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Instituto de Física Teórica
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AN INTRODUCTORY COURSE ON

PHYSICAL COSMOLOGY

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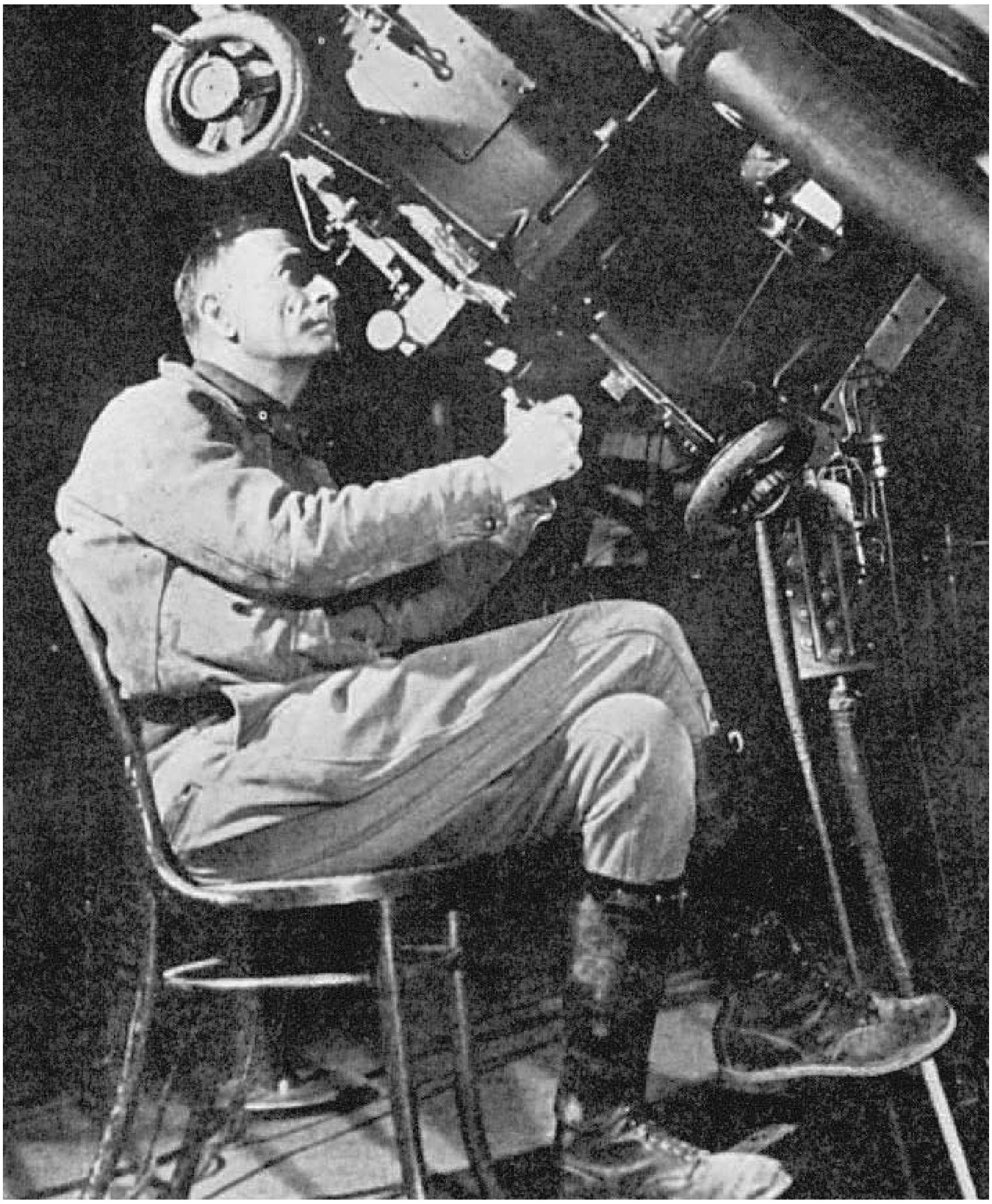

Edwin Hubble at work.

A preliminary note

Cosmology has definitively left its position as a chapter of Philosophy to become a part of Astronomy. This is more or less true since Laplace, but since his times colossal developments have taken place. It is nowadays a preferred arena for applications of the whole of Fundamental Physics. Gravitation does provide the background, but most informations are mediated by the electromagnetic interaction. Weak and strong interactions contribute in one of the most important achievements of the dominating model, the nucleosynthesis of light elements.

Being no astronomers, our presentation has an inevitable theoretical–physicist bias. Gravitation — a lot; Thermodynamics — yes, some; Field Theory — a little, but almost no Astronomy. The mentioned bias will frequently lead to look at Cosmology as an source of motivating examples, through which learning some General Relativity is specially agreeable.

The aim of Physical Cosmology is to describe the Universe in large scale using as far as possible Physics as we know it. That knowledge comes from experiments and observations made basically on Earth and its neighborhood, the solar system. This means that daring extrapolations are inevitable, and that we should be prepared for some surprises. It is remarkable, anyhow, that at least a part of "terrestrial" Physics — atomic spectroscopy, for example — hold strictly both far away in space for billions of light–years and long back in time for billions of years. Success has been so overwhelming that the idea of a possible surprise had actually receded to the outskirts of cosmological thought. And then, in the last few years, surprise did turn up. Observational data of the last few years have given compelling evidence for a large cosmological constant. They mean, in reality, that the dominating contribution to gravitation in large scales has an unknown origin. It is, consequently, good time for a review of our supposed knowledge.

These notes,[1] prepared with a one–term course in view, are intended as a short guide to the main aspects of the subject. The reader is urged to refer

[1] The present is a slightly revised version of the first, rough version dated of 2000. It owes many corrections to our colleagues A.L. Barbosa, R.R. Cuzinatto and L.G. Medeiros.

to the basic texts we have used, each one excellent in its own approach:

- J. V. Narlikar, *Introduction to Cosmology* (Cambridge University Press, Cambridge, 1993).
- S. Weinberg, *Gravitation and Cosmology* (J. Wiley, New York, 1972).
- L.D. Landau and E.M. Lifschitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975).
- P. Coles and G. F. R. Ellis, *Is the universe open or closed?* (Cambridge University Press, 1997).
- Ya. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics II: The Structure and Evolution of the Universe* (University of Chicago Press, Chicago, 1981).
- A. D. Dolgov, M. V. Sazhin and Ya. B. Zeldovich, *Basics of Modern Cosmology* (Editions Frontière, Gif-sur-Ivette, 1990).
- P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, 1993).
- E. Harrison, *Cosmology* (Cambridge University Press, 2nd. ed. 2001).
- A. Liddle, *An Introduction to Modern Cosmology* (J. Wiley, Chichester, 2nd.ed., 2003).
- S. Dodelson, *Modern Cosmology* (Academic Press, San Diego, 2003).

The last reference contains an ample discussion of the recent developments concerning the anisotropies in the Cosmic Microwave Background. Other references not in book form—mostly reviews on that same subject—will be given in the text.

Units and constants from General Physics, Astronomy and Particle Physics are given in Appendix A. The notation and conventions used in the text are summarized in Appendix B, which includes also some formulae from Special Relativity. Appendix C is a formulary on relativistic ideal gases.

Contents

Chapter 1

Physical Cosmology: Object and Method

The object of Cosmology is to describe the Universe in large scales. As its very concept includes the notion that nothing exists outside it, the Universe is taken to be an isolated physical system whose evolution is determined by the interplay of its parts. Though much of such interplay may take the form of collective effects, it ultimately comes from interactions between basic constituents. Four fundamental interactions are known in present-day Physics. Two of them (the strong and the weak interactions) are of very short, subnuclear range, and can only be responsible for evolution in very small scales. Electromagnetism has long range, but the two–signed charges in its sources tend to compensate each other and arrange themselves to produce medium–scale neutrality. Only gravitation has a long range and the same sign for all localized sources. Evolution of the Universe in large scales is therefore governed by this dominating uncompensated long–range interaction, gravitation.

Gravitation, as described by General Relativity, is governed by Einstein's equations. The most general form of Einstein's equations includes a cosmological–constant Λ–term:

$$R_{\mu\nu} - \tfrac{1}{2}R\, g_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}\ . \tag{1.1}$$

The basic idea of Physical Cosmology is to introduce the energy content of the present–day Universe through its energy–momentum and find the solution. It is then possible to proceed towards the past or towards the future. Present–day values are used in this way as starting data (they are indicated by the subscript 0, as in t_0, a_0, H_0, etc).

Einstein's are, however, involved non–linear equations, ten of them and very difficult to solve. In order to find some solution, it is necessary to impose some extra conditions, usually amounting to symmetries, which simplify the equations and reduce their numbers. In Cosmology, the symmetries customarily supposed are homogeneity and isotropy, both considered on 3–space. Notice that what we usually call "Universe" is the space part, the 3–space, or "space section" of spacetime. The Universe is considered to be homogeneous and isotropic. Homogeneity means invariance under displacements from one point to any other, so that all points in 3-space are equivalent. Isotropy around a point means invariance under rotations: all directions taken from that point are equivalent.

Comment 1.0.1 The introductory considerations above serve as guidelines, and should be qualified. Gravitation, as described by General Relativity, is not an attractive interaction in some absolute sense. Certain dynamical effects, as a second–acceleration, can produce repulsion. And the cosmological constant, positive (and dominant!) by recent measurements, does originate a universal repulsive force.

Comment 1.0.2 What has been said above does not mean that weak, strong and electromagnetic interactions are of no interest to Cosmology. There are many sub–nuclear aspects which are "universal". For example, the abundance of the lightest elements seem to be the same everywhere — that is, "universal". Their synthesis is consequently a cosmological problem. The electromagnetic cosmic microwave background is also "universal". Furthermore, almost all information of cosmological interest is brought to us via electromagnetic waves. In the cosmic drama, gravitation provides the stage–set, however Caligari–like, but the other interactions do much of the talking.

Comment 1.0.3 It should be said that much of the terminology used in what follows ("closed" or "open" Universe, "critical density", etc) became standard in the course of the historical development of the theory. During most of this development, the cosmological constant was firmly believed to be zero. These names are no more suitable if $\Lambda \neq 0$. The same is true of notation. It will become clear in the few next pages that introducing 3Λ instead of simply Λ in (1.1) would simplify the ensuing equations. We shall, however, stick to these commonly used nomenclatures to avoid a clash with the current literature.

Chapter 2

The Standard Model

2.1 Introduction

The Standard Cosmological Model has scored a number of fundamental successes. It was believed for some time to be the definitive model, and expected to provide the final answers to all the big questions of Cosmology. It has later been found to have some problems, but remains, as the name indicates, the standard reference with respect to which even alternative models are discussed and presented. Besides some difficulties intrinsic to the model (as the so–called "cosmic coincidence", and the horizon problems), there are questions of more general nature. For instance, there is no generally accepted theory for the origin of the large inhomogeneities as the galaxies and their clusters. Anyhow, it is the best thing we have, and its presentation is a must.

The Model is based on Friedmann's solution of the Einstein equations. This solution represents a spacetime where time, besides being separated from space, is position–independent. And space—the space section of spacetime, which is clearly defined in that solution— is homogeneous and isotropic at each point.

Comment 2.1.1 Once again we are using an inevitably simplified language which would need qualification. Instead of "time is position–independent", for example, we should have said: "there exists a coordinate system in which coordinate time is independent of the coordinates describing position in the space sector". Actually, not every spacetime solving Einstein's equations allows an overall separation of time and space. And counting "time" involves always a lot of convention: it is enough to recall that any monotonous increasing function of an acceptable time is another acceptable time.

2.2 Simplifying Assumptions

§ 2.2.1 Let us examine the Standard Model underlying assumptions.

- In any interval $ds^2 = g_{00}dx^0dx^0 + g_{ij}dx^idx^j$, the second term in the right–hand side represents the 3-dimensional space. If the time component g_{00} of the metric depends on space coordinates x^1, x^2, x^3, time will depend on space position. We shall suppose that this is not the case: g_{00} will be assumed to be space–independent. Coordinates can be chosen so that the time piece is simply c^2dt^2, in which case t will be the "coordinate time". As it will be the same at every point of space, we say that there exists a "universal time".

- The "Universe", that is, the space section, will be supposed to respect the Cosmological Principle, or Copernican Principle. We shall state this principle as follows: the Universe is homogeneous as a whole. Homogeneous means looking the same at each point. Once this is accepted, imposing isotropy around one point (for instance, that point where we are) is enough to imply isotropy around every point. This means in particular that space has the same Gaussian curvature around each point. Due to isotropy, there is an osculating 3-sphere at a generic point p. If it has radius L, the Gaussian curvature at p is equal to $\pm 1/L^2$.

Comment 2.2.1 It is simpler to consider 2-dimensional surfaces. In that case we look for two orthogonal circles tangent at a point p. If they have radii ρ_1 and ρ_1, the Gaussian curvature is $C_G = 1/(\rho_1\rho_2)$. It may happen that the osculating circles touch the surface at different faces (think of a horse's saddle centered at p). One of the radii is then negative, and so is the Gaussian curvature. Isotropy at p will mean $|\rho_1| = |\rho_2| =$ some L. Homogeneity will mean that L is the same for all points p. There is a general characterization of Gauss curvature in terms of the Riemann curvature (see, for instance, Weinberg [2]), which holds for spaces of any dimension (for the 4-dimensional case, see Appendix B, Section B.2.1).

Geometry textbooks (the classical by Eisenhart [12], or the excellent modern text by Doubrovine, Novikov and Fomenko [13]) will teach us that there are only 3 kinds of 3–dimensional spaces with constant curvature:

1. the sphere S^3, a closed space with constant positive curvature;
2. the open hyperbolic space $S^{2,1}$, or (a pseudo–sphere, or sphere with imaginary radius), whose curvature is negative; and
3. the open euclidean space $\mathbb{E}^3$ of zero curvature (that is, flat, with $L \to \infty$).

These three types of space are put together with the help of a parameter κ: $\kappa = +1$ for S^3, $\kappa = -1$ for $S^{2,1}$ and $\kappa = 0$ for $\mathbb{E}^3$. The 3–dimensional line element is then, in convenient coordinates,

$$dl^2 = \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ . \tag{2.1}$$

The last two terms are simply the line element on a 2-dimensional sphere S^2 of radius r — a clear manifestation of isotropy. Notice that these symmetries refer to space alone, and nothing forbids the "radius" L being time-dependent.

- The energy content is given by the energy-momentum of a perfect fluid:

$$T_{\mu\nu} = (p + \rho c^2)\ u_\mu u_\nu - p\ g_{\mu\nu} \ . \tag{2.2}$$

Here u_μ is the four-velocity related to a line of flux and $\rho = \epsilon/c^2$ is the mass equivalent of the energy density ϵ.

§ **2.2.2 Perfect fluid** A perfect fluid is such that an observer following a line of flux will, at each point, see the fluid as isotropic. The pressure p and the energy density $\epsilon = \rho c^2$ are those of matter (the visible matter and that which is not visible, usually called "dark matter") and radiation, besides the so-called "dark energy" and "dark pressure" which parametrize the cosmological constant term appearing in Eq.(1.1). The name "dust" is used for a perfect fluid with $p = 0$. An isotropic fluid will have an energy-momentum tensor density with components

$$\begin{aligned} T_{00} &= \epsilon = \rho c^2 \\ T_{ij} &= p\ \delta_{ij} \\ T_{i0} &= 0 \\ T_{0i} &= 0. \end{aligned} \tag{2.3}$$

This is a tensor density, that is, the values refer to an infinitesimal element of the fluid. The expressions above give the components as seen from a frame solidary with that element, moving with it. Seen from an external, "laboratory" frame, that element will have a 3-velocity $\vec{v}$, corresponding to a contraction factor $\gamma = (1 - \vec{v}^2/c^2)^{-1/2}$, and a 4-velocity $u = \gamma(1, \vec{v}/c)$. In such a frame, the same energy-momentum tensor density has the form (2.2), which reduces to (2.3) when $\vec{v} = 0$. The signs turning up depend on the metric signature adopted. Recall that in our conventions the Lorentz metric is $\eta = \text{diag}(1, -1, -1, -1)$.

An important point on nomenclature: in some contexts, the words "perfect" and "ideal" are used interchangeably for a fluid with no interaction between its constituents. This is not the case here: the "perfect" fluid can have interactions between constituents, provided it remains isotropic for a "comoving" observer.

Comment 2.2.2 An *observer* is any timelike worldline, one whose tangent velocity u^μ satisfies $u^2 = u_\mu u^\mu > 0$ in our conventions. It is convenient to attach to it a Lorentzian frame (tetrad) field. Of the four members of the tetrad $\{h_a\}$, one is timelike, h_0. A coordinate system $\{x^\mu\}$ is assimilated to a trivial tetrad, $\{h_a{}^\mu\} = \frac{\partial x^\mu}{\partial y^a}$, where $\{y^a\}$ is a coordinate system on the tangent Minkowski space (preferably the cartesian system). To attach a frame to a timelike line of velocity field u^μ means to take $u = h_0$, that is, $u^\mu = h_0{}^\mu$. Seen from such a frame, a tensor T whose components are $T^{\rho\sigma\cdots}{}_{\mu\nu\ldots}$ in the coordinate system $\{x^\mu\}$ will have components $T^{ab\cdots}{}_{cd\ldots} = h^a{}_\rho h^b{}_\sigma \ldots h_c{}^\mu h_d{}^\nu \ldots T^{\rho\sigma\cdots}{}_{\mu\nu\ldots}$. For example, the velocity itself will be seen as $u^a = h^a{}_\mu u^\mu = h^a{}_\mu h_0{}^\mu = \delta^a_0$. The energy-momentum of a perfect fluid $T^{\mu\nu} = (p + \rho c^2)\, u^\mu u^\nu - p\, g^{\mu\nu}$ will then be seen with components $T^{ab} = (p + \rho c^2)\, \delta^a_0 \delta^b_0 - p\, \eta^{ab}$, just (2.3).

Comment 2.2.3 A real fluid will have timelike flux lines. Contractions of Eq.(2.2) provide (for real perfect fluids):

the trace $T = g_{\mu\nu} T^{\mu\nu} = \epsilon - 3p$;

source energy density $\epsilon = T_{\mu\nu} u^\mu u^\nu$.

By the way, constractions of Eq. (1.1) are also of interest:

Ricci scalar $R = g^{\mu\nu} R_{\mu\nu} = -4\,\Lambda - \frac{8\pi G}{c^4}\, T = -4\,\Lambda - \frac{8\pi G}{c^4}\, (\epsilon - 3p)$

and further $R_{\mu\nu} u^\mu u^\nu = \frac{1}{2} R + \Lambda + \frac{8\pi G}{c^4}\, \epsilon = -\,\Lambda + \frac{4\pi G}{c^4}\, (\epsilon + 3p)$.

Comment 2.2.4 Formula (2.2) can be alternatively written

$$T_{\mu\nu} = \epsilon\; u_\mu u_\nu - p\; [g_{\mu\nu} - u_\mu u_\nu] \; . \tag{2.4}$$

The expression $P_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$, which appears multiplying the pressure, has the mixed tensor version $P^\nu_\mu = \delta^\nu_\mu - u_\mu u^\nu$ and many interesting features: (1) it is a projector, that is, $P_{\mu\nu} P^\nu_\rho = P_{\mu\rho}$; (2) it is transversal to the lines of flux, $P_{\mu\nu} u^\nu = 0$; (3) it has a fixed squared trace $P_{\mu\nu} P^{\mu\nu} = 3$. $P_{\mu\nu}$ defines, at each point of a curve of velocity u^ν, a 3-space which is transversal to the curve. In a perfect fluid, the contribution of pressure to energy-momentum is purely transversal. Notice that $T_{\mu\nu} P^{\mu\nu} = -\,3p$.

§ 2.2.3 The "convenient" coordinates used above can be arrived at as follows. A 3–space with constant nonvanishing curvature can always be defined in terms of 4–space cartesian coordinates (x_0, x_1, x_2, x_3) as

$$\pm x_0^2 + x_1^2 + x_2^2 + x_3^2 = \pm 1$$

The upper signs refer to spaces with positive curvature, the lower signs to spaces with negative curvature. When we say "constant" curvature, we mean constant in space itself. The 3–space curvature can depend on the 4-th (here, 0-th) coordinate, which is time for us. For the upper sign, new coordinates can be introduced as the natural generalization of spherical coordinates:

$$x_0 = \cos\chi;\ x_1 = \sin\chi\cos\theta;\ x_2 = \sin\chi\sin\theta\cos\phi;\ x_3 = \sin\chi\sin\theta\sin\phi\ .$$

The 3-space line element then becomes

$$dl^2 = d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\ .$$

Introducing $r = \sin\chi$ leads to (2.1) with $\kappa = 1$. For the lower sign, which leads to (2.1) with $\kappa = -1$, it is enough to use $\cosh\chi$ and $\sinh\chi$ instead of $\cos\chi$ and $\sin\chi$. For vanishing curvature ($\kappa = 0$), it is enough to notice that (2.1) is just $dx^2 + dy^2 + dz^2$ in spherical coordinates.

Other coordinate systems can be eventually used, if more convenient to exhibit some special feature.

§ 2.2.4 In cosmologists jargon, the sphere S^3 is said to be "finite, but unbounded". This closed space has a finite volume, but has no boundary. The other spaces are "infinite", because they have infinite volume.

§ 2.2.5 Copernicus has taught us that we are in no special position in the Universe. If no observer whatsoever is in special position, homogeneity follows. Looking at scales large enough, distribution of matter in space seems isotropic to us. These observations are at the origin of the ideas of homogeneity and isotropy. By "large enough", we mean regions of linear size of order 10^{26} cm, larger than galaxy clusters (see table 2.1). This is actually a modelling procedure. It is clear that the homogeneity of matter distribution is only a rough approximation. There is evidence for the existence of agglomerates still larger than galaxy clusters, and of large empty regions. What we do is to take "cells" of volume $\approx 10^{78}\ cm^3$ and smear their matter content so as to have a "continuum", or a "gas" distribution. More impressive is the background radiation, remarkably isotropic (to one part in 10000) if our local motions (of Earth, Sun and Galaxy) are discounted.

	stars	galaxies	clusters
mass	10^{33} g	10^{44} g	10^{47} g
size	10^{11} cm	10^{23} cm	10^{25} cm

Table 2.1: *Typical masses and sizes of some astronomical objects*

2.3 The spacetime line element

§ 2.3.1 We can now put together all we have said. Instead of a time–dependent radius, it is more convenient to use fixed coordinates as above, and introduce an overall *scale parameter* $a(t)$ for 3–space, so as to have the spacetime line element in the form

$$ds^2 = c^2 dt^2 - a^2(t)\, dl^2. \tag{2.5}$$

Thus, with the high degree of symmetry imposed, the metric is entirely fixed by the sole function $a(t)$. Using (2.1),

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] . \tag{2.6}$$

This is the Friedmann–Robertson–Walker (in short FRW) interval [1, 2].

§ 2.3.2 There are other current parameterizations for the 3-space. For example, Landau & Lifshitz [3] prefer

$$ds^2 = c^2 dt^2 - a^2(t) \left[dr^2 + \left\{ \begin{array}{c} \sin^2 r \ (\text{if } \kappa = +1) \\ r^2 \ (\text{if } \kappa = 0) \\ \sinh^2 r \ (\text{if } \kappa = -1) \end{array} \right\} \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] . \tag{2.7}$$

In order to have $r = \sin\chi$ or $\sinh\chi$ as in §2.2.3, or still as eventual arguments of sin and sinh as just above, it is more convenient to take by convention the variable r as dimensionless. In that case the expansion parameter $a(t)$ has the dimension of length. This is advisable in any approach putting together the three possible values of κ.

§ 2.3.3 Expression (2.1) represents, we repeat, the interval on 3–dimensional space. Besides being scaled by $a^2(t)$ in (2.6), it differs from the interval on $\mathbb{E}^3$, which is

$$dl^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (\kappa = 0) \tag{2.8}$$

(included in (2.6) and (2.7) as the case $\kappa = 0$) by the presence of curvature. In the euclidean case, it is more convenient to use cartesian coordinates (x, y, z).

The path leading from a given metric to the detailed expression of Eq.(1.1) is given in Appendix B, section B.2. We shall first state the Friedmann equations, which result from (2.6), and discuss some of its general consequences. Some detail on how these equations are arrived at will be given in Sections 2.6 and 2.7.

2.4 The Friedmann equations

§ 2.4.1 Up to this point, the dynamical equations (1.1) have not been used. The above line element is a pure consequence of symmetry considerations. The extreme simplicity of the model is reflected in the fact that those 10 partial differential equations reduce to 2 ordinary differential equations (in the variable t) for the scale parameter.

In effect, once (2.2) and (2.6) are used, equations (1.1) reduce to the two Friedmann equations for a(t):

$$\dot{a}^2 = \left[2\left(\frac{4\pi G}{3}\right)\rho + \frac{\Lambda c^2}{3}\right] a^2 - \kappa c^2 \; ; \tag{2.9}$$

$$\ddot{a} = \left[\frac{\Lambda c^2}{3} - \frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)\right] a(t) \; . \tag{2.10}$$

The second equation determines the concavity of the function $a(t)$. This has a very important qualitative consequence when $\Lambda = 0$. In that case, for normal sources with $\rho > 0$ and $p \geq 0$, $\ddot{a}$ is forcibly negative for all t and the general aspect of $a(t)$ is that of Figure 2.1. It will consequently vanish for some time $t_{initial}$. Distances and volumes vanish at that time and densities become infinite. This moment $t_{initial}$ is taken as *the* beginning, the "Big Bang" itself. It is usual to take $t_{initial}$ as the origin of the time coordinate: $t_{initial} = 0$. If $\Lambda > 0$, there is a competition between the two terms. It may even happen that the scale parameter be $\neq 0$ for all finite values of t (see, for example, solution (2.24) below).

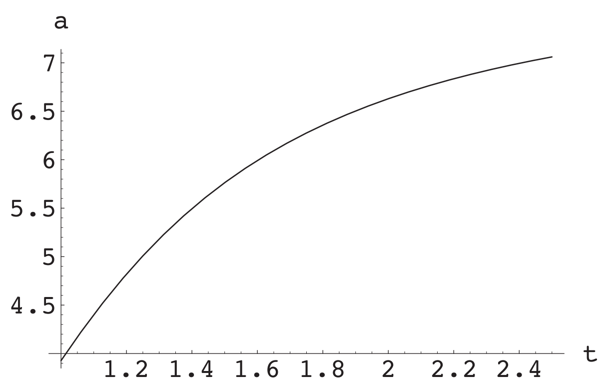


Figure 2.1: *Concavity of $a(t)$ for $\Lambda = 0$.*

§ 2.4.2 Taking the time derivative of (2.9) and using (2.10), we find

$$\frac{d\rho}{dt} = -\,3\;\frac{\dot{a}}{a}\;\left(\rho + \frac{p}{c^2}\right)$$

or

$$\frac{d\epsilon}{dt} = -\,3\;\frac{\dot{a}}{a}\;\left(\epsilon + p\right), \tag{2.11}$$

which is equivalent to

$$\frac{d}{da}(\epsilon a^3) + 3\;p\;a^2 = 0. \tag{2.12}$$

This equation reflects the energy–momentum conservation: it can be alternatively obtained from ${T^{\mu\nu}}_{;\nu} = 0$.

Comment 2.4.1 We have written Einstein's equation (1.1) with the term $\Lambda g_{\mu\nu}$ (usually called "the cosmological term") in the left-hand side. This seems to give it a "kinematical" role, but is a matter of convention. Nobody really knows its nature. It is equivalent to suppose, in the right hand side, a fluid with the exotic equation of state $p = -\,\epsilon = -\,\rho/c^2$. In this case Eq.(2.11) enforces $\epsilon =$ constant. The cosmological term is consequently equivalent to a source with constant energy $\epsilon_\Lambda = \frac{c^4\Lambda}{8\pi G}$. We shall repeatedly come back to this "dark energy".

§ 2.4.3 Equation (2.11) is sometimes called the "adiabaticity condition". The first law of Thermodynamics,

$$dE = TdS - pdV$$

written in terms of the energy and entropy densities $\epsilon = E/V$ and $s = S/V$ assumes the form

$$d\epsilon = Tds + (Ts - \epsilon - p)\frac{dV}{V}\,.$$

Notice that $V = V_0\,\frac{a^3}{a_0^3}$ implies $\frac{dV}{V} = \frac{3}{a}\,da$. Equation (2.11) becomes, in consequence,

$$\frac{1}{s}\,\frac{ds}{dt} = -\,\frac{3}{a}\frac{da}{dt} \quad \text{or} \quad \frac{d}{dt}\ln[sa^3)] = 0.$$

This means $sa^3 = s_0 a_0^3$, or

$$\frac{d}{dt}\,S = 0.$$

The Friedmann expansion is adiabatic.

Comment 2.4.2 Use of $E = \epsilon V$ and $V(t) = V_0\frac{a_0^3}{a(t)^3}$ shows that Eq.(2.11) is just $\frac{dE}{dt} = -\,p\frac{dV}{dt}$, consistent with the above result. Other expressions of interest are $dE = 3pV\frac{da}{a}$ and $dE = -\,3pV\frac{dz}{1+z}$.

§ **2.4.4** It is convenient to introduce two new functions, in terms of which the equations assume simpler forms. The first is the Hubble function

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{d}{dt} \ln a(t) \ , \tag{2.13}$$

whose present-day value is the *Hubble constant*

$$H_0 = 100 \ h \ km \ s^{-1} \ Mpc^{-1} = 3.24 \times 10^{-18} \ h \ s^{-1} \ .$$

The parameter h, of the order of unity, encapsulates the uncertainty in present-day measurements, which was very large ($0.45 \le h \le 1$) up to 1999. Recent values, more and more confirmed and accepted by consensus, are $h = 0.72 \pm 0.07$.[1] The second is the deceleration function

$$q(t) = - \ \frac{\ddot{a}a}{\dot{a}^2} = - \ \frac{\ddot{a}}{\dot{a}H(t)} = - \ \frac{1}{H^2(t)} \ \frac{\ddot{a}}{a} \ . \tag{2.14}$$

Equivalent expressions are

$$\dot{H}(t) = -H^2(t) \, (1 + q(t)) \ ; \quad \frac{d}{dt} \, \frac{1}{H(t)} = 1 + q(t) \ . \tag{2.15}$$

Notice that a constant H implies $q = -1$. The present–day value $q_0 = q(t_0)$ has been called the *deceleration parameter*, because the first matter+radiation models showed a decreasing expansion. Data seemed consistent with $q_0 \approx 0$ up to 1999. Recent data have brought forward the great surprise: preferred values are now $q_0 = -0.67 \pm 0.25$, negative. The "deceleration" keeps this name for historic reasons, but is actually an acceleration: the rate of expansion is increasing.

The Hubble constant and the deceleration parameter are basically integration constants, and should be fixed by initial conditions. As previously said, the present–day values are used as "initial".

§ **2.4.5** There are a few other basic numerical parameters, internal to the theory. The most important is the critical mass density, a simple function of the Hubble constant:

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} = 1.878 \times 10^{-26} \ h^2 \ kg \times m^{-3} \ . \tag{2.16}$$

[1] For recent values of parameters, see W.L. Freedman & M.S. Turner, Rev.Mod.Phys. **75** (2003) 1433. Data from the Wilkinson Microwave Anisotropy Probe (WMAP) are directly available from http://lambda.gsfc.nasa.gov.

We shall see later why this density is critical: in the $\Lambda = 0$ case, the Universe is finite or infinite if (the mass equivalent of) its energy content is respectively larger or lesser than ρ_{crit}. Still another function can be defined, the baryon density function

$$\Omega_b(t) = \frac{\rho_{baryon}(t)}{\rho_{crit}} = \frac{8\pi G \rho_{baryon}(t)}{3H_0^2}. \tag{2.17}$$

Now, the $\Lambda = 0$ Universe is finite (closed) or infinite (open) if the present value $\Omega_{b0} = \Omega_b(t_0)$ (the "baryon density parameter") is respectively > 1 or ≤ 1. Observational data give

$$0.0052 \ \leq \ \Omega_{b0}\ h^2 \ \leq \ 0.026 \tag{2.18}$$

for the matter contained in visible objects. There is a strong evidence for the existence of invisible matter, usually called "dark matter". This is indicated by an enlarged parameter

$$\Omega_m = \frac{\rho_m}{\rho_{crit}} = \Omega_{b0} + \Omega_{dark} + \Omega_\gamma + ... \tag{2.19}$$

encompassing all kinds of matter (visible, radiation, invisible and who knows else). The present-day value Ω_{m0} is in principle measurable through its gravitational effects, and is believed to be at most ≈ 0.3. In any case matter, visible or dark, seems insufficient to close the Universe.

2.5 Particular models

The Standard model has not been attained in one night. Several models have paved the way to it. Some of them can nowadays be seen as special cases.

Einstein static model Historically the first model of modern Cosmology: $a(t) = 1/\Lambda^2$ is constant, $\kappa = +1$, $\Lambda = 4\pi G\rho/c^2$, $p = 0$; everything is constant in time; this model showed that no static solution existed without a $\Lambda \neq 0$; nevertheless, the solution was shown to be unstable by Eddington;

Milne model $a(t) = t$, $\kappa = -1$, $\epsilon + p = 0$; this last "equation of state" is equivalent to a cosmological constant (see Comment 2.4.1, page 10) and, consequently, to an empty Universe;

de Sitter Universe $a(t) = e^{Ht}$, $\kappa = 0$, H a constant; empty, steady–state, constant curvature; shall be examined in detail later on (Chapter 4);

Einstein–de Sitter model $a(t) = C\, t^{2/3}$, $\kappa = 0$, $\Lambda = 0$; when only dust is considered, C is a constant; this is the simplest non-empty expanding Universe; $\Omega = \Omega_0 = 1$; gives for the Universe an age $2/(3H_0)$ (age will be defined in Section 2.8.3); see more in Section 3.2, in particular subsection 3.2.2;

$\Lambda = 0$ models with ordinary matter and radiation; these are the "historic" models: the Universe expands indefinitely for $\kappa = -1$ and $\kappa = 0$; if $\kappa = +1$, it expands up to a maximum radius (placed, for us, in the future) and then contracts back towards the "Big Crunch".

2.6 The flat Universe

§ 2.6.1 Let us, as an exercise, examine in some detail the particular flat case, $\kappa = 0$. The Friedmann–Robertson–Walker line element is simply

$$ds^2 = c^2 dt^2 - a^2(t) dl^2 \ , \tag{2.20}$$

where dl^2 is the Euclidean 3-space interval. In this case, calculations are much simpler in cartesian coordinates. The metric and its inverse are

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix} \ ;$$

$$(g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^{-2}(t) & 0 & 0 \\ 0 & 0 & -a^{-2}(t) & 0 \\ 0 & 0 & 0 & -a^{-2}(t) \end{pmatrix} .$$

Of the metric derivatives appearing in the Christoffel symbols $\overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\nu}$, only those with respect to x^0 are nonvanishing. Consequently, only Christoffel symbols with at least one index equal to 0 will be nonvanishing. For example, $\overset{\circ}{\Gamma}{}^{k}{}_{ij} = 0$. Actually, the only Christoffels $\neq 0$ are:

$$\overset{\circ}{\Gamma}{}^{k}{}_{0j} = \delta^k_j \, \frac{1}{c} \, \frac{\dot a}{a} \quad ; \quad \overset{\circ}{\Gamma}{}^{0}{}_{ij} = \delta_{ij} \, \frac{1}{c} \, a \, \dot a .$$

The nonvanishing components of the Ricci tensor are

$$R_{00} = - \, \frac{3}{c^2} \, \frac{\ddot a}{a} = 3 \, \frac{H^2(t)}{c^2} \, q(t) \quad ;$$

$$R_{ij} = \frac{\delta_{ij}}{c^2}\ [a\ddot{a} + 2\dot{a}^2] = \frac{\delta_{ij}}{c^2}\ a^2 H^2(t)[2 - q(t)]\ .$$

In consequence, the scalar curvature is

$$R = g^{00}R_{00} + g^{ij}R_{ij} = -\ 6\ \left[\frac{\ddot{a}}{c^2 a} + \left(\frac{\dot{a}}{ca}\right)^2\right] = 6\ \frac{H^2(t)}{c^2}\ [q(t) - 1]\ .$$

The nonvanishing components of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\ Rg_{\mu\nu}$ are

$$G_{00} = 3\left(\frac{\dot{a}}{ca}\right)^2 \quad ; \quad G_{ij} = -\ \frac{\delta_{ij}}{c^2}\ [\dot{a}^2 + 2a\ddot{a}]\ .$$

§ 2.6.2 Sourceless case De Sitter caused a commotion at the beginnings of General Relativity, when he showed that there are far-from-trivial solutions without matter and/or radiation sources, provided a cosmological constant is present. Let us consider the flat de Sitter solution. The Einstein equations are then

$$G_{00} - \Lambda g_{00} = 3\left(\frac{\dot{a}}{ca}\right)^2 - \Lambda = 0\ ;\ G_{ij} - \Lambda g_{ij} = -\ \frac{\delta_{ij}}{c^2}\ [\dot{a}^2 + 2a\ddot{a} - \Lambda c^2 a^2] = 0\ .$$

Subtracting 3 times one equation from the other, we arrive at the equivalent set

$$\dot{a}^2 - \frac{\Lambda c^2}{3}\ a^2 = 0\ ;\quad \ddot{a} - \frac{\Lambda c^2}{3}\ a = 0\ . \tag{2.21}$$

These are just the Friedmann equations (2.9) and (2.10) for the case $\rho = 0$, $p = 0$, $\kappa = 0$. They are the same as

$$H^2(t) = \frac{\Lambda c^2}{3}\ \ ;\ H^2(t)\ q(t) = -\ \frac{\Lambda c^2}{3}, \tag{2.22}$$

or to

$$H^2(t) = \frac{\Lambda c^2}{3}\ \ ;\ q(t) = -\,1\ . \tag{2.23}$$

Both parameter–functions are actually constants, so that $H_0 = \sqrt{\frac{\Lambda c^2}{3}}$ and $q_0 = -1$. Deceleration is negative, that is, actually an acceleration. Of the two solutions, $a(t) = a_0 e^{\pm H_0(t-t_0)}$, only

$$a(t) = a_0\ e^{H_0(t-t_0)} = a_0\ e^{\sqrt{\frac{\Lambda c^2}{3}}(t-t_0)} \tag{2.24}$$

would be consistent with pure expansion. Expansion is a fact well established by observation. This is enough to fix the sign, and the model implies an everlasting exponential expansion. This kind of solution is said to be "inflationary". We shall meet it again in Section 4.4 and in Chapter 5, dedicated to inflation. Notice that the scalar curvature is $R = -4\Lambda$, as is always the case in the absence of sources (compare with what has been said in Comment 2.2.3, page 6).

Notice that (2.24) gives a non-vanishing initial value for $a(t)$. We shall use notation $A = a(0) = a_0\ e^{-H_0 t_0}$ and eventually write

$$a(t) = A\ e^{H_0 t}\,. \tag{2.25}$$

2.7 The general case

§ 2.7.1 When $\kappa \neq 0$, the metric corresponding to (2.6) is, now keeping the Friedmann–Robertson–Walker coordinates,

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & -a^2(t)\, r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)\, r^2 \sin^2\theta \end{pmatrix}\ . \tag{2.26}$$

The only nonvanishing Christoffel symbols are

$$\Gamma^i{}_{0i} = \tfrac{1}{c}\, H(t)\ \ ;\ \ \Gamma^1{}_{11} = \frac{\kappa r}{1-\kappa r^2}\ \ ;\ \ \Gamma^2{}_{12} = \Gamma^3{}_{13} = \frac{1}{r}\ \ ;$$

$$\Gamma^0{}_{11} = \frac{a^2(t)}{c(1-\kappa r^2)} H(t)\ ;\ \Gamma^0{}_{22} = \frac{a^2(t) r^2}{c} H(t)\ ;\ \Gamma^0{}_{33} = \frac{a^2(t) r^2 \sin^2\theta}{c}\, H(t)$$

$$\Gamma^1{}_{22} = -\,r\,(1-\kappa r^2)\ \ ;\ \ \Gamma^1{}_{33} = -\,r\,(1-\kappa r^2)\,\sin^2\theta\ \ ;$$

$$\Gamma^2{}_{33} = -\ \sin\theta\,\cos\theta\ \ ;\ \ \Gamma^3{}_{23} = \cot\theta\ \ .$$

The nonvanishing components of the Ricci tensor and the Ricci scalar are

$$R^0{}_0 = \frac{3}{c^2}\frac{\ddot a}{a}\ \ ;\ \ R^i{}_i = \frac{1}{c^2}\left[\frac{\ddot a}{a} + \frac{2\dot a^2 + 2\kappa c^2}{a^2}\right]\ \ ;\ \ R = \frac{6}{c^2}\left[\frac{\ddot a}{a} + \frac{\dot a^2 + \kappa c^2}{a^2}\right]\ \ .$$

Finally, the Einstein tensor components:

$$G^0{}_0 = -\,\frac{3}{c^2}\,\frac{\dot a^2 + \kappa c^2}{a^2}\ \ ;\ \ G^i{}_i = -\,\frac{1}{c^2}\left[2\,\frac{\ddot a}{a} + \frac{\dot a^2 + \kappa c^2}{a^2}\right]\ \ .$$

These expressions lead, once used in Einstein's equations, to the Friedmann equations (2.9) and (2.10).

2.8 Kinematic results

The Standard Model has two kinds of results. Those of the first kind may be called "kinematical", because they come from the Friedmann–Robertson–Walker line element only and, consequently, from the symmetries it summarizes. Those of the second kind are "dynamical": they presuppose the insertion of detailed expressions for ρ and for p which determine, through the Friedmann equations, the time behaviour of the scale parameter. Let us examine first some of the kinematic consequences of the model.

2.8.1 The red–shift

The red-shift z is given by

$$1 + z = \frac{a(t_0)}{a(t)} . \tag{2.27}$$

This formula represented the first great success of the model. For a light ray $ds = 0$, so that, from (2.5), $dl = \frac{c\,dt}{a(t)}$. Suppose we observe light coming from a distant galaxy at fixed θ and ϕ. If it is emitted at a point with radial coordinate r_1, the coordinate distance down to us will be

$$d(r_1) = \int_0^{r_1} \frac{dr}{\sqrt{1-\kappa r^2}} = \begin{cases} \arcsin\ r_1 & (\kappa = 1) \\ r_1 & (\kappa = 0) \\ \text{arcsinh}\ r_1 & (\kappa = -1)\ . \end{cases} \tag{2.28}$$

But this is also

$$d(r_1) = \int_{t_1}^{t_0} \frac{cdt}{a(t)} , \tag{2.29}$$

where t_1 and t_0 are respectively the emission and the reception times. It is sometimes convenient to use the definition of $H(t)$ in the form $da = aH(a)dt$, and write the above formula as

$$d(r_1) = \int_{t_1}^{t_0} \frac{cdt}{a(t)} = \int_{a_1}^{a_0} \frac{da}{a^2 H(a)} . \tag{2.30}$$

Consider a wave maximum which departs at t_1 from a point of coordinate r_1 and arrives here at t_0; and the next maximum with depart from that same point at $t_1 + \delta t_1$ (so that δt_1 is the wave period at emission) and arrival at $t_0 + \delta t_0$ (so that δt_0 is the wave period at reception). Suppose (a fantasticaly good approximation for any observable object) we can neglect the expansion of the Universe during one wave period, so that the distance can be considered the same for the two peaks. Then,

$$d(r_1) = \int_{t_1}^{t_0} \frac{cdt}{a(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{cdt}{a(t)} = \int_{t_1}^{t_0} \frac{cdt}{a(t)} - \int_{t_1}^{t_1+\delta t_1} \frac{cdt}{a(t)} + \int_{t_0}^{t_0+\delta t_0} \frac{cdt}{a(t)}.$$

It follows that

$$\int_{t_1}^{t_1+\delta t_1} \frac{cdt}{a(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{cdt}{a(t)} \ .$$

As the wave periods δt_1 and δt_0 are small in comparison with the time scales involved,

$$\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)} \ .$$

In terms of the frequency, we obtain the law

$$\frac{\nu_1}{\nu_0} = \frac{a(t_0)}{a(t_1)} \ , \tag{2.31}$$

or

$$\nu(t)\ a(t) = \text{constant}. \tag{2.32}$$

The red–shift is defined as $z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\nu_1}{\nu_0} - 1$, from which follows (2.27). Hubble's discovery, in 1929, of a consistent red–shift of light coming from distant galaxies is one of the greatest landmarks in Cosmology. This has, since then, been systematically and extensively confirmed for larger and larger numbers of objects. The astrophysicist analyses the spectrum from some distant object, recognizes the lines emitted by some atom (say, Calcium) and compares with those found in laboratory. She/he finds a systematic shift of the lines, given by (2.27) or (2.32).

Equation (2.32) is important for another reason. The cosmic radiation background has a Planck distribution. This would not be the case if the frequencies evolved according to another law (see section 3.7).

2.8.2 Hubble's law

If the distance between two objects is $l(t_0)$ today, it was

$$l(t) = l(t_0)\ \frac{a(t)}{a(t_0)}$$

at some $t < t_0$. This distance will change in time according to

$$\dot{l}(t) = v(t) = l(t_0)\ \frac{a(t)}{a(t_0)}\ \frac{\dot{a}(t)}{a(t)} \ ,$$

which we write

$$v(t) = H(t)\ l(t). \tag{2.33}$$

At present time, this recession velocity is given by Hubble's law

$$v(t_0) = H_0\ l_0. \tag{2.34}$$

The velocity of recession v between two objects is proportional to their distance. In the "static" form (2.34), the law holds for objects rather close to us, for which $H(t) \approx H_0$. For larger distances the time-dependent form (2.33) must be used.

The first observation that distant galaxies do tend to exhibit redshifts was made by V.M. Slipher in 1914.[2] The linear law was established by Hubble from 1929 on. Figure 2.2 shows his 1936 data on galaxies, restrited to a few Megaparsecs (1 Mpc = 3.24×10^6 light-years). Figure 2.3 shows 1996 data using far away supernovae of type Ia, whose proper luminosity is fairly known.

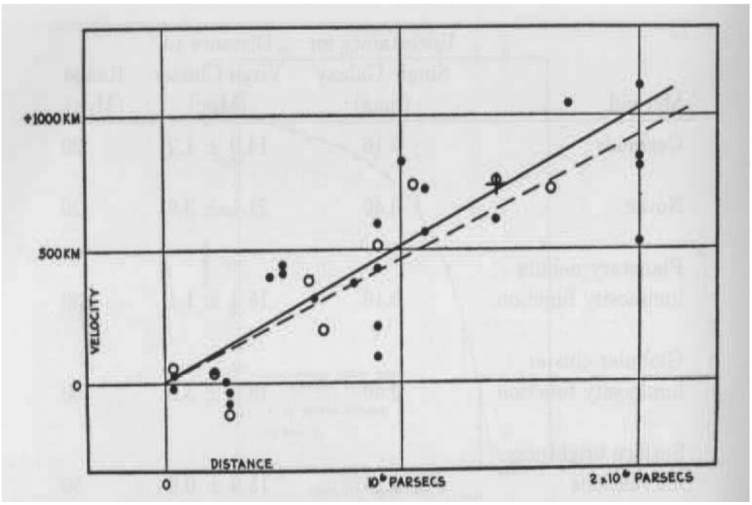


Figure 2.2: *Hubble's data (1936). Notice that the distances did not span more than a few Mpc. And that the law fails for short distances. Source: burro.astr.cwru.edu.*

[2] Nearby galaxies can exhibit blueshifts. The best example is that of Andromeda ($M31$): is at a distance of 0.7 Mpc and moves towards us at a speed of 100 km per second.

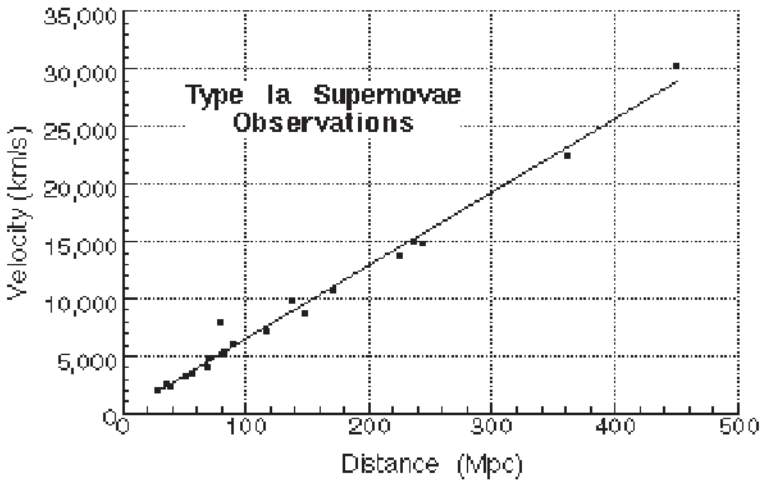


Figure 2.3: *Velocities of supernovae (A. Reise, W. H. Press and R. P. Kirshner, 1996) at distances of hundreds of Mpc. Source: eqseis.geosc.psu.edu.*

2.8.3 Age of the Universe

From (2.13) in the form $dt = \frac{1}{a}\frac{da}{H}$, we can calculate the total time from initial time t_i (which we take as $= 0$),

$$\int_0^{t_0} dt = \int_{a_i}^{a_0} \frac{da}{aH[t(a)]} \quad . \tag{2.35}$$

This "age of the Universe" must be larger than the ages of its somehow structured constituents. It would be a disaster for the model if Earth, for example, were found to be older than the Universe. The formula shows a strong dependence on the behaviour of $H(t)$. We shall see later that a dust–filled Universe, for example, gives an age $= 2/(3H_0)$. This would mean 6.5 billion years for $h = 1$ and 9.0 billion years for the favored value $h = 0.72$. Each model gives, of course, a different age. What we can say is that different energy-momentum contents and parameter values lead to ages in the range 5 — 20 billion years. The present-day favored value, coming from observations, is 13.0 ± 0.5 Gyr (1 Gigayear = one billion years).

There are independent methods to determine the ages of stars, of certain star clusters and of the Earth. There are, for the time being, large uncertainties in these numbers, but the numbers seem consistent.

2.8.4 Energy and red-shift

Given an equation of state in the form $p = p(\rho)$, equation (2.12) can be integrated to give

$$1 + z = \frac{a_0}{a(t)} = e^{\frac{1}{3}\int_{\epsilon_0}^{\epsilon} \frac{d\epsilon}{\epsilon + p(\epsilon)}} \, . \tag{2.36}$$

For example, for a pure radiation content the equation of state is $p = \frac{1}{3}\epsilon$, so that

$$\epsilon_\gamma(z) = \epsilon_0 \; (1 + z)^4 \, . \tag{2.37}$$

The energy density of dust matter, with $p = 0$, will behave according to

$$\epsilon_{dust}(z) = \epsilon_0 \; (1 + z)^3 \, . \tag{2.38}$$

Recall that Eq.(2.12) is a mere consequence of energy conservation. These results are independent of the parameters κ and Λ.

The relationship between z and the Hubble function is easily found by taking time derivatives in Eq.(2.12):

$$\frac{dz}{dt} = - \; H(t)(1 + z),$$

which integrates to

$$1 + z = e^{- \; \int_{t_0}^{t} H(t) dt} \, . \tag{2.39}$$

Notice that, from (2.36) and (2.39), a new form of the energy conservation condition turns up:

$$\frac{1}{3} \int_{\epsilon_0}^{\epsilon} \frac{d\epsilon}{\epsilon + p(\epsilon)} = - \; \int_{t_0}^{t} H(t) dt \, .$$

2.9 Friedmann equations, simpler version

§ 2.9.1 In terms of $H(t)$, the Friedmann equations (2.9, 2.10) can be written

$$H^2 = 2 \left(\frac{4\pi G}{3} \right) \rho - \frac{\kappa c^2}{a^2} + \frac{\Lambda c^2}{3} \; ; \tag{2.40}$$

$$\dot{H} = \frac{\ddot{a}}{a} - H^2 = - \; 4\pi \; G \left(\rho + \frac{p}{c^2} \right) + \frac{\kappa c^2}{a^2} \; , \tag{2.41}$$

the same as the set

$$H^2(t) q(t) = \; \frac{4\pi \; G}{3} \; \left(\rho + 3 \; \frac{p}{c^2} \right) - \frac{\Lambda c^2}{3} \; \; ; \tag{2.42}$$

$$\dot{a}(t) = H(t) \; a(t) \; , \quad \frac{d\rho}{dt} = -3 H(t) \left(\rho + \frac{p}{c^2} \right) \; . \tag{2.43}$$

§ **2.9.2** It is convenient (see Comment 2.4.1, page 10) to attribute to Λ an energy density

$$\epsilon_\Lambda = \frac{c^4 \Lambda}{8\pi G} \ , \tag{2.44}$$

usually called "dark energy" density. The mass equivalent is, of course, $\rho_\Lambda = \epsilon_\Lambda / c^2$. We can use ϵ_s for the total energy density which is introduced as source in Einstein's equation, including detected matter + radiation + undetected matter ("dark matter"), and p_s for the corresponding pressures. Eqs. (2.40), (2.41) take then the forms

$$\begin{aligned} H^2(t) &= \frac{8\pi G}{3c^2}\,[\epsilon_s + \epsilon_\Lambda] - \frac{\kappa c^2}{a^2(t)} \\ \dot{H}(t) &= \frac{\kappa c^2}{a^2(t)} - \frac{3}{2}\,\frac{8\pi G}{3c^2}\,[\epsilon_s + p_s]\,. \end{aligned}$$

We shall see later (in chapter 4) that there are non-trivial solutions of Einstein's equations with a cosmological constant and no matter sources ($\epsilon_s = 0$, $p_s = 0$). The solution with positive Λ is equivalent to that generated by a source with the exotic equation of state $p_\Lambda = -\,\epsilon_\Lambda$ (see Section 4.4). We can therefore define the total (matter + cosmological) energy $\epsilon = \epsilon_s + \epsilon_\Lambda$ and pressure $p = p_s + p_\Lambda$ and rewrite the above equations as (the t-dependence of $a(t)$ will be left implicit from now on)

$$H^2 = \frac{8\pi G}{3c^2}\,\epsilon - \frac{\kappa c^2}{a^2} \tag{2.45}$$

$$\dot{H} = \frac{\kappa c^2}{a^2} - \frac{3}{2}\,\frac{8\pi G}{3c^2}\,[\epsilon + p] \ . \tag{2.46}$$

Alternatively,

$$\begin{aligned} H^2 &= \frac{8\pi G}{3c^2}\,\epsilon - \frac{\kappa c^2}{a^2} \\ \dot{H} + H^2 &= -\,\frac{8\pi G}{3c^2}\,\frac{\epsilon + 3p}{2} \ . \end{aligned}$$

Notice that $\dot{H} + H^2 = \ddot{a}/a = -\,q H^2$ and the last equation is just another form of Eq.(2.10). Taking derivatives and comparing the equations, we get again Eq.(2.12), which we rewrite

$$\dot{\epsilon} + 3\,H\,(\epsilon + p) = 0$$

or

$$a\,d\epsilon + 3\,(\epsilon + p)\,da = 0. \tag{2.47}$$

§ **2.9.3** Actually, the Friedmann equations acquire their most convenient form in terms of dimensionless variables, obtained by dividing by H_0^2. We have then [see Eqs.(2.17), (2.19) and (2.44)]:

$$\Omega_b(t) = \frac{\rho(t)}{\rho_{crit}} \quad ; \quad \Omega_s = \frac{\rho_s}{\rho_{crit}} \tag{2.48}$$

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} = \frac{\rho_\Lambda}{\rho_{crit}} \quad ; \quad \Omega_\kappa(t) = -\ \frac{\kappa c^2}{a^2 H_0^2}\ . \tag{2.49}$$

Only the last one is really new.

§ **2.9.4** It is always good to have numeric expressions at hand, allowing easy tests of order–of–magnitude. The above expressions are

$$\Omega_\Lambda = 2.86 \times 10^{55}\ h^{-2}\ \Lambda\ \ (\text{cm}^2)\ \ ;\ \Lambda = 3.5 \times 10^{-56}\ h^2\ \Omega_\Lambda\ \ (\text{cm}^{-2})\ \ ;$$

$$\Omega_\kappa(t) = -\ \frac{\kappa}{a^2(t)}\ 8.57 \times 10^{55}\ h^{-2}\ .$$

§ **2.9.5** With the total matter mass-equivalent density ρ_s introduced in Eq.(2.19), the first Friedmann equation (2.40) becomes

$$\frac{H^2}{H_0^2} = \frac{\rho_s}{\rho_{crit}} - \frac{\kappa c^2}{a^2 H_0^2} + \frac{\Lambda c^2}{3H_0^2} \equiv \Omega_s(t) + \Omega_\kappa(t) + \Omega_\Lambda, \tag{2.50}$$

which gives on present-day values the constraint $1 = \frac{\rho_{s0}}{\rho_{crit}}$ - $\frac{\kappa c^2}{a_0^2 H_0^2} + \frac{\Lambda c^2}{3H_0^2}$, or

$$\Omega_{s0} + \Omega_{\kappa 0} + \Omega_\Lambda = 1. \tag{2.51}$$

Expression $C_a = \frac{\kappa c^2}{a_0^2}$ has the sense of a Gaussian curvature. The scheme of Figure 2.4 shows the possible domains of curvature in terms of Ω_{s0} and Ω_Λ. Figure 2.5 shows real experimental points. Data coming from supernovae and from the cosmic microwave background are consistent only in the superposition of the two bands there shown.

§ **2.9.6** Another form of the above equation gives its present–day value in terms of other parameters:

$$\frac{\kappa c^2}{a_0^2} = \frac{\Lambda c^2}{3} + H_0^2(\Omega_{s0} - 1). \tag{2.52}$$

Other, analogous manipulations lead to

$$q_0 = \frac{\Omega_{s0}}{2} - \Omega_\Lambda + \frac{3}{2}\ \frac{p_{s0}}{\rho_{crit} c^2}.$$

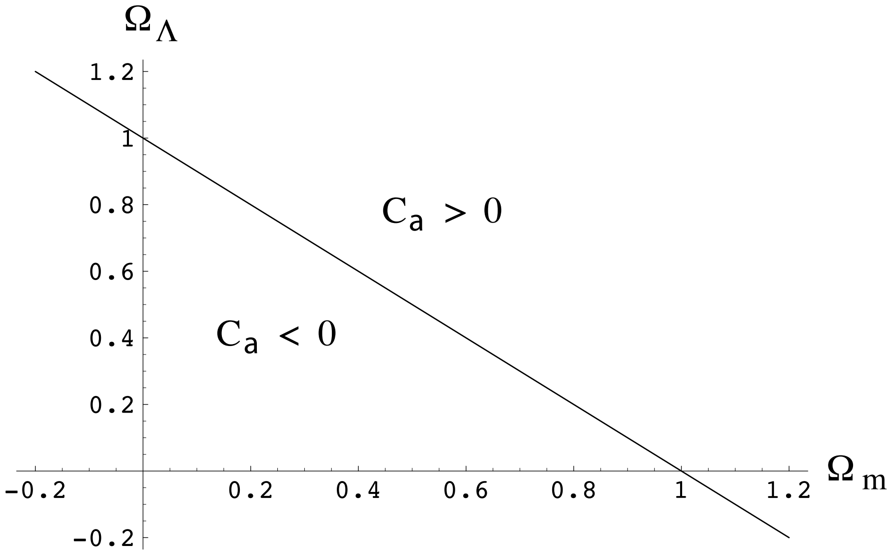


Figure 2.4: *Domains of curvature. The line represents zero curvature cases.*

There is general agreement about the present–day negligible value of the last term, so that

$$q_0 \approx \frac{\Omega_{s0}}{2} - \Omega_\Lambda \ . \tag{2.53}$$

§ 2.9.7 Equation (2.52) is used to discuss the relation between curvature and matter content. If $\rho_{s0} > \rho_{crit}$, that is, if $\Omega_{s0} > 1$, the present–day Gaussian curvature is positive, like that of a sphere. In the opposite case, the sign depends on the value of Λ. If $\Lambda = 0$, then it follows that $\kappa c^2/a_0^2 < 0$ like that of a hyperboloid branch. That is, of course, the reason for the value ρ_{crit} being called "critical". If the mass density ρ is larger than ρ_{crit}, the gravitational field it engenders is strong enough to make of the Universe a self–bound, closed system. If ρ is smaller than ρ_{crit}, as present–day observational data say it is [4], the gravitational field is not strong enough to make of the Universe a bound system.

On the other hand, equation(2.51) and the scheme shown in Figure 2.4 provide the simplest plots of observational data. The recent results on supernovae [14], shown in Figure 2.5, give a domain of possible values for Ω_Λ and Ω_s (the total amount of matter, visible and dark) which is a strip between two straight lines: one going from point $(0, 0)$ to point $(0.9, 1.1)$, the other from $(0, 0.7)$ to $(0.3, 1.1)$. A wide range of values is still possible. Nevertheless, the still more recent data from the background radiation [15] gives a rather narrow strip along the $\kappa = 0$ line. The intersection of both strips is centered at the point ($\Omega_\Lambda = 0.7$, $\Omega_s = 0.3$). These are, since April 2000, the favoured values. Dark energy dominates the energy content. This dominance of the cosmological term, whose real physical origin is unknown (hence the epithet "dark"), is one of the greatest problems of contemporary Cosmology.

§ 2.9.8 Equation (2.47) is particularly interesting to analyse the energy behavior with expansion. As seen in section 2.8.4, it is necessary to add an extra-cosmological input, an equation of state. It is convenient to introduce the "barotropic" equation

$$p = (\gamma - 1)\,\epsilon, \tag{2.54}$$

with γ a parameter[3] ranging from $\gamma = 0$ (for "dark energy", or the cosmological constant) through $\gamma = 1$ (for dust, a zero pressure "gas") and $\gamma = 4/3$ [for ultrarelativistic (UR) gases] to $\gamma = 5/3$ [ideal non-relativistic gas (NR) with $p = nkT$ and energy equipartition, $\epsilon = \frac{3}{2}nkT$] (see table 2.2).

CONTENT	γ	w	exponent n in $\epsilon = \epsilon_0(1+z)^n$
dust	1	0	3
radiation	4/3	1/3	4
UR gas	4/3	1/3	4
NR ideal gas	5/3	2/3	5
dark energy	0	-1	0

Table 2.2: *Values for the barotropic parameterization of the equation of state* $p = (\gamma - 1)\,\epsilon = w\,\epsilon$.

Equation (2.12) or (2.47), $a\,d\epsilon + 3\,(\epsilon + p)\,da = 0$, has then the general solution

$$\frac{\epsilon_2}{\epsilon_1} = e^{-3\int_{a_1}^{a_2} \gamma(a) d \ln a} \ .$$

If γ is supposed constant,

$$\frac{\epsilon_2}{\epsilon_1} = \left[\frac{a_1}{a_2}\right]^{3\gamma} \ ,$$

which gives $\epsilon(t)a^3(t) =$ constant for dust, $\epsilon(t)a^4(t) =$ constant for an ultra-relativistic gas and $\epsilon(t)a^5(t) =$ constant for an ideal non-relativistic gas. The latter, as we shall see later (Section 3.2), is totally unrealistic because the energy densities above are relativistic and contain the particle masses.

[3] The notation $w = \gamma - 1$ is also very frequent in the literature.

§ **2.9.9** In terms of $H(t)$, equation (2.41) becomes

$$\frac{\dot{H}}{H_0^2} = -\frac{3}{2}\frac{\rho + p/c^2}{\rho_{crit}} + \frac{\kappa c^2}{a^2 H_0^2} \ . \tag{2.55}$$

Let us choose for time and length the units

$$H_0^{-1} = 3.086 \times 10^{17}\ h^{-1}\ [sec] = 0.9798 \times 10^{10}\ h^{-1}\ [\text{years}] \ \ ;$$

$$\frac{c}{H_0} = 9.258 \times 10^{25} h^{-1}\ [m] \ . \tag{2.56}$$

These units are natural in Cosmology. The first is of the same order of magnitude of the Universe age, in whatever model; the second, of its "causal size" (distance which light would travel during the age). Their use imply, of course, also $c = 1$. The equations acquire the simpler forms

$$H^2 = \frac{\rho}{\rho_{crit}} - \frac{\kappa}{a^2} + \frac{\Lambda}{3} = \Omega_s(t) + \Omega_\kappa(t) + \Omega_\Lambda \tag{2.57}$$

$$\dot{H} = -\ \frac{3}{2}\ H^2 + \ \frac{3}{2}\ \left(\Omega_\Lambda - \frac{p}{\rho_{crit} c^2}\right) - \ \frac{1}{2}\ \frac{\kappa}{a^2} \ . \tag{2.58}$$

Notice that the density is absent in the last expression. In fact, it is hidden in the first term of the right–hand side.

§ **2.9.10** Notice, by the way, that matter and radiation are *not* in thermal equilibrium with each other at present time. There was, however, thermal equilibrium before the time of hydrogen recombination. In order to establish contact to thermalize two media, some interaction, however tiny, must exist between them. At temperatures higher than $\approx 3000\ ^oK$ there is no neutral hydrogen: protons and electrons move freely. Photons couple to these charges by Compton scattering. For kT much smaller than the masses of e^- and p, the cross–section is a constant proportional to the inverse square mass. Thus, it is the electrons which stop the photons. The cross–section involved[4] is Thomson's [16]:

$$\sigma_T = \frac{8\pi r_e^2}{3} = 0.665 \text{ barn } = 0.665 \times 10^{-24}\ \text{cm}^2 \, .$$

[4] Here some well-known quantities of electrodynamics are worth remembering: the fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = 7.297 \times 10^{-3} = \frac{1}{137.036}$; the electron Compton length $\lambda_{Ce} = \frac{h}{m_e c} \approx 2.42 \times 10^{-10}$ cm; and the "classical electron radius" $r_e = \alpha \lambda_{Ce} = \frac{e^2}{4\pi\epsilon_0 m_e c^2}$ $= 2.817 \times 10^{-13}$ cm. Some authors define the Compton length with $\hbar$ intead of h. The necessary Particle Physics data are given in Appendix A, section A.4.

The mean free path of a photon will be

$$\lambda_\gamma = \frac{1}{n_e \sigma_T} \approx 1.5 \times 10^{23} \, \frac{1}{n_e} \quad \text{(cm)}. \tag{2.59}$$

The calculation of n_e is a rather intricate problem which we shall discuss later. For the time being, let it only be said that the electrons do stop the photons very effectively. After recombination there is only neutral hydrogen. The cross–section photon–hydrogen is practically zero, so that the photon mean free path becomes practically infinite. This means that the photons become free. This is the origin of the background radiation.

§ 2.9.11 Matter and radiation density and pressure are introduced in the energy–momentum tensor (2.2) through their expressions for *ideal* gases. Interactions are only taken into account through the "chemical" reactions supposed to take place in due conditions. As examples, a weak Thomson scattering lies behind thermal equilibrium before recombination, and pair production will be responsible for a huge number of electrons and positrons when kT is higher than $\approx 0.5 MeV$.

§ 2.9.12 The values $\Lambda = 0$, $\kappa = 0$ lead to very simple solutions and are helpful in providing a qualitative idea of the general picture. They will be used as reference test cases. We shall exhibit later the exact analytic general solution for $H(z)$ and an implicit solution for $a(t)$. Nevertheless, in order to get a firmer grip on the relevant contributions and the role of each term, let us shortly review the so-called "thermal history" of the Universe.

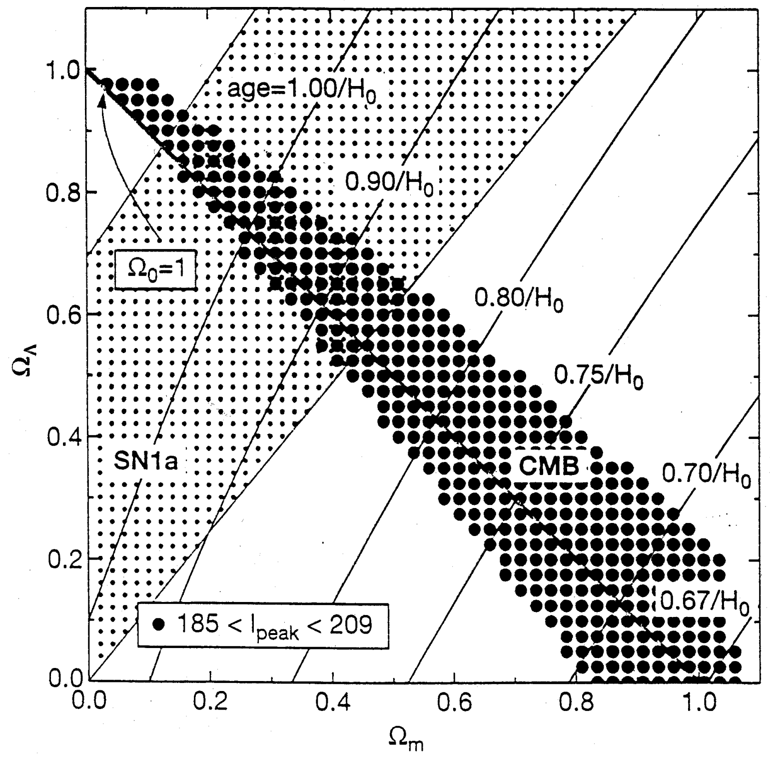


Figure 2.5: *Data on $\Omega_\Lambda \times \Omega_m$, taken from of the Boomerang project report [15]. The northwest-to-southeast band comes from the cosmic microwave background, while the other band comes from previous supernova studies [14].*

Chapter 3

Thermal History

3.1 Overview

§ 3.1.1 If we leave aside the cosmological constant, the present–day energy content of the Universe consists of matter (visible or not) and radiation, the last constituting the cosmic microwave background. The energy density of the latter is very small, much smaller than that of visible matter alone. As repeatedly announced, we take this content and help ourselves of the equations to travel back into the past, that is, toward higher red-shifts. A very simplified scheme is given in Table 3.1, whose numbers are very rough (model-dependent) and which we now proceed to describe from bottom to top. We start by studying what would happen if only matter (essentially in the form of nucleons) and radiation are taken into account. The equations of state for an ideal gas of nucleons and for a black-body are used and compared. Energy densities increase according to Eqs.(2.37) and (2.38). Temperature increases as we proceed to the past. Now, it comes from the equations of state that radiation energy increases faster than matter energy with the temperature. Thus, though matter dominates the energy content of the Universe at present time, this dominance ceases at a "turning point" time (also called "the changeover") given below [equation (3.36)]. At that point radiation takes over. At about the same time, hydrogen — the most common form of matter — ionizes. The photons of the background radiation establish contact with the electrons (via Thomson scattering), and the whole system is thermalized: above that point, there exists a single temperature. And, above the turning point, the dominating photons increase progressively in number while their concentration grows by contraction. The opportunity for interactions between them becomes larger and larger. When they approach the mass of an electrons, pair creation sets up as a stable process. Radiation is now more than a gas of photons: it contains more and more electrons and positrons. Concomitantly, nucleosinthesis stops. As we insist in going up the temperature ladder, the photons, which are more and more energetic, break the composite nuclei. The nucleosynthesis period is the most remote time from which we have reasonably sure information nowadays.

	Hic	*sunt*	*leones*
? $z \approx 10^{12}$; $kT \approx 1 GeV$	hadron era	$p^{\mp}$, n, $\bar{n}$, $\pi^{0,\mp}$ dominate energy content	
$kT \approx 1 GeV$	lepton era	$e^{\mp}$, $\mu^{\mp}$ dominate energy content	
$kT \approx 4\ MeV$ $z \approx 2\ \times 10^{10}$;			nucleo– synthesis
$kT \approx 0.5 MeV$		$e^{\mp}$ annihilation	
$kT \approx 0.5 MeV$ $z \approx 10^7$; $kT \approx 2 eV$	radiative era	radiation dominates energy content	
$t \approx 10^4$ years $z \approx 10^4$; (3233)	turning point (or cross-over)	radiation ceases to dominate	
$kT \approx 3 \times 10^3\ {}^oK$ $kT \approx 10 eV$ $z \approx 10^3$ (1089) $t \approx 5 \times 10^5$ years ($t \approx 3.79 \times 10^5$ years)	recombination time	electrons & protons combine into hydrogen	microwave background formed
	matter era		galaxies ? clusters ?

Table 3.1: *Rough scheme of the thermal history. Temperatures, red-shifts and time values — shown in the left column — are model-dependent. Some data from WMAP (2003; http://lambda.gsfc.nasa.gov) are icluded (underlined).*

§ 3.1.2 But we can go on with our journey into the past. At temperatures around 100 MeV muons begin to be formed by pairs. Then pions appear, and a mesons (ρ, ω) with masses in the hundreds of MeV. Around 1 GeV, nucleons make their appearance. These nucleons have nothing to do with those with which we have started. They are extra nucleons, belonging to the radiation. Below 1 GeV stands a period dominated by $e^{\mp}$ and $\mu^{\mp}$, the so–called lepton era. Above that, baryons dominate to define the hadron era. Still above, honestly, we know nothing. It is frequently claimed that quarks and gluons become free at very high temperature. Laboratory experiments seem to indicate that such "deconfinement" does not happen at energies below 150 GeV. Theory is not of any more help. The theory which describes successfully other aspects of these quark–gluon interactions, Quantum Chromodynamics, has not (yet ?) provided a mechanism for neither confinement nor deconfinement. Of course, the domain of very high energies is a favorite for speculation. For the time being, it stands beyond the scope of Physical Cosmology.

Comment 3.1.1 As a backslash, Cosmology may eventually come to provide information on basic Physics. Whether or not our universal constants $(c, e, \hbar, G, ...)$ are really constants—have always had their present-day values—is a fascinating question. This is becoming more and more interesting with the development of high-precision cosmological measurements. See J. Magueijo, J.D. Barrow and H.B. Sandvika, *Is it e or is it c? Experimental Tests of Varying Alpha*, arXiv:astro-ph/0202374 (2002) and J.D. Barrow, *Cosmological Bounds on Spatial Variations of Physical Constants*, arXiv:astro-ph/0503434 (2005).

§ 3.1.3 It is an instructive exercise to repeat the above in the natural order of time, that is, taking Table 3.1 from top to bottom. There is an initial period on which we know next to nothing. Then follows a period in which the energy content of the Universe is dominated by the hadronic component of the radiation. Hadrons are, by definition, particles able to interact strongly. The equation of state of hadron–dominated radiation is, consequently, unknown. After that comes the lepton era, and toward the end of that period nucleosynthesis of light elements (deuterium, helium, lithium) takes place. Despite their subsequent recycling by the stars, these elements do bring us news of that time. Positrons vanish at the end of this era, and radiation in form of photons dominate. When the Universe was about 10 thousand years old, the matter which is nowadays present surpasses radiation, and dominates since then. This brings to the limelight one of the greatest mysteries of Cosmology. This matter was quite negligible at the beginning. It was there, as the lifetime of the proton is many orders of magnitude larger than any age we can possibly attribute to the Universe.[1] There was, consequently, a tiny excess of matter at the beginning, which in the long run has assumed

[1] Proton lifetime $\geq 2.1 \times 10^{29}$ years, according to S. Eidelman *et al* (Particle Data Group), Phys.Lett. **592** (2004) 1. For data on Particle Physics, see the Particle Data Group site, http://pdg.lbl.gov. The Universe age is nowadays believed to be $(13.7 \pm 0.2) \times 10^{10}$ years.

the control. The origin and the amount of that excess is one of the great unsolved problems.

§ 3.1.4 We shall talk of the radiation-dominated era and of the matter-dominated era. Recent data indicate that the cosmological term actually dominates all, at least in the present stage. It is also believed to have dominated during the "initial period on which we know next to nothing" (see Chapter 5). Consequently, statements that radiation or matter "dominated" must be qualified. In the radiation-dominated era the energy content of radiation is more important then the energy content of non-relativistic matter, and vice-versa.

Let us now see how the above eras and periods come out from the equations.

3.2 Matter–dominated era

3.2.1 General aspects

§ 3.2.1 All during this period $kT << 1\ GeV$, so that protons are non-relativistic. Matter pressure is supposed to be given by the ideal non–degenerate gas expression $p_b = n_b k T_b$. It appears in Friedmann's equations only in the combination

$$\rho_b + p_b/c^2 = n_b\ [m + kT_b/c^2] = \frac{n_b}{c^2}\ [mc^2 + kT_b]\ .$$

Matter is overwhelmingly formed by neutrons and protons. As for them $mc^2 \approx 1\ GeV$, we see that p_b is quite negligible in the prevailing non-relativistic regime. Dust pervades the Universe. Putting $p = 0$, the equations (2.57) and (2.58) become very simple for the reference test case $\Lambda = 0$, $\kappa = 0$:

$$\dot{H} = -\ \frac{3}{2}\ H^2 = -\ \frac{3}{2}\ \frac{\rho_b}{\rho_{crit}}\ . \tag{3.1}$$

The general solution is

$$\frac{1}{H\,(t)} = \frac{3}{2}\ t + C\ . \tag{3.2}$$

3.2.2 The dust singular Universe

§ 3.2.2 Let us here make an exercise, studying what is called "the dust Universe". It is an unrealistic model which supposes matter domination all the time. We fix at the “beginning” $H_{t=0} = \infty$, then the integration constant C vanishes and the exact solution is

$$H(t) = \frac{2}{3t} \qquad \text{(dust)}.$$

This means that

$$\frac{da}{a} = \frac{2}{3}\,\frac{dt}{t} \qquad \text{(dust)}.$$

The expressions relating the Hubble function, the expansion parameter, the red-shift, and the density follow immediately (we reinsert H_0 for convenience):

$$\frac{H^2}{H_0^2} = (1+z)^3 \tag{3.3}$$

$$\frac{a(t)}{a(t_0)} = \left(\frac{t}{t_0}\right)^{2/3} ; \tag{3.4}$$

$$1+z = \left(\frac{t_0}{t}\right)^{2/3} = \left(\frac{2}{3H_0\,t}\right)^{2/3} ; \tag{3.5}$$

$$\frac{\rho_b}{\rho_{crit}} = \frac{H^2}{H_0^2}\,\Omega_{b0} = (1+z)^3\,\Omega_{b0} . \tag{3.6}$$

Using the value (2.16) of the critical density, we find

$$\rho_b = 1.878 \times 10^{-26}\,(1+z)^3\,\Omega_{b0}\,h^2\ [kg\ m^{-3}] \tag{3.7}$$

Dividing by the proton mass, the number density is

$$n_b = 11.2 \times\ (1+z)^3\,\Omega_{b0}\,h^2\ \ [m^{-3}] \tag{3.8}$$

Actually, $\Omega_{b0} = 1$ in the reference case we are considering. This gives a few nucleons per cubic meter at present time. The age of the Universe can be got from (3.5), by putting $z = 0$. One obtains $t_0 = 2/(3H_0)$, which is $\approx 6.5 \times 10^9$ years, a rather small number. Always in the reference case, this dust–dominated Universe is just the Einstein–de Sitter Universe. The general profile of $a(t)/a_0$ is shown in Figure 3.1.

There are, anyhow, serious flaws in this exercise-model: it supposes that matter dominates down to $t \approx 0$. This is far from being the case: radiation dominates at the early stages. Furthermore, the protons cannot, of course, be non-relativistic at the high temperatures of the ”beginning”.

Figure 3.1: $a(t)/a_0$ *for the Einstein–de Sitter Universe. Time is measured in units* $1/H_0$.

3.2.3 Matter–domination: $\kappa = 0, \Lambda = 0$

§ **3.2.3** A bit less unrealistic would be to fix the constant C in Eq.(3.2) by present-day values. This would be valid for the period not too far from present time, where matter at least dominates over radiation. This model remains unrealistic because all the evidence points nowadays to the dominance of the cosmological term, which is neglected. Let us anyhow take it as another exercise and fix the integration constant by the present value

$$\frac{1}{H\,(t_0)} = \frac{1}{H_0} = \frac{3}{2}\;t_0 + C\;.$$

The solution will now be

$$H(t) = \frac{H_0}{1 + \frac{3}{2}H_0(t - t_0)}. \tag{3.9}$$

Integrations of $\frac{da}{a} = H(t)dt$ leads then to

$$a(t) = a_0\,[1 + \frac{3}{2}H_0(t - t_0)]^{2/3} \tag{3.10}$$

$$1 + z = \frac{a_0}{a(t)} = \frac{1}{[1 + \frac{3}{2}H_0(t - t_0)]^{2/3}}\;. \tag{3.11}$$

Equation (3.6), and its consequeces keep holding. The age of the Universe is basically the same: we look for the time t corresponding to $z \to \infty$, and find $t_0 - t = 2/(3H_0)$. Finally, for time ranges $t_0 - t$ small in comparison to H_0^{-1}, we find a linear relation between red-shif and time:

$$z \approx H_0\;(t_0 - t)\;. \tag{3.12}$$

This expression is valid for nearby objects. As to (3.9) and (3.11), they hold from the turning point down to present times (provided, we insist, $\kappa = 0$ and $\Lambda = 0$).

3.2.4 Matter–domination: $\kappa = 0, \Lambda \neq 0$

§ 3.2.4 Recent evidence for $\kappa = 0$ and a nonvanishing cosmological constant at present time gives to this case a prominent role.

Let us insert (2.39) into (3.6), to get

$$\rho = \rho_0 \; e^{-\;3\int_{t_0}^{t} H(t)dt}$$

and then insert this expression into (2.40):

$$H^2 = 2\left(\frac{4\pi G}{3}\right)\rho_0 \; e^{-\;3\int_{t_0}^{t} H(t)dt} + \frac{\Lambda c^2}{3}\,. \tag{3.13}$$

This gives, as it would be expected, (2.51) for the present case with $\Omega_{k0} = 0$:

$$H_0^2 = 2\left(\frac{4\pi G}{3}\right)\rho_0 + \frac{\Lambda c^2}{3}\,.$$

Equation (3.13) in an involved integral equation for $H(t)$. It is simpler to make it into a differential equation, by taking its time derivative. One gets (3.1) corrected for $\Lambda \neq 0$:

$$\frac{dH}{dt} = \frac{3}{2}\left(\frac{\Lambda c^2}{3} - H^2\right).$$

Integration leads to a more involved solution,

$$H(t) = \sqrt{\frac{\Lambda c^2}{3}}\;\frac{\left(\sqrt{\frac{\Lambda c^2}{3}} + H_0\right)e^{3\sqrt{\frac{\Lambda c^2}{3}}(t-t_0)} - \left(\sqrt{\frac{\Lambda c^2}{3}} - H_0\right)}{\left(\sqrt{\frac{\Lambda c^2}{3}} + H_0\right)e^{3\sqrt{\frac{\Lambda c^2}{3}}(t-t_0)} + \left(\sqrt{\frac{\Lambda c^2}{3}} - H_0\right)}\,. \tag{3.14}$$

This expression for $H(t)$ gives $H = H_0$ when $t \to t_0$ and tends to (3.9) when $\Lambda \to 0$. To have it in terms of parameters more accessible to observation, we may use $\Lambda c^2/3 = H_0^2\Omega_\Lambda$ to rewrite it as

$$H(t) = H_0\sqrt{\Omega_\Lambda}\;\frac{\left(\sqrt{\Omega_\Lambda}+1\right)e^{3H_0\sqrt{\Omega_\Lambda}(t-t_0)} - \left(\sqrt{\Omega_\Lambda}-1\right)}{\left(\sqrt{\Omega_\Lambda}+1\right)e^{3H_0\sqrt{\Omega_\Lambda}(t-t_0)} + \left(\sqrt{\Omega_\Lambda}-1\right)}\,. \tag{3.15}$$

To get the relation with z, we notice that (2.40), which is the same as

$$H^2 = 2\left(\frac{4\pi G}{3}\right)\rho_0(1+z)^3 + \frac{\Lambda c^2}{3} = H_0^2\left[\Omega_b(1+z)^3 + \Omega_\Lambda\right] \ ,$$

gives

$$H^2 - \frac{\Lambda c^2}{3} = 2\left(\frac{4\pi G}{3}\right)\rho_0(1+z)^3$$

and therefore

$$H_0^2 - \frac{\Lambda c^2}{3} = 2\left(\frac{4\pi G}{3}\right)\rho_0 \ ,$$

which together imply

$$\frac{H^2 - \frac{\Lambda c^2}{3}}{H_0^2 - \frac{\Lambda c^2}{3}} = (1+z)^3 \ .$$

Alternatively,

$$(1+z)^3 = \frac{H^2 - H_0^2\ \Omega_\Lambda}{H_0^2\ (1-\Omega_\Lambda)} \ . \tag{3.16}$$

Remember that in the present case $\Omega_b + \Omega_\Lambda = 1$. It remains to use (3.14) to obtain

$$1+z = \left(4\ \frac{\Lambda c^2}{3}\right)^{1/3} \frac{e^{\sqrt{\frac{\Lambda c^2}{3}}(t-t_0)}}{\left[\left(\sqrt{\frac{\Lambda c^2}{3}} + H_0\right)\ e^{3\sqrt{\frac{\Lambda c^2}{3}}(t-t_0)} + \left(\sqrt{\frac{\Lambda c^2}{3}} - H_0\right)\right]^{2/3}} \ , \tag{3.17}$$

which is the same as

$$1+z = (4\ \Omega_\Lambda)^{1/3} \frac{e^{H_0\sqrt{\Omega_\Lambda}(t-t_0)}}{\left[\left(\sqrt{\Omega_\Lambda}+1\right)\ e^{3H_0\sqrt{\Omega_\Lambda}(t-t_0)} + \left(\sqrt{\Omega_\Lambda}-1\right)\right]^{2/3}} \ . \tag{3.18}$$

Comment 3.2.1 The scale parameter $a(t)/a_0$ is just the inverse. A few manipulations lead to

$$a(t) = a_0\left\{\cosh\left[\frac{3}{2}H_0\sqrt{\Omega_\Lambda}(t-t_0)\right] + \frac{1}{\sqrt{\Omega_\Lambda}}\ \sinh\left[\frac{3}{2}H_0\sqrt{\Omega_\Lambda}(t-t_0)\right]\right\}^{2/3} \ . \tag{3.19}$$

See another form below, Comment 4.5.1, page 85.

This relationship is expected to be valid during the whole matter-dominated period, that is, from the turning point to present time. We can anyhow calculate the age the Universe would have if matter had dominated all time, by fixing $t \to 0$ when $z \to \infty$. The denominator above then vanishes for

$$t_0 = \frac{1}{3H_0\sqrt{\Omega_\Lambda}}\ \ln\frac{1+\sqrt{\Omega_\Lambda}}{1-\sqrt{\Omega_\Lambda}} \ . \tag{3.20}$$

This corresponds to $t_0 = 0.964\ H_0^{-1} = 2.975 \times 10^{17}$ seconds $= 9.44 \times 10^9$ years. Once this expression for t_0 is put in (3.18), it becomes

$$1 + z = \left(\frac{4\ \Omega_\Lambda}{1 - \Omega_\Lambda} \right)^{1/3} \frac{e^{H_0 \sqrt{\Omega_\Lambda} t}}{\left[e^{3 H_0 \sqrt{\Omega_\Lambda} t} - 1 \right]^{2/3}} \ . \qquad (3.21)$$

We insist that this formula is not expected to hold in reality — it would be valid only if the matter content dominates over radiation all along.

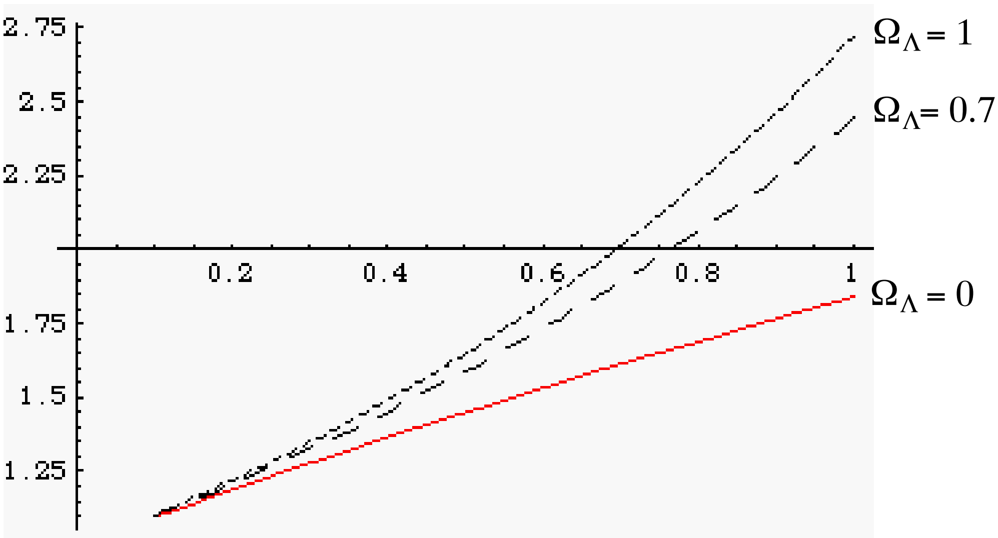


Figure 3.2: *Non-relativistic matter plus cosmological constant, with $\kappa = 0$: comparison of three behaviors.*

Figure 3.2 is the result of an exercise in model behavior. Start at the same lower left point, at one-tenth of H_0^{-1}, which is the time unit used. The lower curve shows the expansion of a pure (non-relativistic) matter Universe. The upper curve shows how Universe driven only by the comological constant would evolve. The medium curve is the closest to present-day data: it represents a Universe in which 30% of the source energy comes from (non-relativistic) matter and 70% consists of 'dark energy".

3.3 Radiation–dominated era

3.3.1 General aspects

§ 3.3.1 For photons the temperature behaves as a frequency. This comes from the equations for the black–body radiation, which give an average energy per photon $< h\nu > \propto kT$. In effect, the energy density is $e_\gamma \propto (kT)^4$, whereas the number density is $n_\gamma \propto (kT)^3$. As the red–shift gives $< h\nu > = < h\nu_0 > (1+z)$, kT must have that same behaviour,

$$T_\gamma = T_{\gamma 0}\ (1+z).$$

For example, hydrogen recombination takes place at $T = T_\gamma = T_b \approx 3000^o K$. This, with the present–day value $T_{\gamma 0} = 2.725 \pm 0.002\ ^o K$ measured for the background radiation, means $(1+z) = 1.1 \times 10^3$. Using the equation of state for the blackbody radiation,

$$e_\gamma = \frac{48\pi}{h^3\ c^3}\zeta(4)\ (kT)^4\ ,$$

the mass-equivalent density $\rho_\gamma = e_\gamma c^2$ is given by

$$\frac{\rho_\gamma}{\rho_{crit}} = 4.46 \times 10^{-7}\ T_\gamma^4\ h^{-2} = 2.46 \times 10^{-5}\ (1+z)^4\ h^{-2}. \tag{3.22}$$

The number appearing as a factor is actually the present-day value of the radiation contribution to Ω_{s0},

$$\Omega_{\gamma 0} = 2.46 \times 10^{-5}\ h^{-2}\ . \tag{3.23}$$

Equation (3.22) is consequently the same as

$$\Omega_\gamma = \Omega_{\gamma 0}(1+z)^4. \tag{3.24}$$

3.3.2 Radiation–domination: $\kappa = 0, \Lambda = 0$

§ 3.3.2 Consider again in the reference case $\kappa = 0$, $\Lambda = 0$ (see page 40 below for the more important case $\Lambda \neq 0$). Because $e_\gamma = 3\ p_\gamma$ and $\rho_\gamma = \frac{e_\gamma}{c^2}$, we have

$$\dot{H} = -\ 2\ H^2 = -\ 2\ \frac{\rho_\gamma}{\rho_{crit}} = -\ 2\ \Omega_{\gamma 0}(1+z)^4\ .$$

Automatically, in units (2.56),

$$H^2 = \Omega_{\gamma 0}(1+z)^4\ . \tag{3.25}$$

Solving the differential equation is only necessary to fix the relation between the time parameter and the red–shift. The solution is

$$H(t) = \frac{1}{2t} \qquad \text{(radiation dominated)} \quad . \tag{3.26}$$

It is far more reasonable to have $H_{t=0} = \infty$ at the "beginning" here than in the dust model. We shall see later that radiation does dominate (at least with respect to matter) at times early enough. On the other hand, these equations are not expected to meet present–day values. Thus, these formulae are expected to hold much before the turning point given below. Thus, for example, the above result implies a much too small age for the Universe: $t_0 = 1/(2H_0) \approx 4.9 \times 10^9$ years. As it is, the only thing we can say about the expansion parameter is that it has the form

$$a(t) = C\, t^{1/2} \qquad \text{(radiation dominated)} \quad , \tag{3.27}$$

with a constant C to be determined. We shall take its value so as to make connection at the turning point [see Eq.(3.38) below].

If we at any rate took seriously this model up to present time, we would have

$$\frac{t}{t_0} = \frac{1}{\sqrt{\Omega_{\gamma 0}}(1+z)^2} \quad ; \quad \frac{a(t)}{a_0} = \sqrt{2\sqrt{\Omega_{\gamma 0}} H_0 \; t} \; . \tag{3.28}$$

To have at hand some numerical expressions (time in seconds, $\kappa = 0$, $\Lambda = 0$):

$$t = \frac{2.2 \times 10^{19}}{(1+z)^2} \;\; ; \; 1 + z = \frac{4.68 \times 10^9}{\sqrt{t}} . \tag{3.29}$$

Before recombination (that is, for higher z's) there is thermal equilibrium between matter and radiation, because electrons are free and the mean free path of the photons is very small. An estimate of the energy per photon at a certain z can be obtained from $kT_{\gamma 0} = 2.3 \times 10^{-10} MeV$, which leads to

$$kT_\gamma = kT_{\gamma 0}(1+z) = 2.32 \times 10^{-10}(1+z) \; MeV. \tag{3.30}$$

For example, an energy of $4MeV$ corresponds to $z \approx 2 \times 10^{10}$.

In Kelvin degrees, the last equation above is

$$T_\gamma = 2.7 \times (1+z) \; . \tag{3.31}$$

Let us make a simple order–of–magnitude estimate concerning recombination. If we take for the hydrogen recombination temperature $T \approx 3000^o K$, we find $z \approx 1100$ for the recombination time. Equation (3.8) gives then

$$n_{bR} \approx 1.5 \times 10^{10}\ h^2\ \Omega_{b0}\ \ [cm^{-3}]$$

for the number density of protons at that time. Suppose now the medium to be neutral as a whole. After recombination, each proton has one electron to neutralize it. Before recombination, the number of free electrons is equal to that of protons. We can consequently use the above expression for the electrons in (2.59) to get an idea of the photon mean free path before recombination:

$$\lambda_\gamma \approx 10^{13}\ \ h^2\ \Omega_{b0}\ \ [cm]. \tag{3.32}$$

§ 3.3.3 By what has been seen, the thermalized state before recombination makes of T_γ, or its corresponding red–shift, a very convenient time parameter. We shall retain for later use the expressions

$$n_\gamma = 2.0287 \times 10^7 T_\gamma^3\ {}^oK^{-3} m^{-3} = 4.22 \times 10^8 (1+z)^3 m^{-3}\ ; \tag{3.33}$$

$$\frac{p_\gamma}{c^2 \rho_{crit}} = 7. \times 10^{-8}\ T_\gamma^4\ h^{-2} = 3.7 \times 10^{-6}\ (1+z)^4\ h^{-2}; \tag{3.34}$$

$$\frac{\rho_\gamma}{\rho_b} = 2.46 \times 10^{-5}\ (1+z)\ \Omega_{b0}^{-1} h^{-2}\ . \tag{3.35}$$

At recombination time,

$$\frac{\rho_b}{\rho_\gamma} \simeq 37\ \Omega_{b0}\ h^2\ .$$

The scale parameter, and consequently the red–shift, behave quite differently in a matter–dominated Universe (3.5) and in a radiation–dominated one (3.28). Eq.(3.35) shows that radiation becomes dominant at high enough z's.

The *turning point*, or change of *régime*, or *changeover*, takes place when $\rho_\gamma \simeq \rho_b$, or

$$1 + z \simeq 4.065 \times 10^4\ \Omega_{b0}\ h^2\ . \tag{3.36}$$

This corresponds to

$$t_{tp} \simeq .9 \times 10^{-8}\ \Omega_{b0}^{-2} H_0^{-1} = 2.75 \times 10^9\ \Omega_{b0}^{-2}\ \text{sec} = 90.6\ \Omega_{b0}^{-2}\ \text{years}. \tag{3.37}$$

§ 3.3.4 To give some numbers: if we use for Ω_{b0} the highest value in Eq. (2.18), $\Omega_{b0} = 0.026 h^{-2}$, and take for h the favored value $h = 0.7$, the changeover time is $t_{tp} \simeq 64570$ years (corresponding to $z \approx 2366$). If we include all kinds of matter in Ω_{b0}, so that $\Omega_{b0} = \Omega_s = 0.3$, then matter takes over much sooner, at $t_{tp} \simeq 1940$ years ($z \approx 13650$; in units of H_0^{-1}: $H_0 t_{tp} \approx 2 \times 10^{-7}$). We shall see that hydrogen recombination takes place at $z \approx 1100$. Only for the lowest value for Ω_{b0} in Eq. (2.18), $\Omega_{b0} = 0.0052 h^{-2}$, does the turning point falls after recombination ($z \approx 230$).

§ **3.3.5** We can use the turning point to get an estimate of the constant C in (3.27). In effect, combining that equation with (3.36) and (3.37) we obtain

$$a(t) = 0.082\, a_0 \sqrt{H_0}\, h^{-2}\, t^{1/2} \quad \text{(radiation dominated}, \kappa = 0, \Lambda = 0). \quad (3.38)$$

Figure 3.3 shows the scale parameter around the turning point.

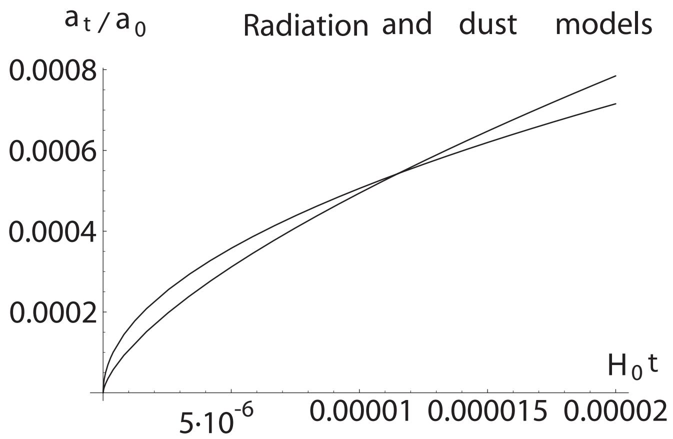


Figure 3.3: $a(t)/a_0$ *for radiation and dust dominated Universes, showing the turning point.*

We have above (page 31) alluded to the problem of the excess of matter at the beginning of the Universe, which is actually the matter nowadays present. This excess is customarily indicated by its relation to radiation. More precisely, a parameter η is defined by

$$\eta = \frac{n_\gamma}{n_b} \,. \quad (3.39)$$

From (3.33) and (3.8),

$$\eta = 3.7 \times 10^7 \; (\Omega_{b0} h^2)^{-1} \,. \quad (3.40)$$

WMAP gives 1.64×10^9 (notice that its notation is inverse to ours). Solving the "great mystery" of page 31 means explaining this number.

3.3.3 Radiation–domination: $\kappa = 0, \Lambda \neq 0$

§ **3.3.6** Recent evidence makes of this case the most important also for the radiation era. The expression for the second Friedmann equation which is the simplest to integrate is

$$\dot{H} = 2\,\frac{\Lambda c^2}{3} - \;2\;H^2 = 2\,\Omega_\Lambda H_0^2 - \;2\;H^2.$$

The main difficulty is choosing convenient integration constants. We shall suppose that $H(t_i) = H_i$ at some "initial" time t_i. The solution is found[2] to be[3]

$$H(t) = \sqrt{\Omega_\Lambda} H_0 \tanh\left[\operatorname{arctanh}\frac{H_i}{\sqrt{\Omega_\Lambda} H_0} + 2\sqrt{\Omega_\Lambda} H_0 (t - t_i)\right] \;,$$

which is[4] the same as

$$H(t) = \sqrt{\Omega_\Lambda} H_0\; \frac{H_i + \sqrt{\Omega_\Lambda} H_0 \tanh\left[2\sqrt{\Omega_\Lambda} H_0 (t - t_i)\right]}{\sqrt{\Omega_\Lambda} H_0 + H_i \tanh\left[2\sqrt{\Omega_\Lambda} H_0\,(t - t_i)\right]} \;.$$

Introducing $\bar{H}(t) = H(t)/H_0$ and $\bar{H}_i = H_i/H_0$,

$$\bar{H}(t) = \sqrt{\Omega_\Lambda}\; \frac{\bar{H}_i + \sqrt{\Omega_\Lambda} \tanh\left[2\sqrt{\Omega_\Lambda} H_0 (t - t_i)\right]}{\sqrt{\Omega_\Lambda} + \bar{H}_i \tanh\left[2\sqrt{\Omega_\Lambda} H_0\,(t - t_i)\right]} \;.$$

If $\bar{H}_i >> 1$, $t >> t_i$, this results in

$$H(t) = \frac{\sqrt{\Omega_\Lambda}\,H_0}{\tanh\left[2\sqrt{\Omega_\Lambda}\,H_0\,t\right]} \;, \tag{3.41}$$

which gives back (3.26) when $\Omega_\Lambda \to 0$. Actually, as this result is only meaningful before the turning point, necessarily $H_0 t < 1.8 \times 10^{-8}\Omega_s^{-2}$. For such values $\tanh\left[2\sqrt{\Omega_\Lambda} H_0\,t\right] \approx 2\sqrt{\Omega_\Lambda} H_0\,t$ is an excellent approximation and so is (3.26). This means that, in what concerns $H(t)$, the presence of the cosmological constant is irrelevant.

If we suppose that the expansion parameter vanishes at $t = 0$, Eq.(3.41) leads[5] to

$$a(t) = \sqrt{\sinh[2\sqrt{\Omega_\Lambda}\,H_0 t]} \quad (\text{radiation dominated}, \kappa = 0, \Lambda \neq 0). \tag{3.42}$$

The relation between time and red-shift is given by the first Friedmann equation,

---

[2] by using $\int \frac{dx}{1-x^2} = \operatorname{arctanh} x$

[3] remember $\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}$

[4] because $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

[5] because $\frac{1}{\tanh x} = \frac{d}{dx} \ln \sinh x$

$$\bar{H}^2(t) = \Omega_\Lambda + \Omega_{\gamma 0}\,(1+z)^4.$$

The solution for $\Omega_\Lambda \neq 0, \Omega_{\gamma 0} \neq 0, \kappa \neq 0$ will be given after we have studied the de Sitter spacetimes — see equation (4.52).

Comment 3.3.1 Causal domains for cosmic lumps The constituents of a inhomogeneity (galaxy, agglomerate) should be causaly related at the time of its creation. This restricts the possible age of present-day matter lumps, or the time of its formation. Only to illustrate the argument in rough brushstrokes, let us apply it to a typical galaxy. In order of magnitude, the mass will be

$$M_{gal} \approx 10^{11} M_{sun} = 2 \times 10^{44} g.$$

The matter mass density is nearly [see Eq.(3.7)]

$$\rho_b \approx 2 \times 10^{-29}(1+z)^3 \Omega_{b0} h^2 \ g\ cm^{-3}.$$

This is actually a minimum; with dark matter, it would be *circa* 70 times larger. The volume containing the matter which is in a galaxy will, consequently, be

$$V \approx \frac{M_{gal}}{\rho_b} = \frac{10^{73}}{(1+z)^3 \Omega_{b0} h^2}\ \mathrm{cm}^3.$$

This means that a causal correlation must be possible in distances of at least

$$L = V^{1/3} \approx 10^{1/3} \frac{10^{24}}{(1+z)[\Omega_{b0} h^2]^{1/3}}\ \mathrm{cm}.$$

Let us choose the maximal value ($= 0.026$) for $\Omega_{b0}h^2$, so that L is minimal: $L \approx 7 \times \frac{10^{24}}{(1+z)}$ cm at least; this should be $\lesssim ct$, so that $t \lesssim 2 \times \frac{10^{14}}{(1+z)}$ sec. If we now use Eq.(3.29), $1+z = \frac{6.8 \times 10^9}{\sqrt{t}}$, so that $\sqrt{t} \lesssim 2 \times \frac{10^{14}}{6.8 \times 10^9} \approx \frac{10^5}{3}$ and

$$t \lesssim 10^9 sec, 1+z \gtrsim 2 \times 10^5.$$

With dark matter, L and t will be smaller by a factor $\frac{1}{70^{1/3}} \approx 0.014$ and $1+z \approx 1.6 \times 10^6$. This rough, model-dependent estimate gives a time a bit before recombination time. Despite its crudeness, it at least indicates that it is difficult to place too early the origin of inhomogeneities.

3.4 The thermalized Universe

§ 3.4.1 Let us now go back to the general equations (2.57) and (2.58). Thermalization makes the whole system dependent of a single temperature. Much of the discussion can be made in terms of energies, by using (3.30) and noticing that, if T is in Kelvin degrees (^oK), then kT is given in MeV's as

$$kT(MeV) = 8.617 \times 10^{-11}\ T(^oK)\ .$$

It is frequently more convenient to use the variable z. Recall that, from Eqs.(2.13) and (2.27),

$$(1+z)H(z) = -\,\frac{dz}{dt}\ . \tag{3.43}$$

Extracting Ω_{s0} from (2.51) and using (2.50) for baryonic matter plus radiation,

$$\frac{H^2(z)}{H_0^2} = \Omega_{\gamma 0}\,(1+z)^4 + \Omega_{\kappa 0}\,(1+z)^2 + \Omega_\Lambda + (1+z)^3(1-\Omega_{\gamma 0}-\Omega_{\kappa 0}-\Omega_\Lambda)\ . \tag{3.44}$$

We shall use the time unit $H_0^{-1} = 1$ and write

$$H^2(z) = \Omega_{\gamma 0}(1+z)^4 + \Omega_{\kappa 0}\,(1+z)^2 + \Omega_\Lambda + (1+z)^3(1-\Omega_{\gamma 0}-\Omega_{\kappa 0}-\Omega_\Lambda)\ . \tag{3.45}$$

Comment 3.4.1 This holds, it is good to keep in mind, if radiation provides all the pressure. We can take in the energy density ϵ the separate contributions of the supposed constituents, say, $\epsilon = \epsilon_b + \epsilon_{Nb} + \epsilon_\gamma$ for baryonic matter, non-baryonic matter and radiation.

An interesting case is that of pure baryonic dust. The colossal proton lifetime ($> 10^{29}$ years, see footnote page 30) is a strong evidence for strict baryon-number conservation. If $N_b = n_b V$ is the number of baryons in volume $V = V_0 a^3/a_0^3$, then $\frac{dN_b}{dt} = V\frac{dn_b}{dt} + n_b\frac{dV}{dt} = 0$ implies

$$\frac{dn_b}{da} + 3\frac{n_b}{a} = 0\ \therefore\ \frac{d}{da}(n_b a^3) = 0. \tag{3.46}$$

For baryon dust $p_b = 0$ and $\epsilon_b = n_b m_b c^2$. Taken into the energy conservation expression (2.12), this leads to an independent conservation of the whole non-baryonic energy content:

$$\frac{d(\epsilon_{Nb}+\epsilon_\gamma)}{da} + 3\,\frac{\epsilon_{Nb}+\epsilon_\gamma+p_{Nb}+p_\gamma}{a} = 0\,.$$

If all matter is supposed to be baryonic, the radiation energy content is conserved:

$$\frac{d\epsilon_\gamma}{da} + 3\,\frac{\epsilon_\gamma + p_\gamma}{a} = \frac{d\epsilon_\gamma}{da} + \frac{4}{a}\,\epsilon_\gamma = 0\ \therefore\ \frac{d}{da}(\epsilon_\gamma a^4) = 0.$$

Thus, in this most simple case, the energies of matter and radiation are separately conserved.

Equation (3.23) gives $\Omega_{\gamma 0} << 1$, so that a good approximation for the $\kappa = 0$ case is

$$H^2(z) = \Omega_{\gamma 0}(1+z)^4 + \Omega_\Lambda + (1+z)^3(1-\Omega_\Lambda)\ . \tag{3.47}$$

Notice that the reference ($\kappa = 0$, $\Lambda = 0$) case

$$H^2 = \Omega_{\gamma 0}(1+z)^4 + (1-\Omega_{\gamma 0})(1+z)^3$$

reduces to (3.3) when the last term dominates the right-hand side, and slightly corrects (3.25) when the first term dominates.

Things are not that simple for the expansion parameter, or for the relation between z and t. Using $1 + z = a_0/a$, Eq.(3.45) can be written—in units (2.56)—as

$$a \frac{da}{dt} = \sqrt{\Omega_{\gamma 0} a_0^4 - \kappa a^2 + \Omega_\Lambda\ a^4 + a_0^3 (1 - \Omega_{\gamma 0} + \kappa a_0^{-2} - \Omega_\Lambda)\ a}, \tag{3.48}$$

whose solution is given by

$$t = t_0 + \int_{a(t_0)}^{a(t)} \frac{y\ dy}{\sqrt{\Omega_{\gamma 0} a_0^4 - \kappa y^2 + \Omega_\Lambda\ y^4 + y\left(1 - \Omega_{\gamma 0} - \Omega_\Lambda + \kappa a_0^{-2}\right) a_0^3}}\ . \tag{3.49}$$

Comment 3.4.2 Equation (3.48) provides a clear vision of the expansion parameter concavity, to be compared with that in the absence of cosmological constant shown in 2.1: its second time-derivative is independent of κ and exhibits a competition in the signs:

$$\ddot{a} = \Omega_\Lambda a - \Omega_{\gamma 0} a_0^4\, a^{-3} - \tfrac{1}{2}\, \Omega_{m0} a_0^3\, a^{-2}$$

where Ω_{m0} represents pure matter. The first term in the right hand side represents an acceleration, the other two are decelerations caused by the attractive gravity engendered by normal sources. We can estimate the point at which acceleration takes over, or $\ddot{a}$ changes its sign: $\ddot{a} = 0$ is the same as $\Omega_\Lambda = \Omega_{\gamma 0}(1+z)^4 + \frac{1}{2}\, \Omega_{m0}(1+z)^3$

For the present-day favored values $\Omega_\Lambda = 0.7, \Omega_{m0} = 0.3, \Omega_{\gamma 0} = 2.2 \times 10^{-5}$, this would mean $z \approx 0.67$ or $a(t) \approx 0.598 a_0$.

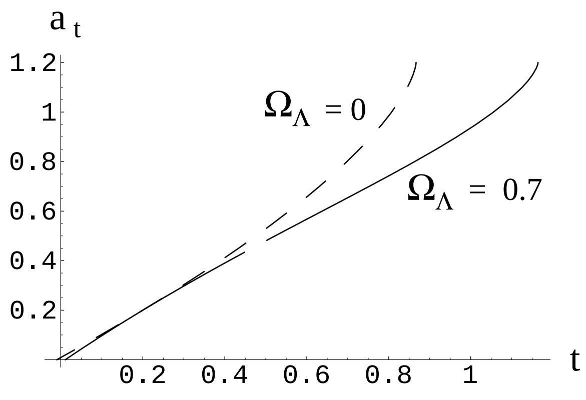


Figure 3.4: *Numerical solutions for $\kappa = 0$: $\Omega_\Lambda = 0$ and $\Omega_\Lambda = 0.7$.*

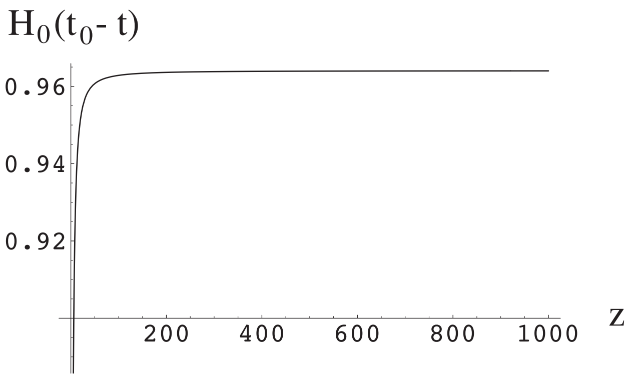


Figure 3.5: *Time versus red-shift with* $\kappa = 0$ *and* $\Omega_\Lambda = 0.7$.

Numerical solutions are shown in the Figures. As $a(t)$ and κ only appear in the forms $a(t)/a_0$ and κ/a_0^2, we can put $a_0 = 1$ and consider relative values. This means that what is numerically solved is

$$t - t_0 = \int_1^{a(t)} \frac{x\, dx}{\sqrt{\Omega_{\gamma 0} - \kappa x^2 + \Omega_\Lambda\ x^4 + x\,(1 - \Omega_{\gamma 0} - \Omega_\Lambda + \kappa)}} \quad .$$

A good approximation its

$$t - t_0 = \int_1^{a(t)} \frac{x\, dx}{\sqrt{x}\,\sqrt{\Omega_\Lambda\ x^3 - \kappa x + (1 - \Omega_\Lambda + \kappa)}} \quad .$$

For the reference value $\kappa = 0$, Figure 3.4 shows the resuls both for $\Omega_\Lambda = 0$ and for the present–day favoured value $\Omega_\Lambda = 0.7$. They should be compared with those for dust and radiation given before. Figure 3.5 shows the time $(t_0 - t)$ in units of H_0^{-1} as a function of z. The turning point [Eq.(3.36)] takes place for z in the interval $\approx 10^3$–10^4.

§ 3.4.2 It will be of interest for later use to have the Friedmann equations for the thermalized period in a simpler notation. Equation (3.45) can be written as

$$\frac{\dot{a}^2 + \kappa c^2}{a^2} = \Omega_{\gamma 0}\,\frac{a_0^4}{a^4} + \Omega_\Lambda + \frac{a_0^3}{a^3}\left(1 - \Omega_{\gamma 0} + \frac{\kappa c^2}{a_0^2} - \Omega_\Lambda\right) \,. \tag{3.50}$$

This is just equation (2.45) for dust plus radiation with $\kappa \neq 0$ and $\Lambda \neq 0$. Equation (2.46), for the time-derivative of H, is

$$\dot{H} = \frac{\kappa c^2}{a^2} - 2\,\Omega_{\gamma 0}\,\frac{a_0^4}{a^4} - \frac{3}{2}\,\frac{a_0^3}{a^3}\left(1 - \Omega_{\gamma 0} + \frac{\kappa c^2}{a_0^2} - \Omega_\Lambda\right) \,. \tag{3.51}$$

We should keep in mind that time is being measured in units of H_0^{-1}. In the right-hand side of (3.50), the first term is the radiation contribution and the third is that of nonrelativistic matter (or dust). It will be convenient to introduce in those terms the notations

$$\gamma c^2 = \Omega_{\gamma 0} a_0^4 \tag{3.52}$$

$$Mc^2 = \left(1 - \Omega_\Lambda - \frac{\gamma c^2}{a_0^4} - \frac{\kappa c^2}{{a_0}^2}\right) {a_0}^3. \tag{3.53}$$

The equations take on the forms

$$\frac{\dot{a}^2 + \kappa c^2}{a^2} = \frac{\gamma c^2}{a^4} + \Omega_\Lambda + \frac{Mc^2}{a^3}\,; \tag{3.54}$$

$$\dot{H} = \frac{\kappa c^2}{a^2} - 2\,\frac{\gamma c^2}{a^4} - \frac{3}{2}\,\frac{Mc^2}{a^3}\,. \tag{3.55}$$

The general solution (3.49) is very complicated. It is actually an implicit expression giving time t in terms of elliptic integrals in the expansion parameter a. We shall postpone further discussion of this topic to the chapter on de Sitter solutions (section 4.5).

3.5 An interlude: chemical reactions

§ 3.5.1 Thermodynamics of chemical reactions is of fundamental importance to Cosmology. Hydrogen ionization–neutralization, so important to the constitution of the microwave background, can be seen as a chemical reaction with neutral hydrogen, electrons and the ion H^+ as reactants. Also the processes involved in nucleosynthesis are reactions, and so are the formation of particle–antiparticle pairs leading to the intricate composition of radiation at very very high temperatures. A parenthesis on some general aspects of chemical reactions is in good place here.[6]

We begin with a short journey into Thermodynamics, and that for two reasons. One is obvious: chemical equilibrium is a particular kind of thermodynamic equilibrium. The second is that thermodynamics helps, by analogy, to clarify some statements we have rather hurriedly made above.

In this Section we apply the thermodynamics of chemical reactions to get Saha's equation for ionization processes. We use it to get an estimate of the hydrogen recombination temperature. Application to nucleosynthesis and to the composition of high–temperature black–body radiation are left to later Sections.

A *caveat*: Thermodynamics, as we shall use it, supposes equilibrium. This means that the equations must be applied with care. We shall try to use them only in conditions for which there are good reasons to suppose that equilibrium is established. For example, to calculate the temperature of half–ionization for Hydrogen. At half–ionization, there is a huge number of free electrons to interact with the radiation photons and establish equilibrium. Near total recombination would mean a small number of free electrons and doubtful equilibrium. In that case, chemical kinetics should be used, or out–of–equilibrium Thermodynamics. Calculations, involving the Boltzmann equation, are then much more elaborate.

Comment 3.5.1 When the system is out of equilibrium, it is not enough to consider the number densities of each species involved (n_e for electrons, n_γ for photons, n_p for protons, n_{H_+} for hydrogen ions, ...), as we are going to do in what follows. Correlations of particles 2–by–2, 3–by–3, etc, must be taken into account. This means that distribution functions for pairs, triads, etc, of particles become relevant, besides the single–particle distributions which the number densities represent. The complete description involves actually correlations to all orders. The general theory for all that does exist in principle, but involves an infinite set of equations, one for each level of correlation: one equation for the distribution of pairs, another for the distribution of triads, and so on. This infinite "hierarchy"

[6] In that topic, we shall follow mainly chapter X of Landau & Lifshitz [18], but chapter 9 of McQuarrie [19] is highly commendable.

of equations (called the *BBGKY hierarchy*)[7] is furthermore complicated by the fact that the equations are coupled (single and pair distributions appear in the equation for triads, and so on for the higher orders). The whole thing is far from being practical, and the art of out–of–equilibrium Statistical Mechanics consists in finding useful approximations. Mostly, these consider systems in which interactions and correlations, though present, are small. The main objective is to get everything written in terms of single–particle distributions. A number of equations turn up for the different types of approximation made: the Vlasov equation (when a mean field approximation is supposed), the Master equation (when the interaction coupling constants involved are small, as in the weak interactions which we will consider below), the Landau equation (when the pair distribution can be obtained from the single particle distribution and small collision terms). They all present a difficulty: most realistic interactions between particles, atoms and molecules are represented by potentials including a hard core, and in that case a weak coupling only gives a reliable approximation for dilute systems, when all constituents spend most of their time in the small tails of each other's potentials. Dilute systems are described by the best known of all such "kinetic equations", the Boltzmann equation. It is expected to hold, for example, at the end of the hydrogen recombination period, when the photons are no more sufficiently coupled to the electrons to ensure thermodynamical equilibrium. We shall see that a Planck distribution for the cosmic microwave is expected from a pure equilibrium treatment (actually, plus some extra assumptions). Some of the very small departures from that distribution, as temperature fluctuations, can be attributed to that non-equilibrium phase, and are approached via the Boltzmann equation.[8]

3.5.1 Some considerations on Thermodynamics

§ 3.5.2 Let us begin with heat. It has been a great achievement of human mind to understand that an infinitesimal quantity δQ of heat transferred to a physical system is not the differential of anything. That there exists no such a thing as a function "heat" Q of which δQ is a differential. And a still greater prowess to perceive that $\delta Q/T$ is the exact differential of a function, entropy. Thus, it is possible to write $TdS = \delta Q$. The same is true of the mechanical work $\delta W = \mathbf{F} \cdot \mathbf{dx}$, which is easier to understand. Suppose we displace the force from a point a to a point b, and want to know the work done. We take the integral $\int_a^b \delta W$, expecting to find $W(b) - W(a)$. However, the integral of $\mathbf{F} \cdot \mathbf{dx}$ is clearly a line integral $\int_{\gamma_{ab}} \mathbf{F} \cdot \mathbf{dx}$, where γ_{ab} is a curve joining a to b. The problem is that there are many such curves and each one gives a different result. Thus, the work necessary to take the force from one point to another depends to the chosen trajectory. Consequently, even if we had chosen some initial value $W(a)$ for W at the point a, it would be impossible to attribute a unique value to W at b. W is not (by far !) a single–valued function. Perception of this point has led to the modern concept of

[7] For Bogoliubov-Born-Green-Kirkwood-Yvon. A remarkable description of all these questions is given by R. Balescu, *Statistical Dynamics*, Imperial College Press, 1997.

[8] See Dodelson, *op.cit.* page iv.

differential form. We would nowadays say that W is not a function of points, but a functional $W[\gamma]$ on the space of curves γ linking a to b. We would also say that δW, like δQ, is not an exact form. Such non–exact differential forms may, in some cases, be integrable with the help of an integrating factor or denominator. This is the case of δQ, for which the temperature is an integrating denominator. The integral of dS between two states leads to a well–defined entropy variation. The same happens for thermodynamic work, for which the pressure is an integrating denominator: dV = - $\delta W/p$. The minus sign comes from the fact that the work done *on* the system by applying a pressure to it would lead to a negative dV. All this will lead to the usual expression of the first law of Thermodynamics. If we furnish heat δQ and work δW to a system, its energy will vary according to

$$dE = \delta Q + \delta W = TdS - pdV \ .$$

This can be generalized to the case in which also particles are supplied to the system, causing a variation dN in their number:

$$dE = \delta Q + \delta W = TdS - pdV + \mu dN \quad .$$

The chemical potential μ is the energy which must be furnished to the system in order that it increases the number of particles by one.

Comment 3.5.2 Now a parenthesis inside the parenthesis. We have previously "integrated" the differential dl, as if an interval function "l" existed. Well, clearly an integral such as $\int_\gamma dl = \int_\gamma \sqrt{dx^2 + dy^2 + dz^2}$ depends on the curve along which it is taken. And here comes the simple explanation: we have, in each case, used a preferred curve, the shortest one. We have in all cases been in 3–space, on which the notion of distance as an infimum,

$$d(a, b) = \inf_{\gamma_{ab}} \int_{\gamma_{ab}} \sqrt{g_{ij} dx^i dx^j}$$

has a sense because the metric, restricted to 3–space, is definite–positive and a geodesic is the shortest path. It would not make sense on 4–dimensional spacetime, on which metric is not definite–positive and the infimum does not exist.

The above expression of the first principle makes of the energy a "thermodynamic potential", in the sense that physical quantities are obtained from it by derivation. Pressure, for example, is minus the derivative of the energy with respect to the volume at constant S and N:

$$p = - \left(\frac{\partial E}{\partial V} \right)_{S,N} .$$

This means that the energy is the convenient potential when the independent variables are S, V and N. In some more detail, the first principle should actually be written

$$dE = T(S, V, N)\ dS - p(S, V, N)\ dV + \mu(S, V, N)\ dN \ . \tag{3.56}$$

We may have reasons, theoretical or experimental, to prefer other variables (entropy, for instance, is difficult to measure directly). And this leads us into the lore of thermal potentials. Changing the variables lead to different potentials, one for each set of variables. In each case, equilibrium takes place when the potential is minimum. Thus, minimizing the potential gives the condition for equilibrium.

The change of variable is made by a Legendre transformation, which corresponds to adding some product to the previous potential. Thus, if we wish to have T, V and N as independent variable, we subtract ST from E. Clearly,

$$d(E - ST) = -SdT - pdV + \mu dN \ .$$

This (T, V, N) potential is the Helmholtz free energy $A = E - TS$.

$$dA = -S(T, V, N)\, dT - p(T, V, N)\, dV + \mu(T, V, N)\, dN \ .$$

If, furthermore, we want p instead of V as independent variable, we add pV. What appears is the Gibbs potential $G(P, T, N)$:

$$dG = d(E - ST + pV) = -\,S(T, p, N)\, dT + V(T, p, N)\, dp + \mu(T, p, N)\, dN \ . \tag{3.57}$$

Some authors call G simply *the* thermodynamic potential. It is the most convenient potential when considering chemical reactions: the independent variables are the rather easily measurable quantities T, p and N.

3.5.2 Chemical equilibrium

§ 3.5.3 Adding sodium oxide to sulfuric acid is a rather expensive method to produce water:

$$Na_2O + H_2SO_4 \longrightarrow H_2O + Na_2SO_4 \ .$$

As long as we add material in the left–hand side, the process is a production reaction, with the arrow pointing to the right. Some time after we stop adding anything, the system attains a kind of equilibrium, the *chemical equilibrium* in which the relative number of molecules of each species is kept constant. The reaction then proceeds both ways with the same velocity, which is indicated as

$$Na_2O + H_2SO_4 \longleftrightarrow H_2O + Na_2SO_4 \ .$$

There is a standard way to indicate reactions as equalities. An equilibrium chemical reaction like

$$2H_2 + O_2 \longleftrightarrow 2H_2O \ ,$$

for example, is written in the symbolic form

$$2H_2 + O_2 - 2H_2O = 0 \ ,$$

with all terms in the left–hand side. Thus, a chemical reaction involving the reatants A_i has a general symbolic form

$$\sum_i \nu_i \ A_i = 0. \tag{3.58}$$

The numbers ν_i fix the reagent proportions in order to maintain equilibrium.

Suppose a system involving one or more reactions. Both the total number of particles N and the numbers N_i of each species can vary. Furthermore, the study of chemical reactions is usually made with the pressure p and the temperature T under external control, that is, as the independent variables. In consequence, the thermal potential appropriate to examine chemical equilibrium is the Gibbs potential $G(p, T, N)$. In the present case, what we have is a mixture of ideal gases, and

$$dG = - \ SdT + Vdp + \sum_i \mu_i dN_i \ .$$

where μ_i is the chemical potential of the component A_i, given by

$$\mu_i = \left(\frac{\partial G}{\partial N_i} \right)_{T,p} \ .$$

Chemical equilibrium, a particular case of thermal equilibrium, is attained for those N_i's which minimize the Gibbs potential. The N_i's are not independent: one of them can be written in terms of the others by using the constraint (3.58). This means that the concentrations must vary consistently: an increase in the concentration of one of the species implies an increase in the others, keeping the proportion fixed by the numbers ν_i. Suppose a variation dN_1 of the first species. In order to keep (3.58) valid, the other concentrations must vary proportionally, according to $dN_i = \frac{\nu_i}{\nu_1}\, dN_1$. Thus, the condition for equilibrium is

$$\left(\frac{\partial G}{\partial N_1}\right)_{T,P} = \sum_i \left(\frac{\partial G}{\partial N_i}\right)_{T,P} \frac{\partial N_i}{\partial N_1} = \sum_i \mu_i \, \frac{\nu_i}{\nu_1} = 0 \; ,$$

which leads to

$$\sum_i \nu_i \, \mu_i = 0 \; . \tag{3.59}$$

This is the chemical equilibrium condition. Substituting the μ_i's for all the reagents leads to the law of mass action.

For a gas of particles with mass m and chemical potential μ, it is convenient to introduce the fugacity variable $z = e^{\mu/kT}$ and the thermal wavelength λ, whose non–relativistic expression is

$$\lambda = \hbar \, \sqrt{\frac{2\pi}{mkT}} = \sqrt{\frac{h^2}{2\pi mkT}} \; . \tag{3.60}$$

A non-degenerate (that is, with no quantum effects) ideal non–relativistic gas has then the pressure and the number density given by

$$p = gkT\frac{z}{\lambda^3}$$

and

$$n = N/V = g\frac{z}{\lambda^3}$$

Here, g is the number of values assumed by the internal degrees of freedom (for example electrons, with their spin 1/2, will have $g = 2$). These are the detailed ("grand-canonical") forms of the pressure and density which lead to the well–known Clapeyron equation $pV = NkT$. From them, two equivalent expressions can be written for the chemical potential:

$$\mu = kT \ln\left(\frac{p\lambda^3}{gkT}\right) = kT \ln\left(\frac{n\lambda^3}{g}\right) \; .$$

These expressions are expected to hold for small values of z. By the way, the expression $n\lambda^3$ is the degeneracy index. It is the number of particles inside a cube whose sides are the thermal wavelength. Quantum effects are negligible when $n\lambda^3 << 1$ and important otherwise. When $z = \frac{n}{g}\, \lambda^3$ is small, the particle wavefunctions do not overlap and quantum effects (boson or fermion statistics) are negligible. The above formula are valid in that case, when $z << 1$ and provided $kT << mc^2$.

Consider a mixture of ideal non–relativistic reacting gases (or a mixture of small relative concentrations $c_i = N_i/N$ of solutes in a liquid). For each species A_j, the partial pressure and the number density will have the forms

$$p_j = c_j p = g_j \; kT \; \frac{z_j}{\lambda_j^3} = g_j \; kT \; \frac{e^{\mu_j/kT}}{\lambda_j^3};$$

$$\bar{n}_j = g_j \; \frac{z_j}{\lambda_j^3} = g_j \; \frac{e^{\mu_j/kT}}{\lambda_j^3} \; .$$

The typical chemical potential follows:

$$\mu_j = kT \; \ln\left[\frac{c_j p \lambda_j^3}{g_j \; kT}\right] = kT \; \ln\left[\frac{c_j n \lambda_j^3}{g_j}\right] = kT \; \ln\left[\frac{n_j \lambda_j^3}{g_j}\right] .$$

As each thermal wavelength is $\lambda_j = \hbar \sqrt{\frac{2\pi}{m_j kT}}$, so that $\lambda_j^3 = \left[\frac{h^2}{2\pi m_j kT}\right]^{3/2}$,

$$\mu_j = kT \; \ln\left[\frac{n_j}{g_j}\left(\frac{h^2}{2\pi m_j kT}\right)^{3/2}\right] = kT \; \ln\left[\frac{n_j}{g_j}\right] + \frac{3}{2} \; kT \; \ln\left[\frac{h^2}{2\pi m_j kT}\right] .$$

Then, (3.59) takes the form

$$\sum_j \nu_j \; \ln\left[\frac{c_j p \lambda_j^3}{g_j \; kT}\right] = 0 \; ,$$

or

$$\prod_j z_j^{\nu_j} = \prod_j \left[\frac{c_j p \lambda_j^3}{g_j \; kT}\right]^{\nu_j} = \prod_j \left[\frac{n_j \lambda_j^3}{g_j}\right]^{\nu_j} = 1 \; ,$$

or still

$$\prod_j c_j^{\nu_j} = p^{-\sum_i \nu_i} \prod_j \left[\frac{h^3}{(2\pi m_j)^{3/2} g_j \; kT^{5/2}}\right]^{-\;\nu_j} .$$

It is customary to introduce "zero" indices to recall that we are concerned with equilibrium values. The law of mass action is then written

$$\prod_i c_{0i}^{\nu_i} = p^{-\sum_i \nu_i} \prod_i p_{0i}^{\nu_i} = K_c(p,T) \; .$$

The c_{0i}'s are the concentrations at equilibrium; $p_i = c_i p = N_i p/N$ are the partial pressures, p_{0i} their equilibrium values; $K_c(p,T)$ and $K_p(T) = \prod_i p_{0i}^{\nu_i}$ are called the "chemical equilibrium constants". The important point is that they are functions only of p and T, and independent of the initial concentrations.

Take for illustration the simplest of all cases: a mixture of two ideal non-relativistic gases $A_1 + A_2$ far from degeneracy, giving two other species $A_3 + A_4$ with analogous characteristics:

$$A_1 + A_2 \longleftrightarrow A_3 + A_4.$$

Then we must have

$$\mu_1 + \mu_2 = \mu_3 + \mu_4.$$

Thus, the equilibrium condition for reaction $A_1 + A_2 \longleftrightarrow A_3 + A_4$ takes the form

$$\frac{n_1}{g_1}\left(\frac{h^2}{2\pi m_1 kT}\right)^{3/2} \frac{n_2}{g_2}\left(\frac{h^2}{2\pi m_2 kT}\right)^{3/2}$$
$$= \frac{n_3}{g_3}\left(\frac{h^2}{2\pi m_3 kT}\right)^{3/2} \frac{n_4}{g_4}\left(\frac{h^2}{2\pi m_4 kT}\right)^{3/2},$$

that is,

$$\frac{n_1 n_2}{n_3 n_4} = \frac{g_1 g_2}{g_3 g_4}\left(\frac{m_1 m_2}{m_3 m_4}\right)^{3/2}.$$

In this simple case, the chemical equilibrium constant (which coincides with right–hand side above) is independent of both pressure and temperature.

For relativistic gases (see Appendix C) things are by far more complicated. To begin with, the thermal wavelength has a fairly involved expression: its cube is

$$\Lambda^3(\beta) = 2\,\pi^2\,\beta mc^2\,\frac{e^{-\beta mc^2}}{K_2(\beta mc^2)}\left(\frac{\hbar c}{mc^2}\right)^3,$$

where $\beta = 1/kT$ is the inverse temperature and $K_2(x)$ is the modified Bessel function of second order. This reduces to (3.60) in the non-relativistic case, and to

$$\Lambda_{UR}(\beta) = \pi^{2/3}\,\beta\,\hbar c$$

in the ultra-relativistic limit. In particular, ultrarelativistic gases have high degeneracy indices $n\lambda^3$ (≈ 0.76 for photons, ≈ 0.57 for electrons), so that the use of complete quantal expressions for p and n is necessary.

3.5.3 Ionization: the Saha formula

§ 3.5.4 A hot gas in equilibrium is composed of all the excited states of the atom or molecule plus photons, in different concentrations. It is as if a series of chemical reactions were simultaneously at work between different species, each species being one of the excited states.

Say, for atomic hydrogen,

$$H \longleftrightarrow H_+ + e^- \ ,$$

with H and H_+ representing neutral and ionized hydrogen.

The chemical potentials are related by the equilibrium condition

$$\mu_H = \mu_+ + \mu_{e^-} \ .$$

A hydrogen atom is a bound state. In the fundamental level, its chemical potential is that given by the usual expression for a free atom minus its binding energy $E_1 = 13.6\ eV$. Using that value and the above expressions for the other chemical potentials, we find

$$\frac{n_H}{g_H}\left[\frac{h^2}{2\pi m_H kT}\right]^{3/2} e^{-\frac{E_1}{kT}} = \left[\frac{n_1}{g_1}\left(\frac{h^2}{2\pi m_H kT}\right)^{3/2}\right]\left[\frac{n_e}{g_e}\left(\frac{h^2}{2\pi m_e kT}\right)^{3/2}\right].$$

From this follows the Saha equation for hidrogen ionization,

$$\frac{n_1 n_e}{n_H} = \frac{g_1 g_e}{g_H}\left[\frac{2\pi m_e kT}{h^2}\right]^{3/2} e^{-\ E_1/kT} \ . \qquad (3.61)$$

The numerical expression is [5]

$$\frac{n_1 n_e}{n_H} = 2.4 \times 10^{15} T^{3/2} e^{-1.58\times 10^5/T} = 2.4 \times 10^{15} T^{3/2} 10^{-6.86\times 10^4/T} \ , \qquad (3.62)$$

with T expressed in ^oK and the number densities in cm^{-3}.

3.5.4 Recombination time

§ 3.5.5 Let us examine the problem of cosmological interest. The detailed calculations are involved, for reasons given below, but we can easily get a rough estimate of the temperature of half–ionization, in which $n_1 = n_e = n_H/2$. The above equation becomes

$$\frac{n_H}{4} = 2.4 \times 10^{15} T^{3/2} 10^{-6.86\times 10^4/T} \ , \qquad (3.63)$$

Use now (3.8),

$$n_H = 11.4 \times \ (1+z)^3 10^{-6}\ \Omega_{b0}\ h^2 \ \ [cm^{-3}]$$

to obtain

$$1.6 \times \ (1+z)^3\ \Omega_{b0}\ h^2 = 10^{21} T^{3/2} 10^{-6.86\times 10^4/T}$$

Use then (3.31), which is

$$T_\gamma = 2.7 \times (1+z)\ ,$$

to get

$$3.15 \times \ T_\gamma^3\ \Omega_{b0}\ h^2 = 10^{20} T^{3/2} 10^{-6.86\times 10^4/T}\ .$$

Now we make our boldest bid: we suppose $T_\gamma = T$. This seems a good assumption for half–ionization: there are already lots of free electrons to "stop" the photons, so that to suppose a thermalized medium is reasonable. After taking $\log_{10}$, we arrive at

$$10 \log[3.15\ \Omega_{b0}\ h^2] - 20 + 15 \log T = 686000/T,$$

which lends itself to simple numerical analysis. We obtain rather close values for the two extreme values of $\Omega_{b0}\ h^2$, $\Omega_{b0}\ h^2 = 0.026$ and $\Omega_{b0}\ h^2 = 0.0052$:

$$T \approx 3430\ ^oK\ .$$

This is the temperature at which half the hydrogen is ionized. Actually, ionization is a more complex process, involving all the hydrogen levels. The atom can go from the fundamental state into the first excited state with emission of a photon whose energy is the difference between the energies of the levels; it can then go to the third level; or it can go directly from the first to the third and so on. The detailed analysis involves a hierarchy of coupled Saha equations. Also the change of regime from radiation–dominated to matter–dominated must be taken into account. All this is solved numerically by huge programs. Hydrogen is found to be neutral at $T \approx 3000\ ^oK$, or $z \approx 1100$, close to the WMAP value given in the Table of page 29.

Comment 3.5.3 Notice that, as $c_j = n_j/n$, the chemical constant is $\frac{c_H}{c_1 c_e} = \frac{n_H n}{n_1 n_e}$.

Comment 3.5.4 Helium atoms, with two electrons, are still more complicated. Saha's formula must be used in a way analogous to that described above, but also allowing for Helium 2nd ionization.

3.6 Nucleosynthesis

§ 3.6.1 In very rough words, nuclei are formed by protons and neutrons kept together by strong interactions mediated by mesons, mainly pions. Astrophysicists have a very good theory for their formation: Starting from the lighter elements (deuterium, tritium, helium–3, helium–4), they are able to show how the heavier nuclei are formed in the natural furnaces of the cosmos, the stars. The problem which remains is the constitution of the initial blocks, the lighter elements. No star has a temperature high enough to produce them.

Consider the deuterium case. The nuclear reaction producing it is

$$n + p \rightarrow d + \gamma \ .$$

A large number of neutrons is necessary. Now, neutrons are unstable particles. They last less than 15 minutes (lifetime = 889 sec), and decay according to

$$n \rightarrow p + e^- + \bar{\nu}_e \ .$$

This decay is favoured by the disponibility of phase space: the mass difference between neutrons and protons is

$$\Delta E_{np} = m_n - m_p = 1.3 MeV \ .$$

Only at temperatures (kT) corresponding to that energy ($T \approx 1.5 \times 10^{10}\ {}^oK$) will the electrons have enough kinetic energy to make of the above reaction an equilibrium reaction

$$n + \nu_e \longleftrightarrow p + e^- .$$

The highest temperatures in the centers of the hottest stars are of the order of $10^8\ {}^oK$. In consequence, stars cannot produce deuterium by lack of neutrons.

Where could we arrive at such fantastic temperatures ? Gamow stepped in with the answer: in the Big Bang. At times remote enough, the Universe attains temperatures as high as we may wish. And this is the most compelling argument in favour of the Standard Model. Gamow, Alpher and Herman showed that the Universe could produce the lightest elements. Actually, they at first used a dust–filled model and found too much He. They added a quantity of radiation in order to break the excess. That radiation would, however, remain as a background. Given the quantity of radiation necessary to leave a good amount of He, they estimated the temperature that the cosmic radiation background should have today. This background has been found later, at a temperature close to the predicted one.

Arguing from luminosity considerations, we can arrive at the conclusion that no more than 1% of the present-day Helium can possibly come from the stars. There is another reason to look for a cosmic origin for He: its abundance, indicated by the parameter

$$Y = \frac{n_{He}}{n_{He} + n_H} ,$$

seems to be universal, that is, the same ($Y \approx 0.23$) everywhere. There are independent methods to determine this abundance in the Sun, in globular clusters of stars and, through the emission lines, in distant objects.

Comment 3.6.1 In what concerns the Sun, there is a well–known puzzle, the "solar neutrino puzzle". The neutrinos produced by the supposed chain of reactions would pass through Earth with a flux six times larger than the detected flux. Now, the latter is biased: detectors are prepared to detect electron-neutrinos. There is now strong evidence that neutrinos oscillate, that is, a good fraction of the solar electron-neutrinos have become muon-neutrinos when arrive at Earth. As a result, a much smaller flux would be detectable. [9]

Comment 3.6.2 Gamow, at that time (around 1947), did not use the expression "Big Bang". This name was invented later by Hoyle, as a mocking joke. A hot primeval Universe was then supported only by a few cosmologists (Lemaitre, Whittaker). The temperature predicted was $\approx 5.5\ ^oK$. There were large errors in the measured cosmological parameters, due to a systematic error in the distance calibration. Hubble himself gave a value 5.5 times higher for his constant. Furthermore, at that time Gamow was actually interested in realizing the nucleosynthesis of *all* elements in the Big Bang. His theory was presented as an alternative to star nucleosynthesis. It was later recognized that stars are better to produce the heavier elements.

Helium production is governed by a series of reactions. First deuterium must be formed through the reaction given above,

$$p + n \longleftrightarrow d + \gamma \, . \tag{3.64}$$

Once enough deuterium $d = H^2$ is available, the other light elements (He^3, He^4, H^3) are produced, the main reactions being

$$d + d \longleftrightarrow He^3 + n \longleftrightarrow H^3 + p \; ; H^3 + d \longleftrightarrow He^4 + n.$$

The equilibrium approach leads to some order–of–magnitude results, helpful in controlling the much more involved (and almost purely numerical) kinetic approach. There is some doubt concerning heavier elements. Some people believe that the Big Bang is responsible for the creation of all light

[9] See F. Wilczek, *The Standard Model transcended*, Nature **394** (1998) 13; S.F. King, *Neutrino Oscillations: Prospects and Opportunities at a Neutrino Factory*, arXiv:hep-ph/0105261 v2; K. Hagiwara et al. (Particle Data Group), Phys.Rev. **D66** (2002) 010001 (URL: http://pdg.lbl.gov).

nuclei up to Li^7, while other people think that cosmological nucleosynthesis stops at He^4. The calculations are subtle and laborious. Including Li^7 processing, there are 12 crucial nuclear cross–sections involved. Our aim here is only to describe qualitatively what is done.[10]

3.6.1 Equilibrium approach

§ 3.6.2 Consider for instance the deuterium-production reaction above, whose equilibrium condition is

$$\mu_p + \mu_n \longleftrightarrow \mu_d + \mu_\gamma \ .$$

As long as we can use the non-relativistic expression

$$\mu_{p,n,d} = kT \ln \frac{n_{p,n,d}\lambda^3_{p,n,d}}{g_{p,n,d}}$$

for the chemical potentials, it follows that

$$n_d = \frac{g_d}{g_p g_n} \left(\frac{\lambda_p \lambda_n}{\lambda_d}\right)^3 e^{-\ \mu_\gamma/kT} = \frac{g_d}{g_p g_n}\ n_p n_n h^3 \left(\frac{m_d}{m_p m_n}\ \frac{1}{kT}\right)^3 e^{-\ \mu_\gamma/kT} \ .$$

Well, photons have vanishing chemical potential. We can use this, provided we take into account separately the binding energies involved. It is simpler to simulate them through a photon chemical potential. In that case, the photon chemical potential must be $\mu_\gamma = \Delta E_{np} - B_d$, the neutron–proton mass difference ($\approx$ 1.3 MeV) minus the deuteron binding energy ($\approx$ 2.2 MeV). We arrive thus at

$$n_d = \frac{g_d}{g_p g_n}\ n_p n_n h^3 \left(\frac{m_d}{m_p m_n}\ \frac{1}{kT}\right)^3 e^{-\ E_{np}/kT} e^{B_d/kT} \ .$$

Analogous treatment must be given to the other equations. A hierarchy of coupled equations turns up. The numbers so obtained are rough estimates which serve, as said, to control the more involved kinetics.

[10] A review can be found in D.N. Schramm and M.S. Turner, Rev.Mod.Phys. **70** (1998) 303. A more recent summing up is S. Eidelman et al., Physics Letters **B592** (2004) 1, arXiv:astro-ph/0406663.

3.6.2 Kinetic approach

§ 3.6.3 A more detailed description can be obtained through a kinetic analysis, taking into account the reaction rates in each case. The reaction rate R for a reaction like

$$n + e^{+} \longrightarrow p + \bar{\nu}_e$$

is the number of reactions per second, roughly velocity/(mean free path):

$$R(n \to p) = v_e \; n_n \; \sigma_{ne} \; .$$

Actually, as each factor depends on the energy (or temperature, or still the momenta), the real expression is given by an integration. In the kinetic approach, the abundance of neutrons

$$X_n = \frac{n_n}{n_n + n_p} \; ,$$

for example, would be given as a function of time by a master equation (or gain-loss equation) like

$$\frac{d}{dt} X_n = X_p \; R(p \to n) - X_n R(n \to p) = (1 - X_n) \; R(p \to n) - X_n R(n \to p). \tag{3.65}$$

Variation in neutron abundance is the abundance of protons times the rate of proton to neutron transformation (this represents the gain) less the neutron abundance times the rate of its disappearance (which represents the loss).

The reaction rates are provided by the theories describing elementary particles interactions. In the case, the Electroweak Theory of Glashow, Weinberg and Salam. Actually, the precision required is not so great at present time. Electrodynamics, and the old "V – A" theory for weak processes are enough. As we are no more supposing equilibrium, all the reactions must be taken into account simultaneously. Thus, in order to calculate the neutron–proton abundance ratio, the three weak processes

$$n + e^{+} \longleftrightarrow p + \bar{\nu} \; ; \; n + \nu \longleftrightarrow p + e^{-} \; ; \; n \longleftrightarrow p + e^{-} + \bar{\nu}$$

are considered. Of course, the expressions involved are rather complicated. Besides the pure weak–interaction transition probability, there are factors taking into account the suppression due to the presence of fermions. An example of rate is

$R(n + \nu \longleftrightarrow p + e^{-}) =$

$$\frac{g_V^2 + 3 g_A^2}{2\pi^3 \hbar^7} \int v_e (\Delta_{np} + p_\nu c)^2 \; \frac{p_\nu^2 dp_\nu}{(e^{-p_\nu c / kT_\nu} + 1) \; (e^{-(\sqrt{\Delta_{np} + p_\nu}) / kT_e} + 1)} \; .$$

Here, $g_V = 1.4 \times 10^{-49}$ erg cm^3 and $g_A = 1.18\, g_V$ are the weak (vector and axial) coupling constants. A set of equations like (3.65) is written, one for each reaction. This system is then solved numerically. A detailed account of all the processes involved, with the expressions coming from Weak–Interaction Theory, is given in Weinberg's book [2].

As said above, Gamow, Alpher and Herman proceeded phenomenologically. They at first found too much deuterium, leading to too much Helium. To break the excess of deuterium, they have found necessary to add a well–chosen amount of radiation in reaction (3.64). The resulting Helium abundance is then in reasonable agreement with the observed value, but a gas of photons remains, which is the origin of the cosmic radiation background. As the amount of photons had been fixed by the necessary amount of deuterium, they were able to say how much there would be of it today and estimate the temperature.[11]

A last point: the neutrinos decouple during these reactions. In a way analogous to the photons during recombination, their mean free path becomes very very large. A neutrino background is predicted, whose detection is a very difficult task.

[11] G. Gamow, Phys.Rev **70** (1946) 572; R.A. Alpher, H. Bethe and G. Gamow, Phys.Rev **73** (1948) 803. The old, classical review on star nucleosynthesis is E.M. Burbidge, G.R. Burbidge, W.A. Fowler & F. Hoyle, Rev.Mod.Phys. **29** (1957) 547.

3.7 The cosmic microwave background

The spectrum of the cosmic microwave background, as we receive it today, is remarkably close to a Planck spectrum.[12] This fact gives important information on its emission, as well as on the processes it suffers (or not) during its travel in space and time. This section has two topics. In the first we analyze, guided by Weinberg's treatment of the subject, the background formation and its subsequent evolution. The idea is to apply the celebrated 1917 Einstein's paper on stimulated emission[13] to the case. In the second topic, we try to understand something about the intricate composition of black–body radiation at very high temperatures, in order to get some ideas on the very early Universe.

3.7.1 Conditions for a Planck spectrum

§ 3.7.1 Suppose again a light ray leaving the source with frequency ν_1 at the moment t_1 and arriving to us at time t_0. It has eventually traversed a large amount of (ionized or not) primeval, intergalactic and interstellar medium. We shall actually concentrate into what happens before and during recombination, up to the so called "surface of last scattering". Let us consider the number density $\mathcal{N}(\nu, t)$ of photons with frequency in the interval $(\nu, \nu + d\nu)$ in the light ray. Photons will be absorbed by the medium along its travel. Let $A(\nu, t)$ be the absorption rate per time unit for photons of frequency ν. This rate is c/(photon mean free path), and is such that $\dot{\mathcal{N}} = -A\mathcal{N}$. If some particles, atoms or molecules are the most efficient "absorbers", then $A = c\ n_a \sigma_{\gamma a}$, where n_a is the concentration of the absorbers and $\sigma_{\gamma a}$ is the absorption cross–section.

There is here a small complication with respect to what would happen in an laboratory experiment: in the cosmological case, the frequency varies during the process. The photon received with frequency ν_0 will have had the frequency $\nu(t)$ at time t. Fortunately that variation is known, given by Eq. (2.32). Thus, the loss of flux of the light ray is, here,

$$\frac{d}{dt}\mathcal{N}(t) = -A\left(\nu_0 \frac{a_0}{a(t)}, t\right)\mathcal{N}(t). \tag{3.66}$$

[12] Up to the 5-th digit. Anisotropies found at that level constitute one of the most important discoveries of the last years. See: W. Hu & S. Dodelson, *Cosmic Microwave Background Anisotropies*, Annu.Rev.Astron.Astrophys. **40** (2002) 171-219; A. Lasenby, *CMB anisotropies: recent measurements and interpretation*, Capetwon Conference, 2002; obtainable from E-mail adress a.n.lasenby@mrac.cam.ac.uk.

[13] Which can be found in B.L. van der Waerden's anthology: *Sources of Quantum Mechanics*, Dover, New York, 1967.

The solution is

$$\mathcal{N}(t_0) = e^{-\tau}\, \mathcal{N}(t_1), \tag{3.67}$$

where τ is the *optical depth*

$$\tau = \int_{t_1}^{t_0} A\left(\nu_0 \frac{a_0}{a(t)}, t\right)\, dt. \tag{3.68}$$

This is the fraction of the photons that were present at t_1 which will have been stopped at t_0. As $A(\nu, t)$ has dimension $(time)^{-1}$, τ is dimensionless.

All this supposes the medium to be purely absorptive—it is as if we had considered only the loss term in (3.65). In principle (here enters Einstein's paper), it will also emit radiation ("induced emission"). Let $\Gamma(\nu, t)$ be the number of emitted photons per unit time, per unit volume and per unit frequency interval at the frequency ν. These photons will be added to the light ray at a rate given by

$$\Omega(\nu, t) = \frac{\Gamma(\nu, t)\, c^3}{8\pi\nu^2}\ . \tag{3.69}$$

Equations (3.66) and (3.68) are corrected to

$$\frac{d}{dt}\mathcal{N}(t) = \left[\Omega\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right) - A\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right)\right]\, \mathcal{N}(t) \tag{3.70}$$

and

$$\tau = \int_{t_1}^{t_0} \left[A\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right) - \Omega\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right)\right]\, dt. \tag{3.71}$$

If the medium is by itself in thermal equilibrium, $\Omega(\nu, t)$ and $A(\nu, t)$ are related by Einstein's formula

$$\Omega(\nu, t) = e^{-\frac{h\nu(t)}{kT_m(t)}}\, A(\nu, t). \tag{3.72}$$

$T_m(t)$ is the medium temperature (if the medium is not in thermal equilibrium, some effective temperature must be introduced). It follows that

$$\tau = \int_{t_1}^{t_0} \left[1 - e^{-\frac{h\nu_0 a(t_0)}{kT_m(t) a(t)}}\right] A\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right)\, dt. \tag{3.73}$$

A few comments:

- to take into account photon scattering, a term $\int_{t_1}^{t_0} \Sigma\left(\nu(t), t\right)\, dt$ should be added to (3.73), $\Sigma\left(\nu(t), t\right)$ being the scattering rate for a photon of frequency $\nu_0 \frac{a(t_0)}{a(t)}$ at time t; it is difficult to take scattering into account for the background; after recombination, Thomson scattering (here, valid because $h\nu \approx kT \ll mc^2$), in particular, produces small effects; for the other possibilities, we suppose scattering to be far less important than absorption;

- notice again that the frequency is, at each time t, given by $\nu(t) = \nu_0 \frac{a(t_0)}{a(t)}$;

- notice that $\Omega \leq A$ and $e^{-\tau} < 1$, unless $T_m(t) < 0$; the latter case may occur when there is some population inversion in the medium; the ray is then amplified; such a maser effect has been observed in the Galaxy;

- the strongest assumption made has been, to the moment, that the medium is in thermal equilibrium.

Up to this point everything would hold for any light ray traversing intergalactic or interstellar media. Consider now the cosmic background. Indicate by $\mathcal{N}(\nu_0, t)\, d\nu_0$ the photon number density at time t which, at time t_0, would have frequencies in the interval between ν_0 and $\nu_0 + d\nu_0$. If neither absorption nor emission took place, $\mathcal{N}(\nu_0, t)$ would be proportional to $a^{-3}(t)$. The change in the number of photons in a volume proportional to $\mathcal{N}(\nu_0, t)a^3(t)$, due to spontaneous emission, will be

$$\Gamma\left(\nu_0\, \frac{a_0}{a(t)}, t\right) a^3(t)\, \frac{a_0}{a(t)}\, d\nu_0\ .$$

The rate of change of $\mathcal{N}(\nu_0, t)a^3(t)\, d\nu_0$ due to stimulated emission and absorption is

$$\left[\Omega\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right) - A\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right)\right]\ \mathcal{N}(\nu_0, t)a^3(t)\ d\nu_0,$$

analogous to (3.70). Thus, with spontaneous emission, stimulated emission and absorption, the time variation will be given by

$$\frac{d}{dt}\ [\mathcal{N}(\nu_0, t)a^3(t)] =$$

$$\Gamma(\nu_0\, \frac{a_0}{a(t)}, t)a^2(t)a_0 + \left[\Omega\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right) - A\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right)\right]\ \mathcal{N}(\nu_0, t)\ a^3(t).$$

If we use equation (3.69),

$$\frac{d}{dt}\ [\mathcal{N}(\nu_0,t)a^3(t)] =$$

$$\frac{8\pi\nu_0^2 a_0^3}{c^3}\ \Omega\left(\nu_0\frac{a_0}{a(t)},t\right) + \left[\Omega\left(\nu_0\frac{a(t_0)}{a(t)},t\right) - A\left(\nu_0\frac{a(t_0)}{a(t)},t\right)\right]\ \mathcal{N}(\nu_0,t)\ a^3(t).$$

The solution is, for an arbitrary t_1 (an integration constant),

$$\mathcal{N}(\nu_0,t)\ a^3(t) = \mathcal{N}(\nu_0,t_1)\ a^3(t_1)\ e^{-\int_{t_1}^{t} dt'\left[A\left(\nu_0\frac{a(t_0)}{a(t')},t'\right)-\Omega\left(\nu_0\frac{a(t_0)}{a(t')},t'\right)\right]}$$

$$+\frac{8\pi\nu_0^2 a_0^3}{c^3}\int_{t_1}^{t} dt'\ \Omega\left(\nu_0\frac{a(t_0)}{a(t')},t'\right)\ e^{-\int_{t'}^{t} dt''\left[A\left(\nu_0\frac{a(t_0)}{a(t'')},t''\right)-\Omega\left(\nu_0\frac{a(t_0)}{a(t'')},t''\right)\right]}\ .$$

Detailing 3.7.1 The above differential equation has the form

$$\frac{d}{dt}f(t) = g(t) + h(t)f(t)\ ,$$

whose solution is

$$f(t) = f(t_1)e^{\int_{t_1}^{t} dt'h(t')} + \int_{t_1}^{t} dt'g(t')e^{\int_{t'}^{t} dt''h(t'')}$$

with t_1 some constant. In effect we find, taking the derivative of this expression,

$$\frac{d}{dt}f(t) = f(t_1)h(t)e^{\int_{t_1}^{t} dt'h(t')} + g(t) + \int_{t_1}^{t} dt'g(t')\ h(t)\ e^{\int_{t'}^{t} dt''h(t'')}$$

$$= g(t) + h(t)\ \left[f(t) - \int_{t_1}^{t} dt'g(t')\ e^{\int_{t'}^{t} dt''h(t'')} + \int_{t_1}^{t} dt'g(t')\ e^{\int_{t'}^{t} dt''h(t'')}\right]$$

$$= g(t) + h(t)f(t)\ .$$

We have made repeated use of the formula for the derivative of an integral with respect to a parameter:

$$\frac{d}{da}\int_{R(a)}^{S(a)} f(x,a)dx = f[S(a),a]\ \frac{dS}{da} - f[R(a),a]\ \frac{dR}{da} + \int_{R(a)}^{S(a)}\frac{df(x,a)}{da}dx\ .$$

The first term in the right–hand side gives the number of photons left over from times before t_1; the second, the number of photons emitted since t_1. Take t_1 remote enough for all background radiation to have been emitted after t_1. And take $t = t_0$. The first term drops out, and the present density of photons per unit frequency interval turns up as

$$n_{\gamma 0}(\nu_0) \equiv \mathcal{N}(\nu_0,t_0)$$

$$= \frac{8\pi\nu_0^2}{c^3}\int_{t_1}^{t_0} dt\ \Omega\left(\nu_0\frac{a(t_0)}{a(t)},t\right)\ e^{-\int_{t}^{t_0} dt'\left[A\left(\nu_0\frac{a(t_0)}{a(t')},t'\right)-\Omega\left(\nu_0\frac{a(t_0)}{a(t')},t'\right)\right]}\ . \qquad (3.74)$$

If, furthermore, the medium is in thermal equilibrium, (3.72) will lead to

$$n_{\gamma 0}(\nu_0) = \frac{8\pi\nu_0^2}{c^3} \times$$

$$\int_{t_1}^{t_0} dt\, A\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right) e^{-\frac{h\nu_0 a_0}{kT_m(t)a(t)}}\, e^{-\int_t^{t_0} dt' \left[1-e^{-\frac{h\nu_0 a_0}{kT_m(t')a(t')}}\right] A\left(\nu_0 \frac{a(t_0)}{a(t')}, t'\right)} \,. \quad (3.75)$$

The expression

$$P(t_0, t; \nu_0) = e^{-\int_t^{t_0} dt' \left[1-e^{-\frac{h\nu_0 a_0}{kT_m(t')a(t')}}\right] A\left(\nu_0 \frac{a(t_0)}{a(t')}, t'\right)} \quad (3.76)$$

is the probability, account taken of stimulated emission, that a photon of frequency $\nu_0 \frac{a(t_0)}{a(t)}$, present at time t, survive up to time t_0. If only hydrogen — ionized and not — is taken into consideration, its general aspect is given in Figure 3.6. It vanishes before the onset of recombination and is equal to 1 after neutralization. The width of the increasing region measures how much the recombination process lasts or, in the language used nowadays, how much recombination is "delayed". A very steep curve means that recombination takes place in a very short interval of time.

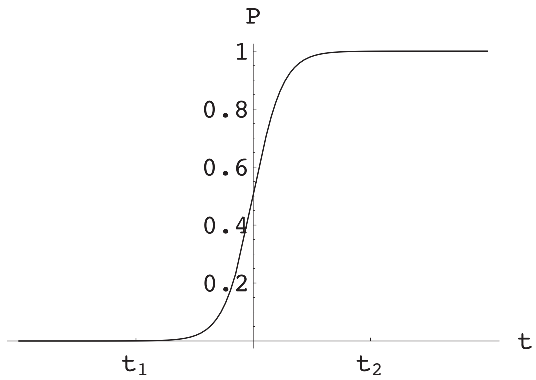


Figure 3.6: *General aspect of* $P(t_0, t; \nu_0)$.

Thus, the proper *energy* density is given by

$$\rho_{\gamma 0}(\nu_0)d\nu_0 = \frac{8\pi h}{c^3}\, \nu_0^3 d\nu_0 \int_{t_1}^{t_0} dt\, e^{-\frac{h\nu_0 a_0}{kT_m(t)a(t)}}\, A\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right) P(t_0, t; \nu_0) \,. \quad (3.77)$$

As

$$\frac{d}{dt} P(t_0, t; \nu_0) = \left[1 - e^{-\frac{h\nu_0 a_0}{kT_m(t)a(t)}}\right] A\left(\nu_0 \frac{a(t_0)}{a(t)}, t\right) P(t_0, t; \nu_0) \,,$$

we arrive at the final expression

$$\rho_{\gamma_0}(\nu_0)d\nu_0 = \frac{8\pi h}{c^3}\,\nu_0^3 d\nu_0 \int_{t_1}^{t_0} dt\,\frac{1}{e^{\frac{h\nu_0 a_0}{kT_m(t)a(t)}}-1}\,\frac{d}{dt}P(t_0,t;\nu_0)\;. \tag{3.78}$$

If recombination takes place in a very short time interval, recombination time will have a well–defined value t_R, P is well approximated by a Heaviside step function $P(t_0,t;\nu_0) = \Theta(t-t_R)$ and $\frac{d}{dt}P(t_0,t;\nu_0) = \delta(t-t_R)$. In this case of "instantaneous recombination",[14] the distribution reduces then to Planck's form:

$$\rho_{\gamma_0}(\nu_0)d\nu_0 = \frac{8\pi h}{c^3}\,\nu_0^3 d\nu_0\,\frac{1}{e^{\frac{h\nu_0}{kT_{\gamma_0}}}-1}\;, \tag{3.79}$$

with

$$T_{\gamma_0} = \frac{T_m(t_R)a(t_R)}{a_0}\;. \tag{3.80}$$

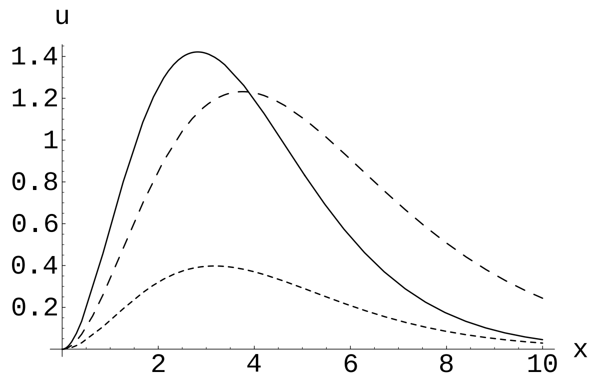


Figure 3.7: *Simulation showing the effect of delaying in recombination: when function $P(t_0,t;\nu_0)$ differs more and more from a step-function, the distribution differs more and more from a Planck distribution (continuum line).*

The cosmic radiation background has been discovered by Penzias and Wilson in the sixties at a particular wavelength ($\lambda = 7.35$ cm), and has since

[14] The possibility of detecting effects of a "delayed recombination" (two cases are illustrated in Figure 3.7) is discussed by V. Mukhanov, *CMB-slow, or How to Estimate Cosmological Parameters by Hand*, arXiv.org/astro-ph/0303072, Section 5. The WMAP project (D.N. Spergel et al., ApJS, **148** (2003) 175 - arXiv:astro-ph/0302209 v3) gives: for recombination ("decoupling"), $z = 1088$ and $t = 372$ kyears; for the width of the period in which our function P increases ("thickness of surface of last scatter"), $\Delta z = 194$ and $\Delta t = 115$ kyears.

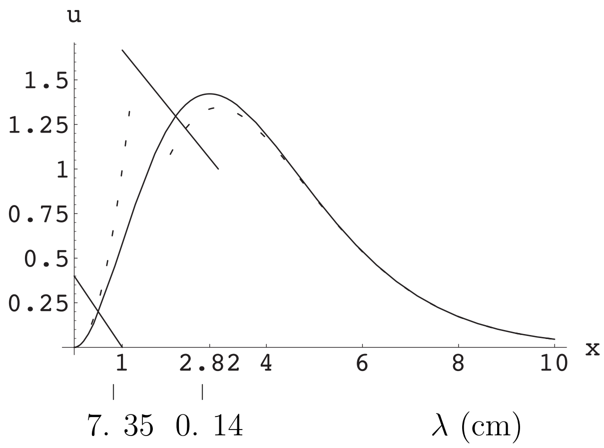


Figure 3.8: *Planck distribution* $u(x) = \frac{x^3}{e^x - 1}$ *, for* $x = h\nu/kT$*. Also are shown the Rayleigh-Jeans (*$u = x^2$*) and the Wien (*$u = x^3 e^{-x}$*) laws.*

then been found to respect formula (3.79) very closely (to five decimal digits). The distribution is shown in Figure 3.8, in terms of the variable $x = h\nu/kT$. The maximum at $x = 2.82$ corresponds to a wavelength $\lambda = 0.14$ cm. Some luck was involved: at ground level, only a limited "window" is detectable, at the left of the maximum (between the two straight lines which cut the curve in the Figure). Actually, only radiation in the so–called microwave domain of wavelengths can attain Earth's surface — and this is the reason for the sanctioned name for the background. The maximum itself, and the whole portion to its right, is screened by atmosphere absorption. Of course, since the time of discovery, much more has been detected by satellite probes. The extreme left region, corresponding to $\lambda >$ some tens of centimeters, is hidden below the emission of radio–galaxies.

Comment 3.7.1 Radio astronomers measure commonly what they call the energy flux, $\phi(\nu) = \rho(\nu)c/4\pi$, in erg cm^{-2} Hz^{-1} sec^{-1} sterad^{-1}.

Notice an overwhelmingly important point: *Only photons in thermal contact with matter have Planck's distribution.* This is clear from Einstein's deduction of Planck's formula in his 1917 paper. Nowadays, the photons of the thermal background have no contact with matter. Their mean free path is larger than any conceivable "size" of the Universe. How is it that a Planck distribution is observed? The Standard model explains this beautifully. The photons were in contact with matter (essentially, with the free electrons) before recombination. They had, consequently, a Planck distribution. After recombination, matter is practically neutral and the photon mean free path acquires a very large value. Their distribution evolves in time ever after, *but always respecting condition $\nu(t)a(t)$ = constant*, condition (2.32). This condition, a consequence of the symmetries of homogeneity and isotropy, has a miraculous property: it ensures the preservation of the functional form of Planck's distribution. A different relationship between frequencies and the scale parameter would lead to a continuous deformation of the distribution with time and no Planck distribution would be observed.

§ 3.7.2 Summing up: the cosmic microwave background will have a strict Planck form for the spectrum if the following conditions are satisfied:

- there is little absorption after recombination;
- photon scattering is irrelevant after recombination;
- the relevant medium *is* in thermal equilibrium or, at least, it is possible to attribute an effective temperature to it.

Observations show a strict Planck spectrum, which is furthermore isotropic, up to the fourth decimal case. Each one of these conditions holds consequently to that level of accuracy. Recent higher-precision data exhibit violations both in the form and isotropy. Such deformations and anisotropies provide information of the utmost importance on the conditions prevailing before, during and after the recombination period (which has been above supposed to be very short, actually instantaneous).[15]

For example, the supposed thermal equilibrium can be expected to fail at the end of the recombination period, when the photons mean free paths become too large to allow them to retain their role of thermalizers. An out-of-equilibrium approach becomes necessary. Fortunately the densities are small enough to make the use of Boltzmann's equation possible.

Though no consensual theory exists for the origin of present-day large inhomogeneities (galaxies, their agglomerates and possible still larger objects), it is a general belief that they cannot have been formed much before recombination (see Comment 3.3.1). If they are formed later, the background radiation will provide "radiographs" of them, which will manifest in anisotropies.

Some at least of the matter lumps are believed to come from density fluctuations, which engender fluctuations in the gravitational field, that is,

[15] Fluctuations of the CMB temperature around its mean value are studied for example by S. Weinberg, Phys.Rev **D** (2001) 123511 and 123512, from which earlier references can be traced.

in the metric. The latter will lead to fluctuations in the radiation background (Sachs-Wolfe effect[16]).

Matter lumps, above all the hot ones, will always include free electrons, however short-lived. Such electrons will scatter the background photons, leadind to further information (rescattering, or Sunyaev-Zel'dovich effect[17]).

3.7.2 High–temperature black body

§ 3.7.3 We intend here to show why the radiation–dominated Universe is a complicated affair at very high energies. At very high temperatures, radiation is a composite system formed by photons and all kinds of particles and antiparticles.

Let us again use Section 3.5 to consider the electron–positron pair creation–annihilation process

$$\gamma + \gamma \longleftrightarrow e^+ + e^- \ . \tag{3.81}$$

The pair creation cross section is, in order of magnitude,[18]

$$\sigma_{\gamma\gamma\to e^+e^-} = \frac{\pi}{2}\left[\frac{e^2}{m_e c^2}\right]^2 = 1.247 \times 10^{-25} cm^2.$$

We can use (3.33), which is

$$n_\gamma = 422\ (1+z)^3 cm^{-3} \ . \tag{3.82}$$

If we measure kT_γ in MeV's,

$$1 + z = 4.3 \times 10^9\ kT_\gamma\ [MeV] \, .$$

Consequently,

$$n_\gamma = 3.37 \times 10^{31}\ (kT_\gamma\ [MeV])^3 cm^{-3} \ . \tag{3.83}$$

The threshold for pair–creation is $\approx 0.5 MeV$. For kT_γ of that order, there is a huge number of photons:

$$n_\gamma \ \approx\ 4.2 \times 10^{30}\ \ \gamma/cm^3.$$

[16] See for example V. Mukhanov, *CMB-slow, or How to Estimate Cosmological Parameters by Hand*, arXiv.org/astro-ph/0303072.

[17] For review on this subject, see: J.E. Carlstrom, G. P. Holder and E. D. Reese, *Cosmology with the Sunyaev-Zel'dovich effect*, Annu. Rev. Astron. Astrophys. **40** (2002) 643–680.

[18] See, for example, Berestetskii, Lifshitz and Pitaevskii [28].

The pair–creation optical length will be

$$\tau_{pair} = n_\gamma \ \sigma_{\gamma\gamma\to e^+e^-} = 5.25 \times 10^5 \ cm^{-1} \ .$$

The corresponding pair–creation mean free path is

$$\lambda_{pair} = \frac{1}{n_\gamma \ \sigma_{\gamma\gamma\to e^+e^-}} = 1.9 \times 10^{-6} \ cm.$$

This means that, on the average, each photon will traverse this distance before meeting another photon and producing a pair, or that it will travel only during $\lambda_{pair}/c = \approx 10^{-16}$ sec before creating an electron and a positron. We can imagine a cylinder of base $\sigma_{\gamma\gamma\to e^+e^-}$ and height λ_{pair}: one e^- (or one e^+) will be created in every volume of that size. But that volume is just $1/n_\gamma$, so that the number density of created e^-'s is the same as that of the photons, $n_{e^\pm} \approx 4.2 \times 10^{30} \ cm^{-3}$. How does this number compare with the density, at that time, of those electrons which exist today ? In the usual travel backwards in time, the latter are produced by hydrogen ionization and, if matter is neutral as a whole, are in the same number as protons, consequently given by equation (3.8):

$$n_{e^-} = 11.4 \times \ (1+z)^3 10^{-6} \ \Omega_{b0} \ h^2 \ \ [cm^{-3}] \approx 10^{21} \ \Omega_{b0} \ h^2 \ \ [cm^{-3}] \ .$$

This is much less than the density of pair–produced electrons $\approx n_\gamma$ previously found. Thus, at $kT \approx 0.5MeV$, the electrons of nowadays are quite negligible. Only those belonging to the radiation count. This means that, at that time, the density of electrons equals that of positrons. If the system is in thermal equilibrium, these densities are given by Statistical Mechanics as

$$n_{e^-} = \frac{2}{h^3} \int \frac{d^3p}{e^{-\beta\mu_{(-)}} e^{\beta \ [\sqrt{p^2c^2+m^2c^4}-mc^2]} + 1}$$

and

$$n_{e^+} = \frac{2}{h^3} \int \frac{d^3p}{e^{-\beta\mu_{(+)}} e^{\beta \ [\sqrt{p^2c^2+m^2c^4}-mc^2]} + 1}$$

in terms of the chemical potentials $\mu_{(-)}$, $\mu_{(+)}$. If these two distributions are equal, then necessarily these chemical potentials are also equal,

$$\mu_{(-)} = \mu_{(+)} \ . \tag{3.84}$$

The chemical potential is the minimal energy variation when a particle is created. In a black body, the number of photons is ill–defined. Photons of nearly zero energy can exists in great numbers and can be created without energy variation. This means that the photon chemical potential μ_γ in a black body is zero. The condition of chemical equilibrium for reaction (3.81) is

$$\mu_{(-)} + \mu_{(+)} = 2\ \mu_\gamma\ ,$$

which says that $\mu_{(-)} + \mu_{(+)} = 0$. Combined with (3.84), this gives

$$\mu_{(-)} = \mu_{(+)} = 0\ . \tag{3.85}$$

The densities are consequently

$$n_{e^-} = n_{e^+} = \frac{2}{h^3} \int \frac{d^3p}{e^{\beta\ [\sqrt{p^2c^2+m^2c^4} - mc^2]} + 1}\ .$$

These integrals have relatively simple expressions in the two limits $kT >> mc^2$ and $kT << mc^2$ (see Appendix C). When $kT >> mc^2$,

$$n_{e^-} = 0.183\ \frac{\tau^3}{\lambda_C^3}\ ,$$

where $\tau = \frac{kT}{mc^2}$ and $\lambda_C = \frac{\hbar c}{mc^2}$ is the Compton wavelength. This is to be compared with the density of photons,

$$n_\gamma = 0.244\ \frac{\tau^3}{\lambda_C^3} = \frac{4}{3}\ n_{e^-}\ .$$

Notice, nevertheless, that at such high energies (recall, we have supposed $kT >> mc^2$) the cross–section also is energy–dependent and the analysis is more complicated. A numerical treatment becomes necessary. The qualitative result, however, remains: the electrons have a number density comparable to that of the photons.

The same reasoning, with analogous results, can be applied to pair production of more massive particles at higher temperatures. Provided we suppose thermal equilibrium, a black–body at very high temperature kT contains pairs of all particles and antiparticles with masses lesser than kT. When the masses are much smaller than kT, the concentrations of each kind of particle is of the same order of magnitude of that of the photons. The Universe is dominated by this intricate soup of particle–antiparticle pairs. As the densities increase with kT, interactions between these particles become more and more important. At present there is no known way to take such interactions into account in the equation of state.

This picture of the primeval Universe enhances the puzzle mentioned at the end of Section 3.3. All kind of present–day existing matter was quite negligible at that time, in comparison with matter contained in radiation. But it was already there, a small amount of dust, and remained to dominate the large scale behaviour from the "turning point" on.

Well, the Standard Model, despite all its achievements, is surely not the final description of our Universe. It faces some other problems, which will be examined later. Before tackling them, it is of interet to study the de Sitter solutions of Einstein's equation.

Chapter 4

de Sitter spacetimes

4.1 Introduction

de Sitter spacetimes are hyperbolic spaces of constant curvature. They are solutions of vacuum Einstein's equation with a cosmological term. There are two different kinds of them (see Figure 4.1): one with positive scalar Ricci curvature, and another one with negative scalar Ricci curvature.

We shall denote by $\mathcal{R}$ the de Sitter pseudo-radius, by $\eta_{\alpha\beta}$ $(\alpha, \beta, \cdots = 0, 1, 2, 3)$ the Lorentz metric of the Minkowski spacetime, and ξ^A $(A, B, \ldots = 0, \ldots, 4)$ will be the Cartesian coordinates of the pseudo-Euclidean 5–spaces. There are two types of spacetime named after de Sitter:

1. **de Sitter spacetime** $dS(4,1)$: hyperbolic 4-surface whose inclusion in the pseudo–Euclidean space $\mathbb{E}^{4,1}$ satisfies
$$\eta_{AB}\, \xi^A \xi^B = \eta_{\alpha\beta}\, \xi^\alpha \xi^\beta - \left(\xi^4\right)^2 = -\mathcal{R}^2 \; . \tag{4.1}$$
It is a one-sheeted hyperboloid with topology $R^1 \times S^3$, and — within our conventions — negative scalar curvature. Its group of motions is the pseudo–orthogonal group $SO(4,1)$

2. **anti–de Sitter spacetime** $dS(3,2)$: hyperbolic 4-surface whose inclusion in the pseudo–Euclidean space $\mathbb{E}^{3,2}$ satisfies
$$\eta_{AB}\, \xi^A \xi^B = \eta_{\alpha\beta}\, \xi^\alpha \xi^\beta + \left(\xi^4\right)^2 = \mathcal{R}^2 \; . \tag{4.2}$$
It is a two-sheeted hyperboloid with topology $S^1 \times R^3$, and positive scalar curvature. Its group of motions is $SO(3,2)$. With the notation $\eta_{44} = s$, both de Sitter spacetimes can be put together in
$$\eta_{AB}\, \xi^A \xi^B = \eta_{\alpha\beta}\, \xi^\alpha \xi^\beta + s\left(\xi^4\right)^2 = s\mathcal{R}^2 \; , \tag{4.3}$$
where we have the following relation between s and the de Sitter spaces:

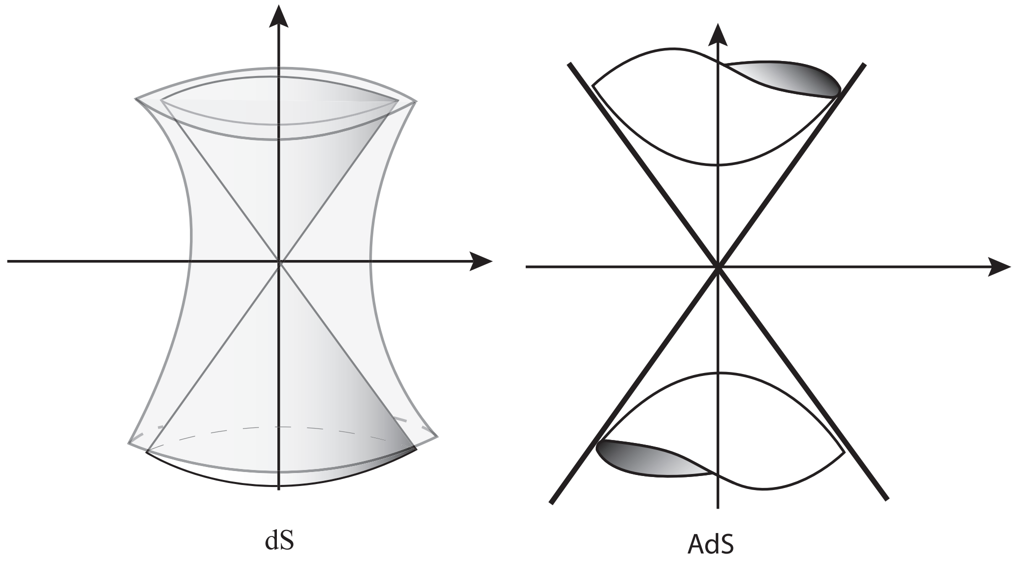


Figure 4.1: *De Siter (dS) space is like a deformed cylinder, with a single sheet which is topologically non-trivial. Anti-de Sitter space has two unconnected sheets, each one topologically trivial. Both kinds tend asymptotically to the same 4-dimensional cone-space, dS space from outside, AdS from inside. Of course, two dimensions have been excluded to make the picture possible.*

$$s = -1 \text{ for } dS(4,1)$$
$$s = +1 \text{ for } dS(3,2).$$

4.2 The de Sitter line element

§ 4.2.1 Let us find now the line element of the de Sitter spaces. The most convenient coordinates are the stereographic conformal. The passage from the Euclidean ξ^A to the stereographic conformal coordinates x^α ($\alpha, \beta, \cdots = 0, 1, 2, 3$) is done by the transformation:

$$\xi^\alpha = \Omega x^\alpha \; ; \; \xi^4 = \mathcal{R}(1 - 2\Omega), \tag{4.4}$$

(a sign in the last expression would have no consequence for what follows) with $\Omega(x)$ a function of x^α which we shall determine. Two expressions preparatory to the calculation of the line element can be immediately obtained by taking differentials:

$$d\xi^\alpha = x^\alpha d\Omega + \Omega dx^\alpha \; , \tag{4.5}$$

and

$$\left(d\xi^4\right)^2 = 4\mathcal{R}^2 d\Omega^2. \tag{4.6}$$

Let us introduce $\rho^2 = \eta_{\alpha\beta}\, x^\alpha x^\beta$ and rewrite the defining relation (4.3) as

$$\Omega^2 \rho^2 + s\left(\xi^4\right)^2 = s\mathcal{R}^2\ . \tag{4.7}$$

Equating $\left(\xi^4\right)^2$ got from (4.4) and (4.7), we find

$$\Omega = \frac{1}{1 + s\ \frac{\rho^2}{4\mathcal{R}^2}}\ . \tag{4.8}$$

Notice that from this expression it follows that

$$d\Omega = -\ s\ \frac{\Omega^2}{2\mathcal{R}^2}\rho d\rho = -s\ \frac{\Omega^2}{4\mathcal{R}^2} 2\eta_{\alpha\beta}\, x^\alpha dx^\beta\ , \tag{4.9}$$

from which another preparatory result is obtained:

$$2\eta_{\alpha\beta}\, x^\alpha dx^\beta = -\ s\ \frac{4\mathcal{R}^2}{\Omega^2} d\Omega\ . \tag{4.10}$$

Now, the de Sitter line element is

$$d\Sigma^2 = \eta_{AB}\, d\xi^A d\xi^B = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta + s\ \left(d\xi^4\right)^2\ ,$$

or, by using (4.5),

$$d\Sigma^2 = \eta_{\alpha\beta}\ \left(x^\alpha d\Omega + \Omega dx^\alpha\right)\left(x^\beta d\Omega + \Omega dx^\beta\right) + s\ \left(d\xi^4\right)^2\ .$$

Expanding and using (4.6),

$$d\Sigma^2 = \eta_{\alpha\beta} x^\alpha x^\beta d\Omega^2 + 2\eta_{\alpha\beta} x^\alpha dx^\beta \Omega d\Omega + \Omega^2 \eta_{\alpha\beta} dx^\alpha dx^\beta + s\ 4\mathcal{R}^2 d\Omega^2\ .$$

Now, using (4.10),

$$d\Sigma^2 = \Omega^2 \eta_{\alpha\beta} dx^\alpha dx^\beta + \left[\rho^2 -\ s\ \frac{4\mathcal{R}^2}{\Omega}\right] d\Omega^2 + s\ 4\mathcal{R}^2 d\Omega^2\ ,$$

and then (4.8),

$$d\Sigma^2 = \Omega^2 \eta_{\alpha\beta} dx^\alpha dx^\beta + \left[-\ s\ 4\mathcal{R}^2\right] d\Omega^2 + s\ 4\mathcal{R}^2 d\Omega^2\ ,$$

so that finally

$$d\Sigma^2 = g_{\alpha\beta} dx^\alpha dx^\beta\ , \tag{4.11}$$

where the metric $g_{\alpha\beta}$ is

$$g_{\alpha\beta} = \Omega^2\ \eta_{\alpha\beta}\ = \frac{1}{\left[1 + s\ \frac{\rho^2}{4\mathcal{R}^2}\right]^2}\ \eta_{\alpha\beta}\ . \tag{4.12}$$

The de Sitter spaces are, therefore, conformally flat, with the conformal factor given by $\Omega^2(x)$.

§ 4.2.2 The Christoffel symbol corresponding to a conformally flat metric $g_{\mu\nu}$ with conformal factor $\Omega^2(x)$ has the form

$$\overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\nu} = \left[\delta^{\alpha}_{\beta}\delta^{\sigma}_{\nu} + \delta^{\alpha}_{\nu}\delta^{\sigma}_{\beta} - \eta_{\beta\nu}\eta^{\alpha\sigma}\right] \partial_\sigma \ln \Omega(x) \; . \tag{4.13}$$

Taking derivatives in (4.8), we find for the de Sitter spaces

$$\overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\sigma} = - \; s \; \tfrac{\Omega}{2\mathcal{R}^2} \left[\delta^{\alpha}_{\beta}\eta_{\gamma\sigma} + \delta^{\alpha}_{\sigma}\eta_{\gamma\beta} - \eta_{\beta\sigma}\delta^{\alpha}_{\gamma}\right] x^{\gamma} \; . \tag{4.14}$$

The Riemann tensor components can be found by taking the following steps. First, take the derivative of the de Sitter connection $\overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\sigma}$:

$$\partial_\rho \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\sigma} = - \; s \; \tfrac{\Omega}{2\mathcal{R}^2} \left[\delta^{\alpha}_{\beta}\eta_{\gamma\sigma} + \delta^{\alpha}_{\sigma}\eta_{\gamma\beta} - \eta_{\beta\sigma}\delta^{\alpha}_{\gamma}\right] \delta^{\gamma}_{\rho} + \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\sigma} \; \partial_\rho \ln \Omega$$

$$= - \; s \; \tfrac{\Omega}{2\mathcal{R}^2} \left[\delta^{\alpha}_{\beta}\eta_{\rho\sigma} + \delta^{\alpha}_{\sigma}\eta_{\rho\beta} - \eta_{\beta\sigma}\delta^{\alpha}_{\rho}\right] - \frac{s x_\rho}{2\mathcal{R}^2\Omega} \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\sigma}$$

$$= - \; s \; \tfrac{\Omega}{2\mathcal{R}^2} \left[\delta^{\alpha}_{\beta}\eta_{\rho\sigma} + \delta^{\alpha}_{\sigma}\eta_{\rho\beta} - \eta_{\beta\sigma}\delta^{\alpha}_{\rho}\right] + \tfrac{x_\rho x^\gamma}{4\mathcal{R}^4} \left[\delta^{\alpha}_{\beta}\eta_{\gamma\sigma} + \delta^{\alpha}_{\sigma}\eta_{\gamma\beta} - \eta_{\beta\sigma}\delta^{\alpha}_{\gamma}\right]$$

$$= - \; s \; \tfrac{\Omega}{2\mathcal{R}^2} \left[\delta^{\alpha}_{\beta}\eta_{\rho\sigma} + \delta^{\alpha}_{\sigma}\eta_{\rho\beta} - \eta_{\beta\sigma}\delta^{\alpha}_{\rho}\right] + \tfrac{x_\rho x^\gamma}{4\mathcal{R}^4\Omega^2} \left[\delta^{\alpha}_{\beta}g_{\gamma\sigma} + \delta^{\alpha}_{\sigma}g_{\gamma\beta} - g_{\beta\sigma}\delta^{\alpha}_{\gamma}\right]$$

$$= - \; s \; \tfrac{\Omega}{2\mathcal{R}^2} \left[\delta^{\alpha}_{\beta}\eta_{\rho\sigma} + \delta^{\alpha}_{\sigma}\eta_{\rho\beta} - \eta_{\beta\sigma}\delta^{\alpha}_{\rho}\right] + \tfrac{1}{4\mathcal{R}^4\Omega^2} \left[\delta^{\alpha}_{\beta}x_\sigma x_\rho + \delta^{\alpha}_{\sigma}x_\beta x_\rho - g_{\beta\sigma}x^\alpha x_\rho\right] \; .$$

Indicating by $[\rho\sigma]$ the antisymmetrization (without any factor) of the included indices, we get

$$\partial_\rho \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\sigma} - \partial_\sigma \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\rho} = - \; s \; \tfrac{\Omega}{2\mathcal{R}^2} \left(\delta^{\alpha}_{[\sigma}\eta_{\rho]\beta} - \eta_{\beta[\sigma}\delta^{\alpha}_{\rho]}\right) + \tfrac{1}{4\mathcal{R}^4\Omega^2} \left(x_\beta\delta^{\alpha}_{[\sigma}x_{\rho]} - x^\alpha g_{\beta[\sigma}x_{\rho]}\right)$$

$$= - \; s \; \tfrac{\Omega}{\mathcal{R}^2}\delta^{\alpha}_{[\sigma}\eta_{\rho]\beta} + \tfrac{1}{4\mathcal{R}^4\Omega^2} \left(x_\beta\delta^{\alpha}_{[\sigma}x_{\rho]} - x^\alpha g_{\beta[\sigma}x_{\rho]}\right) .$$

This is the contribution of the derivative terms.

The product terms are

$$\overset{\circ}{\Gamma}{}^{\alpha}{}_{\lambda\rho}\overset{\circ}{\Gamma}{}^{\lambda}{}_{\beta\sigma} = \tfrac{\Omega^2}{4\mathcal{R}^4} \left[\delta^{\alpha}_{\lambda}\eta_{\gamma\rho} + \delta^{\alpha}_{\rho}\eta_{\gamma\lambda} - \eta_{\lambda\rho}\delta^{\alpha}_{\gamma}\right] \; \left[\delta^{\lambda}_{\beta}\eta_{s\sigma} + \delta^{\lambda}_{\sigma}\eta_{s\beta} - \eta_{\beta\sigma}\delta^{\lambda}_{s}\right] x^\gamma x^s;$$

$$\overset{\circ}{\Gamma}{}^{\alpha}{}_{\lambda\rho}\overset{\circ}{\Gamma}{}^{\lambda}{}_{\beta\sigma} - \overset{\circ}{\Gamma}{}^{\alpha}{}_{\lambda\sigma}\overset{\circ}{\Gamma}{}^{\lambda}{}_{\beta\rho} = \; \tfrac{1}{4\mathcal{R}^4\Omega^2} \left[x_\beta\delta^{\alpha}_{[\rho}x_{\sigma]} + x^\alpha x_{[\rho}g_{\sigma]\beta} + \Omega^2\rho^2 g_{\beta[\rho}\delta^{\alpha}_{\sigma]}\right] .$$

A provisional expression for the curvature is, therefore,

$$\overset{\circ}{R}{}^{\alpha}{}_{\beta\rho\sigma} = \partial_\rho \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\sigma} - \partial_\sigma \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\rho} + \overset{\circ}{\Gamma}{}^{\alpha}{}_{\lambda\rho}\overset{\circ}{\Gamma}{}^{\lambda}{}_{\beta\sigma} - \overset{\circ}{\Gamma}{}^{\alpha}{}_{\lambda\sigma}\overset{\circ}{\Gamma}{}^{\lambda}{}_{\beta\rho}$$

$$= \; - \; s \; \tfrac{\Omega}{\mathcal{R}^2}\delta^{\alpha}_{[\sigma}\eta_{\rho]\beta} + \tfrac{1}{4\mathcal{R}^4\Omega^2} \left(x_\beta\delta^{\alpha}_{[\sigma}x_{\rho]} - x^\alpha g_{\beta[\sigma}x_{\rho]}\right)$$

$$+ \tfrac{1}{4\mathcal{R}^4\Omega^2} \left[x_\beta\delta^{\alpha}_{[\rho}x_{\sigma]} + x^\alpha x_{[\rho}g_{\sigma]\beta} + \Omega^2\rho^2 g_{\beta[\rho}\delta^{\alpha}_{\sigma]}\right] .$$

The first two terms in the last line just cancel the last two in the line above them. Therefore,

$$\begin{aligned}
\overset{\circ}{R}{}^{\alpha}{}_{\beta\rho\sigma} &= -\ s\ \tfrac{\Omega}{\mathcal{R}^2}\delta^{\alpha}_{[\sigma}\eta_{\rho]\beta} + \tfrac{1}{4\mathcal{R}^4\Omega^2}\Omega^2\rho^2 g_{\beta[\rho}\delta^{\alpha}_{\sigma]} \\
&= -\ s\ \tfrac{\Omega}{\mathcal{R}^2}\eta_{\beta[\rho}\delta^{\alpha}_{\sigma]} + \tfrac{1}{4\mathcal{R}^4}\Omega^2\rho^2\eta_{\beta[\rho}\delta^{\alpha}_{\sigma]} \\
&= \left[-\ s\ \tfrac{\Omega}{\mathcal{R}^2} + \tfrac{1}{4\mathcal{R}^4}\Omega^2\rho^2\right]\eta_{\beta[\rho}\delta^{\alpha}_{\sigma]} \\
&= s\ \tfrac{\Omega}{\mathcal{R}^2}\left[\tfrac{s}{4\mathcal{R}^2}\Omega\rho^2 -\ 1\right]\eta_{\beta[\rho}\delta^{\alpha}_{\sigma]}
\end{aligned}$$

Using (4.8), we find that the bracketed term is $= -\,\Omega$. We get finally

$$\overset{\circ}{R}{}^{\alpha}{}_{\beta\rho\sigma} = -\ s\ \tfrac{\Omega^2}{\mathcal{R}^2}\eta_{\beta[\rho}\delta^{\alpha}_{\sigma]} \ = \ -\ \tfrac{s}{\mathcal{R}^2}\left[\delta^{\alpha}_{\sigma}g_{\beta\rho} - \delta^{\alpha}_{\rho}g_{\beta\sigma}\right]\ . \tag{4.15}$$

The Ricci tensor will be

$$\overset{\circ}{R}_{\mu\nu} = \ \tfrac{3\ s}{\mathcal{R}^2}\ g_{\mu\nu} \tag{4.16}$$

and the scalar curvature,

$$\overset{\circ}{R} = \frac{12\ s}{\mathcal{R}^2}. \tag{4.17}$$

§ 4.2.3 We can now make contact with the cosmological term. From the expressions above, we find that

$$\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\ g_{\mu\nu}\ \overset{\circ}{R} + \frac{3\ s}{\mathcal{R}^2}\ g_{\mu\nu} = 0. \tag{4.18}$$

Comparison with (1.1) shows that de Sitter spaces are solutions for the sourceless Einstein's equations with a cosmological constant

$$\Lambda = \ -\ \frac{3\ s}{\mathcal{R}^2}. \tag{4.19}$$

We find again relation (B.20). Notice the relationships to the de Sitter and the anti-de Sitter spaces:

$s = -1$ for the de Sitter space $dS(4,1) \longrightarrow\ \Lambda > 0$
$s = +1$ for the anti-de Sitter space $dS(3,2) \longrightarrow\ \Lambda < 0$.

Notice further that, in terms of $\mathcal{R}$, the dark energy density parameter is

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} = -\ \frac{s}{H_0^2}\frac{c^2}{\mathcal{R}^2}. \tag{4.20}$$

For the de Sitter space and using H_0^{-1} as the unit of time, Ω_Λ is simply $\frac{c^2}{\mathcal{R}^2}$.

Comment 4.2.1 We have been using carefully two coordinate systems. The most convenient system for cosmological considerations is the so–called comoving system, in which the Friedmann equations, in particular, have been written. In that system the scale parameter appears in its utmost simplicity. We shall use it for the de Sitter case below, starting at §4.4.1. The stereographic coordinates are of special interest for de Sitter spaces. We could perform a transformation between the two systems, but that is not really necessary: we have only taken scalar parameters from one system into the other. The only exception, Eq. (4.18), is a tensor which vanishes in a system and, consequently, vanishes also in the other.

Expression (4.12) for the metric is very different from the original one. De Sitter has found it in another coordinate system, in the form

$$ds^2 = \left(1 - \frac{\Lambda}{3}\, r^2\right) c^2 dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{dr^2}{1 - \frac{\Lambda}{3}\, r^2} \quad . \tag{4.21}$$

There are many other metric coordinates expressions, each of interest for a different aim.[1]

4.3 On the cosmological constant

The origin of the cosmological constant, or of the corresponding dark energy, remains a (fittingly dark) mystery. We shall here only briefly comment on some of its aspects: first, the significance of its presence; second, its effect on the motion of a relativistic particle; third, on the non-relativistic limit.

4.3.1 Presence in the Universe

§ 4.3.1 A first point: the Einstein equations with a cosmological constant

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = \tfrac{8\pi G}{c^4} \left[T_{\mu\nu} + \tfrac{c^4}{8\pi G}\, \Lambda g_{\mu\nu}\right], \tag{4.22}$$

takes on, with the energy-momentum $T_{\mu\nu} = (p + \epsilon)\; u_\mu u_\nu - p\; g_{\mu\nu}$ of a perfect fluid, the form

$$G_{\mu\nu} = \tfrac{8\pi G}{c^4} \left[\left(\left[p - \tfrac{c^4}{8\pi G}\, \Lambda\right] + \left[\epsilon + \tfrac{c^4}{8\pi G}\, \Lambda\right]\right)\; u_\mu u_\nu - \left(p - \tfrac{c^4}{8\pi G}\, \Lambda\right) g_{\mu\nu}\right]. \tag{4.23}$$

We see that the cosmological-constant term $\tfrac{c^4}{8\pi G}\, \Lambda$ adds positively to the energy density and negatively to the pressure. Let us interpret $\epsilon_\Lambda = \tfrac{c^4}{8\pi G}\, \Lambda$ as the energy density associated to the cosmological constant. The Friedmann equations (2.9), (2.10) can then be rewritten as

$$\dot{a}^2 = \tfrac{8\pi G}{3c^2} \left[(\epsilon + \epsilon_\Lambda)\, a^2 - \tfrac{3\kappa c^4}{8\pi G}\right]; \tag{4.24}$$

[1] A few, included that given below, are given by R.C. Tolman, *Relativity, Thermodynamics and Cosmology*, Dover, New York, 1987, § 142.

$$\ddot{a} = -\tfrac{4\pi G}{3c^2}\left[(\epsilon + \epsilon_\Lambda) + 3(p - \epsilon_\Lambda)\right] a(t) \ . \tag{4.25}$$

Their aspect corroborate the statement above: ϵ_Λ appears added to ϵ and subtracted from p. When $\Lambda > 0$ (a de Sitter-like contribution), it represents at the same time a positive energy and a negative pressure. The opposite would occur if $\Lambda < 0$ (anti-de Sitter case). By the way, this shows also why the cosmological term can be important, despite the smallness of Λ: it appears multiplied by $\frac{c^4}{8\pi G}$, a very very large factor.

Comment 4.3.1 The observational data give $\Lambda = 3.52 \times 10^{-56}\, \Omega_\Lambda h^2\, [cm^{-2}]$; if we take the present-day favored values $\Omega_\Lambda = 0.75$ and $h = 0.7$, Λ is very small indeed: $\Lambda \approx 1.32 \times 10^{-56}\, [cm^{-2}]$. Nevertheless, the effective energy density is

$$\epsilon_\Lambda = \rho_\Lambda c^2 = \frac{\Lambda c^4}{8\pi G} = \frac{\Lambda c^2}{3}\,\frac{3}{4\pi G}\,\frac{c^2}{2} \approx 10.5 \times (\Omega_\Lambda h^2)\,\frac{eV}{mm^3} \approx 3.8\frac{eV}{mm^3}.$$

This is an impressive number: each cubic millimeter of the Universe contains, according to the most recent data, 3.8 eV in the form of dark energy. As an order-of-magnitude reminder, each electron in the fundamental state of an hydrogen atom has a binding energy = – 13.6 eV.

4.3.2 Effect on test particles

§ 4.3.2 A positive scalar curvature tends to make curves to close to each other. A negative curvature does just the contrary. The relative signs in (4.17,4.19) show that the cosmological constant has the opposite effect: $\Lambda > 0$ leads to diverging curves, $\Lambda < 0$ to converging ones. This actually depends on the initial conditions. Let us look at the geodetic deviation equation,

$$\frac{D^2 X^\alpha}{Du^2} = \overset{\circ}{R}{}^\alpha{}_{\beta\rho\sigma} U^\beta U^\rho X^\sigma \ . \tag{4.26}$$

Using Eq.(4.15),

$$\frac{D^2 X^\alpha}{Ds^2} = -\tfrac{s}{\mathcal{R}^2}\left[\delta^\alpha_\sigma g_{\beta\rho} - \delta^\alpha_\rho g_{\beta\sigma}\right] U^\beta U^\rho X^\sigma$$
$$= -\tfrac{s}{\mathcal{R}^2}\left[\delta^\alpha_\sigma - U^\alpha U_\sigma\right] X^\sigma = -\tfrac{s}{\mathcal{R}^2}\, h^\alpha{}_\sigma X^\sigma \ . \tag{4.27}$$

We have recognized the transversal projector $h^\alpha{}_\sigma$. By the geodesic equation, the component of X along U will have vanishing contributions to the left-hand side. Consequently, only the transversal part $X_\perp$ will appear in the equation, which is now

$$\frac{D^2 X^\alpha_\perp}{Ds^2} + \tfrac{s}{\mathcal{R}^2}\, X^\alpha_\perp = \frac{D^2 X^\alpha_\perp}{Ds^2} + \tfrac{\overset{\circ}{R}}{12}\, X^\alpha_\perp = \frac{D^2 X^\alpha_\perp}{Ds^2} - \tfrac{\Lambda}{3}\, X^\alpha_\perp = 0 \ . \tag{4.28}$$

Negative Λ leads to oscillatory solutions. Positive Λ can lead both to contracting and expanding congruences. If two lines are initially separating, they will separate indefinitely more and more.

4.3.3 Local effects

§ **4.3.3** For another purpose, let us recast the Friedmann equations (2.9), (2.10) in the forms

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\,\rho = \frac{\Lambda}{3}\,c^2 - kc^2 \tag{4.29}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3}\,c^2 - \frac{4\pi G}{3}\,(\rho + \frac{3p}{c^2})\ . \tag{4.30}$$

Consider then the following simple model. Take a small space region bounded by a sphere of radius "r_0". The distance between the centre and any point on its surface is determined by the *physical distance* $a(t)r_0$, which satisfies the equation

$$\frac{d^2(ar_0)}{dt^2} = \frac{\Lambda}{3}\,ar_0 - \frac{4\pi G}{3}\,(\rho + \frac{3p}{c^2})ar_0\ . \tag{4.31}$$

or

$$\frac{d^2(ar_0)}{dt^2} = \frac{\Lambda}{3}c^2\,ar_0 - \frac{GM}{(ar_0)^2} \equiv F_\Lambda + F_N\ , \tag{4.32}$$

where

$$M = \frac{4\pi}{3}\,(\rho + \frac{3p}{c^2})(ar_0)^3 \tag{4.33}$$

is the *total mass* inside the sphere.

The second term of (4.32) is the ordinary Newtonian force

$$F_N = -\frac{GM}{(ar_0)^2}\ .$$

The first term, caused by the cosmological constant Λ, is also a gravitational force, but with the rather unusual form

$$F_\Lambda = \frac{\Lambda}{3}\,c^2\,ar_0\ .$$

This field of force is global and homogeneous. The intensity of the force depends on the distance between the interacting particles as (ar_0). The further away are the particles, the larger is the force.

Comment 4.3.2 Analogous results comes from the Newtonian limit for the interval

$$ds^2 = c^2dt^2\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}\,r^2\right) - \left[\frac{dr^2}{1-\frac{2m}{r}-\frac{\Lambda}{3}\,r^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]\ , \tag{4.34}$$

which describes the Schwartzschild-de Sitter case: a source point-mass with a cosmological constant ($c = 1$).

4.4 de Sitter inflationary solution

§ **4.4.1** The space sections of de Sitter spacetimes are homogeneous and isotropic. Friedmann equations (2.9-2.11) hold consequently for de Sitter cosmology, with vanishing ρ and p:

$$\dot{a}^2 = \frac{\Lambda c^2}{3}\, a^2 - \kappa c^2 \; ; \tag{4.35}$$

$$\ddot{a} = \frac{\Lambda c^2}{3}\, a(t) \; . \tag{4.36}$$

With $\kappa = 0$, we have just (2.21), whose solution is

$$a(t) = A\; e^{\sqrt{\frac{\Lambda}{3}}\, c\, t}. \tag{4.37}$$

Comment 4.4.1 We can look for a solution of the general form

$$a(t) = A e^{\alpha t} + B e^{\beta t} \; .$$

The equations imply $(\dot{a})^2 = a\; \ddot{a}$, which requires $\alpha = \beta$. Thus, the general expression reduces to $a(t) = \mathrm{c}\; e^{\alpha t}$. But then $\dot{a}/a = \alpha = H = \pm\sqrt{\frac{\Lambda c^2}{3}} = \pm\frac{c}{\mathcal{R}}$. If the constant H is taken to be positive, the above solution is the only one. The deceleration function is constant, actually an acceleration, as $q = -1$ follows from (2.15). When $\kappa \neq 0$, it follows directly from the Friedmann equations that $q(t) = \frac{a^2(t)}{\kappa\mathcal{R}^2 - a^2(t)}$.

§ **4.4.2** Let us repeat once again: a solution of type (4.37) is called "inflationary". More precisely, it is called "inflationary of exponential type", to differentiate it from other explosive solutions. There are two supposedly equivalent simplest scenarios for exponential inflation:

1. no sources, and cosmological constant. Eq.(4.37) shows then that for inflation necessarily $\Lambda > 0$ if $\kappa = 0$.

2. no cosmological constant, source with special equation of state

$$p = -\,\epsilon \tag{4.38}$$

   Eq.(2.11) shows then that ϵ and p are constant. Eq.(B.14) shows that $T > 0$ and (B.20) that $R < 0$. It is as if there was an effective cosmological constant $\Lambda = 8\pi G T/(4c^4) = 8\pi G\epsilon/c^4$. Or we may say that a cosmological constant is equivalent to a source fluid with an effective pressure (4.38). This would also agree with (4.17). This brings the problem back to a pure de Sitter case.

Notice that $\Lambda > 0$ is essential for de Sitter inflation. Consequently, we must have, from (4.19), the sign $s = -1$. Therefore, only the de Sitter space $dS(4,1)$ can lead to inflation. Expression (4.37) becomes more suggestive:

$$a(t) = A\, e^{\frac{c}{\mathcal{R}} t}. \tag{4.39}$$

Another requirement: in the standard treatment we have largely considered solutions with initial value $a(0) = 0$. Notice, however, that there is no inflation if the initial value of $a(t)$ is zero.

§ **4.4.3** For $\kappa \neq 0$ the solution is given by

$$a(t) = A \cosh\left[\sqrt{\frac{\Lambda}{3}}\, c(t - t_0)\right] \pm \sqrt{A^2 - 3\kappa/\Lambda}\, \sinh\left[\sqrt{\frac{\Lambda}{3}}\, c(t - t_0)\right]. \tag{4.40}$$

Only the upper sign can lead to the inflationary solution when $\kappa = 0$. Using (4.19), that case becomes

$$a(t) = A \cosh\left[\sqrt{-s}\, \frac{c(t - t_0)}{\mathcal{R}}\right] + \sqrt{A^2 + s\, \kappa \mathcal{R}^2}\, \sinh\left[\sqrt{-s}\, \frac{c(t - t_0)}{\mathcal{R}}\right]. \tag{4.41}$$

Again: only the case DS(4,1), $s = -1$, which can lead to the inflationary solution when $\kappa = 0$. Consequently, the solution allowing for inflation is the DS(4,1) case

$$a(t) = A \cosh\left[\frac{c(t - t_0)}{\mathcal{R}}\right] + \sqrt{A^2 - \kappa \mathcal{R}^2}\, \sinh\left[\frac{c(t - t_0)}{\mathcal{R}}\right], \tag{4.42}$$

or

$$a(t) = A \cosh\left[\sqrt{\Omega_\Lambda}\, H_0(t - t_0)\right] + \sqrt{A^2 - \kappa\, \frac{c^2}{H_0^2\, \Omega_\Lambda}}\, \sinh\left[\sqrt{\Omega_\Lambda}\, H_0(t - t_0)\right]. \tag{4.43}$$

Notice that $\kappa = +1$ imposes $A \geq \mathcal{R}$. Finally, the equivalent expression

$$a(t) = \tfrac{1}{2}\left[\left(A - \sqrt{A^2 - \kappa\, \mathcal{R}^2}\right) e^{-\frac{c\,t}{\mathcal{R}}} + \left(A + \sqrt{A^2 - \kappa\, \mathcal{R}^2}\right) e^{\frac{c\,t}{\mathcal{R}}}\right] \tag{4.44}$$

exhibits clearly the inflationary behavior when $\kappa = 0$.

§ **4.4.4** If we introduce the function

$$f(t) = \left(A + \sqrt{A^2 - \kappa\, \mathcal{R}^2}\right) e^{\frac{c\,t}{\mathcal{R}}} \tag{4.45}$$

and use the relation

$$A - \sqrt{A^2 - \kappa\,\mathcal{R}^2} = \frac{\kappa\,\mathcal{R}^2}{A + \sqrt{A^2 - \kappa\,\mathcal{R}^2}} \qquad (\kappa \neq 0), \tag{4.46}$$

solution (4.44) takes on the form

$$a(t) = \tfrac{1}{2}\left[f(t) + \frac{\kappa\,\mathcal{R}^2}{f(t)}\right]. \tag{4.47}$$

Summing up, the solution is

$$\Lambda = \frac{3}{\mathcal{R}^2} \neq 0, \kappa \neq 0 : \begin{cases} a(t) = \tfrac{1}{2}\left[f(t) + \frac{\kappa\mathcal{R}^2}{f(t)}\right] \\ f(t) = \left(A + \sqrt{A^2 - \kappa\,\mathcal{R}^2}\right)\, e^{\frac{c\,t}{\mathcal{R}}}. \end{cases} \tag{4.48}$$

The Friedmann–Robertson–Walker form of the inflationary de Sitter line element will consequently be

$$ds^2 = c^2dt^2 - \tfrac{1}{4}\left[f(t) + \frac{\kappa\,\mathcal{R}^2}{f(t)}\right]^2 \left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{4.49}$$

4.5 Beyond de Sitter: adding radiation

4.5.1 Radiation–domination: $\kappa \neq 0, \Lambda \neq 0$

§ 4.5.1 Let us go back to a more realistic view. From all the evidence we have today, the Universe has had an initial stage — the radiation–dominated era of Section 3.3 — during which, besides the cosmological term, ultrarelativistic matter or radiation provided a significant contribution as a source in Einstein's equations.

This era lies entirely in the thermalized period described in Section 3.4, from which we learn which extra terms are to be added to the Friedmann equations (4.35, 4.36). We shall preserve the notation of the previous paragraphs and introduce notation (3.52),

$$\gamma = \frac{\Omega_{\gamma 0} H_0^2 a_0^4}{c^2}\ ,$$

with $\Omega_{\gamma 0}$ given by Eq.(3.23) and a_0 the present-day value of the expansion parameter. The equations turn out to be

$$\frac{\dot{a}^2 + \kappa c^2}{a^2} - \frac{c^2}{\mathcal{R}^2} - \frac{\gamma\,c^2}{a^4} = 0 \tag{4.50}$$

$$\dot{H} - \frac{\kappa c^2}{a^2} + \frac{2\gamma c^2}{a^4} = 0 \ . \tag{4.51}$$

Just as in the case without radiation, there are many solutions, inflationary and not. The solution which reduces to (4.44) in the absence of radiation ($\gamma = 0$) is

$$a(t) = \tfrac{1}{2}\sqrt{e^{\frac{2ct}{\mathcal{R}}}\left(A+\sqrt{A^2-\kappa\mathcal{R}^2}\right)^2 + e^{-\frac{2ct}{\mathcal{R}}}\mathcal{R}^2(\kappa^2\mathcal{R}^2-4\gamma)\left(A+\sqrt{A^2-\kappa\mathcal{R}^2}\right)^{-2} + 2\kappa\mathcal{R}^2} \ . \tag{4.52}$$

§ **4.5.2** Let us repeat function (4.45) and write the solution as

$$f(t) = \left(A + \sqrt{A^2 - \kappa\mathcal{R}^2}\right) e^{\frac{ct}{\mathcal{R}}}$$

$$a(t) \ = \tfrac{1}{2}\sqrt{(f + \kappa\mathcal{R}^2 f^{-1})^2 - 4\,\gamma\mathcal{R}^2 f^{-2}}. \tag{4.53}$$

We see immediately that, when $\gamma = 0$, this expression reduces indeed to (4.47) and, equivalently, (4.52) reduces to (4.44). The value at $t = 0$ is

$$\frac{\sqrt{A^2\kappa^2\mathcal{R}^2 - \left(A - \sqrt{A^2 - \kappa\mathcal{R}^2}\right)^2\gamma}}{\kappa\mathcal{R}} ,$$

which reduces dutifully to A when $\gamma = 0$. The solution above involves, besides the cosmological constant and κ, two other quantities: the initial value A and the present-day value a_0 (hidden in γ) of the expansion parameter.

§ **4.5.3** The case $\kappa = 0$ is given by

$$a(t) \ = \sqrt{e^{\frac{2ct}{\mathcal{R}}}A^2 - \frac{\gamma\mathcal{R}^2}{4A^2}e^{-\frac{2ct}{\mathcal{R}}}}. \tag{4.54}$$

Comment 4.5.1 With the above notation, the solution for matter plus cosmological constant (with $\kappa = 0$) of Comment 3.2.1, page 35, becomes

$$a(t) = a_0\left\{\cosh\left[\frac{3}{2}\frac{c}{\mathcal{R}}(t-t_0)\right] + \frac{\mathcal{R}}{c}\sinh\left[\frac{3}{2}\frac{c}{\mathcal{R}}(t-t_0)\right]\right\}^{2/3} \ . \tag{4.55}$$

Notice that the integration constant used is the present-day value a_0, not to the initial value A.

Comment 4.5.2 To add also matter to the above case, it is convenient to use parameter (3.53) (recall that we are using H_0^{-1} as the unit of time),

$$Mc^2 = (1 - \Omega_\Lambda - \Omega_{\gamma 0} - \Omega_\kappa)\,{a_0}^3 = \left(1 - \frac{c^2}{\mathcal{R}^2} - \frac{\gamma c^2}{a_0^4} - \frac{c^2\kappa}{{a_0}^2}\right){a_0}^3.$$

Then, the equations are, instead of (4.50–4.51), the complete case (3.54–3.55),

$$\frac{\dot{a}^2 + \kappa c^2}{a^2} - \frac{c^2}{\mathcal{R}^2} - \frac{\gamma c^2}{a^4} - \frac{Mc^2}{a^3} = 0 \quad \text{and} \quad \dot{H} - \frac{\kappa c^2}{a^2} + \frac{2\gamma c^2}{a^4} + \frac{3Mc^2}{2a^3} = 0, \tag{4.56}$$

or

$$\ddot{a}(t) - \frac{c^2}{\mathcal{R}^2}\, a + \frac{\gamma\, c^2}{a^3} + \frac{M\, c^2}{2\, a^2} = 0. \qquad (4.57)$$

Equation (4.55) solves the case $\gamma = 0, \kappa = 0, M \neq 0$. As then $a_0 = \left(\frac{Mc^2}{1-\frac{c^2}{\mathcal{R}^2}}\right)^{1/3}$,

$$a(t) = \left\{\sqrt{\frac{M}{\mathcal{R}^2 - c^2}}\; c\mathcal{R} \cosh\left[\frac{3}{2}\frac{c}{\mathcal{R}}(t-t_0)\right] + \sqrt{\frac{M}{\mathcal{R}^2 - c^2}}\; \mathcal{R}^2 \sinh\left[\frac{3}{2}\frac{c}{\mathcal{R}}(t-t_0)\right]\right\}^{2/3}. \qquad (4.58)$$

Equation (4.52) solves the case $\gamma \neq 0, \kappa \neq 0, M = 0$.

§ 4.5.4 The Friedmann–Robertson–Walker form of the inflationary de Sitter line element will now, with radiation, be

$$ds^2 = c^2 dt^2$$
$$-\tfrac{1}{4}\left[\left(f + \frac{\kappa\mathcal{R}^2}{f(t)}\right)^2 - 4\,\gamma\mathcal{R}^2 f^{-2}\right]\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \qquad (4.59)$$

4.6 Beyond de Sitter: adding matter

4.6.1 Nonrelativistic matter: $\Lambda \neq 0, \kappa = 0$

Dismissing any scruples about avoiding repetition, the general set of equations to be solved in the thermalized (or pre-recombination) period is

$$\dot{a}^2 + \kappa c^2 - \frac{\gamma c^2}{a^2} - \frac{Mc^2}{a} - \frac{a^2 c^2}{\mathcal{R}^2} = 0$$
$$\ddot{a} + \frac{\gamma c^2}{a^3} + \frac{Mc^2}{2\, a^2} - \frac{ac^2}{\mathcal{R}^2} = 0.$$

The usual technique is to solve — if possible — the second-order equation, getting a solution dependent on two arbitrary constants. The first-order equation is then used on that solution to fix the value of κ. The other constant remains, and can be used to fix the initial value $a(0) = A$. In the absence of radiation and with $\kappa = 0$,

$$\dot{a}^2 - \frac{Mc^2}{a} - \frac{c^2\, a^2}{\mathcal{R}^2} = 0$$
$$\ddot{a} + \frac{Mc^2}{2\, a^2} - \frac{c^2\, a}{\mathcal{R}^2} = 0.$$

A general solution is

$$a(t) = \left(\frac{M\,\mathcal{R}^2}{4(\mathcal{R}^2 - c^2)}\right)^{\frac{1}{3}} \left(\frac{c-\mathcal{R}}{K} e^{-\frac{3\,c\,t}{2\,\mathcal{R}}} + e^{\frac{3\,c\,t}{2\,\mathcal{R}}}\,K\;(c+\mathcal{R})\right)^{\frac{2}{3}}$$

for any value of the constant K. K can then be chosen so as to give $a(0) = A$. Two possibilities turn up,

$$K = \sqrt{\frac{A^3}{\mathcal{R}^2\,M}}\left(1 \pm \sqrt{1 + \frac{\mathcal{R}^2\,M}{A^3}}\right)\sqrt{\frac{\mathcal{R}-c}{\mathcal{R}+c}}.$$

The lower sign leads, in the $M \to 0$ limit, to a non-inflationary solution $Ae^{-\,ct/\mathcal{R}}$. The second leads to the usual inflationary solution. Introducing another convenient function, this solution is

$$j(t) = \sqrt{\frac{A^3}{\mathcal{R}^2\,M}}\left(1 + \sqrt{1 + \frac{\mathcal{R}^2\,M}{A^3}}\right)\sqrt{\frac{\mathcal{R}-c}{\mathcal{R}+c}}\,e^{-\frac{3\,c\,t}{2\,\mathcal{R}}}$$

$$a(t) = \left(\frac{M\,\mathcal{R}^2}{4(\mathcal{R}^2 - c^2)}\right)^{\frac{1}{3}} \left(\frac{c-\mathcal{R}}{j(t)} + (c+\mathcal{R})\,j(t)\right)^{\frac{2}{3}}.$$

In the $\Lambda \to 0$ limit, this tends to

$$a(t) = \left(A^{3/2} + \frac{3}{2}\sqrt{M}ct\right)^{\frac{2}{3}}.$$

In this case $Mc^2 = \left(1 - \frac{c^2}{\mathcal{R}^2} - \frac{\gamma c^2}{a_0^4} - \frac{c^2\,\kappa}{{a_0}^2}\right)\,{a_0}^3 = a_0^3$, so that we get back the usual "matter-dominated" solution when $A \to 0$:

$$a(t) = a_0\left(\frac{3}{2}\,c\,t\right)^{\frac{2}{3}}.$$

Notice that that "usual" case given in Eq.(3.4) is normalized to today's values and does not suppose overall thermalization — it simply supposes dust to be the only source.

4.6.2 Nonrelativistic matter: $\mathbf{\Lambda = 0}, \kappa \neq \mathbf{0}$

The equations are in this case

$$\dot{a}^2 + \kappa c^2 - \frac{Mc^2}{a} = 0$$

$$\ddot{a} + \frac{c^2\,M}{2\,a^2} = 0.$$

The first equation is $\dot{a} = \sqrt{\frac{Mc^2}{a(t)} - \kappa c^2}$, equivalent to

$$\int \frac{a da}{\sqrt{Ma - \kappa a^2}} = c \int dt.$$

<u>Case $\kappa = 0$</u> can be used as a test: integration gives $a^{3/2} - A^{3/2} = \frac{3}{2}\sqrt{M}ct$, just the result of the previous section.

<u>Case $\kappa > 0$, actually $\kappa = 1$:</u> the integral takes up the form

$$\int_A^a \frac{\sqrt{y}\, dy}{\sqrt{1 - \frac{y}{M}}} = \sqrt{M}c \int_0^t dt.$$

Integration gives $a(t)$ in an implicit way:

$$ct \;=\; \sqrt{MA - A^2} \;-\; \sqrt{Ma - a^2} \;+\; M \left[\arcsin\left(\sqrt{\tfrac{a}{M}}\right) - \arcsin\left(\sqrt{\tfrac{A}{M}}\right)\right]. \tag{4.60}$$

A sketch is given in Figure 4.2.

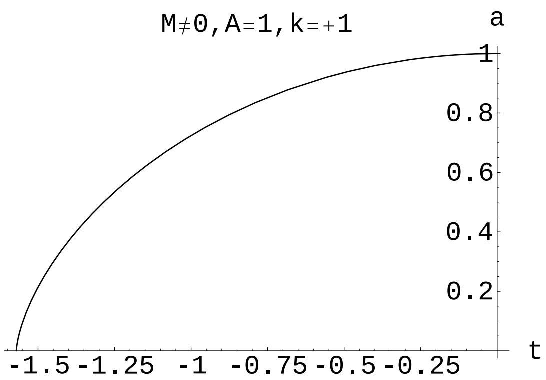


Figure 4.2: *General aspect of the expansion parameter for pure dust, with* $\kappa = +1$.

<u>Case $\kappa < 0$, actually $\kappa = -1$:</u> integration now gives

$$ct = \sqrt{Ma + a^2} - \sqrt{MA + A^2} + M \ln \left[\frac{\sqrt{A} + \sqrt{A + M}}{\sqrt{a} + \sqrt{a + M}}\right]. \tag{4.61}$$

Notice that, when $M \to 0$, $a(t) = A + ct$. When $M > 0$, a function of $a(t)$ turns up:

$$\frac{e^{\sqrt{\frac{a}{M}+\left(\frac{a}{M}\right)^2}}}{\sqrt{\frac{a}{M}}+\sqrt{1+\frac{a}{M}}} = \frac{e^{\sqrt{\frac{A}{M}+\left(\frac{A}{M}\right)^2}}}{\sqrt{\frac{A}{M}}+\sqrt{1+\frac{A}{M}}} e^{\frac{ct}{M}} \ .$$

This can be rewritten as

$$F(x) = \frac{e^{\sqrt{x+x^2}}}{\sqrt{x}+\sqrt{1+x}} \tag{4.62}$$

$$F[a(t)/M] = F[A/M]\, e^{\frac{ct}{M}} \ . \tag{4.63}$$

Function $F(x)$ is actually very simple and monotonic, and so is $a(t)$ (see Figure 4.3).

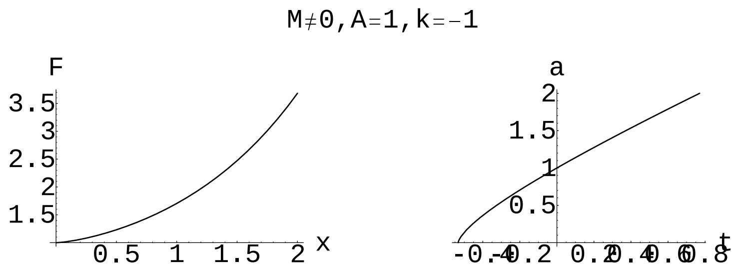


Figure 4.3: *Function (4.62) and the expansion parameter for pure dust, with* $\kappa = -1$*.*

We have above solved Friedmann's equations with a dust source ($M \neq 0$), choosing solutions of inflationary type. Actually, only the cases with *either* $\Lambda \neq 0$ *or* $\kappa \neq 0$ have been presented. The case with both $\Lambda \neq 0$ *and* $\kappa \neq 0$ is much more involved. It is as involved as the general case including radiation.

4.7 General: $\mathbf{\Lambda \neq 0, \gamma \neq 0, M \neq 0, \kappa \neq 0}$

This case, we once more repeat, is described by

$$\dot{a}^2 + \kappa c^2 - \frac{\gamma c^2}{a^2} - \frac{Mc^2}{a} - \frac{c^2}{\mathcal{R}^2} a^2 = 0.$$

It is convenient to part from the usual technique and proceed directly from this first order equation, which is equivalent to

$$a(t)\dot{a} = \sqrt{\frac{c^2\, a^4}{\mathcal{R}^2} + \gamma c^2 + M c^2 a - \kappa c^2 a^2}$$

or

$$\frac{c\, dt}{\mathcal{R}} = \frac{a\, da}{\sqrt{a^4 + \gamma \mathcal{R}^2 + M \mathcal{R}^2 a - \kappa \mathcal{R}^2 a^2}} \quad .$$

The solution is implicit, and involves an intricate elliptic integral:

$$\frac{c}{\mathcal{R}} \int_0^t dt = \frac{c}{\mathcal{R}}\, t = \int_A^a \frac{a\, da}{\sqrt{a^4 + \gamma \mathcal{R}^2 + M \mathcal{R}^2 a - \kappa \mathcal{R}^2 a^2}} \quad .$$

The procedure[2] to arrive at the solution is long and rather cumbersome. Let us go step by step:

1. rewrite the integral above in terms of the roots $\{r_i\}$ of the denominator:

$$\int \frac{a\, da}{\sqrt{a^4 + \gamma \mathcal{R}^2 + M \mathcal{R}^2 a - \kappa \mathcal{R}^2 a^2}} = \int \frac{a\, da}{\sqrt{(a - r_1)(a - r_2)(a - r_3)(a - r_4)}}.$$

2. introduce some constants, in terms of which the roots will be expressed later on:

$$W = \sqrt{\mathcal{R}^2\ (27\, M^4 + 144\, \kappa\, M^2\, \gamma + 128\, \kappa^2\, \gamma^2) - 256\, \gamma^3 - 4\, \kappa^3\, \mathcal{R}^4\ (M^2 + 4\, \kappa\, \gamma)}$$
$$V = 3\, \sqrt{3}\, \mathcal{R}^3\, W - \mathcal{R}^4\, \left(2\, \kappa^3\, \mathcal{R}^2 - 27\, M^2 - 72\, \kappa\, \gamma\right)$$
$$U \ = \ \frac{\kappa\, \mathcal{R}^2}{3} - \frac{V^{\frac{1}{3}}}{12\, (2^{\frac{1}{3}})} - \frac{\mathcal{R}^2\ (\kappa^2\, \mathcal{R}^2 + 12\, \gamma)}{6\, (2^{\frac{2}{3}})\, V^{\frac{1}{3}}}$$

3. the roots will then be:

$$r_1 = \sqrt{\tfrac{\kappa\, \mathcal{R}^2}{2} - U} - \sqrt{U - \frac{M \mathcal{R}^2}{4 \sqrt{\frac{\kappa\, \mathcal{R}^2}{2} - U}}} \ ; \quad r_2 = \sqrt{\tfrac{\kappa\, \mathcal{R}^2}{2} - U} + \sqrt{U - \frac{M L^2}{4 \sqrt{\frac{\kappa\, \mathcal{R}^2}{2} - U}}}$$
$$r_3 = -\sqrt{\tfrac{\kappa\, \mathcal{R}^2}{2} - U} - \sqrt{U + \frac{M \mathcal{R}^2}{4 \sqrt{\frac{\kappa\, \mathcal{R}^2}{2} - U}}} \ ; \ r_4 = -\sqrt{\tfrac{\kappa\, \mathcal{R}^2}{2} - U} + \sqrt{U + \frac{M \mathcal{R}^2}{4 \sqrt{\frac{\kappa\, \mathcal{R}^2}{2} - U}}}$$

4. Solutions will be implicit, and involve elliptic functions [32] of two kinds:

[2] It has been discussed at length by Coquereaux and Grossmann [29] for closed Universes, and by Dabrowski and Stelmach [30] for the open case. We shall here follow the simpler, unified approach given in Ref. [31].

- the elliptic integral of first kind with parameter m and amplitude ϕ:

$$F[\phi, m] = \int_0^\phi \frac{1}{\sqrt{1 - m\sin^2\theta}}\, d\theta$$

- the elliptic integral of third kind with parameter m, characteristic n and amplitude ϕ:

$$\Pi[\phi, n, m] = \int_0^\phi \frac{1}{(1 - n\sin^2\theta)\sqrt{1 - m\sin^2\theta}}\, d\theta$$

5. the characteristic and the parameter turning up wil be:

$$n = \frac{r_2 - r_4}{r_1 - r_4}\ ;\ m = \frac{r_1 - r_3}{r_2 - r_3}\, n$$

6. the amplitude turning up will be

$$\phi(a) = \arcsin\sqrt{\frac{a - r_2}{n(a - r_1)}}$$

7. the implicit solution will then be

$$\frac{ct}{2\mathcal{R}} = \frac{r_1 F[\phi, m] - (r_1 - r_2)\,\Pi[\phi, n, m]}{\sqrt{(r_2 - r_3)(r_1 - r_4)}}\ .$$

We see from their definititons that both elliptic functions vanish when $\phi = 0$, that is, when $a = r_2$. An exceptional case appears when $r_1 = r_2$ and $\phi(a) = \arcsin 1$. A detailed checking shows that this corresponds to $\mathcal{R} \to \infty$. For all other values of L, the initial value is fixed, $A = a(0) = r_2$. Anyhow, it is clear that the roots are generically distinct. The solution above holds in that case.

Chapter 5

Inflation

5.1 Introduction

From the considerations made so far, it may appear that the standard cosmological model explains all features of the present, as well as of the early Universe. However, this is in fact not the case [1]. Despite presenting a lot of successes, the standard cosmological model presents also a lot of different problems. The major successes of the model are:

- It explains the Hubble redshift–distance relation in terms of the expansion of the Universe, with the inverse Hubble constant $H_0^{-1} \approx 10^{10}$ years, determined by observations of distant galaxies, agreeing closely with the ages determined for stars and galaxies by completely different observations.

- It predicts the hot Big Bang early phase of the Universe in which non-equilibrium processes take place, explaining the 3 K Cosmic Microwave Background Radiation (CMWBR) and the nucleosynthesis of light elements, with predict abundance agreeing well with observations.

The major problems of the standard model are: the expansion law problem, the horizon problem, the flatness problem, the cosmological constant problem, and the monopole problem. In the following we are going over some of the above mentioned problems. In this chapter we will use unities for which the velocity of light $c = 1$.

5.2 Problems of the standard model

5.2.1 Expansion law and the equation of state

In a homogeneous and isotropic model, general relativity allows to find explicitly the expansion law of the Universe if the energy density ρ and the pressure p are known. The energy density ρ can be defined by the parameter Ω, and the pressure p is given by the equation of state

$$p = p(\rho) \ .$$

As we have already seen, the expansion of the Universe is described by the scale factor $a(t)$, which characterizes distance between objects as a function of time. The spacetime interval can be written as in Eq.(2.7),

$$ds^2 = dt^2 - a^2(t) \left[dr^2 + f(r)(d\theta^2 + \sin^2\theta d\phi^2)\right] \ , \tag{5.1}$$

where $f(r)$ depends on the topological properties of the Universe as a whole:

For a spatially flat Universe	$f(r) = r^2$
For a closed Universe	$f(r) = \sin^2 r$
For an open Universe	$f(r) = \sinh^2 r$.

We recall that r is a dimensionless parameter, whereas $a(t)$ has dimension of length.

The $a(t)$ dependence is described, in the isotropic and homogeneous case, by the Friedmann equations:

$$\left(\frac{da}{dt}\right)^2 - \frac{8\pi G\epsilon}{3} \, a^2 = -\frac{\kappa}{2} \tag{5.2}$$

$$\frac{d^2a}{dt^2} = -\frac{4\pi G}{3} \, (\epsilon + 3p) \, a \ . \tag{5.3}$$

If the dependence $p = p(\rho)$ is known, then all three unknown functions a, ρ and p can be found. By combining Eqs.(5.2) and (5.3), we get law (2.11) for the energy density variation in the expanding Universe,

$$\frac{d\epsilon}{dt} = -\,3H(\rho + p) \ . \tag{5.4}$$

Therefore, we see that the change in the energy density is caused by two factors: by the expansion of the Universe, and by the work of the pressure forces.

Let us now review the different expansion laws corresponding to different equations of state. We consider only the case in which $\kappa = 0$. For an ideal relativistic gas,

$$p = \frac{\rho}{3} \, .$$

In this case, the scale factor depends on t as

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/2} \, .$$

In the Friedmann cosmology, this expansion law is assumed to be valid from the "beginning" to about 10^{11}s, when non-relativistic particles began to dominate the energy density. After this moment, the expansion is determined by the non-relativistic gas state, for which $p \ll \rho$. With good accuracy one can then put $p = 0$, which leads to the following expansion law:

$$a(t) = a_1 \left(\frac{t}{t_1}\right)^{2/3} \, .$$

One of the main difficulties of the classical cosmology is connected with the very slow growth of the scale factor: $a \sim t^{1/2}$ or $a \sim t^{2/3}$. If one starts with the size we know the Universe has today, and go back in time for small values of t, the scale factor $a(t)$, or the size of the Universe at these times, is still very large. Therefore, the classical Friedmann model as it stands cannot be extrapolated to very early stages. Something different from the above ordinary expansion laws might have occurred in the early Universe.

5.2.2 The horizon problem

One can take any physical system and choose an "initial time" t_i. The state of the system at any later time is affected both by the state at t_i (the initial conditions) and by the subsequent evolution. At any finite time after t_i there are causal limits on how large a scale can be affected by the subsequent evolution (limited ultimately by the speed of light, but really by the actual propagation speeds in that particularly system, which can be much slower). So there are always sufficiently large scales which have not been affected by the subsequent evolution, and on which the state of the system is simply a reflection of the initial conditions.

As with any system, the causal "horizon" of the Universe grows with time. Today, the region with which we are just coming into causal contact (by observing distant points in the Universe) is one "causal radius" in size, which means objects we see in opposite directions are two causal radii apart and have not yet come into causal contact. One can calculate the number of causal regions that filled the currently observed Universe at other times. At the Grand Unification epoch, for example, there were around 10^{80} causally disconnected regions in the volume that would evolve into the part of the Universe we currently can observe.

Let us consider this question in more details. The Universe became transparent to the relic radiation after hydrogen recombination, when protons and electrons bound into neutral hydrogen atoms which almost do not interact with long-wave photons. The hydrogen recombination temperature is 3.000 K, which corresponds to a time $t_r \approx 10^{12} - 10^{13}$s, or equivalently, to $z_r \approx 10^3$. Since that time, relic radiation almost did not interact with anything. Disregarding for a moment the expansion of the Universe, the size of the causally connected region at the moment of recombination (the horizon size) can roughly be set equal to $c \times t_r$. Therefore, parts of the sky separated by an angular distance larger than

$$\theta = (1 + z_r)\,\frac{t_r}{t_0} \approx 10^{-2}$$

should not "be aware" of each other. Nevertheless, the relic radiation is identical everywhere. This mystery of the Friedmann cosmology is called the *horizon problem*.

Actually, the horizon problem is a little bit more involved because the exact formula for the horizon size must take into account the expansion of the Universe. Let us then consider a light ray coming from a point (a particle or a galaxy) of the space to the observer. Let us go back to the spacetime interval of Eq.(5.1),

$$ds^2 = c^2dt^2 - a^2(t)\left[dr^2 + f(r)(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \ .$$

Considering a radial trajectory ($d\theta = d\phi = 0$), and remembering that light rays propagate along null geodesics ($ds = 0$), the equation describing the ray propagation is

$$dr = \frac{cdt}{a(t)} \ .$$

The physical distance $l = a(t)r$ between the source and the observer is, therefore,

$$l = a(t)\int_0^t \frac{c\ dt'}{a(t')} \ .$$

For the radiation-dominated age,

$$a(t) \sim t^{1/2} \ ,$$

and consequently

$$l = 2ct = H^{-1} \ .$$

For the matter-dominated age,

$$a(t) \sim t^{2/3} \ ,$$

which implies

$$l = 3ct = 2H^{-1} \ .$$

We see that $l \sim t$ in both cases, and therefore the physical distance l grows faster than the scale factor. As a consequence, any particle will be inside the horizon in the future. This means that we are able to see different particles — or regions of the Universe — which have never been in causal contact. Nevertheless, the radiation coming from these regions are quite similar. Why would these regions have identical properties?

5.2.3 The flatness problem

The "critical density" ρ_c is defined by the condition $\kappa = 0$, which implies

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_c \ . \tag{5.5}$$

Therefore, a Universe with $\kappa = 0$ has $\rho = \rho_c$, and is said to be flat. When the parameter $\Omega \equiv \rho/\rho_c$ is close to unity, the ρ term dominates in the Friedmann equation, and the Universe is nearly flat. If Ω deviates significantly from unity the κ term (the "curvature term") is dominant.

The Flatness Problem arises from the fact that $\Omega = 1$ is an unstable point in the evolution of the Universe. Because $\rho \propto a^{-3}$ or a^{-4} throughout the history of the Universe, the ρ term in the Friedmann equation falls away much more quickly than the κ/a^2 term as the Universe expands, and the κ/a^2 comes to dominate. In fact, if at the Planck time

$$\Omega - 1 \simeq \mathcal{O}(1) \ ,$$

the Universe would either collapse within 10^{-43}s (if it is closed), or it would expand so fast that no stars could form (if it is open). In the latter case, matter density in 10^{10} years would be much smaller than that observed ($\rho_0 \approx 10^{-29}$g/cm^3). This behaviour is illustrated in Fig.5.1

Despite the strong tendency for the equations to drive the Universe away from the critical density, the value of Ω today is remarkably close to unity, even after 15 Billion years of evolution. Today the value of Ω is within an

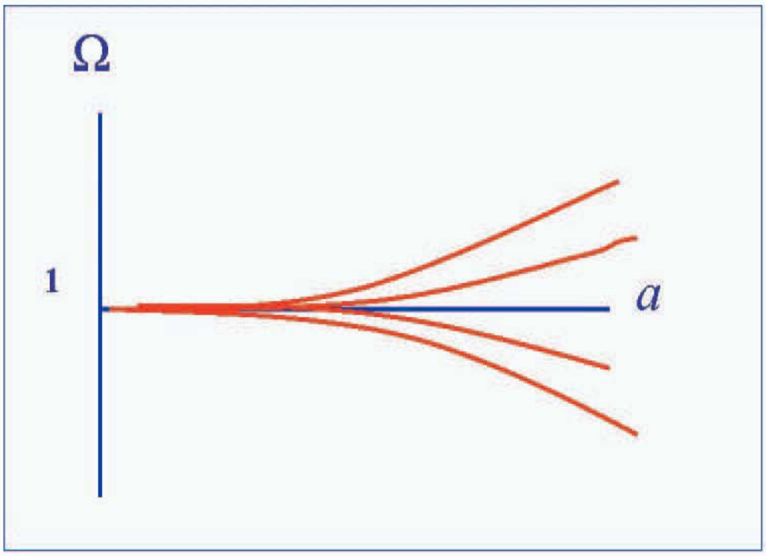


Figure 5.1: *The parameter $\Omega(a)$ tends to evolve away from unity as the Universe expands.*

order of magnitude of unity, and that means that at early times ρ must have taken values that were extremely closely to ρ_c. Let us estimate the necessary fine tuning of Ω. Using the expressions $H = \dot{a}/a$ and $\rho_c = 3H^2/8\pi G$, we can rewrite the equation (2.57) (with $\Lambda = 0$) in the form

$$H^2(1-\Omega) = -\frac{\kappa c^2}{a^2} \,. \tag{5.6}$$

With this equation one can now express Ω as a function of the redshift z and of its present-day value Ω_0:

$$\frac{1-\Omega}{\Omega} = \frac{1-\Omega_0}{\Omega_0}\,\frac{1}{(1+z)^n} \,, \tag{5.7}$$

where $n = 1$ for the matter-dominated Universe, and $n = 2$ for the radiation dominated Universe. One can then estimate how close to unity Ω should have been at early stages, assuming that $(1-\Omega_0)/\Omega_0 \approx 1$. The results are presented in the table below.

Age of the Universe	Period	$\lvert 1-\Omega\rvert$
2.10^{10} years	Contemporary	≤ 1
10^5 years	Recombination	10^{-3}
1 s	Beginning of Nucleonsynthesis	10^{-16}
10^{-5}s	Quark-Gluon Plasma	10^{-21}
10^{-10}s	Electro-Weak Transition	10^{-26}
$t_{\rm Pl} \approx 10^{-43}$s	Universe Creation	10^{-60}

Note that the earlier time, when the initial conditions are fixed, the closer Ω must be to unity in order to obtain the presently observed Universe. The necessity fine tuning is astonishingly small.

These considerations can also be illustrated by comparing the horizon size at the time of the birth of the Universe with the curvature radius in the Friedmann model. The curvature radius $\mathcal{R}$ of the homogeneous and isotropic Universe is defined as the trace of the three-dimensional curvature tensor:

$$\frac{\kappa}{a^2} = \frac{1}{\mathcal{R}^2} \, .$$

We see, therefore, that it is equal to the scale factor: $\mathcal{R} = a$. Since the horizon size is equal to ct and $a \approx (t/t_0)^{1/2}$, then $ct \ll \mathcal{R}$ for $t \ll t_0$. This means that the Universe was almost exactly flat when it was born.

The basic conclusion is that the classical Friedmann model can not be extrapolated to very early stages. Any attempt to apply this model to times near the Planck time leads to wrong results. This means that the law that governs the scale-factor changes might have been completely different for small t. At that period, $a(t)$ must have grown very fast with time. As we are going to see later, if this is in fact the case, the flatness problem can find a reasonable solution.

5.2.4 The cosmological constant problem

The de Sitter solution of Einstein's equation plays an important role in the theory of the early Universe, and consequently in the solutions of the flatness and horizon problems. The de Sitter metric is a homogeneous and isotropic solution of Einstein's equation in vacuum, but with the so called cosmological term. The cosmological constant was first introduced into the gravity equations with the purpose of obtaining a stationary solution which would comply with the cosmological models of the epoch. The idea was to compensate the attraction of matter by a repulsive Λ-term. After the discovery of the expansion of the Universe by Hubble, however, this idea was promptly rejected.

Today, it is a general belief that the expansion of the Universe at the early stages was produced by a Λ-term, or by an equivalent state of matter. Experience has since long shown that the contemporary value of the cosmological constant is either very small or equal to zero. Quite recently, observations of supernovae Ia, however, have change the situation as these results indicate a non-vanishing value for the cosmological constant [14]. On the other hand, quantum field theory as well as analysis of phase transitions in the early Universe predict a very large value for it. Even if it is not zero, as the recent observations seem to indicate, there is still a tremendous contradiction between the predictions and observations. This contradiction is known as the cosmological constant problem.

5.3 The machinery of inflation

5.3.1 Cosmic inflation

The proposed answer to most of these problems is the Inflationary Universe [35, 36, 37]. This model is based on the supposition that, for a temporary period around the GUT epoch (that is, the epoch in which the GUT symmetry was spontaneously broken down), the dynamics of the Universe changed dramatically. As the word *inflation* suggest, during a short period of time the Universe might have experienced a very rapid expansion, which can be described by the following sequence:

$$
\begin{array}{cl}
t < t_1 & : \text{Scale factor } a(t) \sim t^{1/2} \\
t_1 < t < t_2 & : \text{Scale factor } a(t) \sim \exp(\alpha t) \\
t_2 < t & : \text{Scale factor } a(t) \sim t^{1/2}
\end{array}
$$

As we are going to see, for a flat space ($k = 0$), an inflationary solution can be obtained in the absence of sources, but in the presence of a positive cosmological constant. Alternatively, as a cosmological constant is equivalent to some kind of matter which satisfies the exotic equation of state

$$p = -\rho \, ,$$

one can say that such a matter is also able to generate inflation. Independently of what may have cause it, the expansion law is supposed to be exponential,

$$a(t) \sim \exp(\alpha t) \, ,$$

and the relevant spacetime in the short inflationary period becomes a de Sitter spacetime [17]. What is this time range? How do we find α? Is there an alternative model to inflation? These questions, as well as the fundamentals of the inflationary model will be discussed next.

It was after Guth's original paper [35] that Cosmic Inflation became a hope for the solution for the deep mysteries of the Universe. Despite its striking success, there are key unanswered questions about the foundations of inflation which do not allow us to say with certainty that we are in the right track. However, no alternative theory exists which makes of Inflation the only (and consequently the best) available model we have.

Comment 5.3.1 Recently [38], an alternative theory to Inflation has been proposed which is based on a time-varying speed of light. According to this proposal, a very large speed of light in the early Universe could also resolve the same problems Inflation can do.

Over the years numerous inflationary "scenarios" have been proposed. Our goal here is to introduce the basic ideas that underlie all the scenarios. The essentially new idea which allows to solve the above-mentioned problems is to introduce negative pressure. This can be achieved by changing the equation of state from $p = \rho/3$ to

$$p = -\rho \ . \tag{5.8}$$

Remembering that the energy-momentum tensor of a homogeneous fluid is

$$T^{\mu\nu} = (p + \epsilon)\ u^\mu u^\nu - p\ g^{\mu\nu} \ , \tag{5.9}$$

the above equation of state represents a fluid for which

$$T^{\mu\nu} = \rho\ g^{\mu\nu} \ . \tag{5.10}$$

Substituting in Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \ , \tag{5.11}$$

we see that it is equivalent to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda\ g_{\mu\nu} = 0 \ , \tag{5.12}$$

which is the vacuum Einstein's equation but with a cosmological term

$$\Lambda = 8\pi G \rho \ .$$

One may say, therefore, that a cosmological constant is equivalent to a source fluid with an effective pressure given by (5.8), which brings the problem back to a pure de Sitter case.

Let us then take the Friedmann equations (2.9) and (2.10) with $p = \rho = 0$ (alternatively we could have chosen $p = -\rho$ and $\Lambda = 0$):

$$\dot{a}^2 = \frac{\Lambda}{3}\ a^2 - \kappa \tag{5.13}$$

$$\ddot{a} = \frac{\Lambda}{3}\ a \ . \tag{5.14}$$

From Eq.(2.11) we see that ρ and p are constant. This means that the energy decrease due to expansion is compensated by the work of pressure forces.

For a flat space ($\kappa = 0$), the solution to the Friedmann equations is

$$a(t) = a_0 \exp\left[\sqrt{\frac{\Lambda}{3}}\, t\right] . \tag{5.15}$$

A solution of this type is called *inflationary.* Notice that $\Lambda > 0$ is essential for inflation, otherwise the solution would be oscillatory. Consequently, we must have, from (4.19), the sign $s = -1$. Therefore, only the de Sitter space dS(4,1) can lead to inflation. In terms of the de Sitter "radius" $\mathcal{R}$, the solution (5.15) becomes quite suggestive:

$$a(t) = a_0 \exp\left[t/\mathcal{R}\right] . \tag{5.16}$$

A non-vanishing cosmological constant, or equivalently a fluid with the equation of state (5.8), leads to a new phenomenon of gravitational repulsion. This repulsion might serve as the initial push that led to the inflationary expansion of the Universe. It is sufficient for this that the equation of state (5.8) was only approximately valid and only for a finite period. Of course, it remains to explain the physical origin of this equation, as well as when and why the exponential expansion stopped, and the $p = -\rho$ law changed to $p = \rho/3$. It should be remarked that, though the form of this equation is exotic, it "naturally" appears in many cases. These are, for example, phase transitions from symmetric to symmetric-broken phases in primordial plasmas, scalar field dynamics with very specific initial conditions, or theories with dimensions $D > 4$ [39].

5.3.2 Solving the problems

The introduction of an inflationary period in the early Universe, in which the Universe inflated exponentially for a short period of time, has a profound effect on the cosmological problems. Let us then address each one of them.

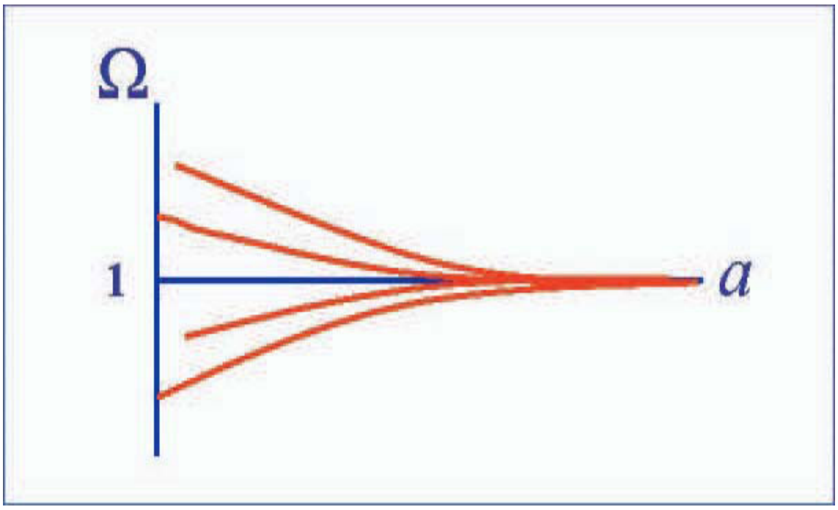


Figure 5.2: *During inflation,* $\Omega = 1$ *is an attractor*

Flatness

The flatness problem is characterized by the tendency of the κ term to dominate the ρ term in the Friedmann equation (2.9). However, during inflation, $H = \dot{a}/a$ does not depend on time, and consequently

$$| 1 - \Omega | \approx \exp[-2Ht] \; .$$

The difference between the expansions

$$a(t) \sim t^{1/2} \quad \text{and} \quad a(t) \sim t^{2/3}$$

and

$$a(t) \sim \exp[-2Ht]$$

is that in the first case $\Omega \to \infty$ for $t \to \infty$, while in the second $\Omega \to 1$ for $t \to \infty$. This behaviour is illustrated in Fig.5.2. Consequently, no fine tuning on Ω is necessary in the inflationary scenario.

Horizon

A period of exponential inflation radically changes the causality structure of the Universe. During inflation, each causally connected region is expanded exponentially. A suitable amount of inflation allows the entire observed Universe to come from a region that was causally connected before inflation. The horizon problem will be solved if the causally connected region, which at the moment of the "creation" of the Universe had the size

$$l \approx m_{Pl}^{-1} = 10^{-33}\text{cm} ,$$

was inflated up to the necessary size of 10^{-3}cm. This region expands later to reach the present horizon size.

Homogeneity

Of course "solving" the Horizon problem does not guarantee a successful picture. Bringing the entire observable Universe inside one causally connected domain in the distant past can give one *hope* that the initial conditions of the standard model can be explained by physical processes. Still, one must determine exactly what the relevant physical processes manage to accomplish. While the result of inflation is clearcut in terms of the flatness, it is much less so in terms of the homogeneity. The good news is that a given inflation model can actually make *predictions* for the spectrum of inhomogeneities that are present at the end of inflation. However, there is nothing intrinsic to inflation that predicts that these inhomogeneities are small. Nonetheless, the mechanics of inflation can be further adjusted to give inhomogeneities of the right amplitude. Interestingly, inflation has a lot to say about other aspects of the early inhomogeneities, and these other aspects are testing out remarkably well in the face of new data.

Monopoles

The original Monopole problem occurs because GUTs produce magnetic monopoles at sufficiently high temperatures. In the standard model these monopoles "freeze out" in such high numbers as the Universe cools that they rapidly dominate over other matter (which is relativistic at that time). Inflation can get around this problem if the reheating after inflation does not reach temperatures high enough to produce the monopoles. During inflation, all the other matter (including any monopoles that may be present) is diluted to completely negligible densities. The ordinary matter is created by the reheating process at the end of inflation. The key difference is that in a non-inflationary model the matter we see has in the past existed at all temperatures, right up to infinity at the initial singularity. With the introduction of an inflationary epoch, the matter around us has only existed up to a finite maximum temperature in the past (the reheating temperature). If this temperature is on the low side of the GUT temperature, monopoles will not be produced and the Monopole Problem is evaded.

Cosmological constant

A very important open question is linked with the cosmological constant problem [40, 41, 42]. Why the cosmological constant is extremely close to zero today (at least from a particle physicist's point of view) is perhaps the deepest problem in theoretical physics. Interestingly, current data is suggesting that there is a non-zero cosmological constant today. Despite not directly contradicting with inflation, the inflationary model does not solve the cosmological constant problem, which remains as an open problem.

5.4 Mechanisms for producing inflation

5.4.1 Scalar fields in cosmology

Scalar fields play a very important role in several branchs of Physics. For example, gauge theories with spontaneously broken symmetry demand the existence of scalar fields in order to ensure renormalizability. From the cosmological point of view, its main interest comes from the fact that it can naturally lead to the equation

$$p = -\rho \tag{5.17}$$

The lagrangian of a complex scalar field has the following form

$$L = |\partial_\mu \Phi|^2 - V(\Phi) \,, \tag{5.18}$$

where the potential $V(\Phi)$, in the case of non-interacting particles, is

$$V_0(\Phi) = m^2 |\Phi|^2 \,. \tag{5.19}$$

The potential of the Higgs field, on the other hand, is

$$V_H(\phi) = \frac{\lambda}{4}\left(\Phi_0^2 - |\Phi|^2\right)^2 \,. \tag{5.20}$$

As a source of gravitation, we are interested in the energy-momentum tensor of the scalar field Φ. Let us write it down for the simpler case of a *real* scalar field, for which the lagrangian is

$$L = \frac{1}{2}\, g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) \,. \tag{5.21}$$

The energy-momentum tensor is defined as [43]

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \,, \tag{5.22}$$

with $\mathcal{L} = \sqrt{-g}L$. Therefore, for the case of the lagrangian (5.21) we obtain

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\left[g^{\lambda\rho}\,\partial_\lambda\Phi\,\partial_\rho\Phi - V(\Phi)\right] , \tag{5.23}$$

where we have used the identity

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\sqrt{-g}\,g_{\mu\nu} .$$

5.4.2 A simple example

As a simple non-realistic example, let us consider the case of a non-interacting scalar field. For a weak gravitational field, and in the case of a homogeneous field depending on time only, $T_{\mu\nu}$ has the form

$$T_{00} \equiv \rho = \frac{1}{2}\left(\dot{\Phi}^2 + m^2\Phi^2\right) \tag{5.24}$$

$$T_{ij} \equiv p\delta_{ij} = \frac{1}{2}\left(\dot{\Phi}^2 - m^2\Phi^2\right)\delta_{ij} . \tag{5.25}$$

It follows from these expressions that, if at some moment t a homogeneous field $\Phi(t)$ satisfies the conditions

$$\dot{\Phi} = 0 \quad \text{and} \quad \Phi = 0 , \tag{5.26}$$

the condition $p = -\rho$ is fulfilled at that moment. A scalar field with these properties is called “Inflaton” because of its hability to induce an inflationary expansion of the Universe.

Let us discuss now the physical meaning of the conditions $\dot{\Phi} = 0$ and $\Phi \neq 0$. The equation of motion of a homogeneous field is

$$\frac{d^2\Phi}{dt^2} = -\frac{\partial V(\Phi)}{\partial\Phi} . \tag{5.27}$$

This is a second order differential equation which is solved with two arbitrary initial conditions. In order to understand its physical meaning, we choose $\dot{\Phi} \neq 0$ and $\Phi = 0$.

Now, Eq.(5.27) is identical to the equation of motion of a massive point particle in classical mechanics. The field Φ plays the role of the coordinate of the particle, and $V(\Phi)$ is the potential in which the particle moves. The potential $V = m^2\Phi^2/2$ is the potential of a harmonic oscillator with frequency m. The initial conditions $\dot{\Phi} = 0$ and $\Phi = \Phi_1$ means that the particle was initially shifted to the distance Φ_1 from the equilibrium point, and did not move. Of course, this initial state is not stable. As soon as the external forces cease to hold, the particle starts moving. The motion of the particle, or in other words, the time dependence of the field Φ is described by the simple oscillator formula:

$$\Phi(t) = \Phi_0 \cos(mt + \psi) \ .$$

The initial conditions determine the phase ψ of the oscillations. It is easy to see that the condition $p = -\rho$ is valid, in this simple case, for a moment after every half-period.

5.4.3 The inflationary scenario

First of all, it should be remarked that the notion of equation of state in this case is rather ambiguous since the system is not relaxed and p is not determined by ρ. The relation between p and ρ changes with time and there is no dynamical equation which would describe the evolution of p and ρ once their initial values are given. For that the equation of motion of the field Φ is necessary, which means that the correct form of the potential $V(\Phi)$ must be known.

Historically, the first inflationary model based on scalar field dynamics used special properties of the Higgs potential (5.20), and in particular its temperature dependence. At nonzero temperature, the Higgs potential is modified in the following way:

$$V(\Phi, T) = \left(bT^2 - \frac{\lambda}{2}\,\Phi_0^2 \right) \Phi^2 + \frac{\lambda}{4}\,\Phi^4 + \frac{\lambda}{4}\,\Phi_0^4 \ . \tag{5.28}$$

At high temperatures the potential has one minimum at $\Phi = 0$. At smaller temperatures an additional minimum in the potential appears. While the temperature goes down, this minimum becomes deeper than that at $\Phi = 0$. Thus, the point $\Phi = 0$ becomes unstable and the system evolves to the state with nonvanishing average value of the field, that is $< \Phi > \neq 0$.

The time the system remains in the metastable state $\Phi = 0$ depends on the value of the constants λ and Φ_0. During this period, the ordinary matter density of the Universe decreases as T^4, while the energy density of the scalar field Φ remains constant:

$$\rho_0 \equiv \rho(\Phi = 0) = \lambda\,\frac{\Phi_0^4}{4} \ . \tag{5.29}$$

Now, for the case of the Higgs potential (5.28), one can prove that

$$T_{\mu\nu} = \frac{\Phi_0^4}{4}\, g_{\mu\nu} \ .$$

This energy-momentum tensor acts effectively like a cosmological constant. If the system remains at the point $\Phi = 0$ long enough, then ρ_0 becomes greater than the energy density of all other forms of matter, and the Universe starts to expand exponentially:

$$a(t) \approx \exp(Ht) \ , \quad H = \frac{\sqrt{\lambda}\, \Phi_0^2}{2m_{\rm Pl}} \ . \tag{5.30}$$

Since the energy density of relativistic matter scales as $\exp(-4Ht)$, and that of nonrelativistic matter scales as $\exp(-3Ht)$, the energy density of other forms of matter soon becomes negligible. The Universe then becomes as isotropic and homogeneous as vacuum can be, and its expansion is determined by the vacuum-like energy (5.29). Note that we are implicitly assuming that there is a nonvanishing cosmological constant

$$\Lambda = \frac{\lambda \Phi_0^4}{32\pi m_{\rm Pl}^2} \ .$$

Note also that it is strictly canceled out by the energy of the condensate when $< \Phi >= \Phi_0$.

According to this picture, the inflaton was trapped inside a classically stable local minimum of the inflaton potential, and only quantum tunneling processes could end inflation. The scalar field tunneling from $\Phi = 0$ to $\Phi = \Phi_1$ (see Fig. 5.3) leads to a phase transition of first order, quite analogous to the boiling of water and formation of bubbles in the overheated fluid. Formation of the bubbles of new phase corresponds to a tunneling transition from point Φ_0 to point Φ_1 (see Fig. 5.3). The estimates of transition probabilities prove that inflation can be sufficiently long to yield the *necessary* inflation. After tunneling the field approaches a stable equilibrium point according to the equation:

$$\ddot{\Phi} + 3H\dot{\Phi} + V'(\Phi) = 0 \ . \tag{5.31}$$

This model, proposed by Guth, however, suffers the following shortcoming. The field Φ quickly approaches its equilibrium. Thus, the bubble size was very small and the visible part of the Universe should contain many bubbles. This leads to large inhomogeneities because the energy density contrast inside a bubble and on its border is huge (basically, all energy is in its wall). Linde [44] and Albrecht and Steinhardt [45] modified this scenario getting rid of the trouble. The potential they proposed is very flat around Φ_1 so the rate of variation of the field Φ is small in comparison with the rate of expansion of the Universe (see the dashed line in the Fig. 5.3):

$$\frac{\dot{\Phi}}{\Phi} \ll H_0 = \frac{\dot{a}}{a} \ .$$

The expansion of the Universe for $\Phi = 0$ and $\Phi = \Phi_1$ differ only slightly. This means that the bubble of the new phase $< \Phi >= \Phi_1$ expands exponentially, and its border goes far outside the present-day horizon. Our Universe, in this model, is a small part of a gigantic bubble. That is why it looks isotropic and homogeneous. This kind of solution for the the original inflation model is called *slow roll* inflation, sometimes also called *new inflation*. Essentially all current models of inflation make use of the *slow roll* mechanism.

The bubbles of the new phase are formed predominantly in the state where the potential of the field Φ is flat (see Fig. 5.3), and grows very slowly. As the field approaches Φ_0, the potential becomes steeper and the field starts to move faster. Having reached the equilibrium point, the field begins to oscillate around it. The oscillations are damped by friction caused by the expansion of the Universe (see Eq. 5.31), and by particle production. In this process, the energy of the field is transformed into particles. If the expansion of the Universe is slower than the rate of particle production and the rate of reactions between them, then these particles are thermalized and we come back to the standard cosmological model. Therefore, the scalar field not only causes inflation, but produces all the matter in the Universe.

5.5 Final remarks

It is far from clear what the Inflaton actually is, and where its potential comes from. It is pretty easy to suppose the existence of a scalar field with the properties we wish; whether Nature has provided such a field is a question to be answered by experiments. In this connection, it could be said that *inflation gives us a set of predictive signatures which lie within scope of realistic tests. Thus we should know in the foreseeable future whether we should continue to embrace inflation and build upon its successes or return to the drawing boards for another try* [46].

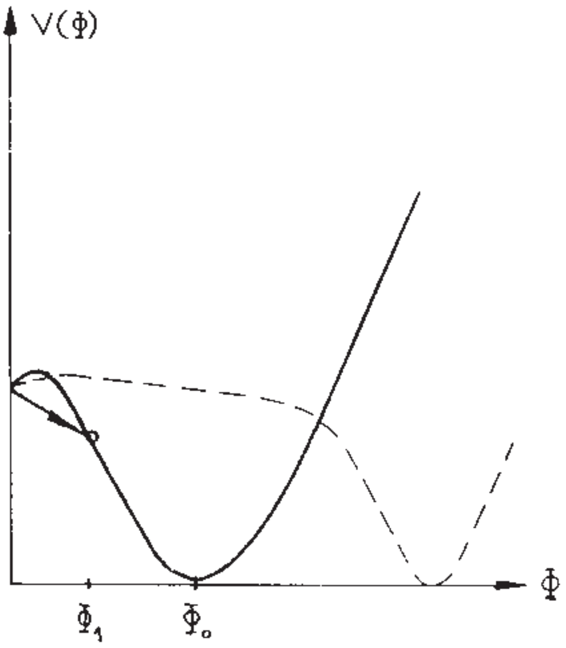


Figure 5.3: *Scalar field tunneling from* $\Phi = 0$ *to* $\Phi = \Phi_0$.

Chapter 6

Formal developments

More theoretically–minded cosmologists – and that includes some of the most illustrious — present their works in a more sophisticated mathematical framework. This chapter is devoted to some frequently used formal notions.

6.1 Fundamental observers

On a pseudo–Riemannian spacetime S, there exists always a family of world–lines which is preferred. They represent the motion of certain preferred observers, the *fundamental observers* and the curves themselves are called the *fundamental world–lines.* The 4-velocities along these lines are $u^\mu = \frac{dx^\mu}{ds}$. Proper time coincides with the line parameter and satisfies $u^2 = u^\mu u_\mu = 1$. The time derivative of a tensor $T^{\rho\sigma\ldots}{}_{\mu\nu\ldots}$ is $\frac{d}{ds}T^{\rho\sigma\ldots}{}_{\mu\nu\ldots} = \frac{d}{dx^\lambda}T^{\rho\sigma\ldots}{}_{\mu\nu\ldots}\frac{dx^\lambda}{ds} = \left(T^{\rho\sigma\ldots}{}_{\mu\nu\ldots}\right)_{,\lambda} u^\lambda$. This is not covariant. The covariant time derivative (sometimes called "the absolute derivative") is

$$\frac{D}{Ds}T^{\rho\sigma\ldots}{}_{\mu\nu\ldots} = \left(T^{\rho\sigma\ldots}{}_{\mu\nu\ldots}\right)_{;\lambda} u^\lambda.$$

For example, the acceleration is

$$a^\mu = \frac{D}{Ds}u^\mu = u^\mu{}_{;\nu}u^\nu.$$

Using the Christoffel connection, it is easily seen that $u_\mu a^\mu = 0$. This property is analogous to that found in Minkowski space, but here only has an invariant sense if acceleration is covariantly defined, as above.

At each point p, and under a condition given below, a fundamental observer has a 3-dimensional space which it can consider to be "hers/his own": its rest–space. Such a space is tangent to S and, as time runs along the fundamental world–line, orthogonal to that line at p (orthogonal to a line means orthogonal to its tangent vector, here u^μ). At each point of a world–line, that 3-space is determined by the projectors

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu .$$

Clearly $h_{\mu\nu}u^\nu = 0$. A projector is an operator P satisfying $P^2 = P$. In the case, the projectors are actually $h^\mu{}_\nu = g^{\mu\rho}h_{\rho\nu}$. They satisfy $h^\mu{}_\nu h^\nu{}_\lambda = h^\mu{}_\nu g^{\nu\rho}h_{\rho\lambda} = h^\mu{}_\nu g^{\nu\rho}(g_{\rho\lambda} - u_\rho u_\lambda) = h^\mu{}_\lambda$. Notice that $h^{\mu\nu}h_{\mu\nu} = h^\mu_\mu = 3$.

The energy–momentum tensor of a fluid is here, instead of (B.4),

$$T_{\mu\nu} = (p+\epsilon)u_\mu u_\nu - p\ g_{\mu\nu} = \epsilon\ u_\mu u_\nu - p\ h_{\mu\nu}. \tag{6.1}$$

The energy density and the pressure can be extracted from $T_{\mu\nu}$ by noticing that $\epsilon = T_{\mu\nu}u^\mu u^\nu$ and $p = \frac{1}{3}T_{\mu\nu}h^{\mu\nu}$. It is convenient to introduce the notations

$$u_{(\mu;\nu)} = \tfrac{1}{2}\ (u_{\mu;\nu} + u_{\nu;\mu})\ \ u_{[\mu;\nu]} = \tfrac{1}{2}\ (u_{\mu;\nu} - u_{\nu;\mu})$$

for the symmetric and antisymmetric parts of $u_{\mu;\nu}$. There are a few important notions to be introduced:

- the vorticity tensor
$$\omega_{\mu\nu} = h^\rho{}_\mu h^\sigma{}_\nu u_{[\rho;\sigma]}\ ;$$
it satisfies $\omega_{\mu\nu} = \omega_{[\mu\nu]} =$ - $\omega_{\nu\mu}$ and $\omega_{\mu\nu}u^\nu = 0$; it is frequently indicated by its magnitude $\omega^2 = \frac{1}{2}\ \omega_{\mu\nu}\omega^{\mu\nu} \geq 0$.

- the expansion tensor
$$\Theta_{\mu\nu} = h^\rho{}_\mu h^\sigma{}_\nu u_{(\rho;\sigma)}\ ;$$
its transversal trace is the volume expansion; $\Theta = h^{\mu\nu}\Theta_{\mu\nu} = \Theta^\mu_\mu = u^\mu_{;\mu}$; in the Friedmann model, Θ turns up as related to the Hubble expansion function by $\Theta = 3H(t)$.

- $\sigma_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{3}\ \Theta\ h_{\mu\nu} = \sigma_{(\mu\nu)}$ is the symmetric trace–free shear tensor; it satisfies $\sigma_{\mu\nu}u^\nu = 0$ and $\sigma^\mu{}_\mu = 0$ and its magnitude is defined as $\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$.

Decomposing the covariant derivative of the 4-velocity into its symmetric and antisymmetric parts and rearranging the terms, a rather involved expression results for it:

$$u_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \tfrac{1}{3}\Theta h_{\mu\nu} - a_\mu u_\nu . \tag{6.2}$$

6.2 Landau–Raychaudhury equation

With the above characterizations of the energy density and the pressure, the Einstein equations reduces to the Landau–Raychaudhury equation:

$$\frac{d}{ds}\,\Theta + \tfrac{1}{3}\,\Theta^2 + 2\left(\sigma^2 - \omega^2\right) - a^{\mu}{}_{;\mu} + \frac{4\pi G}{c^4}\,\left(\epsilon + 3p\right) - \Lambda = 0\;. \tag{6.3}$$

Notice that

$$R_{\mu\nu}u^{\mu}u^{\nu} = \frac{4\pi G}{c^4}\,\left(\epsilon + 3p\right) - \Lambda\;. \tag{6.4}$$

Detailed examination shows that, actually, only when $\omega_{\mu\nu} = 0$ there exists a family of 3-spaces everywhere orthogonal to u^{μ}. In that case, there is a well–defined time which is the same over each 3-space.

From the above equation, we can see the effect of each quantity on expansion: as we proceed along a fundamental world–line, expansion

- decreases (an indication of attraction) with higher values of

  expansion itself

  shear

  energy content

- increases (an indication of repulsion) with higher values of

  vorticity

  second-acceleration

  cosmological constant .

Appendix A

Physical Constants, Units and Observation Data

Taken mainly from [4], [6] and [16].

A.1 Units and constants

1. units

$$1 \;\text{ year} \approx 3.15 \times 10^7 \text{ sec} \; .$$

$$1MeV = 1.6 \times 10^{-13} \text{ Joule} \; ; \; 1 \text{ Joule} = 6.25 \times 10^{12} MeV \; .$$

$$1pc = 3.26 \;\text{ light years} = 3.08 \times 10^{18}\text{cm}; \quad 1 \text{ Mpc} = 3.0857 \times 10^{22}\text{m}$$

   In High-Energy Physics, commonly used system is $\hbar = c = 1$, which leads to $1\,\text{m} = 5.06 \times 10^{12}\,\text{MeV}^{-1}$; $1\,\text{cm}^2 = 2.56 \times 10^{21}\text{MeV}^{-2}$; 1 MeV $= 1.786 \times 10^{-30}$ kg

   for cross-sections, $1\,\text{barn} = 10^{-24}\ \text{cm}^2 = 2.56 \times 10^{-3}\text{MeV}^{-2} = 2.56 \times 10^3\,\text{GeV}^{-2}$;

   $1\,\text{GeV}^{-2} = 3.90625 \times 10^{-4}\text{barn} = 3.90625 \times 10^{-28}\text{cm}^2$.

2. temperature measured in eV or MeV

$$1eV = 11627\ {}^oK \; ; \; kT[MeV] = 8.617 \times 10^{-11}\ T[{}^oK]$$

   for present–day BBR:

$$T_{\gamma 0} = 2.725 \pm 0.002\ {}^oK$$

$$kT_{\gamma 0} = 2.3 \times 10^{-10} \text{ MeV}$$

at any time.

$$kT_\gamma = 2.3 \times 10^{-10}\ (1+z)\ \mathrm{MeV}$$

or

$$1+z = 4.3 \times 10^{9}\ kT_\gamma\ [MeV]$$

3. radio waves: wavelength and frequency

$$300m \longleftrightarrow 1MHz$$

4. constants

Newton gravitational constant

$$G = 6.67390 \times 10^{-11}\ m^3 \times kg^{-1} \times s^{-2} = 0.67 \times 10^{-11}\ \mathrm{N} \times \mathrm{m}^2 \times \mathrm{kg}^{-2}$$

Boltzmann constant (written simply k in the text)

$$k_B = 1.3806568 \times 10^{-23}\ J \times {}^oK^{-1} = 8.6 \times 10^{-5}\ eV \times {}^oK^{-1}$$

Avogadro number and gas constant

$$N_A = 6.023 \times 10^{23}\ ; R = N_A k_B = 8.314 \times 10^7\ \mathrm{erg} \times {}^oK^{-1}.$$

Useful values

$$\frac{k_B}{c^2} = 1.5362 \times 10^{-40}\ \frac{\mathrm{J}}{{}^\mathrm{o}\mathrm{K} \times \mathrm{m}^2}\ \ \mathrm{sec}^2$$

$$c\hbar = 3.16 \times 10^{-26} J \times m\ \ ; (c\hbar)^3 = 3.16 \times 10^{-77} J^3 \times m^3$$

$$(\frac{k_B}{c\hbar})^3 = 8.3285 \times 10^7\ {}^oK^{-3} \times m^{-3}$$

$$\frac{k_B}{c\hbar} = 436.706\ {}^oK^{-1} \times m^{-1} = 4.367 {}^oK^{-1} \times cm^{-1}$$

$$\frac{4\pi G}{3} = 2.795 \times 10^{-10}\ \mathrm{N} \times \mathrm{m}^2 \times \mathrm{kg}^{-2}$$

A.2 Astronomy and Cosmology

1. Earth

   mass $M_\oplus = 5.97223\ (\pm 0.00008) \times 10^{24} kg \approx 6 \times 10^{27}$ g

   equatorial radius $R_\oplus = 6.378 \times 10^6$ m

   surface gravity acceleration $g = GM/R_\oplus^2 = 9.7$ m sec^{-2}

   average orbit radius $\approx 1.5 \times 10^{13}$ cm ; Cf. for Pluto, average orbit radius $\approx 5.5 \times 10^{14}$ cm

2. The Sun

   mass $M_\odot = 1.98843\ (\pm 0.00003) \times 10^{30} kg \approx 2 \times 10^{33}$ g

   radius $R_\odot \approx 7 \times 10^{10}$ cm

   luminosity $L_\odot \approx 4 \times 10^{33}\ erg \times sec^{-1}$

3. our Galaxy

   mass $\approx 1.4 \times 10^{11} M_\odot$

   disc diameter ≈ 30 kpc

   thickness ≈ 1 kpc

   sun is at ≈ 10 kpc from the centre

   sun complete orbit in 200×10^6 years

   differential velocity

   if on the galactic plane, Population I (younger)

   if outside the galactic plane, Population II (older)

4. Hubble constant

$$H_0 = 100\ h\ [km\ sec^{-1}\ Mpc^{-1}] = 3.24 \times 10^{-18}\ h[sec^{-1}]\ ;$$

$$\frac{1}{H_0} = 3.086 \times 10^{17}\ h^{-1}\ sec = 0.9798 \times 10^{10}\ h^{-1}\ years \quad .$$

$$\frac{c}{H_0} = 9.258 \times 10^{27}\ h^{-1}\ cm = 3 \times 10^5\ h^{-1}\ Mpc \quad .$$

5. critical mass density

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} = 1.878 \times 10^{-29}\ h^2\ [g \times cm^{-3}]$$

$$= 1.878 \times 10^{-26}\ h^2\ [kg \times m^{-3}] = 1.054 \times 10^{-5} h^2\ [GeV \times cm^{-3}]$$

6. Planck time, length and energy:

$$t_P = (\frac{\hbar G}{c^5})^{1/2} = 10^{-45}s \ ; L_P = \hbar/cM_P \approx \times 10^{-35}m \ ;$$

$$e_P = M_P c^2 = (c\hbar/G)^{1/2} c^2 \approx 10^{19} GeV$$

7. baryon density

$$\Omega_b = \frac{8\pi G}{3} \frac{\rho_b}{H_0^2} = \frac{\rho_b}{\rho_{crit}};$$

$$0.0052 \leq \Omega_b h^2 \leq 0.026 \ ;$$

$$n_b = 11.4 \times \ \Omega_b \ h^2 \ (1+z)^3 \ \ [m^{-3}]$$

$$\rho_b = 1.82 \times 10^{-26} \ \Omega_b \ h^2 \ (1+z)^3 [kg \ m^{-3}]$$

8. relation to photon density

$$\Omega_b h^2 = 0.004 \eta_{10} \ ; \eta = \eta_{10} \times 10^{-10}$$

9. cosmological constant effective mass density

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \ ; \ \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} = 2.84 \times 10^{51} \ \Lambda \ h^{-2} \ [m^2].$$

cosmological constant effective energy density:

$$\epsilon_\Lambda = \rho_\Lambda c^2 = \frac{\Lambda c^4}{8\pi G} = \frac{\Lambda c^2}{3} \ \frac{3}{4\pi G} \ \frac{c^2}{2} \approx 10.5 \times (\Omega_\Lambda h^2) \ \frac{eV}{mm^3}.$$

useful values:

- $\Lambda = 3.52 \times 10^{-56} \, \Omega_\Lambda h^2 \, [cm^{-2}]$;
  if (present-day favored values) $\Omega_\Lambda = 0.75$ and $h = 0.7$, then
  $\Lambda = 1.32 \times 10^{-56} \, [cm^{-2}]; \sqrt{\Omega_\Lambda} = 0.866$
  $H_0 \sqrt{\Omega_\Lambda} = 1.96 \times 10^{16}$ [sec^{-1}];
  $\frac{H_0 \sqrt{\Omega_\Lambda}}{c} = 9.35 \times 10^{-29}$ [cm^{-1}]
- $\frac{\Lambda c^2}{3} = 1.05 \times 10^{-35} \ \Omega_\Lambda h^2$ [sec^{-2}]
- $\sqrt{\frac{\Lambda c^2}{3}} = \sqrt{- \, s \, \frac{c^2}{\mathcal{R}^2}} = 3.24 \times 10^{-18} \sqrt{\Omega_\Lambda} \, h$ [sec^{-1}]
- $\mathcal{R} = \ = \sqrt{\frac{3}{\Lambda}} = 1.50 \times 10^{28}$ [cm] $\approx 5 \times 10^3$ [Mpc]

10. deceleration parameter:

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{d}{dt}\frac{1}{H} - 1 \ ; \ q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda \ .$$

11. a convenience sometimes used:

$$\Omega_k(t) = -\ \frac{kc^2}{a^2 H_0^2}$$

12. dimensions are as follows:

$$[k] = [0]; \ [a] = L; \ [\Lambda] = L^{-2};$$

$$[\rho] = ML^{-3}; \ [H_0] = T^{-1}.$$

A.3 Statistical Physics & Fluids

13. Black body radiation

    Stefan-Bolzmann constant

$$\sigma = \frac{\pi^2}{60}\ \frac{k^4}{\hbar^3 c^2} = 5.67 \times 10^{-5} g sec^{-3 o} K^{-4} \ .$$

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}\left(\frac{k}{c\hbar}\right)^3 T^3$$

$$n_\gamma = 2.0287 \times 10^7 T^3 [^o K^3 m^{-3}] \ ; \text{at } T = 2.725^o K, n_{\gamma_0} = 410\gamma/cm^3$$

$$n_\gamma = 4.10 \times 10^8 (1+z)^3 m^{-3} \ = 4.10 \times 10^2 (1+z)^3 cm^{-3};$$

$$p_\gamma = \frac{2\zeta(4)}{\pi^2}\frac{1}{(c\hbar)^3}(kT)^4 = 2.52 \times 10^{-16}\ T^4\ J \times m^{-3}$$

$$e_\gamma = \frac{4\sigma}{c}\ T^4 = 3\ p_\gamma = 7.56 \times 10^{-16} T^4 \times J \times m^{-3}$$

$$= 4.72 \times 10^{-9}\ T^4 Mev \times cm^{-3}$$

$$\rho_\gamma = \frac{\pi^2}{15}\ \frac{1}{\hbar^3 c^3}\ (kT_\gamma)^4 = \frac{e_\gamma}{c^2} = 8.42 \times 10^{-33}\ T^4 \times kg \times m^{-3}$$

14. sound velocity: $c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_S$; for air as an ideal gas at $T \approx 300^o K$ with particles of $mc^2 \approx 30\, GeV$, $c_s = \sqrt{\frac{kT}{mc^2}} c \approx \sqrt{\frac{300 \times 8.6 \times 10^{-5} eV}{30 \times 10^9 eV}} c \approx 3 \times 10^2$ m/s.

A.4 Particle Physics

15. photon mass: $m_\gamma < 6 \times 10^{-17}$ [eV] $= 1.0710 \times 10^{-52}$ [kg]

16. protons and nuclei
$$\frac{m_e}{m_p} = 5.45 \times 10^{-4}.$$

    proton mass: $m_P = 1.6726231 \times 10^{-27}$ $[kg]$

    proton lifetime: $\tau_P > 1.6 \times 10^{25} years$;

    proton radius[1]: $r_P = 0.805 f = 0.805 \times 10^{-13}$ cm, or $0.862 f = 0.862 \times 10^{-13}$ cm.

    nuclear matter density: 7.22×10^{17} $[kg \times m^{-3}]$

17. electrons:

    mass $m_e = 0.510998902(21) \frac{MeV}{c^2} = 9.10938188(72) \times 10^{-31} [kg]$

    fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = 7.297 \times 10^{-3} = \frac{1}{137.036}$

    classical electron radius $r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2} = 2.817 \times 10^{-13}$ cm

$$\text{Compton lengths:} \begin{cases} \text{for the electron} & : \frac{h}{m_e c} \approx 2.42 \times 10^{-10} \text{ cm} \\ \text{for the proton} & : \frac{h}{m_P c} \approx 1.32 \times 10^{-13} \text{ cm} \end{cases}$$

18. Thomson cross–section [from ([16])]
$$\sigma_T = \frac{8\pi r_e^2}{3} = 0.665 \text{ barn } = 0.665 \times 10^{-24} \text{ cm}^2$$

19. some other cross–sections [from ([28]), p. 309]
$$\sigma_{\gamma\gamma \to e^+e^-} = \frac{\pi}{2} \left[\frac{e^2}{m_e c^2} \right]^2 = 1.247 \times 10^{-25} cm^2;$$
$$\sigma_{\gamma\gamma \to p\bar{p}} = \frac{\pi}{2} \left[\frac{e^2}{m_p c^2} \right]^2 = 3.71 \times 10^{-32} cm^2;$$

[1] On the lack of agreement on this value, see [27].

Appendix B

Notation and conventions

B.1 Special Relativity

B.1.1 Introduction

When the scene is Minkowski space, there is a (cartesian) coordinate system in which the Lorentz metric is given by

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{B.1}$$

The signature $(+,-,-,-)$ has been chosen. η defines the interval ds, given in those cartesian coordinates as

$$ds^2 = \eta_{ab} dx^a dx^b = dx^0 dx^0 - d\vec{x} \cdot d\vec{x} \tag{B.2}$$

Concerning indices, we shall use $\alpha, \beta, \ldots \mu, \nu, \ldots = 0, 1, 2, 3$ and $i, j, k, \ldots = 1, 2, 3$. The first latin letters $a, b, c, \ldots$ will sometimes run in the range $0, 1, 2, 3, 4$.

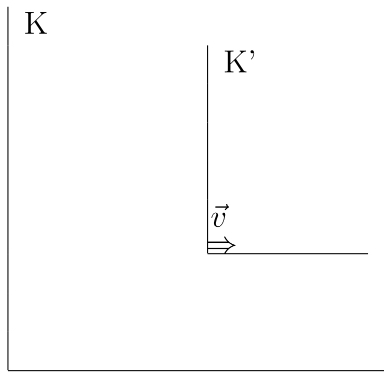
B.1.2 Lorentz transformations

Let K be a reference inertial frame and K' another inertial frame moving with velocity $\vec{v}$ along its x–axis. Call $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ the relative contraction parameter. The measures of an infinitesimal length, in K and K', are related by

$$dx = \gamma(dx' + vdt') \,; dt = \gamma[dt' + \frac{v}{c^2} dx']$$

Velocities are related by

$$v_K = \frac{v_{K'} + v}{1 + \frac{v_{K'} v}{c^2}} .$$



B.1.3 Particle Mechanics

Four–velocities:

$$u^0 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \; ; \quad \vec{u} = \frac{\vec{v}/c}{\sqrt{1 - \frac{v^2}{c^2}}}; \quad u^2 = \eta_{ab} u^a u^b = 1. \tag{B.3}$$

Take the interval (B.2) between two neighboring points on the trajectory γ of a particle. ds is the infinitesimal time along the curve $\gamma(s)$ and the particle four-velocity is the vector $u^a = \frac{dx^a}{ds}$, tangent to γ. $u^2 = 1$ follows from the definition. Taking $\frac{d}{ds}$ of this expression, we have that the acceleration $a = \frac{du}{ds}$ is always orthogonal to the velocity and, consequently, to the curve: $\frac{du}{ds} \cdot u = 0$.

B.1.4 Fluid Mechanics

Energy–momentum tensor of a perfect fluid in Minkowski space is

$$T^{ab} = (p + \epsilon) u^a u^b - p \, \eta^{ab}, \tag{B.4}$$

where ϵ ($= \rho c^2$, with ρ the mass – or equivalent mass – density) is the energy density field and p is the pressure field. Notice

$$T^{00} = \frac{\epsilon + p \, \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} . \tag{B.5}$$

B.2 General Relativity

B.2.1 Introduction

The basic, mediating field is a spacetime Riemannian metric $g_{\mu\nu}$, for which we take the signature $(+,-,-,-)$ and which defines the interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{B.6}$$

The metric $g_{\mu\nu}$ defines a unique torsionless connection $\overset{\circ}{\Gamma}$, whose components is the system $\{x^\mu\}$ are the Christoffel symbols

$$\overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu} = \frac{g^{\lambda\rho}}{2} \left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right]. \tag{B.7}$$

A connection define covariant derivatives of tensors. A useful device is the colon and semicolon notation for usual and covariant derivatives. For example:

$$V_{\mu,\nu} = \partial_\nu V_\mu$$
$$V_{\mu;\nu} = \partial_\nu V_\mu - \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu} V_\lambda$$
$$V^\mu{}_{;\nu} = \partial_\nu V^\mu + \overset{\circ}{\Gamma}{}^\mu{}_{\lambda\nu} V^\lambda$$

Notice that we use *the last* index in $\overset{\circ}{\Gamma}{}^\mu{}_{\lambda\nu}$ as the derivative index.

The curvature of this connection will have as components those of the Riemann tensor

$$\overset{\circ}{R}{}^\alpha{}_{\beta\mu\nu} = \partial_\mu \overset{\circ}{\Gamma}{}^\alpha{}_{\beta\nu} - \partial_\nu \overset{\circ}{\Gamma}{}^\alpha{}_{\beta\mu} + \overset{\circ}{\Gamma}{}^\alpha{}_{\lambda\mu} \overset{\circ}{\Gamma}{}^\lambda{}_{\beta\nu} - \overset{\circ}{\Gamma}{}^\alpha{}_{\lambda\nu} \overset{\circ}{\Gamma}{}^\lambda{}_{\beta\mu}. \tag{B.8}$$

Because $\overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu}$ is symmetric in the two lower indices, the covariant rotational is independent of it:

$$V_{[\mu;\nu]} = V_{\mu;\nu} - V_{\nu;\mu} = \partial_\nu V_\mu - \partial_\mu V_\nu .$$

The curvature Riemann tensor appears in the double covariant derivative of any vector field:

$$V_{\mu;\nu;\rho} - V_{\mu;\rho;\nu} = \overset{\circ}{R}{}^\sigma{}_{\mu\nu\rho} V_\sigma.$$

The Ricci tensor and the scalar curvature are defined as

$$\overset{\circ}{R}_{\mu\nu} = \overset{\circ}{R}{}^\alpha{}_{\mu\alpha\nu} \tag{B.9}$$

$$\overset{\circ}{R} = g^{\mu\nu} \overset{\circ}{R}_{\mu\nu} . \tag{B.10}$$

For 4–dimensional spaces, the Gaussian curvature is

$$K = -\frac{\overset{\circ}{R}}{12}. \tag{B.11}$$

A space has positive (negative) Gauss curvature when R is negative (positive). The Einstein tensor is defined as

$$\overset{\circ}{G}_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} - \frac{1}{2} \overset{\circ}{R} g_{\mu\nu} .$$

A Riemannian space with metric $\eta_{\mu\nu}$ is *flat* if all the components of the corresponding Riemann tensor are zero. In that case also $\eta_{\mu\nu}$ is said to be flat. A Riemannian space is *conformally flat* if the metric can be put into the form $g_{\mu\nu}x) = f(x)\eta_{\mu\nu}$, where f is a scalar function and $\eta_{\mu\nu}$ is flat.

We shall frequently forget the "$\circ$" notation in the rest of this section. It is understood that the connection used is the Christofell symbol.

B.2.2 Field equation

1. Standard Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{B.12}$$

$T_{\mu\nu}$ is the source energy-momentum tensor, G is Newton's constant, and c is the velocity of light in vacuum.

For a homogeneous fluid,

$$T^{\mu\nu} = (p + \epsilon) u^\mu u^\nu - p g^{\mu\nu}, \tag{B.13}$$

whose contraction with the metric is

$$T = \epsilon - 3p. \tag{B.14}$$

Contraction of Standard Einstein's equation with the metric leads to

$$R = -\frac{8\pi G}{c^4} T . \tag{B.15}$$

2. Einstein's equation with a cosmological constant Λ:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{B.16}$$

Contraction gives

$$R = -4\,\Lambda + \frac{8\pi G}{c^4}\,T \tag{B.17}$$

and the Gaussian curvature

$$K = \frac{\Lambda}{3} - \frac{2\pi G}{3c^4}\,T \tag{B.18}$$

3. Sourceless Einstein's equation with cosmological constant

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R - \Lambda g_{\mu\nu} = 0 \tag{B.19}$$

Contraction with the metric gives

$$R = -4\Lambda. \tag{B.20}$$

B.2.3 Particle Dynamics

The interval (B.6) ensures that, like in special–relativistic kinematics, the velocity $u^\mu = \frac{dx^\mu}{ds}$ is a unit vector: $u^2 = u_\mu u^\mu = g_{\mu\nu}u^\mu u^\nu = 1$. Acceleration, however, is different from that of Special Relativity. The object $\frac{du^\mu}{ds}$ is not a well-behaved (that is, is not covariant) under general changes of coordinates. Notice that $\frac{dV^\lambda}{ds} = u^\mu \frac{\partial V^\lambda}{\partial x^\mu}$. To obtain a covariant object, it is necessary to change the last derivative into the covariant derivative

$$\frac{\partial V^\lambda}{\partial x^\mu} + \overset{\circ}{\Gamma}{}^\lambda{}_{\rho\mu}\,V^\rho.$$

Acceleration is therefore given by

$$a^\lambda = \frac{du^\lambda}{ds} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu}\,u^\mu u^\nu. \tag{B.21}$$

A pointlike, spinless inert particle under the only effect of an external gravitational field is postulated to follow a geodesic, a curve whose tangent vector (velocity) at each point is parallel–transported by $\overset{\circ}{\Gamma}$. The equation determining this curve is the geodesic equation:

$$\frac{du^\lambda}{ds} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu}\,u^\mu u^\nu = \frac{d^2x^\lambda}{ds^2} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu}\,\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{B.22}$$

Dynamics reduces to kinematics on a curved space: the above equation just says the the particle goes free, with null acceleration, on the curved space.

B.2.4 Tetrad formalism

The Riemannian metric of any spacetime can always be written in the form

$$g_{\mu\nu} = \eta_{ab}\; h^a{}_\mu h^b{}_\nu \tag{B.23}$$

where η is the flat metric of Minkowski space.

There is always a basis on flat space in which the metric η is a constant matrix. We shall suppose η to be written in that basis. Indices $a, b, c, \ldots$ are lowered and raised by η and its inverse. Indices $\mu, \nu, \rho, \ldots$ are lowered and raised by g and its inverse. On Riemann space, the set of vectors $\{e_a = h_a{}^\mu \partial_\mu\}$ form a basis for vectors (contravariant vectors) at each point. This set of 4 vector fields is called a tetrad field, a four–leg field, or still a vierbein field. Its dual $\{\omega^a = h^a{}_\mu dx^\mu\}$ form a basis for covectors (covariant vectors) at each point. The basis members constitute a Lie algebra of commutation table

$$[e_a, e_b] = C^c{}_{ab} e_c. \tag{B.24}$$

The $C^c{}_{ab}$'s are the structure coefficients of the tetrad basis. Taking the double derivatives of the tetrad field, we find

$$h^a{}_{\mu;\nu;\rho} - h^a{}_{\mu;\rho;\nu} = h^a{}_\sigma R^\sigma{}_{\mu\nu\rho}$$

The Ricci rotation coefficients are

$$\gamma_{acb} = h_{a\mu;\nu} h_b{}^\mu h_c{}^\nu = -\gamma_{bca} \tag{B.25}$$

The expressions

$$\lambda_{abc} = \gamma_{abc} - \gamma_{acb} = h_c{}^\mu h_b{}^\nu \partial_{[\nu} h^a{}_{\mu]} = -C_{cab} \tag{B.26}$$

have the advantage of being independent of the Christoffel symbols. They can be recombined to give back

$$\gamma_{abc} = \tfrac{1}{2}\left[\lambda_{abc} + \lambda_{cab} - \lambda_{bca}\right]. \tag{B.27}$$

Only simple derivatives are then necessary to obtain the Riemann tensor

$$R_{abcd} = \gamma_{abc,d} - \gamma_{abd,c} + \gamma_{abe}[\gamma^e{}_{cd} - \gamma^e{}_{dc}] + \gamma_{aec}\gamma^e{}_{bd} - \gamma_{aed}\gamma^e{}_{bc} \tag{B.28}$$

and the Ricci tensor

$$\begin{aligned} R_{ab} = -\,\tfrac{1}{2}\big(&\lambda_{ab}{}^c{}_{,c} + \lambda_{ba}{}^c{}_{,c} + \lambda^c{}_{ca,b} + \lambda^c{}_{cb,a} + \lambda^{cd}{}_b\lambda_{cda} + \lambda^{cd}{}_b\lambda_{dca} \\ &- \tfrac{1}{2}\;\lambda_b{}^{cd}\lambda_{acd} + \lambda^c{}_{cd}\lambda_{ab}{}^d + \lambda^c{}_{cd}\lambda_{ba}{}^d\big)\;. \end{aligned} \tag{B.29}$$

This means that it is possible, in the detailed calculation of the Riemann tensor from the metric, to short–circuit the calculation of the Christoffel symbols.

An example: the Ricci rotation coefficients for the Friedmann–Robertson–Walker model are: $\gamma^k{}_{0j} = \delta^k_j \frac{\dot{a}}{c}$; $\gamma^0{}_{jk} = \delta_{jk} \frac{\dot{a}}{c}$; $\gamma^k{}_{j0} = 0$; all others are zero. And $\lambda^j{}_{0j} =$ - $\lambda_{j0j} = \frac{\dot{a}}{ca}$, and all others zero.

Another example: we have said in §2.2.2 that a perfect fluid is such that an observer following a line of flux will, at each point, see the fluid as isotropic. We can conceive now that the observer has a frame attached to it, a tetrad whose timelike member will be taken as $h_0 = u = \gamma(1, \vec{v}/c)$. Seen from this solidary comoving frame, the fluid energy-momentum tensor density will have just the components given in Eq.(2.3).

Appendix C

Relativistic Gases

C.1 Introduction

We shall be frequently considering gases at high energies, eventually with effects of quantum nature. What follows is a commented formulary on the Statistical Mechanics of an ideal quantum relativistic gas. We shall profit to rewrite some of the most usual expressions in units specially adequate to the case.

First of all, given a particle of mass m (depending on the range of temperatures of interest, it can be an electron, or a proton), it is natural to use

$$\tau = \frac{kT}{mc^2} \tag{C.1}$$

as the temperature variable. Two lengths are of major interest, the Compton length and the thermal wavelength. The static Compton length is a most natural unit:

$$\lambda_C = \frac{\hbar c}{mc^2} \tag{C.2}$$

For the electron and for the proton, respectively, $\lambda_e = 3.81 \times 10^{-11}$ cm and $\lambda_p = 2.08 \times 10^{-13}$ cm. Thus, a natural volume cell for the proton will be $\lambda_p^3 = 9.0 \times 10^{-39}$ cm^3.

Comment C.1.1 Recall that the Compton length is a measure of the minimum "size" that can be consistently attributed to a particle from the quantum point of view. This comes from the uncertainty principle. The uncertainty in the momentum should be lesser than mc, because a higher value would lead to pair creation. Consequently,

$$mc\Delta x \geq \Delta p \Delta x \approx \hbar$$

and $\Delta x \geq \hbar/mc = \lambda_C$.

If $\beta = 1/kT$ is the inverse temperature, (the cube of) the relativistic generalization [23] of the thermal wavelength is given by (see below)

$$\Lambda^3(\beta) = 2\ \pi^2\ \beta mc^2\ \frac{e^{-\beta mc^2}}{K_2(\beta mc^2)}\ \left(\frac{\hbar c}{mc^2}\right)^3 = \frac{2\ \pi^2}{\tau}\ \frac{e^{-\tau^{-1}}}{K_2(\tau^{-1})}\ \lambda_C^3\ . \tag{C.3}$$

Here $K_2(x)$ is the modified Bessel function of second order. Limits can be found by using the properties

$$K_2(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + \frac{15}{8x} + \dots\right)\ ;$$

$$K_1(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + \frac{3}{8x} + \dots\right) \text{ for x >> 1;}$$

$$K_2(x) \approx 2x^{-2}\ ; K_1(x) \approx x^{-1} \text{ for x << 1.}$$

$K_1(\beta mc^2)$ will appear later, in the energy expression. The non-relativistic limit gives the usual expression

$$\Lambda_{NR}(\beta) = \lambda = \hbar\ \sqrt{\frac{2\pi\beta}{m}} = \sqrt{\frac{2\pi}{\tau}}\ \lambda_C\ . \tag{C.4}$$

For instance, a proton at $kT \approx 4MeV$ will have $\Lambda_{NR} \approx 40\lambda_C$. A fermion, say a proton, will "occupy" a degeneracy-volume $\lambda^3 = \frac{1.42\times 10^{-40}}{\tau^{3/2}}\ cm^3$, from which every other proton will be statistically excluded.

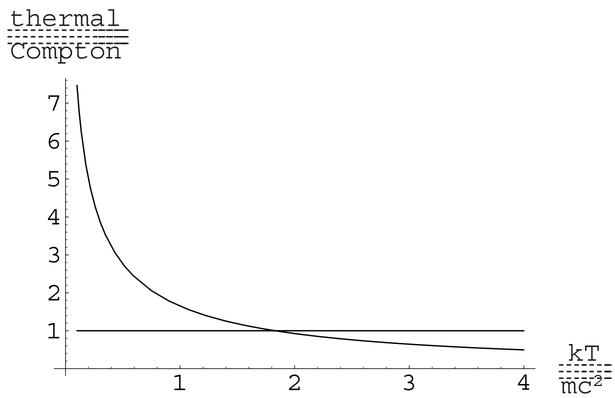


Figure C.1: *Thermal wavelength in units of Compton wavelengths, as a function of $\tau = kT/mc^2$.*

Intuitively, the thermal wavelength is the average "range" of the wavefunction of one particle, or extension along which the wavefunction differs from zero. We see in Figure C.1 how it decreases with the temperature. It reduces to Compton's length at $kT \approx 1.8\, mc^2$. Each particle can be pictured as a bell–like form whose mouth size is the thermal wavelength. The bells shrink more and more at higher and higher temperatures. Quantum effects will turn up when two of such bells superpose. When that happens, we say that the system is degenerate. The Pauli principle acquires a simple description: each fermion is a "hard bell" with respect to other fermions, so that two fermions do not superpose. A convenient measure of quantum effects is the degeneracy index $d = n\lambda^3$, which tells how many particles there are, always on the average, inside a bell. In the fermion case, $d > 1$ means that the wavefunctions must be deformed so that a volume λ^3 can accommodate more particles. Bosons, on the other hand, are positively sociable: quantum effects can be replaced by a small attractive effective potential.

The ultra-relativistic limit turns out to be

$$\Lambda_{UR}(\beta) = \pi^{2/3}\ \beta\hbar c = \pi^{2/3}\ \frac{\lambda_C}{\tau}\ . \tag{C.5}$$

In the ultra-relativistic limit Riemann's zeta function appears frequently. It is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}\ . \tag{C.6}$$

Let us now proceed to the approach appropriate to describe a gas with a variable number of particles, which is the grand-canonical formalism. In particular, it will tells us where (C.3) comes from.

C.2 The grand-canonical formalism

Let us recall that the grand-canonical partition function for an gas of non-interacting quantum particles with chemical potential μ is given by the trace

$$\Xi(V, \beta, \mu) = tr\left[e^{-\beta\sum_i(\epsilon_i-\mu)\hat{n}_i}\right]\ , \tag{C.7}$$

where each $\hat{n}_i$ is the occupation number operator corresponding to the energy level ϵ_i and $\beta = 1/kT$. Calculating the trace means summing over all the energy levels but, before that, summing over all the possible sets $\{n_j\}$ of occupation numbers at each fixed level. Thus,

$$\Xi(V, \beta, \mu) = \sum_{\{n_j\}} \langle n_0 n_1 n_2 \ldots | e^{-\beta\sum_i(\epsilon_i-\mu)\hat{n}_i} | n_0 n_1 n_2 \ldots\rangle =$$

$$\sum_{n_0}\sum_{n_1}\sum_{n_2} \ldots e^{-\beta(\epsilon_0-\mu)n_0}\ e^{-\beta(\epsilon_1-\mu)n_1}\ e^{-\beta(\epsilon_2-\mu)n_2} \ldots =$$

$$\prod_i \sum_n e^{-\beta(\epsilon_i-\mu)n} = \prod_\epsilon \sum_n e^{-\beta(\epsilon-\mu)n} \ .$$

We see that, in this non-interacting case, each level contributes an independent factor. The system can have also internal degrees of freedom, which will likewise contribute separately. Suppose a single degree of freedom (spin, for example) taking g possible values. The partition function will be

$$\Xi(V,\beta,\mu) = \prod_\epsilon \left[\sum_n e^{-\beta(\epsilon-\mu)n}\right]^g \ . \tag{C.8}$$

The kind of statistics appears in the summation, which is over the possible values of the occupation number n, from $n=0$ up to the maximum number of particles allowed in each state: 1 for fermions, ∞ for bosons. The product can be transformed into a summation by using the formal identity

$$\prod_\epsilon \{\ldots\} = \prod_\epsilon [exp(ln\{\ldots\})] = exp \sum_\epsilon ln\{\ldots\}.$$

To treat bosons and fermions at the same time, we adopt the usual convention: upper signs for bosons, lower signs for fermions. The above formulas lead to

$$\ln \Xi^{B,F}(V,\beta,\mu) = \mp g \sum_\epsilon \ln\left[1 \mp e^{-\beta(\epsilon-\mu)}\right] \ . \tag{C.9}$$

It is convenient to use the fugacity variable, either the usual non-relativistic fugacity $z = e^{\beta\mu}$ (where μ is the chemical potential) or its relativistic version $Z = e^{\beta\mu_R} = z\, e^{\beta mc^2}$. If we do not care about zero–energy states, the sum over the energy levels can be replaced by an integral over the momenta through the prescription $\sum_\epsilon \to \frac{\int d^3x d^3p}{h^3}$. This would lead to a difficult integral:

$$\ln \Xi^{B,F}(V,\beta,\mu) = \mp g \frac{4\pi V}{h^3} \int_0^\infty p^2 dp \ln\left[1 \mp Z e^{-\beta(p^2c^2+m^2c^4)^{1/2}}\right] \ .$$

The solution is to expand the logarithm and collect like terms. The partition function acquires the form

$$\Xi^{B,F}(V,\beta,z) = \exp\left\{\frac{gV}{h^3} \sum_{j=1}^{\infty} \frac{(\pm 1)^{j-1}}{j} z^j \int d^3p e^{-j\beta[(p^2c^2+m^2c^4)^{1/2}-mc^2]}\right\} \ . \tag{C.10}$$

The relativistic thermal wavelength (C.3) appears now in the integral

$$\frac{1}{h^3} \int d^3p \;\; e^{-\beta[(p^2c^2+m^2c^4)^{1/2}-mc^2]} = \frac{1}{\Lambda^3(\beta)} \ , \tag{C.11}$$

which leads to the final expression for the grand-canonical partition function for a gas of non-interacting quantum particles,

$$\Xi^{B,F}(V,\beta,z) = \exp\left\{ gV \sum_{j=1}^{\infty} \frac{(\pm 1)^{j-1}}{j} z^j \frac{1}{\Lambda^3(j\beta)} \right\} , \tag{C.12}$$

or its equivalent

$$\Xi^{B,F}(V,\beta,z) = \exp\left\{ g\frac{4\pi V}{h^3 c^3} \frac{(mc^2)^2}{\beta} \sum_{j=1}^{\infty} \frac{(\pm 1)^{j-1}}{j} Z^j K_2(j\beta mc^2) \right\} . \tag{C.13}$$

The pressure and the particle number follow by standard thermal relations:

$$pV = kT \;\ln \Xi = g\; kT\; V \sum_{l=1}^{\infty} \frac{(\pm)^{l-1}}{l} z^l \frac{1}{\Lambda^3(l\beta)} . \tag{C.14}$$

$$\bar{N} = \left[z\frac{\partial}{\partial z}\; ln\; \Xi(V,\beta,z) \right]_{V,\beta} = g \sum_{\epsilon} \frac{1}{z^{-1}e^{\beta\epsilon} \mp 1} \tag{C.15}$$

$$= \frac{gV}{h^3} \int \frac{d^3p}{z^{-1}e^{\beta\;[\sqrt{p^2c^2+m^2c^4}-mc^2]} \pm 1} = gV \sum_{l=1}^{\infty} (\pm)^{l-1} z^l \frac{1}{\Lambda^3(l\beta)} . \tag{C.16}$$

These equations can be seen as a parametric form (with parameter z) of the equation of state. The expressions in terms of integrals or of series are more or less convenient, depending on the application in view. We can extract the density number of particles at energy ϵ, $n_\epsilon = \frac{g}{z^{-1}e^{\beta\epsilon} \mp 1}$. Positivity of this last number implies $0 \leq z \leq 1$, or $\mu \leq 0$, for bosons, but no restriction in the fermion case.

The average energy, including the masses, is

$$\bar{E} = -\left(\frac{\partial}{\partial\beta}\; \ln \Xi(V,\beta,z) \right)_{Z,V} = \sum_{\epsilon} n_\epsilon \epsilon = 3pV +$$

$$4\pi\; g \left(\frac{mc^2}{\lambda_C^3}\right) \left(\frac{kT}{mc^2}\right) \sum_{l=1}^{\infty} \frac{(\pm)^{l-1}}{l} z^l e^{l\beta mc^2} K_1(l\beta mc^2) . \tag{C.17}$$

The degree of degeneracy, which indicates how quantal the gas is, is given by

$$d = \frac{\bar{N}\Lambda^3(\beta)}{V} = g \sum_{l=1}^{\infty} (\pm)^{l-1} z^l \frac{\Lambda^3(\beta)}{\Lambda^3(l\beta)} = g \sum_{l=1}^{\infty} \frac{(\pm)^{l-1}}{l} z^l \frac{e^{-\beta mc^2} K_2(l\beta mc^2)}{e^{-l\beta mc^2} K_2(\beta mc^2)} . \tag{C.18}$$

C.2.1 Massive particles

For a massive particle, the variables $w = p/Mc$ and (C.1), we can write $n = \frac{\bar{N}}{V}$ in the form

$$n = \frac{g}{2\pi^2} \frac{1}{\lambda_C^3} \int_0^\infty \frac{w^2 dw}{z^{-1} e^{\sqrt{1+w^2}/\tau} \pm 1} \ . \tag{C.19}$$

Comment C.2.1 For use in Cosmology: changing to $x = w/\tau$,

$$n = \frac{g}{2\pi^2} \frac{\tau^3}{\lambda_C^3} \int_0^\infty \frac{x^2 dx}{z^{-1} e^{\sqrt{1/\tau^2+x^2}} \pm 1} \ . \tag{C.20}$$

Thus, in units of the proton Compton wavelength λ_C^3 and the proton dimensionless temperature,

$$n_b = 9.9 \times 10^{-47} \Omega_{b0} h^2 \ \frac{(1+z)^3}{\lambda_C^3} = 6.3 \times 10^{-15} \ \Omega_{b0} h^2 \ \frac{\tau^3}{\lambda_C^3} \ . \tag{C.21}$$

With our proton-related variables, the temperature during the thermalized period preceding recombination will be

$$\tau = \frac{kT_\gamma}{mc^2} = 2.5 \times 10^{-13}(1+z) \ . \tag{C.22}$$

For example,

$$n_\gamma = 3.8 \times 10^{-39} \ (1+z)^3 \lambda_C^{-3} \ . \tag{C.23}$$

C.2.2 Massless particles

For a massless particle, the change of variables $x = pc/kT = w/\tau$ can be used directly to give

$$n = \frac{g}{2\pi^2} \frac{\tau^3}{\lambda_C^3} \int_0^\infty \frac{x^2 dx}{z^{-1} e^x \pm 1} \ . \tag{C.24}$$

Gas of photons

For a gas of photons (using $g = 2$, $z = 1$, $\zeta(3) = 1.20206$), there are many expressions of interest:

$$n_\gamma = \frac{\bar{N}_\gamma}{V} = \frac{2}{h^3} \int \frac{d^3 p}{e^{\beta pc} - 1} = 2 \sum_{l=1}^\infty \frac{1}{\Lambda_{UR}^3 (l\beta)} = \frac{16\pi}{h^3 c^3} \zeta(3) \ (kT_\gamma)^3. \tag{C.25}$$

Comment C.2.2 Again for Cosmology:

$$n_\gamma = \frac{1}{\pi^2} \frac{\tau^3}{\lambda_C^3} \int_0^\infty \frac{x^2 dx}{e^x - 1} = 2 \ \frac{\tau^3}{\pi^2 \lambda_C^3} \sum_{l=1}^\infty \frac{1}{l^3} = 2 \ \zeta(3) \frac{\tau^3}{\pi^2 \lambda_C^3} = \frac{2 \ \zeta(3)}{\pi^2} \left(\frac{kT_\gamma}{\hbar c} \right)^3 = 0.244 \ \frac{\tau^3}{\lambda_C^3}. \tag{C.26}$$

The above expression gives the total number of photons per unit volume. The number of photons per unit volume and with the momentum in the interval $(\mathbf{p}, \mathbf{p} + d\mathbf{p})$ is just the first integrand above. Integration on the angular variables gives, for an isotropic distribution, a factor 4π. The energy of a photon of frequency ν is $pc = h\nu$, so that the number of photons per unit volume and with frequency in the interval $(\nu, \nu + d\nu)$ is

$$\frac{8\pi}{c^3} \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1}$$

Using again the energy $h\nu$ of each photon, the energy density in the frequency interval $(\nu, \nu + d\nu)$ is

$$\rho_\gamma(\nu) d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1} , \tag{C.27}$$

which is Planck's formula. Using (C.5), the pressure is found to be

$$p_\gamma = \frac{16\pi}{h^3 \ c^3} \zeta(4) \ (kT)^4 . \tag{C.28}$$

Also

$$p_\gamma = \frac{\zeta(4)}{\zeta(3)} \ n_\gamma \ kT_\gamma = 0.900 \ n_\gamma \ kT_\gamma = \frac{2\zeta(4)}{\pi^2} \left(\frac{k}{\hbar c} \right)^3 \ kT_\gamma^4 \tag{C.29}$$

$(\zeta(4) = \pi^4/90 = 1.08232)$. The energy density is $e_\gamma = 3p_\gamma$, that is,

$$e_\gamma = \frac{48\pi}{h^3 \ c^3} \zeta(4) \ (kT)^4 = 4 \ \frac{\sigma}{c} T^4 , \tag{C.30}$$

which can the taken as the equation of state for the blackbody radiation. The entropy will be $S_\gamma = 4 \, p_\gamma V/T$.

The above expressions are typical of ultrarelativistic boson gases.

Massless fermions

For a gas of fermions with g = 2 (like protons or antiprotons),

$$n_{\bar{p}} = \frac{\bar{N}_p}{V} = \frac{1}{\pi^2} \frac{1}{\lambda_C^3} \int_0^\infty \frac{w^2 dw}{z^{-1} e^{\sqrt{1+w^2}/\tau} + 1} = 2 \sum_{l=1}^{\infty} (-)^{l-1} z^l \frac{1}{\Lambda^3(l\beta)} \tag{C.31}$$

In the ultrarelativistic regime this becomes (we take the antiproton as the standard example), if we put $z = 1$,

$$n_{\bar{p}}^{(UR)} = \frac{1}{\pi^2} \frac{\tau^3}{\lambda_C^3} \int_0^\infty \frac{x^2 dx}{e^x + 1} = \frac{3}{2} \frac{\zeta(3)}{\pi^2} \frac{\tau^3}{\lambda_C^3} = 0.183 \ \frac{\tau^3}{\lambda_C^3} . \tag{C.32}$$

This is the same as

$$n_{\bar{p}}^{(UR)} = \frac{3}{2}\,\frac{\zeta(3)}{\pi^2}\,\left(\frac{kT_\gamma}{\hbar c}\right)^3 \,. \tag{C.33}$$

A factor $\frac{n_\gamma}{n_{\bar{p}}^{(UR)}} = \frac{4}{3}$ comes from the fermion repulsion effect, encapsulated in the sign in the integrand denominator, opposite to that in (C.25). It is more difficult to pack fermions than bosons together. This can be seen also from the limits of the degeneracy index (C.18). In this ultrarelativistic regime, its values are, for photons and antiprotons, respectively, $n_\gamma\,\Lambda_{UR}^3 = 0.244\ \pi = 0.766$ and $n_{\bar{p}}^{(UR)}\,\Lambda_{UR}^3 = 0.183\ \pi = 0.575$. The pressure will be

$$p_{\bar{p}}^{(UR)} = 2\,\tfrac{7}{8}\,\zeta(4)\,\frac{(kT)^4}{\pi^2(\hbar c)^3} \;=\tfrac{7}{8}\,p_\gamma. \tag{C.34}$$

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