1 Integrable systems: Yang-Baxter equation, Bethe ansatz, Yangian, Lax Pairs, Toda and Calogero-Moser systems; application to $\mathcal{N}=4$ SYM; A basic introduction - Notes by Horatiu Nastase

1.1 Introduction, terminology

A hamiltonian mechanical system (i.e. in 0+1 dimensions) with n degrees of freedom is **integrable** if it has n independent integrals of motion, in which case it is completely solvable.

A 1+1 dimensional field theory is integrable if it is solvable by means of the quantum inverse scattering method (derive S matrices through equations imposed on it, and then derive everything from them). It also has an infinite number of conserved quantities ("integrals of motion"), that can sometimes be used to map the system to an infinite dimensional integrable hamiltonian mechanical system. The best known example is the sine-Gordon theory, for a scalar field with equation of motion

$$\Box\phi + m^2 \sin\phi = 0 \tag{1}$$

which is known to be nonperturbatively equivalent to the massive Thirring model (fermionic model).

An integrable 1+1 dimensional field theory can be related by going to Euclidean space to a solvable 2d model of classical statistical physics.

Putting integrable 1+1d models on the lattice, one obtains integrable hamiltonian models that can be interpreted as quantum spin models of magnetic chains ("**spin chains**"). Reversely, limits of various spin chain models give (all?) integrable field theories in 1+1d as continuum limits. Thus spin chains are primers for all integrable models. Among them, the **Heisenberg spin 1/2 chain** is the most studied example, the **XXX** (rotationally invariant) model in particular. One defines more general "spin chains" by generalizing the "spin" from the usual case of a given representation j of $SU(2)$ to an arbitrary representation of a Lie algebra, i.e. the "spin variables" take value in representation $\mathcal{R}$ of a Lie algebra G.

Bethe studied the Heisenberg XXX model and gave an ansatz (parametrization) for the wavefunctions of its solutions in the coordinate representation ("**coordinate Bethe ansatz**"), and equations for them. Nowadays, one uses an algebraic method, where the eigenvalues and eigenfunctions can be derived from algebraic considerations, and one uses a parametrization in terms of abstract creation/annihilation operators ("**algebraic Bethe ansatz**"), leading to algebraic equations, the (algebraic) **Bethe equations**. Solving them is a whole industry.

A 2d theory with a factorizable n-body S matrix (that can be written as products of 2-body S matrices) satisfies the **Yang-Baxter equation** and is thus integrable, as one can go from the Yang-Baxter equation to solving the theory a la Bethe ansatz. (Is the reverse true, that an integrable theory has a factorizable S matrix?). For the lattice models ("spin chains"), the analogue of the S matrix is called transfer matrix or Lax operator $L(z)$.

The algebra corresponding to the R matrix of an XXX Heisenberg model satisfying the Yang-Baxter equation, more exactly to the "monodromy matrix" $T(u)$ is called the **Yangian**. It can be defined abstractly and is the algebra of the infinite (for length $L \to \infty$) set of charges of the integrable system, related to the se of commuting conserved quantities ("integrals of motion") through their common generating function, the monodromy matrix $T(u)$.

Conformal field theories arise as limits of integrable massive QFT's when we take a UV limit (or take the mass to zero). These massive QFT's are described by factorizable S matrices and can be understood as relevant perturbations of CFTs.

The themodynamics of these massive theories in the infinite volume (thermodynamic limit...) can be derived completely from the S matrices. This is known as the **Thermodynamic Bethe ansatz (TBA)** and consists of first writing Bethe equations for an ansatz for the wavefunctions and then taking a thermodynamic limit for deriving the thermodynamics of the physical particles of the theories (real solutions of the S matrix Bethe equations, with real rapidities). One writes down an equation (TBA equation) for quantities $\epsilon_a(\theta)$ (like dispersion relations), that define the density of states, from which thermodynamics can be built.

1.2 Mechanical integrable systems: Lax pairs, Toda and Calogero-Moser systems

A Hamiltonian mechanical system with n degrees of freedom and a Poisson bracket defined is $(i = 1, n)$

$$\begin{aligned} &H(x_i, p_i); \quad \{x_i, p_j\} = \delta_{ij} \\ &\dot{x}_i = \{x_i, H\} = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial x_i} \end{aligned} \tag{2}$$

It is integrable if (and only if) there exist n independent integrals of motion $I_i(x, p), i = 1, n$ such that

$$\dot{I}_i = \{I_i, H\} = 0; \quad \{I_i, I_j\} = 0 \tag{3}$$

For an integrable system, there sometimes exists a **Lax pair**, which is a pair of $N \times N$ matrices $L(x, p)$ and $M(x, p)$ such that the Lax equation $\dot{L} = [L, M]$ is equivalent to the Hamilton equations of the mechanical system, i.e.

$$\dot{L} = [L, M] \Leftrightarrow (\dot{x}_i = \{x_i, H\}; \quad \dot{p}_i = \{p_i, H\}) \tag{4}$$

The dimension N of the matrices is a priori unknown, and there is no algorithm to establish even the existence of a Lax pair, let alone to find it. All we know is that $n \leq N$, in order to get enough integrals of motion, as all

$$I_i = tr(L^{n_i}) \Rightarrow \dot{I}_i = n_i tr(L^{n_i - 1}[L, M]) = 0 \tag{5}$$

are thus integrals of motion, and for $n_i \geq N$ we get functionally dependent I_i's.

The Lax pair is not unique, as there is the "gauge transformation"

$$L^S = S^{-1}LS; \quad M^S = S^{-1}MS - S^{-1}\dot{S} \tag{6}$$

The I_i's are gauge invariant however as we can easily see. If one chooses a basis of linearly independent I_i's, $i = 1, n$, then the Hamiltonian is a function of them

$$H = h(I_1, ...I_n) \tag{7}$$

A stronger form of integrability arises when one has Lax pairs that depend upon an additional (complex) variable, called *spectral parameter*, z, thus we have $L(z), M(z)$, and for all values of z this is a Lax pair. The Lax pair with spectral parameter is still a $N \times N$ matrix, function of (x, p). The mechanical system is independent of z.

$$\dot{L}(z) = [L(z), M(z)] \Leftrightarrow (\dot{x}_i = \{x_i, H\}; \quad \dot{p}_i = \{p_i, H\}) \tag{8}$$

Then one can define a *spectral curve* as

$$\Gamma = \{(k, z) \in \mathbf{C} \times \mathbf{C}; det(kI - L(z)) = 0\} \tag{9}$$

with the one form $d\lambda = kdz$. The spectral curve is time independent, gauge invariant, depends only on the n integrals of motion I_i. Most likely, the integrals of motion can be now recovered from the series expansion of $trL(z) = \sum_n L_n z^n$.

Quantum Mechanics? (Tentative)

What happens for a (nonrelativistic) quantum mechanical system? Instead of the classical Hamiltonian system (2) we have, in the coordinate representation and in the time-independent case

$$\begin{aligned} &\{\hat{H}, \psi(x_i, t)\} \quad [x_i, p_j] = (\hbar) i\delta_{ij} \\ &\hat{H}\psi(x_i, t) = i\hbar\frac{\partial}{\partial t}\psi(x_i, t) = E\psi(x_i, t) \end{aligned} \tag{10}$$

or for some abstract dynamical variables X_i^α, satisfying some general commutation relations (...), i.e. algebra, with α a finite index and i possibly infinite

$$\begin{aligned} &\{\hat{H}, \psi(X_i^\alpha)\} \quad [X_i^\alpha, X_j^\beta] = ... \\ &\hat{H}\psi(X_i^\alpha) = E\psi(X_i^\alpha) \end{aligned} \tag{11}$$

The dynamical variables will define some Hilbert space $\mathcal{H}$. If the algebra of variables is ultralocal, i.e. $[X_i^\alpha, X_j^\beta] = 0$ for $i \neq j$, the Hilbert space is a tensor product

$$\mathcal{H} = \prod_{i=1}^{N} \otimes h_i = h_1 \otimes ... \otimes h_N \tag{12}$$

If not, one can sometimes still define the Hilbert space as a tensor product.

The Lax description in the case with a spectral parameter should involve still an operator $L(z)$ that is also a matrix (operator) in an auxiliary $N \times N$ Hilbert space $\mathcal{H}'$, as well as an operator on the system's Hilbert space $\mathcal{H}$, such that observables (time independent, mutually commuting quantities, i.e. integrals of motion) should be again traces of it over the auxiliary Hilbert space $\mathcal{H}'$, i.e. for fixed z

$$I_i = tr_{\mathcal{H}'}(L^{n_i}) \tag{13}$$

and in the case of $L(z)$ most likely can be retrieved as combinations of the coefficients in the series expansion $tr_{\mathcal{H}'}L(z) = \sum_n L_n z^n$. It is unclear what is the analog of the requirement that the equations of motion are derived from the Lax pair. I suspect that in the Heisenberg picture the classical case should go through unchanged, except that everything is now an operator, time dependent since we are in the Heisenberg picture. The only problem is what happens if we have spin variables (very important case later on), which don't have a classical counterpart. I think one could still define it to be (since $[x,p]/(i\hbar) = 1$ replaces $\{x,p\} = 1$)

$$\dot{\hat{X}}_{H_i}^{\alpha} = \frac{1}{i\hbar}[\hat{X}^{\alpha}_{H,i}, \hat{H}_H] \Leftrightarrow \dot{\hat{L}}_H(z) = \frac{1}{(i\hbar?)}[\hat{L}_H(z), \hat{M}_H(z)] = \frac{i}{\hbar}[M_H(z), L_H(z)]? \tag{14}$$

Here the last line, if the insertion of $i\hbar$ was correct, means that the operator $M(z)$ in the Heisenberg representation acts as the Hamiltonian in the Heisenberg representation on $L(z)$. If not, one just has to reabsorb $i\hbar$ to get H_H. In any case, that equivalence then becomes somewhat irrelevant, and all that one needs is an operator $L(z)$ acting on some auxiliary space that gives the observables through its traces. However, the defining quantum equation for $L(z)$ should still be equivalent to solving the system (i.e. with the Schrodinger equation). For instance, for spin chains there will be an $L(z)$ acting as a connection, i.e. defining a Dirac equation $D\psi = 0$ from which one can find the Schrodinger equation.

Note then that the specification of the Hilbert space defines just the kinematics. The choice of a system (the dynamics) is a particular case of Lax operator $L(z)$ giving the particular Hamiltonian as one of its observables. We will see that at least for spin chains, $L(z)$ satisfies also the Yang-Baxter algebra, but that is also just kinematics, a particular system corresponding to a particular representation for the Lax operator $L(z)$.

Important (classical) cases

The **non-periodic Toda system** is a (nonrelativistic) system of $n+1$

points on a linear chain, with exponential nearest-neighbor interaction, i.e.

$$H = \frac{1}{2}\sum_{i=1}^{n+1} p_i^2 - M^2 \sum_{i=1}^{n} e^{x_{i+1}-x_i} \tag{15}$$

The center of mass $x_0 = \sum_i x_i$ decouples, leaving n degrees of freedom. The system with $n = 1$ reduces to Liouville theory.

The **periodic Toda system** is a (nonrelativistic) system of $n+1$ points on a circular chain with exponential nearest-neighbor interaction, i.e.

$$H = \frac{1}{2}\sum_{i=1}^{n+1} p_i^2 - M^2 \sum_{i=1}^{n+1} e^{x_{i+1}-x_i} \tag{16}$$

with $x_{n+2} = x_1$. The center of mass decouples, leaving n degrees of freedom. For $n = 1$, it reduces to Sine-Gordon theory.

One can generalize further the Toda systems by giving them a Lie algebraic interpretation, and in this way also getting the Lax pairs. View the linear chain as the Dynkin diagram for the Lie algebra $A_n \sim SU(n+1)$, and the circular chain of the periodic Toda system as the Dynkin diagram for the untwisted affine Lie algebra $A_n^{(1)}$. Then the set of simple roots $\mathcal{R}_*$ is

$$\begin{array}{llll} A_n & e_i - e_{i+1} & i = 1, n & \\ A_n^{(1)} & e_i - e_{i+1} & i = 1, n+1; & e_{n+2} = e_1 \end{array} \tag{17}$$

such that the Hamiltonian for both can be written as

$$H = \frac{1}{2}p^2 - M^2 \sum_{\alpha\in\mathcal{R}_*} e^{-\alpha c\dot{x}} \tag{18}$$

and generalize to any finite-dimensional semi-simple Lie algebra $\mathcal{G}$ or one of the associated untwisted or twisted affine Lie algebras, and have the same H with $\mathcal{R}_* \to \mathcal{R}_*(\mathcal{G})$.

All these systems are integrable and have Lax matrices

$$\begin{aligned} L &= p\cdot h + \sum_{\alpha\in\mathcal{R}_*} M e^{-\frac{1}{2}\alpha\cdot x}(E_\alpha - E_{-\alpha}) + \mu^2 e^{\frac{1}{2}\alpha_0\cdot x}(zE_{-\alpha_0} - z^{-1}E_{\alpha_0}) \\ M &= -\frac{1}{2}\sum_{\alpha\in\mathcal{R}_*} M e^{-\frac{1}{2}\alpha\cdot x}(E_\alpha + E_{-\alpha}) + \frac{\mu^2}{2} e^{\frac{1}{2}\alpha_0\cdot x}(zE_{-\alpha_0} + z^{-1}E_{\alpha_0}) \end{aligned} \tag{19}$$

where $h(h_1, ...h_n)$ is the array of Cartan generators of $\mathcal{G}$, with $[h_i, h_j] = 0$, E_α are the generators of $\mathcal{G}$ associated with the root α, $\mathcal{R}_* = \mathcal{R}_*(\mathcal{G})$ is the set of simple roots of $\mathcal{G}$, α_0 is the affine root of $\mathcal{G}$, to be included in $\mathcal{G}^{(1)}$, and we are in an N dimensional representation ρ of $\mathcal{G}$, so that all h_i, E_α are $N \times N$ matrices. Thus N can correspond to any representation, but as the dimension of a representation, is larger than n.

In the non-periodic case ($\mathcal{G}$) we have $\mu \equiv 0$ and there is no spectral parameter, but there is a μ in the periodic case, thus then there is a spectral curve.

The **Calogero-Moser** systems are nonrelativistic mechanical models of $n+1$ particles on a complex line with 2-body interactions not limited to nearest-neighbor.

The *rational Calogero-Moser system*

$$H = \frac{1}{2}\sum_{i=1}^{n+1} p_i^2 - \frac{1}{2}m^2 \sum_{i\neq j}^{n+1} \frac{1}{(x_i - x_j)^2} \tag{20}$$

The *trigonometric Calogero-Moser system*

$$H = \frac{1}{2}\sum_{i=1}^{n+1} p_i^2 - \frac{1}{2}m^2 \sum_{i\neq j}^{n+1} \frac{1}{\sin^2(x_i - x_j)} \tag{21}$$

The *ellyptic Calogero-Moser system*

$$H = \frac{1}{2}\sum_{i=1}^{n+1} p_i^2 - \frac{1}{2}m^2 \sum_{i\neq j}^{n+1} \mathcal{P}(x_i - x_j; \omega_1, \omega_2) \tag{22}$$

In the ellyptic case, the potential is the Weierstrass ellyptic (doubly periodic) function

$$\mathcal{P}(x; \omega_1, \omega_2) \equiv \frac{1}{x^2} + \sum_{(m,n)\neq(0,0)} (\frac{1}{(x + 2m\omega_1 + 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \tag{23}$$

Here $2\omega_1, 2\omega_2$ are periods, and for $\omega_2 \to \infty$ one gets the trigonometric case, and if also $\omega_1 \to \infty$, one gets the rational case.

Again one can write the Lax pairs by giving a Lie algebraic interpretation on $A_n = SU(n+1)$ in terms of the set $\mathcal{R}$ of *ALL the roots* of A_n, which in an orthonormal basis $e_i, i = 1, n+1$ i s

$$\mathcal{R}(A_n) = \{e_i - e_j, \quad i \neq j = 1, n+1\} \tag{24}$$

and then

$$H = \frac{1}{2}p^2 - \frac{1}{2}m^2 \sum_{\alpha\in\mathcal{R}} \mathcal{P}(\alpha\cdot x;\omega_1,\omega_2) \tag{25}$$

One can generalize to other Lie algebras as well, but there are subtleties. The ellyptic Calogero-Moser admits a Lax pair with spectral parameter z,

$$\begin{aligned} L_{ij}(z) &= p_i\delta_{ij} - m(1-\delta_{ij})\Phi(x_i - x_j; z) \\ M_{ij}(z) &= d_i(x)\delta_{ij} + m(1-\delta_{ij})\Phi'(x_i - x_j; z) \end{aligned} \tag{26}$$

where $d_i(x) = m\sum_{k\neq i}\mathcal{P}(x_i - x_k)$, $\Phi'(x;z) = \partial_x\Phi(x;z)$ and $\Phi(x;z)$ is the Lame function

$$\Phi(x;z) = \frac{\sigma(z-x)}{\sigma(x)\sigma(z)} e^{x\zeta(z)} \tag{27}$$

and $\mathcal{P}(z) = -\zeta'(z), \zeta(z) = \sigma'(z)/\sigma(z)$ and $\zeta(z)$ is the Weierstrass function.

Toda systems can be obtained as limits of elliptic Calogero-Moser systems, by taking

$$\omega_1 = -i\pi,\ Re(\omega_2)\to\infty,\ m = Me^{\delta\omega_2};\ x_j = X_j + 2\omega_2\delta j \tag{28}$$

with M, X_j, $0\leq\delta\leq 1/N$ fixed.

The Toda and Calogero-Moser systems are also connected to higher dimensional integrable systems. Some important connections are to:

Toda field theory, for n scalar fields in 1+1 dimensions, $\phi(t,x)$ and

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\cdot\partial^\mu\phi - M^2\sum_{\alpha\in\mathcal{R}_*} e^{\alpha\cdot\phi} \tag{29}$$

Korteveg-de Vries (KdV) equation: single scalar in 1+1 dimensions, with equation

$$\partial_t u = 6u\partial_x u - \partial_x^3 u \tag{30}$$

There are rational solutions, of the form

$$u(t,x) = 2\sum_j \frac{1}{(x - x_j(t))^2} \tag{31}$$

They satisfy KdV if $x_j(t)$ satisfies some constraints and also satisfies the equation

$$\partial_t x_j = \{x_j, trL^3\} \tag{32}$$

where L is the Lax operator for the rational Calogero-Moser system.

Kadomzev-Petviashvili (KP) equation in 1+2 dimensions (one space and two times) for $u(t,t',x)$ with

$$3\partial_{t'}^2 u = \partial_x(\partial_t u - 6u\partial_x u - \partial_x^3 u) \tag{33}$$

The t' independent solutions are governed by KdV, whereas the elliptic solutions of the form

$$u(t,t',x) = 2\sum_j \mathcal{P}(x - x_j(t,t')) \tag{34}$$

are solutions if $x_j(t,t')$ obeys

$$\partial_t x_j = \{x_j, trL^3\}; \quad \partial_{t'} x_j = \{x_j, trL^2\} \tag{35}$$

where L is the ellyptic operator for the Calogero-Moser system.

1.3 Spin chains

The most studied spin chain is the Heisenberg spin 1/2 XXX (rotationally invariant) spin chain, with Hamiltonian

$$H = J\sum_{j=1}^{L}\sum_{\alpha=1}^{3}\sigma_j^\alpha \sigma_{j+1}^\alpha \tag{36}$$

where $\vec{\sigma}_j$ are Pauli matrices (spin 1/2 operators) at site j, with periodic boundary conditions, $\sigma_{L+1} = \sigma_1$. The case introduced by Heisenberg (1928) is $J < 0$, corresponding to ferromagnetism, whereas antiferromagnetism corresponds to $J > 0$. Sometimes one subtracts a constant from this hamiltonian $(-J)$, so that E=0 on the ground state, with all spins parallel. Also, sometimes the normalization is different: sometimes one uses spin operators $\vec{S}_j$ instead of Pauli operators $\vec{\sigma}_j$, thus dividing the hamiltonian (36) by 4, and sometimes one uses just 1/2 of (36). As we see, the interaction is only nearest neighbor. This model is called XXX because the coupling is the same in all directions. One can have different couplings in different directions, $J_x\sigma_j^x\sigma_{j+1}^x$, $J_y\sigma_j^y\sigma_{j+1}^y$ and $J_z\sigma_j^z\sigma_{j+1}^z$, and then we have the XYZ model. If $J_x = J_y \neq J_z$ we have the XXZ model. There is also a higher spin generalization, but it is not the obvious one, replacing the spin 1/2 operators with spin s ones (that hamiltonian is not integrable). We will derive the integrable generalization later. One can also consider variables taking values in a representation $\mathcal{R}$ of a Lie algebra G, instead of spin j representation of $SU(2)$, that will also be called "spin chains".

Coordinate Bethe ansatz (Bethe, 1931)

The basis of states for the Heisenberg XXX hamiltonian is given by the commuting operators $\vec{S}^2, S^z$ at each site j, in other words by spins up $|\uparrow>$ or down $|\downarrow>$. We denote by $|x_1, x_2, .., x_N>$ the state with spins up ("magnons") at sites x_i along the chain of spins down, i.e. e.g. $|1, 3, 4>_{L=5}= |\uparrow, \downarrow, \uparrow, \uparrow, \downarrow>$.

One defines the permutation operator P_{ij} as

$$P_{ij} = \frac{1}{2} + \frac{1}{2}\vec{\sigma}_i \cdot \vec{\sigma}_j = \frac{1}{2} + \frac{1}{2}\sigma_i^z \sigma_j^z + \sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+ \tag{37}$$

Here $\sigma^\pm = (\sigma^1 \pm i\sigma^2)/2$ and $i \neq j$ and the operator interchanges the values of spins (i.e. up or down) at sites i and j. Then the Hamiltonian (36) is

$$H = J \sum_{j=1}^{L} (2P_{j,j+1} - 1) \tag{38}$$

It is easier however to drop the constant in the energy and thus subtract a JL term in the above. We will do it from now on.

The "one magnon" (single pseudoparticle) sector is diagonalized trivially by the Fourier transform

$$|\psi(p_1)>= \sum_{x=1}^{L} e^{ip_1 x}|x> \Rightarrow H|\psi(p_1)>= 8J\sin^2(p_1/2)|\psi(p_1)> \tag{39}$$

where $p_1 = 2\pi k/L$ with $k \in \mathbf{Z}$. However, for finite L, the independent solutions clearly have $n = 1, ..., L$, thus there are L-1 of them.

The "2-magnon" (s pseudoparticle) state in momentum space

$$|\psi(p_1, p_2)>= \sum_{1 \leq x_1 < x_2 \leq L} \psi(x_1, x_2)|x_1, x_2> \tag{40}$$

needs to solve the "position space" Schrodinger equation $H|\psi(p_1, p_2)>= E|\psi(p_1, p_2)>$. It was done with Bethe's ansatz, giving a superposition of an incoming and an outgoing plane wave, with an S matrix $S(p_1, p_2)$ for scattered particles.

$$\psi(x_1, x_2) = e^{i(p_1x_1+p_2x_2)} + S(p_2, p_1)e^{i(p_2x_1+p_1x_2)} \tag{41}$$

thus particles simply exchanged momenta, besides the S matrix.

The Schrodinger equation then determines both the energy as just the sum of the one-particle energies and the (nonrelativistic) S matrix as

$$E = 8J(\sin^2(p_1/2) + \sin^2(p_2/2))$$
$$S(p_1, p_2) = \frac{\phi(p_1) - \phi(p_2) + i}{\phi(p_1) - \phi(p_2) - i} = S^{-1}(p_2, p_1); \quad \phi(p) = \cot(p/2)/2 \tag{42}$$

Here $\phi(p)$ are called **Bethe roots** and will be later denoted by u and be shown to be analogs of rapidities of massive particles for this nonrelativistic theory. Note that this S matrix has poles at $\phi_{12} = \phi(p_1) - \phi(p_2) = i$, denoting a bound state of two magnons (later, we will say that the S matrix has a pole in rapidity $u_{12} = u_1 - u_2$ at i).

That would be all for the infinite length chain, but for the finite length one we need to impose periodicity $\psi(x_1, x_2) = \psi(x_2, x_1 + L)$, giving the *Bethe equations*

$$e^{ip_1L} = S(p_1, p_2); \quad e^{ip_2L} = S(p_2, p_1) \tag{43}$$

which are equations giving the possible values of p_i or $\phi(p_i)$. Note that they imply

$$e^{i(p_1+p_2)L} = 1 \Rightarrow p_1 + p_2 = 2\pi n/L(mod 2\pi), \quad n = 0, .., L-1 \tag{44}$$

In particular, there are real solutions obeying

$$p_1 = -p_2 \in \mathbf{R} \Rightarrow e^{ip_1(L-1)} = 1 \Rightarrow p_1 = \frac{2\pi n}{L-1} \tag{45}$$

and substituting in the wavefunction we find their eigenfunctions,

$$|\psi(n)>= C_n \cos \pi n \frac{2l+1}{L-1} |x_2 + l'; x_2 >; \quad C_n = 2e^{-i\frac{\pi n}{L-1}} \tag{46}$$

Note that p_1, p_2 MUST be different numbers, which is just a reflection of the fermionic nature of the spin chain, since if we would have $p_1 = p_2$ we easily see that we would have $S(p_1, p_2) = -1$ and correspondingly $\psi = 0$, as it should.

Then for this integrable model that is all, since the S matrix is factorizable, thus the M-body problem is found from the 2-body problem: one just scatters elastically p_i, p_j with $S(p_j, p_i)$, thus exchanging momenta. The M-body wavefunction is

$$\psi(x_1, ..., x_M) = \sum_{P \in Perm(M)} exp[i \sum_{i=1}^{M} p_{P(i)} x_i + \frac{i}{2} \sum_{i<j} \delta_{P(i)P(j)}] \tag{47}$$

and the phase shifts $\delta_{ij} = -\delta_{ji}$ are given by

$$S(p_i, p_j) = exp[i\delta_{ij}] \tag{48}$$

and the Schrodinger equations imply again that the energies add up, and from periodicities we get the M Bethe equations

$$e^{ip_k L} = \prod_{i \neq k; i=1}^{M} S(p_k, p_i) \tag{49}$$

Again, this also implies that

$$e^{i(\sum_i p_i)L} = 1 \Rightarrow \hat{P} = \sum_i p_i = 2\pi n / L (mod 2\pi); \quad n = 0, L-1 \tag{50}$$

thus quantizing the total momentum $\hat{P}$, and again we must have the momenta p_i (thus the Bethe roots $\phi(p_i)$) different, else we get a zero wavefunction, because of the fermionic nature of the chain.

The Bethe equations have both real and complex solutions, both of which are relevant, as we will see. The equations generally have sets of solutions $\{p_1, ..., p_N\}_n$ (n labels solutions) for fixed N, L. For $N = 1$ there are $L-1$ (sets of) solutions, for $N = 2$ there are $L(L-3)/2$ (sets of) solutions and the total number for any N is 2^L. Each solution $\{p_i\}$ is characterized by a set $\{n_i\}$ of integers coming from the log of (49), i.e. from

$$p_k L = \sum_{i \neq k, i=1}^{k} \delta_{ki} + 2\pi n_k \tag{51}$$

A real solution $\{p_i\}$ will be characterized by a set of different n_i's while a complex one can have the same n_i's.

1.4 Yang-Baxter equation and algebraic Bethe ansatz

We have seen that in the coordinate Bethe ansatz one thinks of the chain as a discretized coordinate and then one obtains factorizable S matrices. Reversely, one can think of scattering of *massive* particles in 1+1 dimensions in the case that the S matrix is factorizable. In 1+1d, for massive particles the rapidity μ is the only relevant parameter, $E = m\cosh\mu, p = m\sinh\mu$, thus $E^2 - p^2 = m^2$. Since $(p_1 - p_2)^2 = -m_1^2 - m_2^2 + 2m_1m_2\cosh(\mu_1 - \mu_2)$, if the 2-body S matrix is a function of $(p_1 - p_2)^2$ only, it is then a function of $\mu_1 - \mu_2$ only. Let us thus write $S_{\alpha\beta}^{\alpha'\beta'}(\lambda - \mu)$ for the 2-body S matrix of particles with indices $\alpha = 1, n$. Factorizability of the 3-body S matrix into 2-body S matrices can then be done in two ways, first 2,3, then 1,3, then 1,2 or first 1,2, then 1,3, then 2,3, giving the **Yang-Baxter equation** (YBE)

$$\begin{aligned} S_{123}^{(3)} &= S_{12}(\lambda_1 - \lambda_2)S_{13}(\lambda_1 - \lambda_3)S_{23}(\lambda_2 - \lambda_3) \\ &= S_{23}(\lambda_2 - \lambda_3)S_{13}(\lambda_1 - \lambda_3)S_{12}(\lambda_1 - \lambda_2) \end{aligned} \quad (52)$$

Although this equation was derived for scattering of massive particles in 1+1d, it is valid in more general contexts, due to its algebraic nature. Often, the parameters λ, μ are not rapidities per se, but take other values, sometimes complex.

The most general Yang-Baxter relation one can have is for a matrix $\mathcal{R}$ defined as an element in $\mathcal{A} \otimes \mathcal{A}$, where $\mathcal{A}$ is some algebra, and then the Yang-Baxter relation (**Yang-Baxter algebra**) is

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \quad (53)$$

which holds in $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, and one denotes in obvious way

$$\mathcal{R}_{12} = \mathcal{R} \otimes \mathbf{I}; \quad \mathcal{R}_{23} = \mathbf{I} \otimes \mathcal{R} \quad (54)$$

and $\mathcal{R}_{13}$ has the $\mathbf{I}$ stuck in the middle. The algebra $\mathcal{A}$ must have a family of representations $\rho(\lambda, a)$, with λ a continous (complex) parameter and a a discrete label. In the case of the XXX models this algebra $\mathcal{A}$ is called the **Yangian** and we will discuss it later. Also note that we will refer to a different looking relation as the Yang-Baxter algebra shortly, but it can be derived from this general prescription.

C.N. Yang wrote this equation (YBE) for particles on a line with delta function interaction potential ($V = \sum_{i<j} 2c\delta(x_i - x_j)$), and then R.J. Baxter

considered 2d lattices where each vertex has 4 sides, thus one has a matrix with 4 indices, $R_{\alpha_1\alpha_2\alpha_3\alpha_4}(\lambda-\mu)$ and $\lambda-\mu$ characterizes the energy at the vertex and the temperature (rather, βE, I guess).

Heisenberg XXX spin chain

As a specific model, take again the Heisenberg XXX spin chain. One can check that the operator

$$R_{ij}(u,v) \equiv u - v + P_{ij} \tag{55}$$

where u and v are complex numbers and P_{ij} is the permutation operator defined above, satisfies the Yang-Baxter equation

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v) \tag{56}$$

We will pursue an algebraic method to solving the spin chain. The spin chain is a quantum mechanical system of spins, thus falls into the category of integrable mechanical systems, for which one can have Lax pairs, at least classically. Quantum mechanically, as we saw, we can define a Lax operator $L(z)$ that lives both in an auxiliary Hilbert space $\mathcal{H}'$ and in the defining Hilbert space $\mathcal{H}$. Defining the kinematics (Hilbert space $\mathcal{H}$ of spin chains) will be the first step. This approach is an algebraic approach, leading to the **algebraic Bethe ansatz**, in that we will write an algebra for the Lax operator $L(z)$, and then construct a particular representation, corresponding to the XXX Heisenberg spin chain. However, for the sake of clarity, we will derive the algebra from the particular representation for the XXX spin chain. Then we can apply this to other chains like XXZ for instance.

The Hilbert space is

$$\mathcal{H} = h_1 \otimes ... \otimes h_N \tag{57}$$

where h_i is the Hilbert space of X_i^α variables ($SU(2)$ spin representation 1/2 here, in general it can be any representation of any group).

By dividing (56) by $u-v$, setting $w = 1/2$, replacing u, v by iu, iv, denoting 1 by a, 2 by b and 3 by j and multiplying by P_{ab} from the left, we get

$$\begin{aligned}
&\left(\frac{i}{v-u} + P_{ab}\right)\left(u - \frac{i}{2}\sigma_a^\alpha \sigma_j^\alpha \otimes I_b^2\right)\left(v - \frac{i}{2} I_a^2 \otimes \sigma_b^\beta \sigma_j^\beta\right) \\
&= \left(v - \frac{i}{2}\sigma_a^\alpha \sigma_j^\alpha \otimes I_b^2\right)\left(u - \frac{i}{2} I_a^2 \otimes \sigma_b^\beta \sigma_j^\beta\right)\left(\frac{i}{v-u} + P_{ab}\right)
\end{aligned} \tag{58}$$

viewed as an operator acting in the quantum Hilbert space denoted by the spin j and the auxiliary spaces denoted by the spins a and b, with unit operators I_a^2, I_b^2. Denoting

$$\frac{i}{v-u} + P = \check{R}(u,v); \quad u - \frac{i}{2}\sigma^\alpha \sigma_j^\alpha = L_j(u) \tag{59}$$

one obtains the **Yang-Baxter algebra**

$$\check{R}_{12}(u-v)(L_j(u)\otimes I)(I\otimes L_j(v)) = (L_j(v)\otimes I)(I\otimes L_j(u))\check{R}_{12}(u-v) \tag{60}$$

An alternative form of the **Yang-Baxter algebra** is obtained by multiplying it with P_{ab} from the left and obtaining

$$R_{12}(u-v)(L_j(u)\otimes I)(I\otimes L_j(v)) = (I\otimes L_j(v))(L_j(u)\otimes I)R_{12}(u-v) \tag{61}$$

or symbolically

$$R_{12}(u,v)L_j^1(u)L_j^2(u) = L_j^2(v)L_j^1(u)R_{12} \tag{62}$$

but note that $L_j^1(u)L_j^2(v)$ is not a tensor product, since these are also operators in the Hilbert space j, so their order matters (and would be ill defined in the tensor product). This is the Yang-Baxter algebra with R matrix R_{12} (or $\check{R}_{12}$ in the other form), acting like some sort of generalized structure constants (hence the name algebra). Note that both R_{12} and $\check{R}_{12}$ act in the auxiliary space alone, and are not operators on the quantum Hilbert space (j). In the first form of the algebra, one interchanges just u and v, and in the second, the whole $L_j^i(u)$ operators.

Then $L_j(u)$ is a particular representation of the YB algebra with such an R matrix, and is the sought-after Lax operator, whose representations correspond to various physical systems, except that at this moment $L_j(u)$ acts on a single spin Hilbert space h_j, not on the whole Hilbert space $\mathcal{H}$. To do that, we must define the "monodromy matrix" $T(u)$

$$T(u) = L_L(u)...L_2(u)L_1(u) \tag{63}$$

and all matrices on both sides act on the same (2d) auxiliary space, and then $T(u)$ is an operator on the whole $\mathcal{H}$, depending on all the 3L spin operators. The Yang-Baxter algebra is now

$$R_{12}(u,v)T^1(u)T^2(v) = T^2(v)T^1(u)R_{12}(u,v) \tag{64}$$

(and a similar genealization for the $\check{R}_{12}$ relation), and then $T(u)$ is the true analog of the Lax operator (even though $L_j(u)$ is called that). Here as before $R_{12}(u,v)$ is a matrix in the product auxiliary space, i.e. a 4×4 matrix.

Then the Lax operators $L_j(u)$ act as a connection, defining parallel transport (hence it is also called Lax connection), via the Lax equation

$$\psi_{j+1} = L_j\psi_j \tag{65}$$

with ψ_j a 2d auxiliary space vector and an element of h_j. This is the discretized version of the Dirac equation

$$\begin{aligned}&(\partial_x + A_x(x))\psi(x) = 0 \Rightarrow \psi(x+dx) = (1 + A_x(x)dx)\psi(x)\\ &\Rightarrow \psi(x_2) = Pexp[\int_{x_1}^{x_2} A_x(x)dx]\psi(x_1)\end{aligned} \tag{66}$$

where $1 + A_x dx \to L_j$ and $U = Pexp[\oint A_x(x)dx] \to T$ and the fact that A_x (and L_j) is a connection means that $\psi(x)$ (and ψ_j) are locally gauge invariant solutions.

Then it is easy to show from (64) by multiplying with R^{-1} and taking the auxiliary space trace that the traces commute, i.e.

$$trT^1(u)trT^2(v) = trT^2(v)trT^1(u) \tag{67}$$

and one can expand $T(u)$ into a power series, finding in fact that it is a polynomial in u of order L, i.e.

$$T(u) = \sum_{n=0}^{L} t_n u^n \tag{68}$$

with the order L term being proportional to the identity.

Because the traces commute, the objects trt_n are commuting operators, L-1 of them nontrivial (the order L one is trivial and t_{L-1} has zero trace), and the Hamiltonian $\hat{H}$ and the discrete momentum on the lattice $\hat{P}$ belong to this family.

Using that

$$L_j^a(-i/2) = -iP_{j,a};\quad \frac{d}{du}L_j^a(u) = \mathbf{1} \tag{69}$$

one finds that the shift operator $U = P_{12}P_{23}...P_{N-1,N}$, satisfying

$$U^{-1}X_jU = X_{j-1} \tag{70}$$

for any set of operators X_j defined at sites j (like the variables $X_j^\alpha = \sigma_j^\alpha$ themselves), is given by

$$U = i^L trT(-i/2) \equiv e^{i\hat{P}} \tag{71}$$

and then the momentum operator on the discrete lattice is $\hat{P}$.

This was somewhat to be expected, since as we saw, the monodromy $T(u)$ defines parallel transport around the chain on the solution to the Lax equation (analog of the Dirac equation).

Similarly, one finds that the Hamiltonian is

$$\hat{H} = -JL - 2iJ\frac{d}{du}\ln trT(-i/2) \tag{72}$$

or compactly having $\hat{H}$ and $\hat{P}$ together in

$$\tau(u) \equiv -i\ln trT(u) = -\frac{\pi L}{2} + \hat{P} + (u+i/2)(\frac{\hat{H}}{2J} + \frac{L}{2}) + \mathcal{O}(u+i/2)^2 \tag{73}$$

The momentum can be written in terms of U also as

$$\hat{P} = \frac{2\pi}{L}\sum_{n=1}^{L-1}(\frac{1}{2} + \frac{U^n}{e^{-in\pi/L}-1}) \tag{74}$$

Note that now the Lax equation (65), or more precisely the eigenvalue equation derived from it,

$$T(u)\psi = t(u)\psi \tag{75}$$

is in some sense equivalent to the Schrodinger equation (as the hamiltonian is found among the traces of T), thus it deserves the name of Lax equation.

Bethe ansatz equations (algebraic and connection with coordinate)

We had

$$\check{R}(u,v) = \begin{pmatrix} f(v,u) & & & \\ & g(v,u) & 1 & \\ & 1 & g(v,u) & \\ & & & f(v,u) \end{pmatrix}$$

$$L_j(u) = \begin{pmatrix} u - \frac{i}{2}\sigma_j^z & -i\sigma_j^- \\ -i\sigma_j^+ & u + \frac{i}{2}\sigma_j^z \end{pmatrix}$$

$$f(v,u) = 1 + \frac{i}{v-u}; \quad g(v,u) = \frac{i}{v-u} \tag{76}$$

It follows then also that

$$T(u)=\begin{pmatrix}A(u) & B(u)\\ C(u) & D(u)\end{pmatrix};\quad trT(u)=A(u)+D(u) \tag{77}$$

and by solving the Yang-Baxter equations (64) we get the relations

$$\begin{aligned}&A(u)B(v)=f(u,v)B(v)A(u)-g(u,v)B(u)A(v);\quad [B(u),B(v)]=0\\ &D(u)B(v)=f(v,u)B(v)D(u)-g(v,u)B(u)D(v)\end{aligned} \tag{78}$$

Consider now the Fock vacuum

$$|\uparrow>_L\otimes ...\otimes|\uparrow>_1\equiv\begin{pmatrix}1\\ 0\end{pmatrix}^{\otimes L}\equiv|\Omega> \tag{79}$$

and act on it with $T(u)$. One obtains

$$\begin{aligned}&A(u)|\Omega>=a(u)|\Omega>;\quad a(u)=(u-i/2)^L\\ &D(u)|\Omega>=d(u)|\Omega>;\quad d(u)=(u+i/2)^L\\ &C(u)|\Omega>=0;\quad [B(u),B(v)]=0\end{aligned} \tag{80}$$

Then the **algebraic Bethe ansatz** (equivalent of the coordinate Bethe ansatz) is the "assumption" that the $C(u)$ are annihilation operators and the $B(u)$ are creation operators. Then consider the state

$$|\vec{u}>=B(u_1)...B(u_N)|\Omega> \tag{81}$$

Then we get that

$$\begin{aligned}&trT(u)|\vec{u}>=A(u)|\vec{u}>+D(u)|\vec{u}>\\ &=a(u)\prod_{j=1}^N f(u,u_j)|\vec{u}>+d(u)\prod_{j=1}^N f(u_j,u)|\vec{u}>\\ &-\sum_{n=1}^N a(u_n)g(u,u_n)\prod_{j\neq n,j=1}^N f(u_n.u_j)[B(u)\prod_{k\neq n,k=1}^N B(u_k)|\Omega>]\\ &-\sum_{n=1}^N d(u_n)g(u_n,u)\prod_{j\neq n,j=1}^N f(u_j,u_n)[B(u)\prod_{k\neq n,k=1}^N B(u_k)|\Omega>]\end{aligned} \tag{82}$$

which means that the state $|\vec{u}>= B(u_1)...B(u_N)|\Omega>$ is an eigenstate of $trT(u)$ (and thus both of momentum and of Hamiltonian, that can be obtained from it), with eigenvalue

$$\lambda(u) = a(u)\prod_{j=1}^{N} f(u,u_j) + d(u)\prod_{j=1}^{N} f(u_j,u) \tag{83}$$

provided the last two lines in (82) cancel, which means that one needs to satisfy the condition

$$\frac{a(u_n)g(u,u_n)}{d(u_n)g(u_n,u)} + \prod_{j\neq n,j=1}^{N} \frac{f(u_j,u_n)}{f(u_n,u_j)} = 0 \tag{84}$$

and by substituting the explicit for for $f(u,v), g(u,v)$ we get the **algebraic Bethe equations**

$$\left(\frac{u_n - i/2}{u_n + i/2}\right)^L = \prod_{j\neq n,j=1}^{N}\left(\frac{u_n - u_j - i}{u_n - u_j + i}\right); \quad n = 1, N \tag{85}$$

which are exactly equal to the coordinate space Bethe equations (49).

The states $|\vec{u}>$ are the highest-weight states of the spin operator $S^z = \sum_{j=1}^{N}\sigma_j^z/2$, since

$$S^z|\vec{u}>= \frac{1}{2}(L - 2N)|\vec{u}>; \quad S^+|\vec{u}>= 0 \tag{86}$$

Solving the algebraic Bethe equations (85) for N=1, i.e. $((u - i/2)/(u + i/2))^L = 1$ is done by

$$\frac{u + i/2}{u - i/2} \equiv e^{ip} \Rightarrow p = \frac{2\pi n}{L}; \quad n = 1, ..., L \tag{87}$$

thus there are L-1 states with one spin flipped, and the p are the momenta of a free particle with periodic boundary conditions, the same as we obtained in the coordinate Bethe ansatz!

Since we work in a spin 1/2 chain, we obtain that the creation/ annihilation operators $B(u)/C(u)$ are fermionic, thus the Bether roots u_i must be different (otherwise the wavefunction is zero, as we saw in the coordinate Bethe ansatz).

Moreover, since

$$\lambda(u) = trT(u) = a(u)\prod_{j=1}^{N} f(u,u_j) + d(u)\prod_{j=1}^{N} f(u_j,u)$$
$$= (\prod_{j=1}^{N}\frac{1}{u-u_j})[(u-i/2)^L\prod_{j=1}^{N}(u-u_j+i) + (u+i/2)^L\prod_{j=1}^{N}(u-u_j-i)]$$
$$\tau(u) \equiv -i\ln trT(u) = -\frac{\pi L}{2} + \hat{P} + (u+i/2)(\frac{\hat{H}}{2J}+\frac{L}{2}) + ... \tag{88}$$

one gets that

$$U = e^{i\hat{P}} = \prod_{j=1}^{N}\frac{u_j - i/2}{u_j + i/2} \Rightarrow P = \sum_{j+1}^{N} p_j (mod\ 2\pi) \tag{89}$$

where the p_j are the momenta of the free particles just defined in (87). Then one also finds that the energy is

$$\frac{E}{2J} = \frac{L}{2} - \sum_{j=1}^{N}\frac{1}{u_j^2+1/4} = \frac{L}{2} + \sum_{j=1}^{N} 2(\cos p_j - 1) \tag{90}$$

One has now a quasiparticle interpretation (magnons), quasiparticles being created by $B(u)$, decreasing the spin S^z by one unit, has momentum p_j and energy $\epsilon_j = 2(\cos p_j - 1)$.

The reason why u is called rapidity (and we used the derivation of the Yang-Baxter equation where u was a rapidity-like variable) even though we have a *nonrelativistic* theory is the following. We can easily prove that

$$\frac{dp}{du} = -\frac{1}{u^2+1/4} = \epsilon(u); \quad u = \frac{1}{2}\cot(p/2) \tag{91}$$

compared to

$$p = m\sinh\mu; \quad E = m\cosh\mu \Rightarrow E = \frac{dp}{d\mu}; \;\; \mu = arcsinh(p/m) \tag{92}$$

Here u_j's are called **Bethe roots** and as we see are the nonrelativistic analogs of rapidities, in that for both $E = dp/d\mu$. If the "momenta" p_i are real, the Bethe roots u must be real, however note that we don't really need that. All we need is to find solutions with real *energies*, and from the relation (90) we see that for that it is enough to have pairs, for any root u_k, to also have its conjugate u_k^* (or in terms of p, for any p_i, also p_i^*). Thus the sought-for roots can span the whole complex plane.

In fact, one can *prove* that the solutions of the Bethe equations are such that one always has complex conjugate pairs (without needing reality of the energy), i.e. the complex conjugate of a solution (set) $\{u_k\}$ of the Bethe equations is itself.

The Bethe root solutions in general come in sets, and each set is found at the intersection of several curves in the complex plane. For instance, for 2 magnons, the complex roots are situated at the intersection of the circles

$$(2Re(u) - \cot(\frac{n\pi}{L}))^2 + (2Im(u))^2 = \frac{1}{(\sin n\pi/L)^2} \quad (93)$$

where n parametrizes sets of solutions, and the curve

$$[\frac{(2Re(u))^2 + (2Im(u)+1)^2}{(2Re(u))^2 + (2Im(u)-1)^2}]^L = \frac{(2Im(u)+1)^2}{(2Im(u)-1)^2} \quad (94)$$

Generalizations

This algebraic Bethe ansatz method can be applied to other spin chains. In particular, it can be applied to find the integrable Hamiltonian for the Heisenberg XXX spin chain for spin s. The calculation is more involved, in particular because the Yang-Baxter algebra (53) must be taken as a starting point and trying to derive Lax operators $L(u)$ and an R matrix $R(u,v)$ from it, and in fact we have an $R^{1/2,s}(u,v)$ and $R^{1/2,s'}$, $R^{ss'}$ as Lax operators (for s=s' a spin representation other than 1/2), and observables (integrals of motion) can be constructed from both Lax operators. One finds the Hamiltonian among these observables and it is

$$H = \sum_{n=1}^{N} \sum_{\alpha} (S_n^\alpha - S_{n+1}^\alpha - (S_n^\alpha - S_{n+1}^\alpha)^2) \quad (95)$$

One can also find the Lax operator for the XXZ model by considering that it deforms the space like a q-deformation, ($q = e^{i\gamma}$) i.e.

$$x \to [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \prod_{n=-\infty}^{+\infty} (\frac{x + n\pi\gamma^{-1}}{1 + n\pi\gamma^{-1}}) \quad (96)$$

and one has the Lax operator (which can be thought of as another representation of Lax operator on the spin chain) is

$$L_j^{XXZ}(u) = \prod_{k=-\infty}^{\infty} L_j^{XXX}(u + ik\pi\gamma^{-1}) \tag{97}$$

One can also in principle do a similar construction for "spin chains" corresponding to a representation $\mathcal{R}$ of a group G.

Thermodynamic limit of Bethe ansatz

Going back to the XXX spin chain, we want to take the thermodynamic limit of $N, L \to \infty$, N/L=fixed. Then the Bethe equations (85) rewritten by taking the log as

$$L\ln(\frac{u_i + i/2}{u_i - i/2}) = \sum_{k\neq i, k=1}^{N} \ln(\frac{u_i - u_k + i}{u_i - u_k - i}) - 2\pi i n_i \tag{98}$$

where n_i are arbitrary integers for each root u_i, so that the set $\{n_i\}$ can be understood as a quantum number for the multiparticle system $\{u_i\}$, become (assuming self-consistently that $p_i \sim 1/L$, implying that $u_i \sim L$)

$$\frac{1}{x_i} + 2\pi n_i = \frac{2}{L}\sum_{k\neq i; k=1}^{N} \frac{1}{x_i - x_k} \tag{99}$$

where $x_i = u_i/L$ is finite.

We saw that the sought-for roots can be anywhere in the complex plane, except that for any root u_i, we must also have its complex conjugate u_i^*, if we are to have a real energy, this being also found from the Bethe equations. Moreover, in the limit $L \to \infty$, if two magnons have the same real part, i.e. $Re(u_n) = Re(u_j)$, then they must have different imaginary parts (in order to have different rapidities, as needed due to the fermionic nature of the chain, as we saw). But then we see from the Bethe equations (85) that the l.h.s tends to either zero or infinity, meaning the r.h.s. must have either a zero or a pole (i.e. the 2-body S matrix $S(p_1, p_2)$ has a pole), thus $u_n = u_j \pm i$. This corresponds to two magnons of same energy forming a bound state by splitting their rapidities in the complex plane. In the $L \to \infty$ limit then the roots with the same real part will be separated in an array with $u_k = Re(u) + ik$. It is obvious from (99), by analyzing the change under $u_i \to u_{i+1}$, that for roots lying on an array, u_k cannot change, thus $n_i = n_C$ characterizes the array and then the set of k's represents the quantum numbers of the solution $\{u_k\}$.

In terms of x_i the separation $u_{k+1} - u_k$ becomes $x_k - x_{k+1} \sim 1/L \to 0$, and this is valid even if $N \to \infty$, in which case however the arrays are not vertical lines anymore, but can curve.

Thus in the thermodynamic limit the roots u_i accumulate on smooth contours in the complex u plane known as **Bethe strings**, symmetric from the real axis, turning the algebraic Bethe equations into an integral equation. Defining the Bethe root density

$$\rho(x) \equiv \frac{1}{L}\sum_{j=1}^{N} \delta(x - x_j); \quad \int_C dx \rho(x) = \frac{N}{L} \tag{100}$$

where C is the countour that supports the density, we get the singular integral Bethe equation

$$2P\int dy \frac{\rho(y)}{y-x} = -\frac{1}{x} + 2\pi n_{C(u)}; \quad x \in C; \tag{101}$$

where $P\int$ stands for the principal part prescription and $n_{C(u)}$ are integers assumed to be constant on each smooth component C_n of the density support $C = U\ C_n$ in the complex plane. The energy associated with a contour is given by the thermodynamic limit of (90), which is

$$E = \frac{1}{L}\int_C dx \frac{\rho(x)}{x^2} \tag{102}$$

From the $x \to 0$ limit of (101), or the $u \to 0$ limit of the Bethe equations (98), we get a condition for the quantization of the total momentum in the thermodynamic limit,

$$\hat{P} = \sum_i p_i \simeq \sum_i \frac{1}{u_i} = \frac{1}{L}\sum_i \frac{1}{x_i} = 2\pi m \to \int_C dx \frac{\rho(x)}{x} = 2\pi m \tag{103}$$

Note that the total momentum can in fact be $2\pi m/L$ in general, so this statement is equivalent to cyclicity (translational invariance along the chain), i.e. it restricts the types of chain to the ones similar to dual strings.

These equations (100,101,102,103) can be written independently for various Bethe strings (various smooth components C_n), corresponding to various macroscopic solutions in the thermodynamic limit of the XXX spin chain.

1.5 Symmetries and Yangian

We have seen that the Yangian can be defined as the algebra $\mathcal{A}$ that appears in the Yang-Baxter algebra (53) for the XXX spin chain, however this notion can be generalized to a chain with an arbitrary representation $\mathcal{R}$ of an arbitrary Lie algebra G. Its abstract definition is as follows. A Yangian algebra $Y(G)$ is an associative Hopf algebra generated by the elements J^A and Q^B with

$$[J^A, J^B] = f_C^{AB} J^C; \quad [J^A, Q^B] = F_C^{AB} Q^C \tag{104}$$

and the Serre relations

$$\begin{aligned}
&[Q^A, [Q^B, J^C]] + [Q^B, [Q^C, J^A]] + [Q^C, [Q^A, J^B]] \\
&= \frac{1}{24} f^{ADK} f^{BEL} F^{CFM} f_{KLM} \{J_D, J_E, J_F\}
\end{aligned} \tag{105}$$

$$\begin{aligned}
&[[Q^A, Q^B], [J^C, Q^D]] + [[Q^C, Q^D], [J^A, Q^B]] \\
&= \frac{1}{24} (f^{AGL} f^{BEM} f^{KFN} f_{LMN} f_K^{CD} \\
&+ f^{CGL} f^{DEM} f^{KFN} f_{LMN} f_K AB) \{j_G, J_E, J_F\}
\end{aligned} \tag{106}$$

for J^A taking value in the Lie algebra of an arbitrary semi-simple Lie group G (and Lie algebra indices are lowered and raised with an invariant nondegenerate metric tensor g_{AB}). The symbol $\{A, B, C\}$ denotes symmetrized product. The Yangian has then the basis $\mathcal{J}_n^A$, such that $\mathcal{J}_0^A = J^A, \mathcal{J}_1^A = Q^A$ and for $n > 1$ we have an n-local operator arising in the (n-1)-form commutator of Q's. Note that for $SU(2)$ the relation (105) is trivial, and for $SU(N), N \geq 3$, (105) implies (106).

For a two dimensional model with a group G of symmetries, with Lie algebra (defined by generators T_A), given by $[T_A, T_B] = f_{AB}^C T_C$, and the action of G generated by a current j_μ^A that is conserved, i.e. $\partial_\mu j^{\mu A} = 0$, nonlocal charges Q^A arise if, in addition, the Lie algebra valued current $j_\mu = \sum j_\mu^A T_A$ can be interpreted as a flat connection, i.e.

$$\partial_\mu j_\nu - \partial_\nu j_\mu + [j_\mu, j_\nu] = 0 \tag{107}$$

Then as usual the conserved charges that generate the action of G are

$$J^A = \int_{-\infty}^{+\infty} dx j^{0A}(x, t) \tag{108}$$

But then in addition, the quantities

$$Q^A = f^A_{BC} \int_{-\infty}^{+\infty} dx \int_x^{+\infty} dy j^{0B}(x,t) j^{0C}(y,t) - 2 \int_{-\infty}^{+\infty} dx j^A_a(x,t) \tag{109}$$

are also conserved, and J^A and Q^A generate a Yangian algebra.

For the discrete case, of spin systems ("spin chains"), i.e. at site i the "spins" transform in the representation $\mathcal{R}$ of the group G, we also have Yangian symmetry. The total charge generator is obviously

$$J^A = \sum_i J^A_i \tag{110}$$

For a general group G and representation $\mathcal{R}$ there is no satisfactory definition of Q^A, however for $G = SU(N)$ we can define it for any $\mathcal{R}$. In certain representations (see below), it is just the generalization of (109) as

$$Q^A = f^A_{BC} \sum_{i<j} J^B_j J^C_j \tag{111}$$

Indeed, for $SU(N)$, whose Lie algebra is the space of traceless $N \times N$ matrices, i.e. $J^a{}_b$, with $\sum_a J^a{}_a = 0$, and the generators Q^A are also traceless $N \times N$ matrices $Q^a{}_b$, the Yangian is

$$\begin{aligned}
[J^a{}_b, J^c{}_d] &= \delta^c{}_b J^a{}_d - \delta^a{}_b J^c{}_b \\
[J^a{}_b, Q^c{}_d] &= \delta^c{}_b Q^a{}_d - \delta^a{}_b Q^c{}_b
\end{aligned} \tag{112}$$

with the Serre relations (equivalent to the ones before)

$$\begin{aligned}
&[J^a{}_b, [Q^c{}_d, Q^e{}_f]] - [Q^a{}_b, [J^c{}_d, Q^e{}_f]] \\
&= \frac{h^2}{4} \sum_{p,q} ([J^a{}_b, [J^c{}_p J^p{}_d, J^e{}_q J^q{}_f]] - [J^a{}_p J^p{}_b, [J^c{}_d, J^e{}_q J^q{}_f]])
\end{aligned} \tag{113}$$

which form of the Yangian is more common in spin chains.

For a single spin in $SU(N)$, one can represent the Yangian with $Q^A = Q^a{}_b = 0$ for some representations (there is a criterion), for example fundamental and antisymmetric k-th rank tensors, for any k (but not for the adjoint and others). Since the Yangian has a "coproduct," i.e. a map $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$

for an algebra $\mathcal{A}$ that is, a homomorphism of algebras (and obeys some more axioms), explicitly

$$\begin{aligned}\Delta(J^A) &= J^A \otimes \mathbf{1} + \mathbf{1} \otimes J^A \\ \Delta(Q^A) &= Q^A \otimes \mathbf{1} + \mathbf{1} \otimes Q^A + f^A_{BC} J^B \otimes J^C\end{aligned} \tag{114}$$

we can extend the representation on a single spin to representation on a spin chain through repeated application of Δ. Then when $Q^A = 0$ is a representation of the Yangian on a single spin Hilbert space (for the above representations of $SU(N)$), (111) is a representation of Q^A for the spin chain. Moreover, for ANY representation of $SU(N)$, we can still represent the Q^A's on a single spin by

$$Q^a{}_b = \frac{1}{2} J^a{}_p J^p{}_b \tag{115}$$

and using the coproduct Δ one can in principle find the representation for the whole chain, but in general it is difficult.

This whole discussion goes through if we replace $SU(N)$ by $U(N)$ or even the supergroups $U(N|M)$ and $SU(N|M)$ if $N \neq M$, but for $N = M$ it is slightly different.

On a spin chain, operators that are a sum of operators that are local on the chain, and commute with the whole Yangian, are called Hamiltonians of the integrable spin chain. In fact, as we saw, there are exactly L-1 charges Q_k that commute with the Hamiltonian and among themselves, i.e. $[Q_k, Q_l] = 0$, and these are coefficients of the series expansion around $u = 0$ of $trT(u)$. Then Q_k will in fact involve up to k neighbouring interactions. Moreover, these conserved quantities (integrals of motion) are related to the generators of the Yangian algebra, in that the $\mathcal{J}^A_n$ arise from the coefficients in the expansion *around* $u = \infty$ of $T(u)$ (without trace!)

More precisely, at least for $SU(N)$, the monodromy matrix $T(u)$, satisfying the Yang-Baxter algebra

$$\mathcal{R}(u-v)(T(u) \otimes \mathbf{1})(\mathbf{1} \otimes T(v)) = (\mathbf{1} \otimes T(v))(T(u) \otimes \mathbf{1})\mathcal{R}(u-v) \tag{116}$$

admits an expansion around $u = \infty$,

$$T(u) = \mathbf{1} + h \sum_{n=0}^{\infty} u^{-n-1} t^{(n)} \tag{117}$$

Here all objects are $SU(N)$ matrices, thus for instance $T(u) = T^A(u) = T^a{}_b(u)$ and we have

$$P^A = \mathcal{J}_0^A = t^{(0),a}{}_b$$
$$Q^A = \mathcal{J}_1^A = \mathcal{J}_1{}^a{}_b = t^{(1)a}{}_b - \frac{h}{2} t^{(0)a}{}_d t^{(0)d}{}_b \tag{118}$$

and since as we saw the whole Yangian is defined by P^A and Q^A that is enough. (Note that this means that the $t^{(n)}$ are *non-local* objects.) The analog of this statement is that now all the expansion in $1/u$ can be reconstructed from the knowledge of $t^{(0)a}{}_b$ and $t^{(1)a}{}_b$. The fact that the Yangian is obtained from the series expansion of $T(u)$ is what we meant at the begining of this section when we said that the Yangian algebra is the algebra $\mathcal{A}$ in the definition of the Yang-Baxter algebra for the XXX spin chain.

One the other hand as we saw

$$trT(u) = \sum_{n=0}^{L} f^{(n)} u^n \tag{119}$$

and then $f^{(n)}$ are conserved *local* quantities (integrals of motion).

In the continuum version of the XXX $SU(2)$ Hamiltonian,

$$H = -\frac{1}{4}\int_0^L dxtr(\partial_x S \partial_x S); \quad S(x) \equiv S^i(x)\sigma_i, \quad S^i(x)S^i(x) = s^2 = \text{fixed} \tag{120}$$

the monodromy matrix $T(x,u)$ is

$$T(x,u) = P\ exp[-\int_0^x A_x(y,u)dy] \tag{121}$$

where the Lax connection A_μ is

$$A_x = \frac{i}{u} S(x); \quad A_t = -\frac{2is^2}{u^2} S(x) + \frac{1}{2u}[S(x), \partial_x S(x)] \tag{122}$$

One can indeed check that if $S^i(x)S^i(x) = s^2$ H discretizes as

$$\int dx \partial_x S^i \partial_x S^i = -\int dx S^i \partial_x^2 S^i$$
$$\to \sum_n S_n^i (S_{n+1}^i + S_{n-1}^i - 2S_n^i) = -2Ls^2 + 2\sum_n S_n^i S_{n+1}^i \tag{123}$$

that is, to the spin 1/2 XXX Hamiltonian.

The Lax equation is now the Dirac equation

$$(\partial_x + A_x)\psi = 0 \tag{124}$$

and finding the Bethe roots $\{u_i\}$ that diagonalize $trT(u)$ is now equivalent to finding quasiperiodic solutions $\{u, \psi_u(x)\}$ to the Lax equation (Dirac equation), i.e. solutions that diagonalize its monodromy. The "roots" u of the solutions lie on curves= "Bethe strings".

Then the expansion of $trT(u)$ gives the integrals of motion by

$$trT(u) = 2\cos P_0(u); \quad P_0(u) = -\frac{sL}{u} + \sum_{n=0}^{\infty} u^n I_n \tag{125}$$

and I_n are integrals of motion, for instance I_0 is the momentum and I_1 is the energy.

Then the Yangian is generated by charges Q_n^i, with

$$Q^i(u) \equiv \sum_{n=0}^{\infty} u^{-n-1} Q_n^i \tag{126}$$

and then the monodromy matrix is

$$T(u) = \frac{1}{2}W(u)\mathbf{1} - \frac{i}{2}W^{-1}(u)Q^i\sigma_i; \quad W(u) \equiv \sqrt{2 + \sqrt{4 - \vec{Q}^2(u)}} \tag{127}$$

but on the other hand

$$\vec{Q}^2(u) = 4\sin^2(2P_0(u)) \tag{128}$$

and thus its expansion around $u = 0$ generates the integrals of motion.

1.6 Thermodynamic Bethe ansatz (TBA)

The thermodynamic Bethe ansatz is a method for deriving the thermodynamics of massive relativistic integrable 1+1d theories in the large volume limit from their factorizable S matrices. One starts by writing down Bethe ansatz equations, and then one takes a thermodynamic limit, defining densities of states, and writing an integral equation, like it was done in (99)-(101). The most important difference from the usual Bethe ansatz is that one deals with real particles, which have real rapidities, as opposed to the usual Bethe ansatz, where as we saw in (42) the "rapidities" $u_{ik}=u_i-u_k$ of the poles of the two-body S matrix are imaginary ($u_{ik}=i$), corresponding to bound states, and we found that for real energies we only need to have the Bethe roots come in complex conjugate pairs. Also, the theories to which we apply TBA are relativistic. TBA was developped into the present tool by Al.B. Zamolodchikov in 1989, based on a famous paper of Yang and Yang from 1969.

In this subsection we will denote actual rapidities by θ, i.e.

$$E=m\cosh\theta,\quad p=m\sinh\theta \tag{129}$$

A purely elastic scattering theory, i.e. a 1+1d QFT with S-matrix that is factorizable and diagonal (thus automatically satisfies the Yang-Baxter equation), can still have a complicated bound state structure. The S matrix is defined as usual ($\theta_{ab}=|\theta_a-\theta_b|$)

$$|a(\theta_a),b(\theta)b>_{in}=S_{ab}(\theta_{ab})|a(\theta_a),b(\theta_b)>_{out} \tag{130}$$

The requirements of real analiticity, unitarity, crossing symmetry, meromorphicity in θ and polynomial boundedness in momenta implies that the S matrix looks like

$$S_{ab}(\theta)=\prod_{\alpha\in A_{ab}}f_\alpha(\theta);\quad f_\alpha(\theta)=\frac{\sinh((\theta+i\pi\alpha)/2)}{\sinh((\theta+i\pi\alpha)/2)} \tag{131}$$

where the sets A_{ab} of α's depend on the theory. For minimal theories (i.e. which are massive perturbations of nontrivial CFTs), a simple pole of $S_{ab}(\theta)$ at $\theta_{ab}=iu_{ab}^c$ (imaginary relative rapidity) indicates a *bound state* c of a and b with mass

$$m_c^2=m_a^2+m_b^2+2m_am_b\cos u_{ab}^c \tag{132}$$

and similarly for any order pole (but for nonminimal theories the poles cannot be interpreted as bound state poles, says [5]).

If we have an elastic minimal theory scattering theory, defined on a circle of length L (we will take $L\rightarrow\infty$), with N particles, N_a of species a, at positions $x_1,...,x_N$, then asymptotically the wave function is (when the particles are well separated)

$$\psi(x_1,...x_N)=exp(i\sum_j p_jx_j)\sum_{Q\in S_N}A(Q)\Theta(x_Q) \tag{133}$$

where Q are permutations, $\Theta(x_Q) = 1$ only if $x < Q_1 < ...x_{Q_N}$ and zero otherwise and if Q and Q' differ just by permuting i and j we have

$$A(Q') = S_{ij}(\theta_i - \theta_j)A(Q) \tag{134}$$

In terms of the phase shifts $\delta_{ij}(\theta_i - \theta_j) = -i \ln S_{ij}(\theta_i - \theta_j)$, this expression is essentially the one Bethe took in (47). That was for the case of "scattering" in the nonrelativistic spin chain, but now we have a more general relativistic 1+1d massive theory.

Again imposing periodicity conditions on the asymptotic wavefunction as Bethe did, we get the exact analog of the Bethe equations (49), namely

$$e^{iLm_i \sinh\theta_i} \prod_{j\neq i;j} S_{ij}(\theta_i - \theta_j) = (-1)^{F_i} \tag{135}$$

($p = m\sinh\theta$), where the $(-1)^{F_i}$ factors arise for fermions, where we put antiperiodic conditions. Taking the log of this equation gives the Bethe ansatz equation in this case, namely

$$Lm_i \sinh\theta_i + \sum_{j\neq i,j} \delta_{ij}(\theta_i - \theta_j) = 2\pi n_i, \quad i = 1, N \tag{136}$$

where $\{n_i\}$ can be taken to be the quantum number of the state of the multi-particle system, and n_i is integer for bosons and half integer for fermions. From these equations we can calculate the momenta (given by θ_i) of a multi-particle state in the box L. The dynamics of the theory is encoded in the phase shifts $\delta_{ij}(\theta)$, that one can in principle calculate, given the theory. Note that unlike the case of the usual Bethe ansatz, where as we saw, for real energy all we needed was complex conjugate pairs of "rapidities" (Bethe roots) u_i, here all rapidities are necessarily real, as we are interested in real particles!

Unlike the case of the Heisenberg XXX chain, the Bethe ansatz is not necessarily justified (Bethe proved that it solves the XXX Hamiltonian and the energy is just the sum of free energies, thus the theory is in some sense free). For a general theory, at finite L, the equation (136) is only approximate, since the asymptotic wavefunction will be correct only up to 1/L corrections (for Bethe it was exact). However, at $L \to \infty$, the Bethe equations become exact!

So the thermodynamic limit of the above case, with N particles living in a box of length L, with $L, N_a \to \infty$, N_a/L finite will be exact and one has to take a continuum limit of (136). The rapidity density $\rho^{(r)}(\theta)$ is defined analogously to (100), i.e. the number of particles of species a with rapidities in an interval $\Delta\theta$, divided by $L\Delta\theta$.

Defining

$$J_a(\theta) = m\sinh\theta + 2\pi \sum_{b=1}^{n} (\delta_{ab} * \rho_b^{(r)})(\theta)$$

$$(f * g)(\theta) \equiv \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} f(\theta - \theta') g(\theta') \tag{137}$$

the Bethe equations (136) become just $J_a(\theta) = 2\pi n_{a,i}/L$, giving the solutions (*Bethe roots*) $\theta = \theta_{a,i}$. The function $J_a(\theta)$ is monotonically increasing. The Bethe roots (solutions) that are actually occupied (occur in the system) have density $\rho_a^{(r)}(\theta)$, and those that are skipped (the corresponding integers $n_{a,i}$ do not occur) are called *holes*, and have a density $\rho_a^{(h)}(\theta)$. Then the Bethe equations $J_a(\theta) = 2\pi n_{a,i}/L$ imply that the total density of roots and holes is

$$\rho_a(\theta) = \rho_a^{(r)}(\theta) + \rho_a^{(h)}(\theta) = \frac{1}{2\pi}\frac{d}{d\theta} J_a(\theta)$$

$$= \frac{m_a}{2\pi}\cosh\theta + \sum_{b=1}^{n} (\phi_{ab} * \rho_b^{(r)})(\theta); \quad \phi_{ab}(\theta) \equiv \frac{d}{d\theta}\delta_{ab}(\theta) \tag{138}$$

Thermodynamics

Now we are ready to define the thermodynamics of the theory. Define, in analogy with the familiar relation for the fraction of occupied states of species a,

$$\frac{\rho_a^{(r)}(\theta)}{\rho_a(\theta)} = \frac{1}{e^{(E_a(\theta)-\mu_a)/T} + 1} \tag{139}$$

the quantity $\epsilon_a(\theta)$ by

$$\frac{\rho_a^{(r)}(\theta)}{\rho_a(\theta)} = \frac{1}{e^{\epsilon_a} + 1} \tag{140}$$

such that $E_a(\theta) = T\epsilon_a(\theta) + \mu_a$ is the "dressed" one-particle excitation energy.

The entropy per unit length has the standard statistical mechanics expression

$$s(\rho, \rho^{(r)}) = \sum_{a=1}^{n} s_a(\rho_a, \rho_a^{(r)}) = \sum_{a=1}^{n} \int_{-\infty}^{\infty} d\theta$$

$$[\rho_a \ln\rho_a - \rho_a^{(r)} \ln\rho_a^{(r)} - (\rho_a - \rho_a^{(r)}) \ln(\rho_a - \rho_a^{(r)})] \tag{141}$$

and the energy per unit length is

$$h(\rho^{(r)}) = \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta \rho_a^{(r)}(\theta) m_a \cosh\theta \tag{142}$$

The equilibrium thermodynamics is then found by minimizing the free energy per unit length,

$$f(\rho) = h(\rho^{(r)}) - Ts(\rho, \rho^{(r)}) \tag{143}$$

subject to fixed particle densities

$$D_a \equiv \frac{N_a}{L} = \int_{-\infty}^{+\infty} d\theta \rho_a^{(r)}(\theta) \tag{144}$$

which are introduced with Lagrange multipliers= chemical potentials μ_a, as usual. The extremization of $f(\rho)$ leads to the **TBA equations**

$$\epsilon_a(\theta) = -\hat{\mu}_a r + \hat{m}_a r \cosh\theta - \sum_{b=1}^{n} (\phi_{ab} * L_b)(\theta); \quad a = 1, n \tag{145}$$

where $\hat{\mu}_a = \mu_a/m_1, \hat{m}_a = m_a/m_1$ and m_1 is the smallest mass, and $r = Rm_1, R = 1/T$, thus $\hat{\mu}_a r = \mu_a/T, \hat{m}_a r = m_a/T$. Also,

$$L_a(\theta) \equiv \ln(1 + exp(-\epsilon_a(\theta))) \tag{146}$$

Thus, the TBA equations (145) are equations for the $\epsilon_a(\theta)$'s (actually, $\epsilon_a(\theta, T, \mu_a)$), which then from their definition (140) together with the Bethe particle density equations (derived from the Bethe equations) (138), give all the densities $\rho_a(\theta), \rho_a^{(r)}(\theta)$. Then any thermodynamic quantity can be deduced, for instance from the pressure

$$P(T, \mu) = \frac{T}{2\pi} \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta L_a(\theta, m/T, \mu) m_a \cosh\theta \tag{147}$$

(or the free energy $f = -P + \sum_a \mu_a D_a(T, \mu)$) by $dP = sdT + \sum_a D_a d\mu_a$.

1.7 $\mathcal{N}$=4 SYM as an example

For $\mathcal{N}=4$ SYM consider two complex scalars $Z=\phi^1+i\phi^2$ and $W=\phi^3+i\phi^4$ and define the operator with J_1 Z-fields and J_2 W-fields

$$\mathcal{O}_\alpha^{J_1,J_2}=Tr[Z^{J_1}W^{J_2}]+... \qquad (148)$$

where the ... are permutations. At tree leven it has the scaling dimension $\Delta_0^{J_1,J_2}=J_1+J_2$.

Because of mixing, at higher levels one needs to be more careful. One defines the **dilatation operator** $\mathcal{D}$ defined at a point x, whose eigenvalues are the scaling dimensions Δ, by

$$\mathcal{D}\mathcal{O}_\alpha^{J_1,J_2}(x)=\sum_\beta \mathcal{D}_{\alpha\beta}\mathcal{O}_\beta^{J_1,J_2}(x);\quad \mathcal{D}=\sum_{n=0}^\infty \mathcal{D}^{(n)} \qquad (149)$$

constructed as to attach relevant diagrams to the open legs of the "incoming" trace operators, and in perturbation theory, when the n-th term is of order g_{YM}^{2n}, we have (i.e. one can calculate)

$$\mathcal{D}^{(0)}=Tr(Z\check{Z}+W\check{W});\quad \mathcal{D}^{(1)}=-\frac{g_{YM}^2}{8\pi^2}Tr[Z,W][\check{Z}\check{W}];\quad \check{Z}_{ij}\equiv\frac{d}{dZ_{ij}} \qquad (150)$$

The planar sector of the gauge theory corresponds to the free $AdS_5\times S_5$ string (no order g_s string interactions). In this planar case we have

$$\mathcal{D}^{(1)}_{planar}=\frac{\lambda}{8\pi^2}\sum_{i=1}^L(1-P_{i,i+1})=\frac{\lambda}{8\pi^2}H_{XXX_{1/2}} \qquad (151)$$

thus is nothing but the Heisenberg XXX spin chain hamiltonian.

Heisenberg spin chain results

The "one-magnon" excitation has vanishing energy due to the cyclicity of the trace, and the first stringy excitation is the "two-magnon" excitation. The cyclicity of the trace is instated in the Heisenberg spin chain by requiring a vanishing total momentum

$$P=\sum_{j=1}^N p_j=0 \qquad (152)$$

corresponding to vanishing momentum on the dual closed string in $AdS_5\times S_5$.

The anomalous dimension of the operators to first order planar case is

$$\gamma = \Delta - L = \mathcal{D}^{(1)}_{planar} = \frac{\lambda}{8\pi^2}\sum_j \frac{1}{u_j^2 + 1/4} \tag{153}$$

When the number M of magnons is much smaller than L, and the mode number (dual string momentum) is not large (i.e. $n \ll L$) the spectrum (dual to the free string in $AdS_5 \times S_5$) is given by the creation operators

$$a_n^+ = \frac{1}{\sqrt{L}}\sum_{l=1}^{L} e^{\frac{2\pi i n l}{L}}\sigma_l^- \tag{154}$$

corresponding to the BMN operators in SYM and its anomalous dimension (energy of magnons) is the first order expansion in λ/L^2 of the BMN formula $E = \sum_{n_k}(\sqrt{1+n_k^2\lambda/L^2}-1)$, i.e. (from $p_k = 2\pi n_k/(L-1) \rightarrow u_k \sim L/(2\pi n_k)$)

$$\gamma = \frac{\lambda}{2L^2}\sum_{k=1}^{M} n_k^2 \tag{155}$$

However, when $M \sim L$, even though the interaction between magnons cannot be neglected anymore, we saw that the Heisenberg XXX model has still a Fock spectrum, solved by the Bethe ansatz.

The 2-magnon state (M=2), with $p_2 = -p_1$, thus $u_2 = -u_1$, in the case that the mode number n is large has energy given by the usual Bethe ansatz energy (solving the Bethe ansatz equations by $p_i = 2\pi n/(L-1)$)

$$\gamma = \frac{\lambda}{\pi^2}\sin^2\frac{\pi n}{L-1} \tag{156}$$

that reduces to the BMN formula for small n. The eigenfunctions corresponding to it can also be found from (46), giving (here $L \equiv J+2$)

$$\mathcal{O}_n^{(J,2)} = \sum_{l=0}^{J}\cos(\pi n\frac{2l+1}{J+1})Tr[WZ^lWZ^{J-l}] \tag{157}$$

valid for any n, that reduces to the BMN operator

$$\mathcal{O}_n = \sum_{l=0}^{L}\cos\frac{2\pi n}{J}Tr[WZ^lWZ^{J-l}] \tag{158}$$

for small n and $J \to \infty$.

Various macroscopic strings in $AdS_5 \times S_5$ correspond to various Bethe strings in the XXX spin chain, i.e. to the thermodynamic limit N/L=fixed as $N, L \to \infty$ of the system, with a certain distribution of Bethe roots (Bethe string) for a dual macroscopic string.

We saw that in order to have real energies we needed Bethe strings that are symmetric from the real axis (pairs of complex conjugate roots, (u_i, u_i^*)). Also, $-p_i$ corresponds to $-u_i$, so if we have zero total momentum in the simplest way, namely by having matching pairs of $(+p_i, -p_i)$, that would correspond to matching pairs of $(u_i, -u_i)$. However, this happens only in particular cases, like for a folded string. In general, the zero momentum condition becomes in the thermodynamic limit just

$$2M \int_C du \rho(u) \cot^{-1}(2u) = 0 \tag{159}$$

One also finds that in the case of a circular string the roots condense on the imaginary axis.

String theory dual: $AdS_5 \times S_5$

The strings in $AdS_5 \times S_5$ corresponding to large operators with 2 R-charges (for Z and W) live in the middle of AdS_5 and move in an $S^3 \in S^5$. The worldsheet is then parametrized by X^0 and X^i, i=1,4 with $X^i X^i = 1$, defining an $SU(2)$ element $(Z = X^1 + iX^2, W = X^3 + iX^4)$

$$g = \begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix} \tag{160}$$

and the string action in the conformal gauge is

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \int d\tau [\frac{1}{2} Tr j_a^2 + (\partial_a X^0)^2] \tag{161}$$

where

$$j_a = g^{-1} \partial_a g = \sigma^A \frac{j_a^A}{2i} \tag{162}$$

and

$$\{X^0 = k\tau; tr j_\pm^2 = -2k^2\} \tag{163}$$

solves the X^0 equations of motion and Virasoro constraints.

This action in the presence of the (163) is very similar to the continuum Heisenberg action (120), so we expect to find similar Bethe equations.

The equations of motion of S, together with the j_a definition, give

$$\partial_+ j_- + \partial_- j_+ = 0; \quad \partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0 \tag{164}$$

which can be written together as the flat connection condition (valid at any x)

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0; \quad J_\pm \equiv \frac{j_\pm}{1 \pm x} \tag{165}$$

These flat connections $J_\pm$ are the analogs of the Lax connections A_μ for the 1+1d heisenberg action, thus its monodromy

$$\Omega(x) = Pexp[-\int_0^{2\pi} d\sigma J_\sigma] = Pexp[\int_0^{2\pi} d\sigma \frac{1}{2}(\frac{j_+}{x-1} + \frac{j_-}{x-1})] \tag{166}$$

gives both the Yangian charges and the integrals of motion through its expansions at $x = \infty$ and $x = 0$.

Specifically, the $\Omega(x)$ expansion at $x = \infty$ generates the Yangian charges, and the expansion of

$$tr\Omega(x) = 2\cos p(x) \tag{167}$$

at $x = \pm 1$ gives the local conserved charges =integrals of motion ($tr\Omega(x)$ is time independent on shell). The equivalent of the Lax equation is the Dirac equation

$$D_\sigma \psi = [\partial_\sigma - \frac{1}{2}(\frac{j_+}{x-1} + \frac{j_-}{x-1})]\psi = 0 \tag{168}$$

and finding its quasiperiodic solutions $\{x, \psi_x\}$ is equivalent to solving the system (to the Heisenberg picture equations of motion). Indeed, by definition $\psi(\sigma + 2\pi; x) = \Omega(x)\psi(x)$ for a solution to the Dirac equation, and finding $\{x, \psi_x\}$ corresponds to diagonalization, such that $\psi(\sigma + 2\pi; x) = e^{\pm i p(x)}\psi(\sigma; x)$.

The expansion of $p(x)$ around $x = 0$ and $x = \infty$ is governed by the integral charges Q_L, Q_R under $SU(2)_L$, $SU(2)_R$ of the sigma model, together with an integral charge m, specifically

$$\begin{aligned} p(x) &= -\frac{2\pi Q_R}{\sqrt{\lambda}}\frac{1}{x} + ... = -\frac{2\pi(L-2M)}{\sqrt{\lambda}}\frac{1}{x} + ...(x \to \infty) \\ p(x) &= 2\pi m + \frac{2\pi Q_L}{\sqrt{\lambda}}x + ... = 2\pi m + \frac{2\pi L}{\sqrt{\lambda}}x + ..., (x \to 0) \end{aligned} \tag{169}$$

where we identified $Q_L = L, Q_R = L - 2M$ in order to match with the SYM operators under consideration. We also need to identify the generator of scale transformations on the boundary with the generator of time translations, i.e.

$$\Delta = \sqrt{\lambda} 2\pi \int_0^{2\pi} d\sigma \partial_\tau X_0 = \sqrt{\lambda} k \tag{170}$$

The quasi-momentum $p(x)$ also has poles at $x = \pm 1$,

$$p(x) = -\frac{\pi k}{x \pm 1} + ...(x \to \mp 1) \tag{171}$$

and subtracting them we get a function

$$G(x) = p(x) + \frac{\pi k}{x+1} + \frac{\pi k}{x-1} \tag{172}$$

which has only branch cut singularities, thus is determined completely by its discontinuities $G(x + i0) - G(x - i0) \equiv 2\pi i \rho(x)$, and so admits a dispersion representation

$$G(x) = \int_C dy \frac{\rho(y)}{x - y} \tag{173}$$

One then obtains the Bethe-like equations

$$2P \int dy \frac{\rho(y)}{x-y} = \frac{2\pi k}{x-1} + \frac{2\pi k}{x+1} + 2\pi n_k, \quad x \in C \tag{174}$$

and from the expansion of $p(x)$ at 0 and ∞ together with the definition of $\rho(x)$ one gets the normalization conditions

$$\begin{aligned} \int dx \rho(x) &= \frac{2\pi}{\sqrt{\lambda}}(\Delta + 2M - L) \\ \int dx \frac{\rho(x)}{x} &= 2\pi m \\ \int dx \frac{\rho(x)}{x^2} &= \frac{2\pi}{\sqrt{\lambda}}(\Delta - L) \end{aligned} \tag{175}$$

Matching with SYM

Rescaling $x \to 4\pi L x/\sqrt{\lambda}$ one gets

$$
\begin{aligned}
&2P\int dy\frac{\rho(y)}{x-y}=\frac{x}{x^2-\frac{\lambda}{16\pi^2L^2}}\frac{\Delta}{L}+2\pi n_k, \quad x\in C\\
&\int dx\rho(x)=\frac{M}{L}+\frac{\Delta-L}{2L}\\
&\int dx\frac{\rho(x)}{x}=2\pi m\\
&\frac{\lambda}{8\pi^2L}\int dx\frac{\rho(x)}{x^2}=\Delta-L=\frac{\lambda}{8\pi^2}H^{(1)}
\end{aligned}
\tag{176}
$$

We observe then that these equations reduce in the limit $\lambda/L^2 \to 0$ to the thermodynamic Bethe equations for the Heisenberg spin 1/2 chain (100,101,102, 103), which as we found is indeed the 1-loop Hamiltonian.

In the Heisenberg spin chain, the "resolvent" $G(x)$ is

$$
G(x)=\frac{1}{L}\sum_i\frac{1}{x-x_i}=\int_C dy\frac{\rho(x)}{x-y} \tag{177}
$$

and its Taylor expansion at $x=0$ generates the set of conserved charges (integrals of motion). Indeed, from the explicit form of the (log of the) Lax monodromy $\ln\lambda(u)=\ln trT(u)=\sum_j 1/(u-u_j)+...$ one expects that on general grounds, as well as from the fact that in the dual string theory it is also derived from the Lax monodromy $\Omega(x)$.

One can also check explicitly that individual Bethe strings, corresponding to individual macroscopic strings in $AdS_5\times S_5$, match. Specifically, one finds the Bethe curve, calculates the resolvent $G(x)$, then the density $\rho(x)$ and finally derives the energy, and matches it against the one of the strings. The results agree, for folded strings, circular strings.

Higher loops and BDS conjecture

At higher loops, one finds the dilatation operators

$$
\begin{aligned}
\mathcal{D}_{2-loop}&=\sum_{j=1}^{L}-\vec{\sigma}_j\cdot\vec{\sigma}_{j+2}+4\vec{\sigma}_j\cdot\vec{\sigma}_{j+1}-3\cdot\mathbf{1}\\
\mathcal{D}_{3-loop}&=\sum_{j=1}^{L}-\vec{\sigma}_j\cdot\vec{\sigma}_{j+3}+(\vec{\sigma}_j\cdot\vec{\sigma}_{j+2})(\vec{\sigma}_{j+1}\cdot\vec{\sigma}_{j+3})\\
&-(\vec{\sigma}_j\cdot\vec{\sigma}_{j+3})(\vec{\sigma}_{j+1}\cdot\vec{\sigma}_{j+2})+10\vec{\sigma}_j\cdot\vec{\sigma}_{j+2}-29\vec{\sigma}_j\cdot\vec{\sigma}_{j+1}+20\cdot\mathbf{1}
\end{aligned}
\tag{178}
$$

and in general, the k-loop contribution involves $k+1$ neighbouring spins, i.e. the full dilatation operator will correspond to a long-range interacting spin chain Hamiltonian.

This Hamiltonian is *integrable* (note the quartic spin interaction at 3 loops, expected $(\vec{\sigma}\cdot\vec{\sigma})^k$ at $[k/2]$-loop level, needed for integrability) and admits BMN scaling.

In fact, the BMN limit and integrability uniquely fixes the Hamiltonian up to 5-loops. Initially, it was thought that the 3-loop Hamiltonian should be derived from an ellyptic Inozemtsev chain, which it was thought to be the most general integrable model (analog to the case of the ellyptic Calogero-Moser multi-particle system), but the proof relied on the fact that the lowest charge only contains two-spin interactions, which is not valid here. Then there was a dissagreement at 3 loops, since one violated BMN scaling in the Inozemtsev Hamiltonian in the BMN limit. Later, Beisert, Dippel and Staudacher [10] proved that in fact there is an all-loop integrable Hamiltonian, defined through its Bethe equations, that reproduces the unique BMN scaling 5-loop integrable Hamiltonian.

The general form of the Bethe equations is

$$e^{iLp_k} = \prod_{j\neq k, j=1}^{N} \frac{\phi(p_k)-\phi(p_j)+i}{\phi(p_k)-\phi(p_j)-i} = \prod_{j\neq k, j=1}^{N} S(p_k,p_j) \tag{179}$$

and the conserved charges are given as sum of single-particle contributions, i.e.

$$Q_r = \sum_{k=1}^{N} q_r(p_k); \quad H \equiv Q_2 \tag{180}$$

For the XXX spin chain we have

$$\phi(p) = \frac{1}{2}\cot\frac{p}{2}; \quad q_r(p) = \frac{2^r}{r-1}\sin(\frac{1}{2}(r-1)p)\sin^{r-1}(\frac{p}{2}) \tag{181}$$

The BDS proposal is then (supposed to be valid at all orders in $g^2 =$

$\lambda/(8\pi^2)$ and all orders in $J \equiv L-2$)

$$\phi(p) = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+8g^2\sin^2\frac{p}{2}}$$
$$h(p) = q_2(p) = \frac{1}{g^2}(\sqrt{1+8g^2\sin^2\frac{p}{2}}-1); \;\; \mathcal{D} = g^2\sum_k h(p_k)$$
$$q_r(p) = \frac{2\sin(\frac{1}{2}(r-1)p)}{r-1}\left(\frac{\sqrt{1+8g^2\sin^2\frac{p}{2}}-1}{2g^2\sin\frac{p}{2}}\right)^{r-1} \quad (182)$$

thus the "local" part of the transfer matrix is

$$T(x) = exp[i\sum_{r\geq 1} x^{r-1}Q_r] \quad (183)$$

In terms of rapidity (the inverse relations), one has

$$e^{ip} = \frac{x(\phi+i/2)}{x(\phi-i/2)}; \quad x(\phi) \equiv \frac{\phi}{2}+\frac{1}{2}\sqrt{\phi^2-2g^2} \quad (184)$$

Note that $g=0$ is indeed the usual case (XXX spin chain). The Bethe equations become

$$\frac{x(\phi+i/2)^L}{x(\phi-i/2)^L} = \prod_{j\neq k, j=1}^{N}\frac{\phi_k-\phi_j+i}{\phi_k-\phi_j-i} \quad (185)$$

and the charges are

$$q_r(\phi) = \frac{i}{r-1}(\frac{1}{x(\phi+i/2)^{r-1}}-\frac{1}{x(\phi-i/2)^{r-1}}) \quad (186)$$

As these Bethe ansatze were not derived, one needs to check that they give correct results (diagonalize the Hamiltonian in a correct way), and they do. The coordinate space wavefunctions look more complicated, involving not just the modified S matrices, but also non-plane wave wavefunctions.

3-loop disagreement and outside $SU(2)$

This Bethe ansatz generates an expression for the 2 particle (2 magnon) *all loop* anomalous dimension valid up to $1/J^2$ (similarly to what was done for the 1-loop XXX case), and it is

$$D(J,n,\lambda') = J+2\sqrt{1+\lambda' n^2}-\frac{4\lambda' n^2}{J\sqrt{1+\lambda' n^2}}+\frac{2\lambda' n^2}{J(1+\lambda' n^2)}+\mathcal{O}(1/J^2) \quad (187)$$

This comes from the formula for p_k (valid to $\mathcal{O}(1/J)$ and all orders in λ'), coming from the Bethe ansatz,

$$p \simeq \frac{2\pi n}{J} - \frac{2\pi n}{J^2}\frac{2\sqrt{1+\lambda' n^2}-1}{\sqrt{1+\lambda' n^2}} + \mathcal{O}(1/J^2) \tag{188}$$

Unfortunately, this does not match the near-plane wave limit computation on the string side

$$D(J,n,\lambda') = J + 2\sqrt{1+\lambda' n^2} - \frac{2\lambda' n^2}{J} + \mathcal{O}(1/J^2) \tag{189}$$

at 3 loops for the $1/J$ term (the 2-loop term agrees).

The same disagreement is found if one studies the Bethe equations for the all-loop spin chain in the thermodynamic limit and tries to match with the spinning folded and circular string solutions in AdS (and their corresponding Bethe equations).

One possible explanation lies in the fact that the order of limits on the two sides differs: String theory works with $\lambda \to \infty$ with J^2/λ fixed, whereas gauge theory works with $\lambda \ll 1$, takes $J \to \infty$ afterwards, keeping terms scaling as λ/J^2, and the two limits need not commute.

The 1-loop scalar Hamiltonian when not restricting to the $SU(2)$ sector, but consider all the 6 scalars, gives rise to an integrable $SO(6)$ magnetic spin chain, with $SU(2)$ as a subsector, and this was generalized to all local operators, giving the super-integrable spin chain $SU(2,2|4)$. When we leave the $SU(2)$ sector at *higher loop orders* however, the spin chain length starts fluctuating, since for instance two fermions have the same classical dimension as 3 scalars.

References

[1] Peter van Nieuwenhuizen, "Quantum field theory" lecture notes, chapter on "The Yang-Baxter equation and the algebraic Bethe ansatz," unpublished.

[2] M. Karbach and G. Muller, "Introduction to the Bethe ansatz I," cond-mat/9809162.

[3] E. D'Hoker and D.H. Phong, "Lectures on Supersymmetric Yang-Mills theory and integrable systems," hep-th/9912271.

[4] L.D. Faddeev, "How algebraic Bethe ansatz works for integrable model," hep-th/9605187.

[5] T.R. Klassen and E. Melzer, "The thermodynamics of purely elastic scattering theories and conformal perturbation theory," Nucl. Phys. B350 (1991) 635.

[6] J. Plefka, "Spinning strings and integrable spin chains in the AdS-CFT correspondence," hep-th/0507136,

[7] K. Zarembo, "Semiclassical Bethe ansatz and AdS-CFT," hep-th/ 0411191.

[8] L. Dolan, C. Nappi and E. Witten, "Yangian symmetry in D=4 superconformal Yang-Mills symmetry," hep-th/0401243.

[9] D. Bernard, "An introduction to the Yangian," hep-th/9211133.

[10] N. Beisert, V. Dippel and M. Staudacher, "A novel long range spin chain and planar $\mathcal{N} = 4$ super Yang-Mills," hep-th/ 0405001.