag{87}$$

The action is simply the superinvariant measure (unlike in 3d, for example),

$$S = \int d^4x d^4\theta \; sdet \; E_\Lambda^M \tag{88}$$

Its variation (on the constraints) is given by

$$\delta S = \int d^4x d^4\theta \; sdet \; E_\Lambda^M \; [v^m G_m - RU - R^* U^*] \tag{89}$$

where v^m and U are arbtrary superfields, thus the field equations are

$$G_m = R = 0 \tag{90}$$

and these encode the equations of motion of the off-shell supergravity. In particular, the auxiliary fields are given by

$$R(x, \theta = 0) = S + iP \equiv M; \quad G_m(x, \theta = 0) = A_m^{aux} \tag{91}$$

We can make the above formalism look more like we expect from GR and rigid superspace, but we will do it next section.

7 Superspace actions and coupling supergravity to matter

The simplest expression for matter actions is in superspace. We saw that covariant derivatives have the right rigid superspace limit, so we can define chiral fields in the same way, by $\bar{D}_{\dot{A}} \Phi = 0$. We have already seen that R and $G_{A\dot{A}}$ are chiral. All the formulas of rigid superspace follow, remembering to use the new covariant derivatives. We get component fields for ϕ by acting with the covariant derivatives.

We note that now the superspace has to have flat spinor indices (or we can make a transformation to flat spinor indices).

We have again

$$\Phi = \Phi(y, \theta) = \phi(y) + \sqrt{2}\psi(y) + \theta^2 F(y); \quad y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \tag{92}$$

thus

$$\phi(x) = \Phi|_{\theta=0}; \;\; \psi(x) = \frac{D_A \Phi|_{\theta=0}}{\sqrt{2}}; \;\; F(x) = D^2\Phi|_{\theta=0} \tag{93}$$

One can also define YM superfields. As usual, we start with a real gauge superfield $V = V^+$, that has a gauge transformation (for the nonabelian case)

$$e^V \to e^{\bar{\Lambda}} e^V e^{-\Lambda} \tag{94}$$

with Λ chiral $\bar{D}_{\dot{A}}\Lambda = 0$. The only nontrivial thing is the appearance of R in the definition of the gauge invariant field strength:

$$W_A = (\bar{D}^2 - \frac{1}{3}R)e^{-V} D_A e^V \tag{95}$$

since we now have

$$\int d^4x d^2\bar{\theta} = \int (\bar{D}^2 - \frac{1}{3}R)|_{\theta=0} \tag{96}$$

In order to write lagrangians, we need to find invariant measures. We have already found the invariant measure for the total superspace. Thus the Kahler potential in curved space is

$$\int d^4x d^4\theta \; sdet \; E_\Lambda^M K(\Phi, \Phi^+) \tag{97}$$

However, things are clearer in chiral superspace. We need to find the chiral measure, i.e. a chiral curved space-invariant density, which we will call $\mathcal{E}$, with $\bar{D}_{\dot{A}}\mathcal{E} = 0$. It is found to be

$$\mathcal{E} = e[1 + i\theta\sigma^m \bar{\psi}_m - \theta^2(M^* + \bar{\psi}_m \bar{\sigma}^{mn} \bar{\psi}_n)] \tag{98}$$

Thus the superpotential term is written as

$$\int d^4x d^2\theta \mathcal{E} W(\Phi) \tag{99}$$

The superpotential is a chiral superfield, and thus can always be written as $d^2\theta U = (\bar{D}^2 - \frac{1}{3}R)U$, with U a general superfield. We see that if we choose U=1, we get

$$\int d^4x d^2\theta \; \mathcal{E} R = \int d^4x d^2\theta \mathcal{E} d^2\bar{\theta} \tag{100}$$

and we can figure out that the last expression has to be an invariant measure, thus equals the Einstein action

$$-3 \int d^4x d^4\theta \; sdet \; E_\Lambda^M \tag{101}$$

We can check this also explicitly, as we can find that

$$R = M + \theta(\sigma^m \bar{\sigma}^n \psi_{mn} - i\sigma^m \bar{\psi}_m M + i\psi_m A^m) + \theta^2[-\frac{1}{2}R_{mn i}{}^{mn}\bar{\psi}^m \sigma^n \psi_{mn}$$
$$+\frac{2}{3}MM^* + \frac{A_m^2}{3} - ie_m^\mu D_\mu A^m + \frac{1}{2}\bar{\psi}\bar{\psi}M - \frac{1}{2}\psi_m \sigma^m \bar{\psi}_n \sigma^n$$
$$+\frac{1}{8}\epsilon^{mnpq}(\bar{\psi}_m \bar{\sigma}_n \bar{\psi}_{pq})] \tag{102}$$

Thus the most general Lagrangean is

$$S = \int d^4x d^4\theta \; sdet \; E_\Lambda^M [K(\Phi, \Phi^+) + \Phi^+ e^V \Phi]$$
$$+ \int d^4x d^2\theta \mathcal{E}[W(\Phi) + Tr W^A W_A] + h.c.$$
$$= \int d^4x d^2\theta \; \mathcal{E}[\bar{D}^2 - \frac{1}{3}R][K(\Phi, \Phi^+) + \Phi^+ e^V \Phi]$$
$$+ \int d^4x d^2\theta \mathcal{E}[W(\Phi) + Tr W^A W_A] + h.c. \tag{103}$$

Unlike in flat space, now a constant term in the superpotential is nontrivial, since it couples to supergravity. Also, the Kahler potential for a neutral scalar starts in flat space with $\Phi^+\Phi$, but now we can have also $a + c\Phi^+$, and it is still nontrivial. In fact, there should be a term with a, as it contains just the Einstein action, as we have mentioned.

Note that with this Lagrangean, we get the supergravity action multiplied by $a + \phi^+\phi$ for a usual scalar (the $\theta = 0$ component of the Kahler potential), thus we are in a Brans-Dicke parametrization:

$$-\frac{1}{3}\int d^4x d^2\theta \; \mathcal{E} R(a + \phi^+(x)\phi(x)) \tag{104}$$

or in general (if we extract the constant term out of the Kahler potential)

$$-\frac{1}{3}\int d^4x d^2\theta \; \mathcal{E} R(a + K(\phi(x), \phi^+(x))) \tag{105}$$

The choice $a = -3$ gives the usual Einstein action. We redefine K

$$1 - \frac{1}{3}K = e^{-k/3} \tag{106}$$

where k is the modified Kahler potential, and this expression multiplies the off-shell supergravity action. We must perform a Weyl rescaling on the x-space action to get to the Einstein frame and eliminate $e^{-k/3}$.

The scalar potential before the rescaling would be again given by the auxiliary fields, just that now we have also sugra auxiliary fields, multiplied by $e^{-k/3}$:

$$g^2V = \sum_i |F_i|^2 - \frac{g^2}{2}D^aD^a - \frac{1}{3}(|M|^2 + A_m^2)e^{-k/3} \tag{107}$$

but since the θ^2 component of $\mathcal{E}$ is M^++fermions, we get a $M\ W(\phi)$ coupling, as well as a $-M\phi/3dW(\phi)/d\phi$ coupling. Also from the Kahler potential, we get a coupling $\partial K/\partial\phi_i F_i M$, thus

$$M \sim \phi\frac{dW(\Phi)}{d\phi} - 3W(\Phi) \tag{108}$$

and F_i also get modified. The actual formula is quite complicated, and after the Weyl rescaling we get something simpler:

$$\begin{aligned} V = e^k[\sum_{ij} g^{-1}_{i\bar{j}}(\frac{\partial W}{\partial\phi_i} + W\frac{\partial k}{\partial\phi_i})(\frac{\partial W}{\partial\phi_j} + W\frac{\partial k}{\partial\phi_j})^* - 3|W|^2] \\ +\frac{1}{2}f^{-1}_{AB}(\frac{\partial k}{\partial\phi_i}(T_A)_{ij}\phi_j)(\frac{\partial k}{\partial\phi_k}(T_B)_{kl}\phi_l)^* \end{aligned} \tag{109}$$

where

$$g_{i\bar{j}} = \frac{\partial^2 k}{\partial\phi_i\partial\phi_j^*} \tag{110}$$

is the metric on scalar field (ϕ_i) space (as well as on the space of their fermionic partners), thus the kinetic action is

$$\int \sqrt{g}[g_{i\bar{j}}D_\mu\phi^i(D^\mu\phi)^{*\bar{j}} + g_{i\bar{j}}\psi^i D\!\!\!/\bar{\psi}^{\bar{j}}] \tag{111}$$

and f_{AB} is a scalar-dependent metric on YM space (the kinetic term is $-1/4Re[f_{AB}F^A_{\mu\nu}F^{B\mu\nu})$, thus the second term in V comes from the D terms.

One usually defines the "covariant derivative"

$$D_i = \frac{\partial}{\partial\phi_i} + \frac{\partial k}{\partial\phi_i} \tag{112}$$

and thus in the absence of D terms one writes for the scalar potential

$$V = e^k[\sum_{ij} g^{-1}_{i\bar{j}} D_i W (D_j W)^* - 3|W|^2] \tag{113}$$

and the gaugino action (in the presence of D terms) is

$$-\frac{1}{2}Re[f_{AB}\bar{\lambda}^A D\!\!\!/ \lambda^B] + \frac{1}{2}e^{k/2} Re\sum_{ij} g^{-1}_{i\bar{j}} D_i W (\frac{\partial f_{AB}}{\partial \phi_i})^*(\bar{\lambda}^A \lambda^B) \tag{114}$$

8 General dimensions and KK

8.1 On shell

In 4d, $\mathcal{N}=2$ supergravity is obtained by coupling the $(2, 3/2)$ supergravity multiplet to a $(3/2, 1)$ multiplet (gravitino and abelian vector field). The number of gravitinos will correspond in general to the number of supersymmetries (as each one maps the graviton into a different gravitino).

If one minimally couples the gravitinos to the abelian gauge field, obtaining what is called a *gauged supergravity*, one is forced to introduce a cosmological constant term. One finds the new gravitino law

$$\delta\psi^i_\mu = D_\mu(\omega(e,\psi))\epsilon^i + g\gamma_\mu\epsilon^i + g\epsilon^{ij}A_\mu\epsilon^j \tag{115}$$

and in general, the cosmological constant manifests itself by the nonlinear susy term in the gravitino law, $g\gamma_\mu\epsilon^i$, with g being the cosmological constant also (thus having AdS supergravity).

The $\mathcal{N}=3$ model couples the multiplets (2, 3/2), 2(3/2, 1), (1, 1/2), thus the field content is $e^a_\mu, \psi^i_\mu, A^i_\mu, \lambda$ (i=1,3).

Again, coupling gravitinos minimally to abelian gauge fields to obtain a *gauged supergravity* implies a cosmological constant (thus having AdS supergravity), but moreover, one also makes the gauge fields nonabelian.

The $\mathcal{N}=4$ model also has scalar fields, for a field content (e^a_μ, ψ^i_μ, A^k_μ, B^k_μ, λ^i, ϕ, B), where A_μ are vectors, B_μ are axial vectors, ϕ is a scalar and B a pseudoscalar. It can be derived by dimensional reduction from the $\mathcal{N}=1$ supergravity in d=10. It also can be gauged, and again the gauged model is nonabelian and AdS.

Finally, one has the maximal $\mathcal{N}=8$ supergravity model, that can be derived from reduction from the $\mathcal{N}=2$ model in d=10 or from the unique

$\mathcal{N} = 1$ model in 11d. It also can be gauged, obtaining a nonabelian AdS model.

By requiring that we have at most spin 2, (higher spins are not coupled consistently to gravity), we can have at most the $\mathcal{N} = 8$ model, which can be obtained from the unique d=11 supegravity, which has thus a lot of appeal. Indeed, M theory is supposed to be 11d sugra at low energies.

One can find a lot of supergravities in many dimensions, but all can be obtained by dimensional reductions of the 11d supergravity and the IIB model in d=10. For ungauged models, that has generally been proven, as one needs to compactify on tori (usual dimensional reduction). To obtain gauged models, one needs to compactify on nontrivial spaces, so that was done case by case (and there are cases that have not been solved).

The field content of 11d supergravity is graviton, gravitino and 3-form, $e^a_\mu, \psi_\mu, A_{\mu\nu\rho}$, and the only nontrivial bosonic term is a CS coupling, ϵFFA.

8.2 Off-shell and superspace

The set of auxiliary fields is not known in general, except in a few cases, like the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ models.

One does know however in a few cases a superspace formulation that gives, when imposing constraints (and maybe Bianchi identities), the on-shell supergravity (i.e. its equations of motion).

Most notably, this is the case for the d=11 $\mathcal{N} = 1$ supergravity. Brink and Howe and independently Cremmer and Ferrara have found that one can have a superspace formulation in terms of a supervielbein E_Λ^M and super-spin connection Ω_Λ^{MN} (again, with only Ω_Λ^{mn} independent, for local Lorentz invariance). But one finds that one also needs a super-3 form $A_{\Lambda\Sigma\Omega}$. Thus in general, one needs more superspace fields than the supervielbein and super-connection, and what kind of fields needs to be treated case by case.

For the 11d sugra, one defines super-torsion and curvature in the usual way (as in 4d), and also $H = dA$ as the curvature of $A = dz^\Lambda dz^\Sigma dz^\Omega A_{\Lambda\Sigma\Omega}$ ($dz^\Lambda = (dx^\mu, d\theta)$). Then one defines (as for torsion and curvature) flat index curvature by multiplying with inverse supervielbeins (here $E^M = dz^\Lambda E_\Lambda^M$):

$$H = E^M E^N E^P E^Q H_{MNPQ} \tag{116}$$

One can write down a set of constraints on torsion, curvature and H_{MNPQ} that, together with the usual Bianchi identities and the Bianchi identity of H, $dH = 0$, imply the 11d sugra equations of motion.

8.3 KK reduction

Metrics and generalities

Kaluza-Klein reduction (KK) treats the case of a space that breaks into a noncompact space M times a compact space S. There are 3 metrics that one sometimes calls *KK metrics*. The first one is the *KK background metric*, that has to be a solution of the supergravity theory. Generically, it is (in the case $M \times S$)

$$g_{\Lambda\Sigma} = \begin{pmatrix} g^{(0)}_{\mu\nu}(x) & 0 \\ 0 & g^{(0)}_{mn}(y) \end{pmatrix} \tag{117}$$

where x are coordinates on the noncompact space M and y are coordinates on the compact space S.

The second one is the equivalent of Fourier decomposition on a circle or spherical harmonic decomposition on S_2. It is the *KK expansion*. Every field in the theory, including every component of the metric is expanded in the complet set of (generalized) *spherical harmonics* on the compact space S, called $Y^I_n(y)$ (we have put explicitly an index n for the degree of the spherical harmonic, e.g. l on S_2, although it is implicit in the representation index I). Thus generically, we write for a field $\phi(x,y)$

$$\phi(x,y) = \sum_n \phi^I_n(x) Y^I_n(y) \tag{118}$$

The third one is the *KK (reduction) ansatz* metric and corresponds to having the dimensional reduction ansatz of keeping only the fields in the first representation (n=1), what we would call independent of the coordinate y on the compact space (although generically, there will be the given y dependence of the first spherical harmonic $Y_1(y)$. Also generically, it is not necessarilly the first expansion element that is kept for all fields, some might keep only n=2, say). Thus generically, this would be

$$\phi(x,y) = \phi_1(x) Y_1(y) \tag{119}$$

Thus let's see what happens for a KK reduction on a p-torus T^p (i.e. on a background $M \times T^p$). The KK expansion of the metric would be

$$g_{\Lambda\Sigma} = \begin{pmatrix} g_{\mu\nu}(x,y) = g^{(0)}_{\mu\nu}(x) + \sum_n h^{(n)}_{\mu\nu}(x) e^{in\frac{y}{R}}; g_{\mu m}(x,y) = \sum_n B^{m,(n)}_{\mu}(x) e^{in\frac{y}{R}} \\ g_{\mu m}(x,y) \qquad\qquad g_{mn}(x,y) = \delta_{mn} + \sum_q h^{(q)}_{mn}(x) e^{iq\frac{y}{R}}(y) \end{pmatrix} \tag{120}$$

In this case the spherical harmonics for different components are the same (just Fourier exponentials), but in general they are not. Also, in general, there would be a nontrivial background metric for the compact metric (but a torus is flat).

The KK reduction ansatz is then

$$g_{\Lambda\Sigma} = \begin{pmatrix} g_{\mu\nu}(x) = g^{(0)}_{\mu\nu}(x) + h^{(0)}_{\mu\nu}(x) & g_{\mu m}(x) = B^{m,(0)}_{\mu}(x) \\ g_{\mu m}(x) & g_{mn}(x) = \delta_{mn} + h^{(0)}_{mn}(x) \end{pmatrix} \tag{121}$$

And we can see the general behaviour: $g_{\mu\nu}(x)$ will give the metric, $g_{\mu m}$ gives vectors, and g_{mn} gives scalars in the lower dimension. Similarly, for a vector A_μ will still be a vector, but A_m will be scalars, and similarly for antisymmetric tensors, etc. A $M \times S$ spinor will split into many spinors in the lower dimension (M).

Consistent truncation and nonlinear ansatz

We should note that whereas the KK expansion is always valid (it being the result of a generalized Fourier theorem- that is, the spherical harmonics form a complet set over the compact space), the KK reduction ansatz is not in general, with the exception of the torus reduction. It is in general valid only at the *linearized level*.

Indeed, making a truncation to the lowest modes must be a solution to the equations of motion, i.e. it must be a *consistent truncation*.

To see what can go wrong, note that we can have a term in the KK expanded action of a ϕ^3 coupling, of the type

$$(...)\int d^dx\sqrt{detg^{(0)}_{\mu\nu}}\phi_n^{I_n}\phi_0^{I_0}\phi_0^{J_0}(x)\int d^{(D-d)}y\sqrt{detg^{(0)}_{mn}}Y^{I_n}Y^{I_0}Y^{J_0}(y) \tag{122}$$

If the y integral is nonzero, the equation of motion for ϕ_n will be like

$$(\Box + ...)\phi_n^{I_n}(x) = (...)\phi_0^{I_0}\phi_0^{J_0}(x) \tag{123}$$

and we see that it's *inconsistent*, that is, not a solution of the equations of motion, to put the mode ϕ_n to zero and keep only the lowest mode ϕ_0.

For the torus reduction, this is not a problem, since then $Y^{I_0}(y) = 1$ and $\int dyY^{I_n} = \int dye^{iny/R} = 0$, so there are no nonzero couplings as the one above.

A generalization of this case is if we have a global symmetry under a group G and by the reduction ansatz we keep ALL the singlets under G. Then

$\phi_0^{I_0}$ is a singlet and so is $\phi_0^{I_0}\phi_0^{J_0}$, but $\phi_n^{I_n}$ is not, and by spherical harmonic orthogonality the integral of their product is zero.

We also see what we could do to make the KK truncation consistent. We have to make *nonlinear redefinitions* of fields, of the type

$$\begin{aligned} \phi_n' &= \phi_n + a\phi_0^2 + ... \\ \phi_o' &= \phi_0 + \sum_{mn} c_{mn}\phi_m\phi_n + ... \end{aligned} \quad (124)$$

This corresponds to making from the begining a *nonlinear KK ansatz*, which we have to remember that comes from the KK expansion only after a nonlinear redefinition.

The simplest example of *nonlinear KK ansatz* is the fact that in order to get the correct d-dimensional Einstein action (in Einstein frame) from the D-dimensional we need to rescale the metric as follows.

$$g_{\mu\nu}(x,y) = g_{\mu\nu}(x)\left[\frac{det\ g_{mn}(x,y)}{det\ g_{mn}^{(0)}(y)}\right]^{-\frac{1}{d-2}} \quad (125)$$

We can easily check that then we don't get extra factors of the compact metric in front of the d-dimensional Einstein action.

Original Kaluza-Klein

As an example, let us look at the original Kaluza-Klein reduction, of 5d gravity to 4d (on a circle). The *linearized KK reduction ansatz* is

$$g_{\Lambda\Sigma} = \begin{pmatrix} g_{\mu\nu}(x) & B_\mu(x) \\ B_\mu(x) & \phi(x) \end{pmatrix} \quad (126)$$

where $g_{\mu\nu}(x)$ is the 4d metric, $B_\mu(x)$ is a vector ("electromagnetism" were hoping K&K), and ϕ is a scalar. Incidentally, note that if we put the scalar $\phi = 1$ as K&K wanted, we get an inconsistent ansatz! A fact noticed by K&K, who noted that one needs a scalar. As it is, the ansatz is still inconsistent, but now we can make a nonlinear redefinition of fields, or equivalently, write from the begining the *consistent nonlinear KK ansatz*

$$g_{\Lambda\Sigma} = \begin{pmatrix} g_{\mu\nu}(x)\phi^{-1/2}(x) & B_\mu(x)\phi(x) \\ B_\mu(x)\phi(x) & \phi(x) \end{pmatrix} = \bar{\phi}^{-1/3}(x)\begin{pmatrix} g_{\mu\nu}(x) & B_{\mu\nu}(x)\bar{\phi}(x) \\ B_{\mu\nu}(x)\bar{\phi}(x) & \bar{\phi}(x) \end{pmatrix} \quad (127)$$

For general compact spaces, the linearized KK ansatz for the off-diagonal metric is

$$g_{\mu m}(x,y) = B_\mu^{AB}(x) V_m^{AB}(y) \tag{128}$$

where $V_m^{AB}(y)$ are the Killing vectors of the space (corresponding to isometries). Thus for avery independent Killing vector, we get a vector field.

Vielbein and spinors. Killing spinors

All of this was in terms of the metric, but we saw that one uses the vielbein in supergravity. Let us use α, a for flat indices on the noncompact and compact spaces, respectively. Then one can fix the off-diagonal part of the local Lorentz group (from $SO(D-1,1)$ to $SO(d-1,1)\times SO(D-d)$) by imposing the gauge choice $E_m^\alpha = 0$. Then one has the nonlinear KK ansatz

$$\begin{aligned} E_\mu^\alpha(x,y) &= e_\mu^\alpha(x)\left[\frac{detE_m^a(x,y)}{dete_m^{(0)a}(y)}\right]^{-\frac{1}{d-2}} \\ E_\mu^a(x,y) &= B_\mu^m(x) E_m^a(x,y); \quad B_\mu^m(x,y) = B_\mu^{AB}(x) V_m^{AB}(y) \end{aligned} \tag{129}$$

and the nonlinear ansatz for E_m^a has to be determined separately (case by case).

Spin 1/2 fermions on a torus just split up into many d dimensional fermions. The D-dimensional fermionic index A splits up into the d dimensional M and the compact space i, so that the linearized KK ansatz is of the type $\lambda_A(x,y) = \lambda_M^i(x)$. In general, on a product space, we will have a product of spinors:

$$\lambda_A(x,y) = \lambda_M(x)^I \eta_i^I(y) \tag{130}$$

where I counts the number of compact space spinors (their representation). Howver since both λ_A and λ_M need to be anticommuting, we need to take $\eta_i^I(y)$ to be a *commuting spinor*.

On a space with symmetries, $\eta_i^I(y)$ are so-called *Killing spinors*. The Killing spinors are the "square root" of Killing vectors in the following sense. A Killing vector satisfies the Killing equation

$$D_{(\mu} V_{\nu)}^{AB} = 0 \tag{131}$$

(in the case of gravity theories, we always mean the equation in the background metric), whereas a Killing spinor satisfies the equation

$$D_\mu \eta_i^I = c e_\mu^\alpha \gamma_\alpha \eta_i^I \tag{132}$$

On a sphere, the two are related by (where we have been a bit cavalier about the index conventions)

$$V_\mu^{AB} = \bar{\eta}^I \gamma_\mu \eta^J (\gamma^{AB})_{IJ} \tag{133}$$

Notice that $D_\mu \eta$ is the susy variation of the $\mathcal{N} = 1$ supergravity vielbein.

In more general gravitational theories (supergravity) on more general spaces, one defines therefore Killing spinors as spinors that preserve some susy, meaning that one takes $\delta_{susy}\lambda_A(x, y) = 0$, which in general will imply for the compact space η^I something of the type

$$D_\mu \eta^I = (\text{fields} \quad \times \quad \text{gamma} \quad \text{matrices})_\mu|_{background} \eta^I \tag{134}$$

and one still makes the ansatz

$$\lambda_A(x, y) = \lambda_M(x)^I \eta_i^I(y) \tag{135}$$

but one keeps only as many spinors as Killing spinors: since they preserve susy, the d-dimensional spinors they multiply will be massless, whereas the other spinors will become massive in the limit, so for the KK reduction one keeps only the massless modes.

From the Killing spinors one can build up all the other "massless spherical harmonics", Killing vectors, scalars, conformal Killing vectors, and vector-spinor (for the gravitino), thus completing the linear KK reduction.

Symmetries

When one dimensionally reduces supergravity on a q-torus T^q, one gets many scalars (from D-dimensional scalar ϕ_I, from the compact metric g_{mn}, compact vectors A_m^I, compact antisymmetric tensors, e.g. A_{mn}), many vectors (from vectors A_μ^I, off-diagonal metric $g_{\mu m}$, off-diagonal antisymmetric tensors $A_{\mu m}$, etc.), and so on. Usually the fields of same spin (even if coming from different D dimensional fields) combine to form multiplets of some global symmetry group G (there doesn't seem to be a systematic of which group, basically is trial and error, counting the number of fields in each spin and trying to arrange them in multiplets of some group).

When one dimensionally reduces on a nontrivial compact space instead, e.g. a sphere S_q, the torus abelian Killing spinors turn into a nonabelian Killing spinors (satisfying nonabelian commutation relations), and so the abelian vector fields transforming in a global symmetry group G turn into nonabelian fields of part or all of the group G, thus *gauging the global symmetry*, getting a *gauged supergravity*.

To get the gauged supergravity, one generically needs however to use a *nonlinear KK ansatz*, in order to have a *consistent truncation.*

9 Calabi-Yau compactifications and special geometry; flux compactifications

Kahler and Calabi-Yau

If we compactify 10d string theory on a space K_6, we would like to preserve $\mathcal{N}=1$ susy in 4d. The holonomy group, defined as the group formed by paralel transport of a spinor along closed paths in a n-dimensional space, is an element of SO(n).

If we have $\mathcal{N}=1$ susy in 4d, and *if we don't have any fluxes*, the Killing spinor equation will be just the condition for covariantly constant spinor

$$D_\mu \eta = 0 \tag{136}$$

(for instance, for compactification on a sphere, the γ_μ term on the r.h.s. of the Killing spinor equation comes from a constant flux in the susy law, that is needed to keep the curvature of the sphere).

But if there is one (and only one) covariantly constant spinor, the holonomy of a space is SU(n/2).

A complex manifold is a manifold over which we can find an *almost complex structure* $J^i{}_j$ that obeys $J^2=-1$ and can be diagonalized at any point over complex numbers. If when we diagonalize J at a point we have the Nijenhuis tensor =0 (analog to having the Riemann tensor =0 when we put the metric to 1), we have a complex manifold, with coordinates z^i (and $\bar{z}^{\bar{j}}$).

If a manifolld has $U(N)$ (or subgroups) holonomy, with N=n/2, then it is called a Kahler manifold , and then $J^i{}_j$ is covariantly constant (which is an equivalent definition of the Kahler manifold). On a Kahler manifold we can always find locally a *Kahler potential* such that the metric is

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{j}}} \tag{137}$$

The spin connection on a Kahler manifold is a $U(N) \sim SU(N) \times U(1)$ gauge field. If the holonomy is SU(N), the U(1) part is topologically trivial (pure gauge, with zero YM curvature), or equivalently, the *first Chern class* $c_1(K)=0$.

Conversely, Calabi and Yau proved that for a Kahler manifold of $c_1(K) = 0$ there exists a unique Kahler metric of SU(N) holonomy. We saw that means that there is a covariantly constant spinor, $D_i\eta = 0$, and from $[D_i, D_j]\eta = 0$ one gets that the manifold is also Ricci-flat, $R_{i\bar{j}} = 0$, thus obeying both the susy and Einstein equations needed for a good compactification to 4d. It is called a Calabi-Yau space.

Note now that the general chiral multiplets in 4d also have a Kahler potential, and it is not a coincidence. Indeed, the general $\mathcal{N} = 1$ chiral multiplet model has a *Kahler metric on* the scalar field space. The scalar fields in $\mathcal{N} = 2$ vector multiplets have a metric that is of a type called *special Kahler*, sometimes called **special geometry**. And the scalar fields in $\mathcal{N} = 2$ hypermultiplets have a *quaternionic manifold.*

When we compactify string theory on a Calabi-Yau (or any other space for that matter), the low energy theory has (massless) scalars that correspond to deformations of the compact space (not to the coordinates of the space!!), called *moduli.* Thus the *moduli space* of Calabi-Yau compactifications is also Kahler (since it preserves $\mathcal{N} = 1$ susy, which implies a Kahler potential for the scalars), just like the Calabi-Yau itself!

We know however that in 4d we can't break $\mathcal{N} = 2$ susy, so for phenomenology we need $\mathcal{N} = 1$, so why study it? However, a startling result found in the early 90's (see e.g. the Strominger paper in 1990 that defined special geometry rigorously) is that in $\mathcal{N} = 1$ *string compactifications* we still have the $\mathcal{N} = 2$ structure of special geometry on the vector moduli= scalars in the low energy vector multiplets (but no quaternionic structure for the hypermultiplet scalars). This was found first for heterotic compactifications that were preserving $\mathcal{N} = 2$ susy on the CY, and then extended to type II compactifications through dualities.

So the geometry of *moduli space of Calabi-Yau's* is actually not only Kahler, but *special Kahler*, or *special geometry.*

Before we study that though, we will say a few things about topology.

Topology

Betti numbers. On a real manifold one can define p-forms (antisymmetric tensors) and the differential operator d acting on them. Exact forms: $\psi = d\phi$. Closed: $d\psi = 0$. Cohomology, as usual= equivalence classes of closed forms, modulo exact forms: for p-forms, we have the p-th cohomology group, $H^p(M; R)$, whose dimension is called the p-th Betti number b_p (the number of linearly independent p-forms on the manifold that are closed, but not exact). The number b_p of M is also= the number of independent closed p-dimensional surfaces on M that are topologically nontrivial, and is = the number of linearly independent harmonic p-forms. This real cohomology is called *de Rham cohomology.*

On a compactification on $M_4 \times K_6$, a p-form with n indeices in M_4 and p-n in K_6 will be temporarily called a $(n, p-n)$ form (not to be confused with the complex (p,q) forms below). The number of zero eigenvalues of the laplacean on K_6, Δ_K, on such forms is $b_{p-n}(K)$ and is= the number of massless n-forms in M_4.

For fermions, the *index of the (gauged) Dirac operator*,

$$index D_K\!\!\!/ = n_+ - n_- \tag{138}$$

is the number of positive chirality zero eigenvalues- the number of negative chirality eigenvalues and is a topological invariant (nonzero eigenvalues cancel). The Atiyah-Singer theorem relates it to other topological invariants. On a 6d manifold K_6, we have (for spinors in the Q representation)

$$index_Q D_K\!\!\!/ = \frac{1}{48(2\pi)^3}\int_K [tr_Q F\wedge F\wedge F - \frac{1}{8} tr_Q F\wedge trR\wedge R] \tag{139}$$

Hodge numbers On Kahler manifolds, one defines *Dolbeault cohomology*, where the differential operator is $\bar\partial$, and cohomology is defined as closed ($\bar\partial\psi = 0$), modulo exact ($\psi = \bar\partial\phi$) forms again. But now we have z^i and $\bar z^{\bar j}$ coordinates, and correspondingly we have (p,q)-forms, thus we have the cohomology groups $H^{(p,q)}(K)$, of dimensions $h^{p,q}$, called *Hodge numbers.* The Hodge decomposition theorem states that

$$H^n_D = \oplus_{p+q=n} H^{p,q} \Rightarrow b_n = \sum_{p+q=n} h^{p,q} \tag{140}$$

The Euler characteristic of a manifold is given by

$$\chi = \sum_n (-1)^n b_n = \sum_{p,q} (-1)^{p+q} h^{p,q} \tag{141}$$

Relations:

$$\begin{aligned} &b_p = b_{n-p}(real\ dim.n) \Rightarrow h^{p,q} = h^{N-p,N-q}(complex\ dim.N)\\ &(Poincare\ \ duality); h^{p,q} = h^{q,p};\ \ b_0 = 1 \Rightarrow h^{0,0} = 1 \end{aligned} \tag{142}$$

Holomorphic form. On a Calabi-Yau space of complex dimension N, there exists a unique holomorphic, everywhere nonzero N-form (more precisely, one (0,N) and one (N,0) form). Let's call it Ω. In fact, the Calabi-Yau definition involves the condition $c_1(K) = 0$, which is equivalent to the existence of this unique holomorphic form. Thus also $h^{N,0} = h^{0,N} = 1$.

Calabi-Yau moduli space

The moduli of Calabi-Yau spaces in string theory (deformations that are not changing the topology, thus don't affect the low energy theory, thus being zero energy deformations, or *moduli*) are of two types.

The *complex structure* moduli are defined as follows.

A CY space X has b_3 topologically nontrivial 3-surfaces (see before). There exists a basis for it, (A_I, B^J), $I, J = 1, b_3/2$, of surfaces with intersection numbers

$$A_I \cap B^J = -B^J \cap A_I = \delta_I^J; \quad A_I \cap A_J = B^I \cap B^J = 0 \tag{143}$$

which is unique up to an $Sp(b_3, Z)$ transformation that acts on the vector (A_I, B^J) and preserves the intersection matrix above. This basis is dual to the basis of 3-forms for $H^3(X, R)$, given by (α_I, β^J) (i.e. $\int_{A_I} \beta^J = \delta_I^J$, etc.).

The complex structure moduli are the b_3 periods of the holomorphic N-form (3-form for CY_3) Ω, i.e.

$$F_I = \int_{A_I} \Omega; \quad Z^J = \int_{B^J} \Omega \tag{144}$$

They will give $\mathcal{N} = 2$ vector multiplets and thus live in a special Kahler manifold. The two periods are related by the period matrix N^{IJ}: $Z^J N_{IJ} = F_I$, or rather

$$N_{IJ} = \frac{\partial F_I}{\partial Z^J} \tag{145}$$

The *Kahler structure moduli* are defined as follows. The Kahler form on a Calabi-Yau is $k_{ij} = g_{i\bar{k}} J^{\bar{k}}{}_j$, or compactly (denoted also J to confuse people), the 2-form

$$J = g_{i\bar{j}} dz^i \wedge dz^{\bar{j}} \tag{146}$$

In string theory, one also has the NS-NS 2-form field $B_{\mu\nu}$, defining another 2-form B, and then the *complexified Kahler class* on X is $K = J + iB$. Similarly to the complex structure then, one has b_2 topologically nontrivial 2-surfaces, and one can find a basis $(A'_{I'}, B'^{J'})$, $I', J' = 1, b_2/2$ on them and define the Kahler moduli as integrals of K on them

$$X_{I'} = \int_{A_{I'}} K; \quad X^{J'} = \int_{B^{J'}} K \tag{147}$$

They will give $\mathcal{N} = 2$ hypermultiplets, thus have a *quaternionic structure (hyper-geometry)*.

For the simplest Calabi-Yau, namely the torus T^2, the Kahler structure modulus is the overall volume (plus B-field integral), while the complex structure is τ, the complex parameter defining the ratio of torus cycles (the complex periodicities of the torus are 1 and τ). Thus Kahler class determines "size" and complex structure determines "shape".

Mirror symmetry exchanges complex structure moduli with Kahler moduli, thus also exchanging the topological numbers b_2 and b_3 for CY_3 and vector multiplets with hypermultiplets.

Conifold transitions. Moduli spaces of CY have sometimes singularities, for instance when one of the periods, say Z^1, vanishes, and then the Calabi-Yau becomes singular, and is known as a *conifold*. Strominger showed in 1995 that string theories resolves these singularities, that one can associate with black hole multiplets becoming massless and condensing (the same way Seiberg and Witten resolved $\mathcal{N} = 2$ SYM moduli space singularities by condensation of monopoles). Then Greene, Morrison and Strominger (also in 1995) showed that this string-smoothed transition in moduli space signals a smooth transition between topologically different Calabi-Yau's (with different Euler characteristic and Hodge numbers).

Special geometry (see Fre; deWit and van Proeyen 1995 reviews)

On the space of n_v $\mathcal{N} = 2$ vector multiplets coupled to $\mathcal{N} = 2$ supergravity, one has special geometry, defined by a $Sp((n_v + 1), Z)$ vector bundle.

The metric on the moduli space of the Calabi-Yau's will be the scalar field moduli space, thus the Kahler potential of the CY moduli space is the modified supergravity Kahler potential k of the scalars in chiral multiplets:

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} k \tag{148}$$

The complex dimension of the moduli space gives the number of vector multiplets n_v (since each vector multiplet has a complex scalar). In the presence of $\mathcal{N} = 2$ supergravity though, we have also the graviphoton, i.e. the photon superpartner of the graviton, hence the n_v+1 in the vector bundle.

This is called *local special geometry.* If we don't have supergravity, the 1 is absent and we have *rigid special geometry.*

For type IIB on a CY_3, we have the $\mathcal{N}=2$ multiplets: supergravity, h_{21} vectors coming from the complex structure, and $b_2+1=h_{1,1}+1$ hypermultiplets coming from the Kahler structure and

the complexified coupling

$$\tau = a + ie^{-\phi} \tag{149}$$

Special geometry for the moduli space of CY's means that (Z^I, F_J) are projective sections (coordinates) of an $Sp(b_3, Z)$ vector bundle over the moduli space. They are defined *locally*, and then one has

$$k(Z^I, \bar{Z}^{\bar{J}}) = -ln(iF_I\bar{Z}^I - iZ^I\bar{F}_I) \tag{150}$$

The *global definition* for CY_3 involves the holomorphic 3-form Ω. Defining the inner product

$$< A|\bar{B} >= \int_X d^6x A \wedge \bar{B} \tag{151}$$

one has

$$k = -ln < \Omega|\bar{\Omega} > \tag{152}$$

and the relation to the special coordinates from before is given by the local decomposition of the holomorphic 3-form in the 3-form basis

$$\Omega = Z^I\alpha_I + F_J\beta^J \tag{153}$$

which implies the previous definition.

Remembering that special geometry is $\mathcal{N}=2$ theory on vector multiplets, for which the chiral action is

$$\int d^4xd^4\theta\mathcal{F}(\Psi) \tag{154}$$

the Kahler potential is the above if

$$F_J(Z^I) = \frac{\partial\mathcal{F}}{\partial Z^J} \tag{155}$$

and the metric on *vector field space* is

$$f_{IJ} = \partial_I\partial_J\mathcal{F} \tag{156}$$

so that the YM kinetic action is

$$\int d^4x Im(f_{IJ}F^I_{\mu\nu}F^J_{\mu\nu}) \tag{157}$$

We see that the metric on vector field space is the same as the period matrix.

Also the Yukawa couplings in the $\mathcal{N}=2$ vector multiplets are

$$F_{IJK} = \partial_I\partial_J\partial_K\mathcal{F} \tag{158}$$

Flux compactifications and inflation

In the presence of IIB G-flux, defined as the 3-form

$$G = F^{RR} - \tau H^{NS} \tag{159}$$

(see Giddings, Kachru and Polchinski= GKP) one has the Gukov-Vafa-Witten (GVW) superpotential

$$W = \int_{K_6} \Omega \wedge G_{(3)} \tag{160}$$

where Ω is the holomorphic 3-form.

The F-theory generalization is (F theory is by definition IIB with varying complex coupling τ, interpreted as the complex structure modulus of a T^2 fibration of the compactification manifold, i.e. τ is the ratio of cycle periodicities of a torus that varies from point to point)

$$W = \int_{X_8} \Omega_4 \wedge G_{(4)} \tag{161}$$

where Ω_4 is the holomorphic 4-form over the Calabi-Yau X_8 and $G_{(4)}$ is the previous flux viewed as a 4-form when τ is geometric.

GKP consider a single Kahler modulus, a radius ρ, as well as the complex structure moduli Z^α and the dilaton τ, with tree-level Kahler potentials

$$\begin{aligned} K(\rho) &= -3ln[-i(\rho-\bar{\rho})] \\ K(\tau, Z^\alpha) &= -ln(-i(\tau-\bar{\tau})) - ln(-i\int_{K_6}\Omega\wedge\bar{\omega}) \end{aligned} \tag{162}$$

Tadpole cancellation (generalized D3 charge conservation on the compact space) requires for F theory

$$L \equiv \frac{1}{2}\int_{X_8} G_4 \wedge G_4 = \frac{\chi}{24} - N_{D3} \tag{163}$$

where N_{D3} is the number of D3's minus the number of anti-D3's transversal to X_8. Here N_{D3} is D3 charge, L is D3 charge induced by the flux (in IIB language, $2L = \int H_{(3)} \wedge F_{(3)}$, thus this charge contribution comes from a Chern-Simons coupling in the II action, of the type $\int H_{(3)}^{NS} \wedge F_{(3)}^{RR} \wedge A_{(4)}^{RR} = \int B_2^{NS} \wedge F_{(3)}^{RR} \wedge H_{(5)}^{+,RR}$), and $\chi/24$ is gravitational charge, which in IIB language is negative charge of O3 orientifolds and D3 charge induced on D7's.

Kachru, Kallosh, Linde and Trivedi (KKLT) showed that the tree level Kahler potential from before, together with the nonperturbative superpotential

$$W = W_0 + Ae^{ia\rho} \tag{164}$$

(which can come either from euclidean D3 brane instantons which live in X_8, or from a purely 4d phenomenon, the $\mathcal{N} = 1$ SYM effective theory having gluino condensation), are enough to create a stable AdS minimum (the potential $V(\sigma = Im\rho)$ has a negative minimum). Then addition of $\bar{D}3$ branes (and correspondingly extra flux, to satisfy charge conservation), breaks susy and gives a potential of type

$$V = \frac{D}{\sigma^3} \tag{165}$$

that lifts the AdS vacuum to a dS vacuum. Its moduli are fixed, so the $\bar{D}3$ is stuck at a fixed position. This creates a scenario where one needs to tunnel out of the potential barrier to zero energy (old inflation).

Then Kachru, Kallosh, Maldacena, McAllister and Trivedi (KKLMMT) introduced also an extra moving D3 brane, and one has inflation from the $D3 - \bar{D}3$ potential in the flux compactification geometry (the inflaton is the distance between the fixed D3 and the moving $\bar{D}3$). But generically, moduli stabilization is too constraining, and the potentials are too steep for good inflation. At special points, these problems might be aleviated.