Teleparallelism: A New Insight into Gravity

J. G. Pereira
Instituto de Física Teórica - UNESP
São Paulo, Brazil

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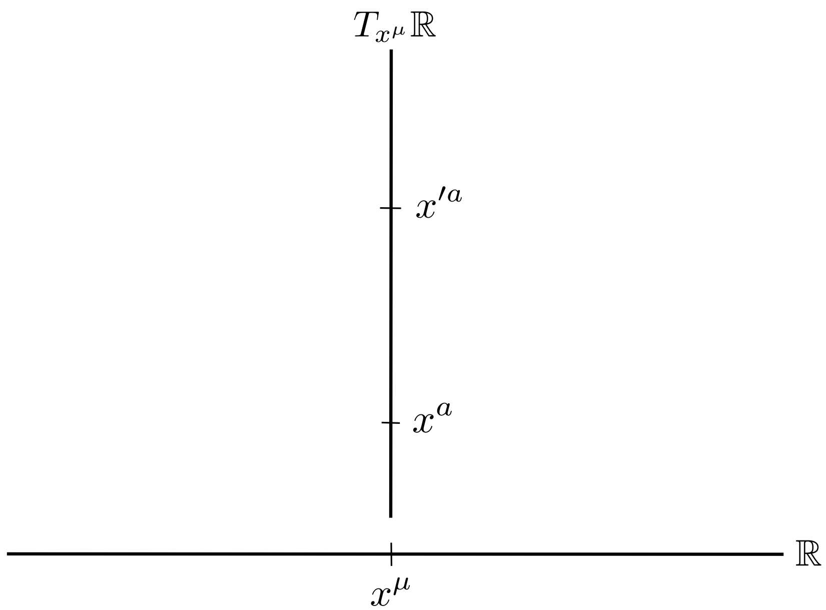
Plan of the Seminar*

1. Preliminaries
   - Tangent bundle
   - Lorentz connections
2. General relativity: a recall
   - Levi-Civita connection
   - Interaction and geometry
3. Teleparallel gravity
   - Teleparallel spin connection
   - Equivalence with general relativity
4. Achievements of teleparallel gravity

* *Teleparallelism: A New Insight into Gravity*, chapter in *Springer Handbook of Spacetime* Ed. by A. Ashtekar and V. Petkov (Springer, Berlin, 2014), arXiv:1302.6983 [gr-qc]

The Tangent Bundle

At each point of spacetime — the base space of the bundle — there is a tangent space attached to it — the fiber of the bundle



- The Latin alphabet $a, b, c, \ldots$ denote Minkowski tangent spaces indices
- The Greek alphabet $\mu, \nu, \rho, \ldots$ denote spacetime indices

The set formed by the base space and all tangent spaces is called the tangent bundle

The Tangent Bundle is Soldered

Spacetime (the base space of the bundle) and the tangent space (the fiber of the bundle) are connected to each other

$$\Downarrow$$

This connection is made by the *solder form*, whose components are the ...

... **Tetrad Field** $h^a{}_\mu$

$$\Downarrow$$

$$g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \, \eta_{ab}$$

Absence of gravitation: Trivial Tetrad $e^a{}_\mu = \partial_\mu x^a$

$$\Downarrow$$

$$\eta_{\mu\nu} = e^a{}_\mu e^b{}_\nu \, \eta_{ab}$$

Lorentz Connections

A Lorentz connection A_μ is a 1-form assuming values in the Lie algebra of the Lorentz group:

$$A_\mu = \frac{1}{2} A^{ab}{}_\mu \, S_{ab}$$

$$\Downarrow$$

Connection $A^{ab}{}_\mu$, usually referred to as **spin connection**, can be written with spacetime indices $\Gamma^\rho{}_{\nu\mu}$

$$\Downarrow$$

$$\Gamma^\rho{}_{\nu\mu} = h_a{}^\rho A^a{}_{b\mu} h^b{}_\nu + h_a{}^\rho \partial_\mu h^a{}_\nu$$

$$A^a{}_{b\mu} = h^a{}_\nu \Gamma^\nu{}_{\rho\mu} h_b{}^\rho + h^a{}_\nu \partial_\mu h_b{}^\nu$$

$$\Downarrow$$

$A^a{}_{b\mu}$ and $\Gamma^\rho{}_{\nu\mu}$ are two ways of writing the same connection

Curvature and Torsion

A general Lorentz connection $A^a{}_{b\mu}$ has curvature and torsion:

$$R^a{}_{b\nu\mu} = \partial_\nu A^a{}_{b\mu} - \partial_\mu A^a{}_{b\nu} + A^a{}_{e\nu}\, A^e{}_{b\mu} - A^a{}_{e\mu}\, A^e{}_{b\nu}$$

$$T^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + A^a{}_{e\nu}\, h^e{}_\mu - A^a{}_{e\mu}\, h^e{}_\nu$$

$$\Downarrow$$

Through contraction with tetrads, these tensors can be written in spacetime–indexed forms:

$$R^\rho{}_{\lambda\nu\mu} \equiv h_a{}^\rho\, h^b{}_\lambda\, R^a{}_{b\nu\mu}$$

$$T^\rho{}_{\nu\mu} \equiv h_a{}^\rho\, T^a{}_{\nu\mu}$$

GENERAL RELATIVITY

Gravitation is universal

Particles with different masses and compositions feel gravitation in such a way that all of them acquire the same acceleration

$$\Updownarrow$$

Provided the initial conditions are the same, all of them will follow the same trajectory

Universality of Free Fall

$$\Updownarrow$$

Weak Equivalence Principle

$$m_g = m_i$$

General relativity is grounded on the *equivalence principle*

Lorentz Connection of General Relativity

The fundamental connection of general relativity is the Levi-Civita (or Christoffel) connection

$$\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} = \tfrac{1}{2}\, g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

$$\Downarrow$$

The corresponding spin connection is

$$\boxed{\overset{\circ}{A}{}^{a}{}_{b\mu} = h^{a}{}_{\nu} \overset{\circ}{\Gamma}{}^{\nu}{}_{\rho\mu} h_b{}^{\rho} + h^{a}{}_{\nu} \partial_\mu h_b{}^{\nu}}$$

$$\Downarrow$$

It is a connection with vanishing torsion, but non-vanishing curvature:

$$\overset{\circ}{T}{}^{a}{}_{\nu\mu} = 0 \qquad \text{and} \qquad \overset{\circ}{R}{}^{a}{}_{b\nu\mu} \neq 0$$

Particle Equation of Motion

In general relativity, the equation of motion of a (spinless) particle is the geodesic equation

$$\frac{du^a}{ds} + \overset{\circ}{A}{}^a{}_{b\mu}\, u^b\, u^\mu = 0$$

The left-hand side is the particle four-acceleration

$\Downarrow$

The right-hand side represents then the gravitational force acting on the particle

$\Downarrow$

Its vanishing means that in general relativity ...

... there is no the concept of gravitational force

How does general relativity describe the gravitational interaction?

When restricted to general relativity, only the torsionless Levi–Civita connection is present

$\Downarrow$

In this case, its curvature can be interpreted, together with the metric, as part of the spacetime definition

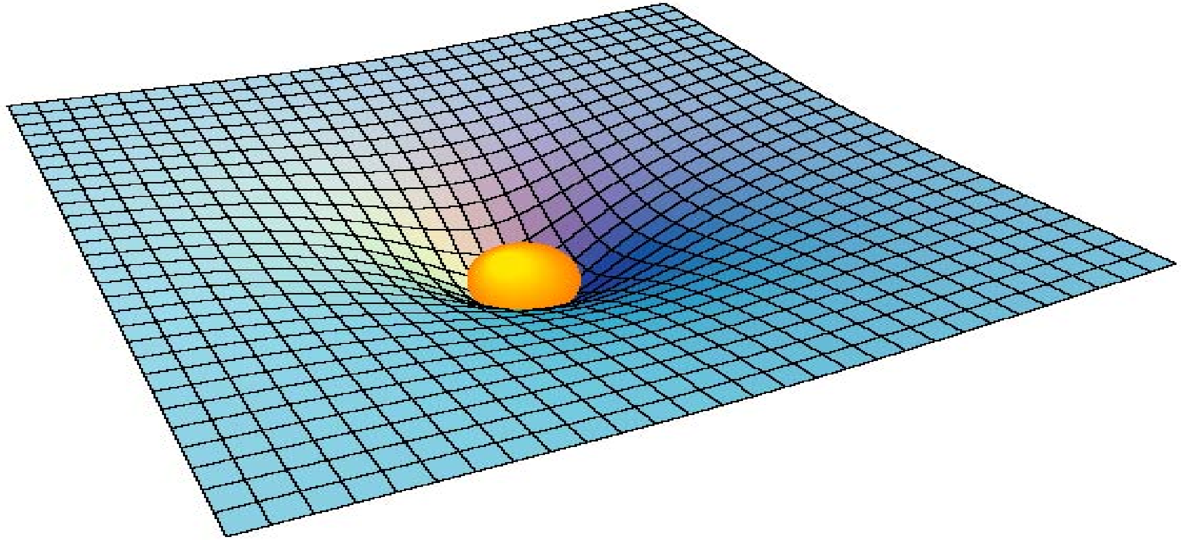
$\Downarrow$

One can then talk about ...

... curved spacetime

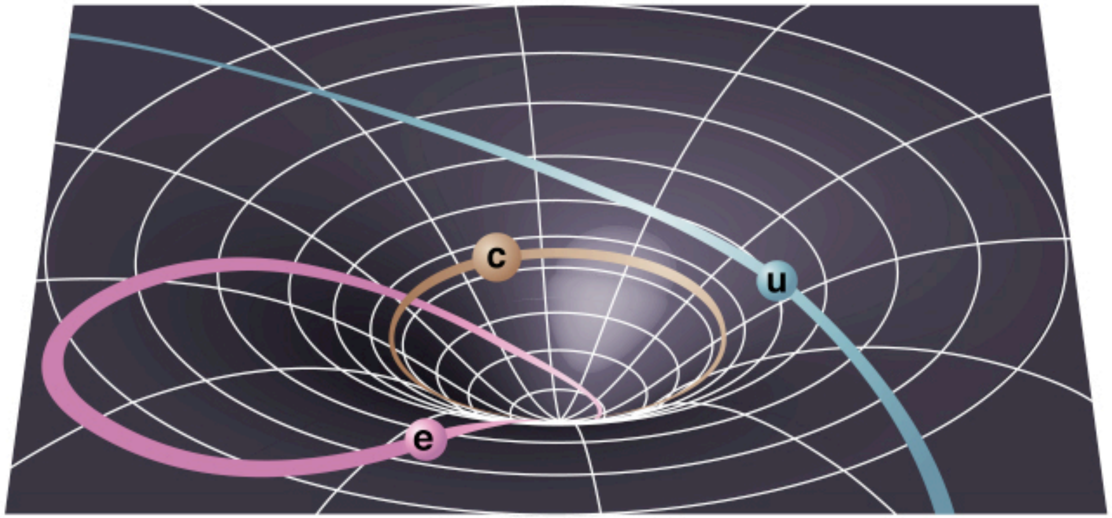
Geometrizing the Interaction

The presence of a gravitational field was supposed by Einstein to produce a *curvature* in spacetime



... Geometrizing the Interaction

A particle in a gravitational field simply follows the geodesics of the curved spacetime



In general relativity, the responsibility of describing the gravitational interaction is transferred to spacetime

$\Downarrow$

Geometry replaces the concept of force

How does matter curve spacetime?

The answer is given by Einstein equation

$$\overset{\circ}{R}{}^{a}{}_{\mu} - \tfrac{1}{2}\, h^{a}{}_{\mu}\, \overset{\circ}{R} = k\, \Theta^{a}{}_{\mu} \qquad (k = 8\pi G/c^{4})$$

$\Downarrow$

Given a matter distribution represented by the energy-momentum tensor $\Theta^{a}{}_{\mu}$

$\Downarrow$

One can solve Einstein equation and obtain the Riemann curvature tensor $\overset{\circ}{R}{}^{a}{}_{b\lambda\mu}$

TELEPARALLEL GRAVITY

The fundamental connection of teleparallel gravity
is a vanishing connection

$$\overset{\bullet}{A}{}^{a}{}_{b\mu} = 0$$

$$\Downarrow$$

Performing a local Lorentz transformation, it acquires the form

$$\overset{\bullet}{A}{}^{a}{}_{b\mu} = \Lambda^{a}{}_{e}(x)\,\partial_{\mu}\Lambda_{b}{}^{e}(x)$$

which represents a purely inertial connection

$$\Downarrow$$

In teleparallel gravity, Lorentz connections keep their
special relativity of representing inertial effects only

The spacetime-indexed linear connection corresponding to the inertial connection $\overset{\bullet}{A}{}^a{}_{b\mu}$ is

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} = h_a{}^{\rho}\, \overset{\bullet}{A}{}^a{}_{b\nu}\, h^b{}_{\mu} + h_a{}^{\rho}\, \partial_\nu h^a{}_{\mu}$$

which is known as the Weitzenböck connection

$\Downarrow$

The gravitational theory that follows from choosing this connection is just

Teleparallel Gravity

$\Downarrow$

It is a gauge theory for the translation group!!!

One may wonder why the translation group

The answer follows from playing the gauge game

$\Downarrow$

The source of gravitation is energy and momentum

$\Downarrow$

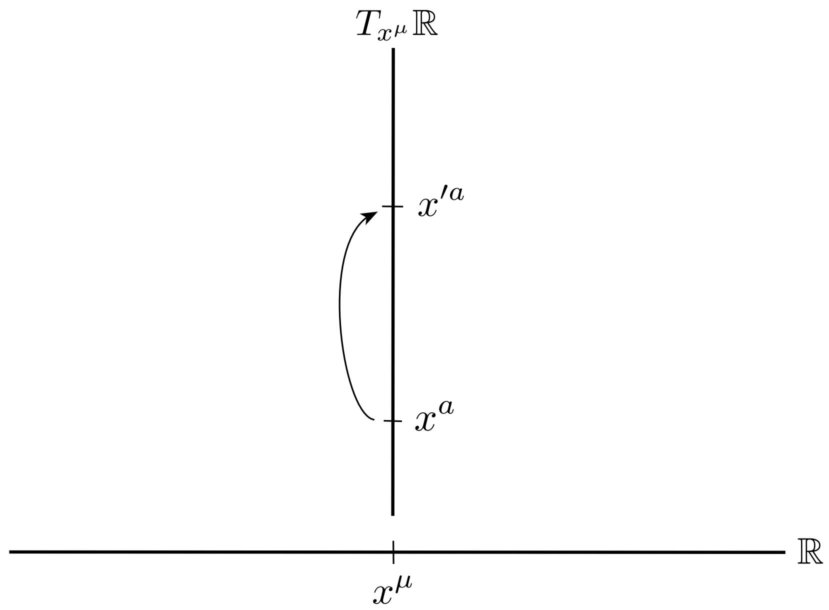
From Noether's theorem, the energy-momentum tensor is conserved provided the source lagrangian is invariant under spacetime translations

$\Downarrow$

If gravity is to be described by a gauge theory with energy-momentum as a source, it must be a gauge theory for the translation group

Gauge Transformations ...

... are local translations in the Minkowski tangent space, the fiber of the tangent bundle



$$x^a \to x'^a = x^a + \epsilon^a(x^\mu)$$

Fundamental Field

The fundamental field of teleparallel gravity is the translational gauge potential $B^a{}_\mu$, a 1-form assuming values in the Lie algebra of the translation group

$$B_\mu = B^a{}_\mu \, P_a$$

$$P_a = \partial/\partial x^a \rightarrow \text{generators of infinitesimal translations}$$

$$\Downarrow$$

It appears as the nontrivial part of the tetrad field $h^a{}_\mu$

$$h^a{}_\mu = \partial_\mu x^a + B^a{}_\mu$$

$$\Downarrow$$

By non-trivial part we mean

$$B^a{}_\mu \neq \partial_\mu \epsilon^a$$

Curvature and Torsion

The inertial connection has vanishing curvature:

$$\overset{\bullet}{R}{}^{a}{}_{b\nu\mu} \equiv \partial_\nu \overset{\bullet}{A}{}^{a}{}_{b\mu} - \partial_\mu \overset{\bullet}{A}{}^{a}{}_{b\nu} + \overset{\bullet}{A}{}^{a}{}_{e\nu}\, \overset{\bullet}{A}{}^{e}{}_{b\mu} - \overset{\bullet}{A}{}^{a}{}_{e\mu}\, \overset{\bullet}{A}{}^{e}{}_{b\nu} = 0$$

$$\Downarrow$$

However, for a non-trivial tetrad, it has non-vanishing torsion

$$\overset{\bullet}{T}{}^{a}{}_{\nu\mu} \equiv \partial_\nu h^{a}{}_{\mu} - \partial_\mu h^{a}{}_{\nu} + \overset{\bullet}{A}{}^{a}{}_{e\nu}\, h^{e}{}_{\mu} - \overset{\bullet}{A}{}^{a}{}_{e\mu}\, h^{e}{}_{\nu} \neq 0$$

$$\Downarrow$$

It can alternatively be written in the form

$$\overset{\bullet}{T}{}^{a}{}_{\nu\mu} \equiv \partial_\nu B^{a}{}_{\mu} - \partial_\mu B^{a}{}_{\nu} + \overset{\bullet}{A}{}^{a}{}_{e\nu}\, B^{e}{}_{\mu} - \overset{\bullet}{A}{}^{a}{}_{e\mu}\, B^{e}{}_{\nu}$$

$$\Downarrow$$

Torsion is the field-strength of teleparallel gravity

Lagrangian and Field equation

The Lagrangian of teleparallel gravity, like any gauge Lagrangian, is quadratic in the field strength, or torsion:

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{4k}\left[\frac{1}{4}\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu}\overset{\bullet}{T}{}_{\rho}{}^{\mu\nu} + \frac{1}{2}\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu}\overset{\bullet}{T}{}^{\nu\mu}{}_{\rho} - \overset{\bullet}{T}{}_{\rho\mu}{}^{\rho}\overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right]$$

$$h = \det(h^{a}{}_{\mu}) = \sqrt{-g}$$

$$\Downarrow$$

Variation in relation to the gauge field $B^{a}{}_{\rho}$ yields the sourceless teleparallel field equations

$$\boxed{\partial_{\sigma}(h\overset{\bullet}{S}{}_{a}{}^{\rho\sigma}) - k\,(h\overset{\bullet}{\jmath}{}_{a}{}^{\rho}) = 0}$$

$\overset{\bullet}{S}{}^{\rho\mu\nu} = -\overset{\bullet}{S}{}^{\rho\nu\mu}$ → Superpotential

$h\overset{\bullet}{\jmath}{}_{a}{}^{\rho}$ → Energy-momentum pseudo-current

Connection Decomposition

According to a theorem by Ricci, given a general Lorentz connection $A^a{}_{b\nu}$, it can always be decomposed according to

$$A^a{}_{b\nu} = \overset{\circ}{A}{}^a{}_{b\nu} + K^a{}_{b\nu}$$

$$K^a{}_{b\nu} = \tfrac{1}{2}\left(T_b{}^a{}_\nu + T_\nu{}^a{}_b - T^a{}_{b\nu}\right) \;\rightarrow\; \text{Contortion tensor}$$

$$\Downarrow$$

In the specific case of the teleparallel spin connection $\overset{\bullet}{A}{}^a{}_{b\mu}$, the decomposition has the form

$$\boxed{\overset{\bullet}{A}{}^a{}_{b\mu} = \overset{\circ}{A}{}^a{}_{b\mu} + \overset{\bullet}{K}{}^a{}_{b\mu}}$$

Equivalence with General Relativity

Using the above decomposition, the teleparallel lagrangian is found to be equivalent to the general relativity lagrangian

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{2k} \overset{\circ}{R} - \partial_\mu \omega^\mu$$

$$\Downarrow$$

As a consequence, the corresponding field equations are also equivalent

$$\partial_\sigma (h \overset{\bullet}{S}_a{}^{\rho\sigma}) - k^2 (h \overset{\bullet}{\jmath}_a{}^{\rho}) \equiv h \left(\overset{\circ}{R}_a{}^{\rho} - \tfrac{1}{2} h_a{}^{\rho} \overset{\circ}{R} \right)$$

$$\Downarrow$$

Teleparallel gravity is equivalent to general relativity

Why Gravitation Has Two Alternative Descriptions?

Like any other interaction of nature, gravitation has
a description in terms of a gauge theory

$\Downarrow$

Teleparallel gravity corresponds to a
gauge theory for the translation group

$\Downarrow$

On the other hand, universality of gravitation allows a description
in terms of a geometrization of the interaction

$\Downarrow$

General Relativity is a geometric theory for gravitation

$\Downarrow$

Universality of the gravitational interaction is thus the responsible
for the existence of the alternative geometric description

ACHIEVEMENTS OF TELEPARALLEL GRAVITY

Although equivalent theories ...

... there are conceptual differences between GR and TG

$\Downarrow$

For example, GR describes gravitation in terms of curvature,

whereas TG describes gravitation in terms of torsion

$\Downarrow$

This means that curvature and torsion are related

to the same degrees of freedom of gravity:

no new physics is associated to torsion

$\Downarrow$

In what follows we are going to explore other

properties of teleparallel gravity

1. Separating Inertia from Gravitation

Let us consider again the connection decomposition

$$\overset{\circ}{A}{}^{a}{}_{b\rho} = \overset{\bullet}{A}{}^{a}{}_{b\rho} - \overset{\bullet}{K}{}^{a}{}_{b\rho}$$

$$\Downarrow$$

Since $\overset{\bullet}{A}{}^{a}{}_{b\rho}$ represents inertial effects only, this expression corresponds to a separation between inertial effects and gravitation

$$\Downarrow$$

In fact, in the local frame in which $\overset{\circ}{A}{}^{a}{}_{b\rho} \doteq 0$, the above identity becomes

$$\overset{\bullet}{A}{}^{a}{}_{b\rho} \doteq \overset{\bullet}{K}{}^{a}{}_{b\rho}$$

$$\Downarrow$$

This expression shows explicitly that, in such a local frame, inertial effects exactly compensate gravitation

2. Particle Equation of Motion

In general relativity, the equation of motion of a spinless particle is the geodesic equation

$$\frac{du^a}{ds} + \overset{\circ}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = 0$$

$$\Downarrow$$

Substituting the connection decomposition

$$\overset{\circ}{A}{}^a{}_{b\rho} = \overset{\bullet}{A}{}^a{}_{b\rho} - \overset{\bullet}{K}{}^a{}_{b\rho}$$

$$\Downarrow$$

we obtain the corresponding equation of motion in teleparallel gravity

$$\frac{du^a}{ds} + \overset{\bullet}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = \overset{\bullet}{K}{}^a{}_{b\rho}\, u^b\, u^\rho$$

$$\Downarrow$$

This is the teleparallel version of the equation of motion

... Particle Equation of Motion

$$\frac{du^a}{ds} + \overset{\bullet}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = \overset{\bullet}{K}{}^a{}_{b\rho}\, u^b\, u^\rho$$

$$\Downarrow$$

This is a force equation, with contortion playing the role of gravitational force

$$\Downarrow$$

This means that, though mostly unnoticed ...

... torsion has already been detected

$$\Downarrow$$

It is responsible for all gravitational effects, including the physics of the solar system, which can be reinterpreted in terms of a force equation, with contortion playing the role of force

3. Gravitational Energy Localization

As is well-known, in general relativity the complex defining the energy-momentum density of gravitation is a pseudotensor

$\Downarrow$

The reason is that the spin connection of general relativity $\overset{\circ}{A}{}^a{}_{b\mu}$ includes both gravitation and inertial effects

$\Downarrow$

Since inertial effects are non-tensorial by definition, the resulting energy-momentum density is necessarily a pseudotensor

$\Downarrow$

This property has been considered for a long time an inherent characteristic of the gravitational field

... Gravitational Energy Localization

However, on account of the possibility of separating inertial effects from gravitation, in teleparallel gravity the energy-momentum pseudotensor can be split according to

$$\overset{\bullet}{\jmath}_a{}^\rho \equiv \overset{\bullet}{\imath}_a{}^\rho + \overset{\bullet}{t}_a{}^\rho$$

$\overset{\bullet}{\imath}_a{}^\rho \rightarrow$ EM of inertial effects: non-tensorial

$\overset{\bullet}{t}_a{}^\rho \rightarrow$ EM of gravitation only: tensorial

$$\overset{\bullet}{t}_a{}^\rho = \frac{1}{k} h_a{}^\lambda \, \overset{\bullet}{S}_c{}^{\nu\rho} \, \overset{\bullet}{T}{}^c{}_{\nu\lambda} - \frac{h_a{}^\rho}{h} \overset{\bullet}{\mathcal{L}}$$

$$\Downarrow$$

In teleparallel gravity it is possible to define a tensorial energy-momentum density for gravitation

4. A Genuine Gravitational Variable

The spin connection of general relativity includes both gravitation and inertial effects

$\Downarrow$

It is always possible to find a local frame in which inertial effects exactly compensate gravitation, and the connection vanishes

$$\overset{\circ}{A}{}^{a}{}_{b\mu} \doteq 0$$

$\Downarrow$

Since there is gravitational field at that point, such connection cannot be considered a genuine gravitational variable

$\Downarrow$

Any approach to quantum gravity based on this connection will include a quantization of inertial effects, whatever this may come to mean

... A Genuine Gravitational Variable

In teleparallel gravity, on the other hand, the gravitational field is represented by a translational gauge potential

$$B_\mu = B^a{}_\mu P_a$$

$$\Downarrow$$

It represents gravitation only, and cannot be made to vanish at a point

$$\Downarrow$$

$B^a{}_\mu$ is a genuine gravitational variable

$$\Downarrow$$

It is, therefore, the variable to be quantized in any approach to quantum gravity

5. Gravitation and the Equivalence Principle

In the lack of universality, the geometric description
of general relativity breaks down

$\Downarrow$

On the other hand, like Newtonian gravity, the gauge description
of teleparallel gravity remains a consistent theory

$\Downarrow$

On has just to reinsert m_i and m_g in the equations of motion

$\Downarrow$

In this sense, teleparallel gravity can be
considered a more robust theory

6. Curved Spacetime Paradigm

The advent of teleparallel gravity breaks dow
the paradigm of the curved spacetime

$\Downarrow$

The statement that spacetime is curved depends on the theory
used to describe the gravitational interaction

$\Downarrow$

Whether spacetime is curved or not turns out
to be a matter of convention!