1 Susy, sugra, strings, branes and duality in a nutshell- notes by Horatiu Nastase

Susy:

Simplest: WZ model in 1+1d. Susy uses Majorana fermions-it's easier. ($\bar{\psi} = \psi^C \equiv \psi^T C$).

1) on-shell susy. A Majorana fermion ψ has 2 real d.o.f. (general spinor has $2^{d/2}$ complex comp.), so 1 d.o.f. on-shell. Susy relates bosons and fermions, so the number of d.o.f. needs to match. Thus on-shell we need one scalar.

The action

$$S = \frac{1}{2}\int d^2x((\partial\phi)^2 + \bar{\psi}\partial\!\!\!/\psi) \tag{1}$$

implies the dimensions $[\phi] = 0, [\psi] = 1/2$. Susy should start as

$$\delta\phi = \bar{\epsilon}\psi \tag{2}$$

which means that $[\epsilon] = -1/2$, and then by dimensional reasons

$$\delta\psi = \partial\!\!\!/\phi\epsilon \tag{3}$$

We can check that the action is invariant on-shell under this symmetry.

2) off-shell susy. To match d.o.f. off-shell we need to add an auxiliary scalar field F, so that

$$S = \frac{1}{2}\int d^2x((\partial\phi)^2 + \bar{\psi}\partial\!\!\!/\psi + F^2) \tag{4}$$

and then $[F] = 1$. By dimensional reasons and because the extra terms added to the susy rules can only be eq. of motion, we have

$$\delta F = \bar{\epsilon}\partial\!\!\!/\psi; \quad \delta\psi = \text{previous} + F\epsilon \tag{5}$$

General statement: off-shell- add e.o.m. terms to susy rules.

3) Superspace: Grassman variable θ has property that $\int d\theta = d/d\theta$, meaning that the Taylor expansion of a general function $\phi(x,\theta)$ is finite:

$$\phi(x,\theta) = \phi(x) + \theta\lambda(x) \tag{6}$$

4 dimensions Reduce to 2d by using 2-component notation. A Majorana spinor in 4d has 4 real comp. and can be expanded as

$$\psi^\alpha = \begin{pmatrix} \psi^A \\ \bar{\psi}_{\dot{A}} \end{pmatrix} \tag{7}$$

The C matrix in 2 component notation is just

$$C_{\alpha\beta} = \begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & \epsilon_{\dot{A}\dot{B}} \end{pmatrix} \tag{8}$$

One uses the convention $\psi_B = \psi^A \epsilon_{AB}; \zeta^{\bar{A}} = \epsilon^{\dot{A}\dot{B}}\zeta_{\dot{B}}$. Note that $a^A b_A = -a_A b^A$, e.g.

Thus 4d superspace has 4 fermions, written as $\int d^4\theta \equiv \int d^2\theta d^2\bar{\theta}$. A superfield is a function

$$\phi(x,\theta) = c(x) + \theta^A \lambda_A(x) + \bar{\theta}^{\dot{A}}\bar{\lambda}_{\dot{A}}(x) + ... + \theta^A\theta_A\bar{\theta}^{\dot{B}}\bar{\theta}_{\dot{B}} d(x) \tag{9}$$

Supergravity $\equiv$ susy theory of gravity $\equiv$ theory of local susy. Is a susy theory of e_μ^a (vielbein) and $\psi_{\mu\alpha}$ (gravitino). In d=4 this is enough to get an on-shell susy multiplet (N=1 supergravity), whereas in higher dim. or for extended susy algebras one will have more propagating fields.

It is written in the vielbein formulation of gravity. Here a= flat indices, acted on by the local Lorentz symmetry and μ= curved indices, acted on by general coordinate transformation. $g_{\mu\nu} = e_\mu^a e_\nu^a$ and one needs to introduce also the spin connection ω_μ^{ab}, that is the connection associated with the Lorentz group action on the spinors, namely

$$D_\mu \lambda = \partial_\mu \lambda + \frac{1}{4}\omega_\mu^{ab}\Gamma^{ab}\lambda \tag{10}$$

In the absence of spinors, the spin connection is related to the vielbein through the 2-form no-torsion constraint (no YM curvature for the vielbein)

$$T^a = De^a = 0 \to \partial_{[\mu} e^a_{\nu]} - \omega^{ab}_{[\mu} e^b_{\nu]} = 0 \Rightarrow \omega = \omega(e) \tag{11}$$

The Riemann tensor is just the YM curvature of the spin connection:

$$R^{ab}(\omega) = d\omega^{ab} + \omega^{ac} \wedge \omega^{cb} \tag{12}$$

and the Ricci scalar is the contraction with inverse vielbeins:

$$R(e,\omega) = R^{ab}_{\mu\nu} e_a^{-1\mu} e_b^{-1\nu} \tag{13}$$

whereas the covariant derivative acting on vectors is given by the Christoffel symbol, defined in terms of the metric

$$\begin{aligned} D_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda \\ \Gamma^\lambda_{\mu\nu}(g) &= \frac{1}{2} g^{\lambda\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \end{aligned} \tag{14}$$

Then the Einstein-Hilbert action action is

$$S_{EH} = -\frac{1}{2k^2} \int (det(e)) R(e,\omega) \tag{15}$$

and the unique action for spin 3/2 is the Rarita-Schinger action, in 4d is

$$S_{RS} = -\frac{1}{2} \int \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma \tag{16}$$

and in a general dimension is (we can chack that in 4d they are the same)

$$S_{RS} = -\frac{1}{2} \int \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \tag{17}$$

A general supergravity, having $S_{EH} + R_{RS} + ...$ in general dimension, will have a local susy transformation of the type

$$\begin{aligned} \delta e^a_\mu &= \frac{k}{2} \bar{\epsilon} \gamma^a \psi_\mu + ... \\ \delta \psi_\mu &= \frac{1}{k} D_\mu(\omega(e,\psi))\epsilon + ... \end{aligned} \tag{18}$$

Strings The Nambu-Goto action for bosonic string is just the area spanned by the string worldsheet

$$S_{NG} = (-\frac{1}{2\pi\alpha'}) \int d\tau d\sigma \sqrt{\det(h_{ab})} \tag{19}$$

where the worldsheet metric is the pull-back of the spacetime metric onto the worldheet (induced)

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \tag{20}$$

The first order action corresponding to S_{NG} is obtained by making the worldsheet metric independent, and is just the nonlinear sigma model action for a 2d scalar, called Polyakov action

$$S_P = (-\frac{1}{4\pi\alpha'}) \int d\tau d\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \sim \int (\partial\phi)^2 \tag{21}$$

Superstrings The superstring action in the Green-Schwarz formulation, namely with spacetime susy, is obtained by adding a worldsheet scalar and spacetime fermion θ^A. One replaces the derivative of the scalar $\partial_a X^\mu$ with the spacetime susy invariant derivative

$$\Pi_a^\mu = \partial_a X^\mu - i\bar{\theta}^A \Gamma^\mu \partial_a \theta^A \tag{22}$$

invariant under

$$\delta X^\mu = -\bar{\epsilon}^A \Gamma^\mu \partial_a \theta^A; \quad \delta\theta^A = \epsilon^A \tag{23}$$

One also adds a topological term called Wess-Zumino term, which in flat background is purely fermionic, thus

$$S = (-\frac{1}{4\pi\alpha'}) \int d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}\Pi_a^\mu \Pi_b^\nu g_{\mu\nu} + \int d\tau d\sigma \epsilon^{ab}\Pi_a^\mu \Pi_b^\nu B_{\nu\nu} \tag{24}$$

where $B_{\mu\nu}$ is the antisymmetric tensor field, but is zero in flat space. In fact, there are more compomnents to the WZ term than written here, and we will explain it shortly.

There is also a fermionic "kappa" symmetry on the worldsheet, and by fixing a gauge for it, the spacetime fermions θ^A become also worldsheet spinors.

(Super-) branes

The generalization to p space dimensional extended objects, called "p-branes" is given by the action

$$S_d = T_d \int d^d\xi (-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\Pi_i^A \Pi_j^B \eta_{MN} e^{\frac{a(d)\phi}{d}}$$
$$+\frac{d-2}{2}\sqrt{-\gamma} - \frac{1}{d!}\epsilon^{i_1 \dots i_d}\Pi_{i_1}^{A_1} \dots \Pi_{i_d}^{A_d} A_{A_1 \dots A_d}) \tag{25}$$

where the bosonic p-brane has $\Pi_i^A \to \Pi_i^a = \partial_i X^\mu E_\mu^a$ (E_μ^a is the spacetime vielbein: the spacetime metric is $g_{\mu\nu} = E_\mu^a E_\nu^b \eta_{ab}$). For the superbrane, one introduces spacetime supervielbeins E_M^A (A= flat superindex= a for bosonic and α for fermionic, M= curved superindex= μ for bosonic and m for fermionic). Using the superspace spacetime coordinates $Z^M(\xi)$, one pulls them back to the p-brane worldvolume, obtaining the induced

$$\Pi_i^A \equiv E_i^A = (\partial_i Z^M) E_M^A \tag{26}$$

For the spacetime superspace description, one also needs a super d-form A, with curvature (field strength) $H = dA$. The requirement of "kappa" symmetry invariance, in general of the type

$$\delta E^a = 0;\ \ \delta E^\alpha = (1+\Gamma)^\alpha{}_\beta k^\beta$$
$$\delta g_{ij} = 2[X_{ij} - g_{ij} X^k_k/(d-1)] \tag{27}$$

where $X_{ij} = X_{ij}(E_i^A)$ and Γ is a given function of gamma matrices and vielbeins, puts constraints on H, fixing it to be

$$H = E^\alpha E^\beta E^{a_1}...E^{a_n}(\gamma_{a_1...a_n})_{\alpha\beta} \tag{28}$$

For the M2 brane, these constraints are solved by the 11d supergravity superspace constraints (thus the super M2 can only propagate consistently, i.e. with kappa symmetry, only in 11d supergravity background).

Also, the previous H is closed (i.e. dH=0) only if a certain gamma matrix identity is satisfied

$$(\gamma^a)_{(\alpha\beta}(\gamma^{aa_1...a_{d-1}})_{\gamma\delta)} = 0 \tag{29}$$

This identity can hold only for a certain number of spacetime dimensions D and worldvolume dimensions d=p+1. These cases form the *brane scan*, i.e. the allowed p-branes in every dimension.

The spacetime metric for the above p-brane action is the canonical (Einstein frame) metric. One can make conformal transformations to different frames $g^{(f)}_{\mu\nu} = e^{\alpha(f)\phi} g^{can}_{\mu\nu}$, in particular in string theory one often uses the *string frame* $g^s_{\mu\nu}$ for which there is no dilaton factor in the string action.

P-branes as solutions to supergravity equations of motion

Generically, one has a supergravity action involving a d-form potential, of the type

$$I_D(d) = \frac{1}{2k^2}\int d^D x\sqrt{-g}(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(d+1)!} e^{-a(d)\phi} F^2_{d+1}) \tag{30}$$

One has two types of solutions:

- elementary (electrically charged), which are solutions to $I_D(d) + S_d$, thus taking the previous bosonic p-brane action as a source, the same way that an electron is a solution to

$$\int \frac{F^2_{\mu\nu}}{4} + \int j^\mu A_\mu \tag{31}$$

  with j_μ being a delta function source ($j^0 = e\delta^3(x)$). Indeed, when we use the S_d source, we get a $\delta^{D-d}(x)$. The solution has nonzero electric charge, $\int *F$.

- solitonic (magnetically charged), which are solutions to I_D alone, in the same way a 't Hooft monopole is a solution to the SO(3) gauge theory. However, these solutions are somewhat in between the 't Hooft and Dirac monopole (with singular, i.e. delta function source for the magnetic field) type. The solution has nonzero magnetic charge, $\int F$.

Duality

The point is the following: For the electric solution, the equations of motion and Bianchi identities are

$$d*(e^{-a(d)\phi}F) = 2k^2(-)^{d^2} * J, \quad dF = 0 \tag{32}$$

but the delta function singularity is needed only here. If we look at the coefficients of the delta function source on the r.h.s. of the Einstein and dilaton equations, on the solution, they go to zero at x=0. In other words, the source was not needed in the Einstein and dilaton equations, was needed only for the F equation. The ansatz for A is

$$A_{\mu_1\dots\mu_d} = -\frac{1}{g^d}\epsilon_{\mu_1\dots\mu_d}e^C \Rightarrow F_{m\mu_1\dots\mu_d} = -\frac{1}{g^d}\epsilon_{\mu_1\dots\mu_d}\partial_m e^C \tag{33}$$

The magnetic solution has no source, but it still has a magnetic field which is singular at the origin. The ansatz for F is

$$\frac{1}{\sqrt{2}k}F_{d+1} = g_{\tilde{d}}\frac{\epsilon_{d+1}}{\Omega_{d+1}} \Rightarrow F_{m_1\dots m_{d+1}} = \frac{1}{g^d}\epsilon_{m_1\dots m_{d+2}}\frac{y^{m_{d+2}}}{|y|^{d+2}}\frac{g_{\tilde{d}}\sqrt{2}k}{\Omega_{d+1}} \tag{34}$$

where ϵ_{d+1} is the volume form, and $\tilde{d} = D - d - 2$ is the dual (magnetic) worldvolume dimension, $g_{\tilde{d}}$ is the magnetic charge. Now $d*F = 0$ (no source), but $dF \neq 0$ (nonzero magnetic charge $\int dF$), and we have a $\tilde{d}-1$ solitonic brane solution.

Duality works, because we didn't actually need the source in the Einstein and dilaton equations. If we put a $F_{\tilde{d}+1}$ and $a(\tilde{d}) = -a(d)$, implying that

$$(\tilde{g}_{(\tilde{d})})^{\tilde{d}} \sim e^{a(\tilde{d})\phi} = e^{a(d)\phi} \sim (1/g_{(d)})^d \tag{35}$$

we have a strong-weak coupling duality. Duality means that $\tilde{I}_D(\tilde{d})$ has as solitonic solution the precious elementary d-1 brane. And $\tilde{I}_D(\tilde{d}) + \tilde{S}_{\tilde{d}}$ has as elementary solution the previous $\tilde{d}_1$ brane. What is the same in the canonical (Einstein frame) metric.