An Introduction to

TELEPARALLEL GRAVITY

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Preface

Soon after General Relativity was given its final presentation as a new theory for the gravitational field, an attempt to unify gravitation and electromagnetism was made by H. Weyl in 1918 [1]. His beautiful proposal did not succeed, but introduced for the first time the notions of *gauge transformations* and *gauge invariance*, and can be considered as the seed of what is known today as gauge theory [2, 3]. Another attempt in the same direction was made by A. Einstein [4], about ten years later. This attempt was based on the mathematical structure of teleparallelism, also referred to as distant or absolute parallelism. Roughly speaking, the idea was the introduction of a tetrad field, a field of orthonormal bases on the tangent spaces at each point of the four-dimensional spacetime. The tetrad has sixteen components, whereas the gravitational field, represented by the spacetime metric, has only ten. The six additional degrees of freedom of the tetrad was then supposed by Einstein to be related to the six components of the electromagnetic field.[1] This attempt of unification did not succeed either, because the additional six degrees of freedom of the tetrad are actually eliminated by the 6–parameter local Lorentz invariance of the theory. Like Weyl's work, however, it introduced concepts that remain important to the present day.

After this initial period, which included also discussions with Weyl and Pauli on the failure in the unifying aspiration, the notion of teleparallelism experienced no new advances in the following three decades. Already in the nineteen–sixties, Møller [6] revived Einstein's original idea, no more for unification purposes, but in the pursuit of a gauge theory for gravitation. Following this work, Pellegrini & Plebanski [7] found a lagrangian formulation for teleparallel gravity, a problem that Møller reconsidered later [8]. In 1967, Hayashi & Nakano [9] formulated a gauge theory for the translation group, which was further developed by Hayashi [10]. A few years later, Hayashi [11] pointed out the connection between this theory and teleparallelism, and an attempt to unify these two developments was made by Hayashi & Shirafuji [12] in 1979. According to this approach, General Relativity — a theory that involves only curvature — was supplemented by Teleparallel Gravity — a theory that involves only torsion, and presents three free parameters which should be determined by experiment. This theory, called New General Relativity, represented a new way of including torsion in General

[1] A historical account of the teleparallel–based Einstein's unification theory can be found in Ref. [5].

Relativity, actually an alternative to the scheme previously provided by the Einstein–Cartan approach [13].

For a specific choice of the free parameters, Teleparallel Gravity is found to be completely equivalent to General Relativity. In this case it is sometimes referred to as the *Teleparallel Equivalent of General Relativity.* Although frequently reserved for the three-parameter theory, the name Teleparallel Gravity will be used here as a synonymous for the teleparallel equivalent of General Relativity. A fundamental property embodied in Teleparallel Gravity is that, due to the equivalence with General Relativity, curvature and torsion are able to provide equivalent descriptions of the gravitational interaction. In General Relativity, curvature is used to *geometrize* the gravitational interaction. That is to say, geometry replaces the concept of gravitational force, and the trajectories are determined, not by force equations, but by geodesics. Teleparallel Gravity, on the other hand, attributes gravitation to torsion, but not through a geometrization: it acts as a *force.* In consequence, there are no geodesics in Teleparallel Gravity, only force equations quite analogous to the Lorentz force equation of electrodynamics [14].

The reason for gravitation to present two equivalent descriptions lies in its most peculiar property: *universality.* Like the other fundamental interactions of Nature, gravitation can be described in terms of a gauge theory. In fact, Teleparallel Gravity is a gauge theory for the translation group. Universality of free fall, on the other hand, makes it possible a second, geometrized description, based on the equivalence principle, just General Relativity. As the sole universal interaction, it is the only one to allow also a geometrical interpretation, and hence two alternative descriptions. From this point of view, curvature and torsion are simply alternative ways of representing the same gravitational field, accounting for the same degrees of freedom of gravity.

There has been many contributions from different authors to Teleparallel Gravity. Differently from General Relativity, it is a collective creation. Although it can be considered a finished theory by now, there are still some different views and interpretations of the theory. In this sense, it is important to keep in mind that the ideas presented here are strongly biased by the authors' point of view on the subject, and are essentially based on the research developed by them along many years. This book is actually an expanded version of the relevant publications, and includes also many corrections to the original texts. Although the authors are the solely responsible for the contents of the book, they owe very much to their former and present collaborators: V. C. de Andrade, H. I. Arcos, T. V. Aucalla, A. L. Barbosa, P. B. Barros, M. Calçada, T. Gribl Lucas, L. C. T. Guillen, R. A. Mosna, D. J. Rezende, R. da Rocha, G. Rubilar, K. H. Vu, P. Zambianchi and C.

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Contents

Chapter 1

Basic Notions

A general spacetime is a 4-dimensional differentiable manifold whose tangent space is, at each point, a Minkowski space. A tetrad field establishes a relationship between the manifold and its tangent spaces. Connections are essential to produce derivatives with a covariant meaning. Different connections lead to different curvatures and torsions, as well as to distinct accelerations. These notions are essential for the study of the gravitational interaction.

1.1 Linear Frames and Tetrads

Spacetime is the common arena on which the four presently known fundamental interactions manifest themselves. Electromagnetic, weak and strong interactions are described by gauge theories, involving transformations taking place in "internal" spaces, by themselves unrelated to spacetime. Gravitation, on the other hand, is deeply linked with the very spacetime structure.

All these theories have a strong geometrical flavor. For gauge theories, the basic settings are the principal bundles[1] with a copy of the corresponding gauge group at each spacetime point. The geometrical setting of any theory for gravitation is the tangent bundle, a natural construction always present in any spacetime. In fact, at each point of spacetime — the base space of the bundle — there is always a tangent space attached to it — the fiber of the bundle, which is seen as a vector space.

Comment 1.1 Mathematicians use an *invariant* language, stating and proving results without any use of explicit bases or coordinates. Physicists use a *covariant* language, in part because they are used to, but mainly because they have to prepare for experiments, which are always performed in a particular frame, using apparatuses which suppose a

[1]Bundles will be discussed in some more detail in Chapter 2.

particular coordinate system. Also, physicists have a more pictorial discourse, frowned upon by mathematicians, in which immediate intuition plays a dominant role. For example, tangent spaces are spoken of as "touching" a manifold at a point, internal spaces are "attached" to the manifold at a point, etc. We shall, of course, follow this practice.

We are going to use the Greek alphabet ($\mu, \nu, \rho, \ldots = 0, 1, 2, 3$) to denote indices related to spacetime, and the first half of the Latin alphabet ($a, b, c, \ldots = 0, 1, 2, 3$) to denote indices related to the tangent space, a Minkowski spacetime whose Lorentz metric is assumed to have the form

$$\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1). \tag{1.1}$$

The middle letters of the Latin alphabet ($i, j, k, \ldots = 1, 2, 3$) will be reserved for space indices. A general spacetime is a 4-dimensional differential manifold (indicated $\mathbf{R}^{3,1}$ from now on) whose tangent space is, at any point, a Minkowski spacetime. Each one of these tangent spaces can be acted upon by the Poincaré group — the semi-direct product $SO(3,1) \oslash T^{3,1}$ of the Lorentz by the translation group — or any of its subgroups.

Spacetime coordinates will be denoted by $\{x^\mu\}$, whereas tangent space coordinates will be denoted by $\{x^a\}$. Such coordinate systems determine, on their domains of definition, local bases for vector fields, formed by the sets of gradients

$$\{\partial_\mu\} \equiv \left\{\frac{\partial}{\partial x^\mu}\right\} \quad \text{and} \quad \{\partial_a\} \equiv \left\{\frac{\partial}{\partial x^a}\right\}, \tag{1.2}$$

as well as bases $\{dx^\mu\}$ and $\{dx^a\}$ for covector fields, or differentials. These bases are dual, in the sense that

$$dx^\mu\,(\partial_\nu) = \delta^\mu_\nu \quad \text{and} \quad dx^a\,(\partial_b) = \delta^a_b\,. \tag{1.3}$$

On the respective domains of definition, any vector or covector can be expressed in terms of these bases, which can furthermore be extended by direct product to constitute bases for general tensor fields.

We are going to use the notation $\{e_a, e^a\}$ for general linear frames. A *holonomic* (or coordinate) base like $\{e_a\} = \{\partial_a\}$, related to a coordinate system, is a very particular case of linear base. Any set of four linearly independent fields $\{e_a\}$ will form another base, and will have a dual $\{e^a\}$ whose members are such that $e^a(e_b) = \delta^a_b$. Notice that, on a general manifold, vector fields are (like coordinate systems) only locally defined — and linear frames, as sets of four a such fields, are only defined on restricted domains.

Comment 1.2 The rather special manifolds on which a vector field can be defined everywhere are called *parallelizable*. Of all the spheres $\mathbf{S}^n$ only $\mathbf{S}^1$, $\mathbf{S}^3$ and $\mathbf{S}^7$ are parallelizable. Lie groups are parallelizable manifolds, which means for instance that no Lie group can have $\mathbf{S}^2$ for its underlying manifold. Also, all toruses are parallelizable. A vector field on a non–parallelizable manifold will always vanish at some point, which is quite unacceptable for the member of a vector base.

These frame fields are the general linear bases on the spacetime differentiable manifold $\mathbf{R}^{3,1}$. The set of these bases, under conditions making of it also a differentiable manifold, constitutes the *bundle of linear frames*. A frame field provides, at each point $p \in \mathbf{R}^{3,1}$, a base for the vectors on the tangent space $T_p\mathbf{R}^{3,1}$. Of course, on the common domains they are defined, the members of a base can be written in terms of the members of the other. For example,

$$e_a = e_a{}^{\mu}\, \partial_{\mu} \quad \text{and} \quad e^a = e^a{}_{\mu}\, dx^{\mu} \tag{1.4}$$

and conversely

$$\partial_{\mu} = e^a{}_{\mu}\, e_a \quad \text{and} \quad dx^{\mu} = e_a{}^{\mu}\, e^a. \tag{1.5}$$

We can consider general transformations taking any base $\{e_a\}$ into any other set $\{e'_a\}$ of four linearly independent fields. These transformations constitute the linear group $GL(4, \mathbb{R})$ of all real 4×4 invertible matrices. Notice that these frames, with their bundles, are constitutive parts of spacetime. They are automatically present as soon as spacetime is taken to be a differentiable manifold [16].

Comment 1.3 A *caveat*: there is no consensus on the topology of spacetime. This means that frequently used notions like "neighborhood", "coordinate" and "continuity" are actually not well-defined from the mathematical point of view. The Lorentz metric, being non-positive definite, does not define any topology [15]: its role is actually to introduce causality. Many proposals have been made to fix that topology [**?**], but none has obtained general acceptance. In practice, physicists make implicitly a purely operational option: they use an underlying euclidean $\mathbb{E}^4$ when eventually using global coordinates, or when talking about "continuous" fields. In order to have causality, they then superpose an *additional* Lorentz metric, making of $\mathbb{E}^4$ a Minkowski $\mathbf{M} \equiv \mathbb{E}^{3,1}$; this is the causal space. They finally deform $\mathbb{E}^{3,1}$ into a riemannian space (indicated by $\mathbf{R}$ or $\mathbf{R}^{3,1}$) of the same signature, so as to locally preserve causality. This $\mathbf{R}^{3,1}$ has, at each point, a tangent space which is identical to the causal Minkowski $\mathbf{M}$.

Consider now the spacetime metric $\mathbf{g}$, with components $g_{\mu\nu}$, in some dual holonomic base $\{dx^{\mu}\}$:

$$\mathbf{g} = g_{\mu\nu} dx^{\mu} \otimes dx^{\nu} = g_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{1.6}$$

A tetrad field

$$h_a = h_a{}^{\mu}\, \partial_{\mu}, \tag{1.7}$$

also frequently called *vierbein*, meaning four-legs, will be a linear base which relates $\mathbf{g}$ to the tangent–space metric

$$\eta = \eta_{ab}\, dx^a \otimes dx^b = \eta_{ab}\, dx^a dx^b \tag{1.8}$$

by

$$\eta_{ab} = g(h_a, h_b) = g_{\mu\nu}\, h_a{}^\mu h_b{}^\nu. \tag{1.9}$$

This means that a tetrad field is a linear frame whose members h_a are pseudo–orthogonal by the pseudo–riemannian metric $g_{\mu\nu}$. We shall see later how two of such bases are related by the Lorentz subgroup of the linear group $GL(4, \mathbb{R})$. The components of the dual base members $h^a = h^a{}_\nu dx^\nu$ satisfy

$$h^a{}_\mu h_a{}^\nu = \delta^\nu_\mu \quad \text{and} \quad h^a{}_\mu h_b{}^\mu = \delta^a_b, \tag{1.10}$$

so that Eq. (1.9) has the converse

$$g_{\mu\nu} = \eta_{ab}\, h^a{}_\mu h^b{}_\nu. \tag{1.11}$$

Notice that the Lorentz metric (1.1) fixes the signature

$$s = |(\text{number of positive eigenvalues}) - (\text{number of negative eigenvalues})|$$

as $s = 2$ for all metrics given by (1.11). The determinant

$$g = \det(g_{\mu\nu}) \tag{1.12}$$

is negative because of the signature of metric η_{ab}. We shall also be using the notation

$$h = \det(h^a{}_\mu) = \sqrt{-g}. \tag{1.13}$$

Anholonomy — the property by which a differential form is not the differential of anything, or of a vector field which is not a gradient — is commonplace in many chapters of Physics. Heat and work, for instance, are typical anholonomic coordinates on the space of thermodynamic variables, and the angular velocity of a generic rigid body in euclidean space $\mathbf{E}^3$ is a classical example of anholonomic velocity. Take a dual base h^a such that $dh^a \neq 0$, that is, not formed by differentials. Apply the anholonomic 1-forms h^a to ∂_μ. The result,

$$h^a{}_\mu = h^a\,(\partial_\mu), \tag{1.14}$$

give the components of each

$$h^a = h^a{}_\mu dx^\mu \tag{1.15}$$

along dx^μ. The procedure can be inverted when the h^a's are linearly independent, defining vector fields

$$h_a = h_a{}^\mu \partial_\mu \tag{1.16}$$

which are not gradients. Because closed forms (ω such that $d\omega = 0$) are locally exact (that is, $\omega = d\alpha$ for some α), holonomy/anholonomy can be given a trivial criterion: a form is holonomic *iff* its exterior derivative vanishes.

A holonomic tetrad, on the other hand, will always be of the form $h^a = dx^a$ for some coordinate set $\{x^a\}$. The important point is that, for such a tetrad, the metric tensor (1.11) would simply exhibit the components of the "trivial" Lorentz metric (1.8) transformed to the coordinate system $\{x^\mu\}$. We can think of a change of coordinates $\{x^a\} \leftrightarrow \{x^\mu\}$ represented by

$$dx^a = (\partial_\mu x^a)\ dx^\mu \quad \text{and} \quad dx^\mu = (\partial_a x^\mu)\ dx^a. \tag{1.17}$$

The 1-form dx^a is holonomic, just the differential of the coordinate x^a, and the objects $\partial_\mu x^a$ are the components of the holonomic form dx^a written in base $\{dx^\mu\}$, with $\partial_a x^\mu$ its inverse. Thus, such a coordinate change is just a change of holonomic bases of 1-forms. For the dual base we have the relations

$$\partial_\mu = (\partial_\mu x^a)\ \partial_a \quad \text{and} \quad \partial_a = (\partial_a x^\mu)\ \partial_\mu. \tag{1.18}$$

An anholonomic basis $\{h_a\}$ satisfies the commutation relation

$$[h_a, h_b] = f^c{}_{ab}\ h_c, \tag{1.19}$$

with $f^c{}_{ab}$ the so-called structure coefficients, or coefficients of anholonomy, or still the anholonomy of frame $\{h_a\}$. The frame $\{\partial_\mu\}$ has been presented above as holonomic precisely because its members commute with each other. The dual expression of the commutation relation above is the Cartan structure equation

$$dh^c = -\tfrac{1}{2}\, f^c{}_{ab}\ h^a \wedge h^b = \tfrac{1}{2}\,(\partial_\mu h^c{}_\nu - \partial_\nu h^c{}_\mu)\ dx^\mu \wedge dx^\nu. \tag{1.20}$$

The structure coefficients represent the curls of the base members:

$$f^c{}_{ab} = h^c{}_\mu[h_a(h_b{}^\mu) - h_b(h_a{}^\mu)] = h_a{}^\mu h_b{}^\nu(\partial_\nu h^c{}_\mu - \partial_\mu h^c{}_\nu). \tag{1.21}$$

Notice that $f^c{}_{ab} = 0$ would mean $dh^a = 0$: h^a is a closed differential form. If this holds at a point p, then there is a neighborhood around p on which functions (coordinates) x^a exist such that $h^a = dx^a$. We say that a closed differential form is always locally integrable, or exact.

Comment 1.4 In Classical Mechanics, a force is of this type when it is the gradient of a potential, $F = -\,dV$. In this case, its integral (the work W) from a point a to a point b is independent on the path taken from a to b — it is "integrable", simply the difference $W(b) - W(a)$. This is not what happens for an anholonomous force, one not coming from a potential. The criterion for a force to come from a potential is well–known: $dF = 0$, that is to say, its curl is zero. Analogously, the tetrads are *trivial*, or 4–dimensional gradients of some coordinates, when their 4–rotationals vanish.

In the presence of gravitation, $f^a{}_{cd}$ includes both inertial and gravitational effects. In this case, the spacetime metric (1.11) represents a general (pseudo) riemannian spacetime. In absence of gravitation, on the other hand, the anholonomy of the frames is entirely related to the inertial forces present in those frames. In this case, $h^a{}_\mu$ becomes trivial and $g_{\mu\nu}$ turns out to represent the Minkowski metric in a general coordinate system. The preferred class of inertial frames, denoted h'_a, is characterized by absence of inertial forces, and consequently represented by frames for which

$$f'^a{}_{cd} = 0. \tag{1.22}$$

They are called, for this reason, holonomic frames. Different classes of frames are obtained by performing *local* (point-dependent) Lorentz transformations. Inside each class, different frames are related through *global* (point-independent) Lorentz transformations.

1.2 Connections

Objects with a well-defined behavior under point-dependent transformations (coordinate and gauge transformations, for example) are rather loosely called *covariant* under those transformations. It is a remarkable fact that usual derivatives of such covariant objects are not themselves covariant. In order to define derivatives with a well-defined tensor behavior (that is, which are covariant), it is essential to introduce connections A_μ, which behave like vectors in what concerns the spacetime index, but whose non-tensorial behavior in the algebraic indices just compensates the non-tensoriality of ordinary derivatives. Gauge potentials are connections, introduced to produce derivatives which are covariant under gauge transformations — which, as previously said, take place in "internal" spaces (of isospin, of color, etc). Connections related to the linear group $GL(4,\mathbb{R})$ and its subgroups — such as the Lorentz group $SO(3,1)$ — are called *linear* connections. They have a larger degree of intimacy with spacetime because they are defined on the bundle of linear frames, which is a constitutive part of its manifold structure. That bundle has some properties not found in the bundles related to *internal* gauge theories. Mainly, it exhibits soldering, which leads to the existence of *torsion* for every connection [16]. Linear connections — in particular, Lorentz connections — always have torsion, while internal gauge potentials do not have. It is worth noticing that a vanishing torsion is quite different from a non-existent (non-defined) torsion.

Comment 1.5 We shall see below that the zero torsion which appears in General Relativity has an important consequence for the theory: the Bianchi identity of Eq. (1.67), for which no analog exists in internal gauge theories.

A spin connection (or Lorentz connection) A_μ is a 1-form assuming values in the Lie algebra of the Lorentz group,

$$A_\mu = \tfrac{1}{2}\, A^{ab}{}_\mu\, S_{ab}, \tag{1.23}$$

with S_{ab} a given representation of the Lorentz generators. This connection defines the Fock–Ivanenko covariant derivative [17]

$$\mathcal{D}_\mu = \partial_\mu - \tfrac{i}{2} A^{ab}{}_\mu\, S_{ab}, \tag{1.24}$$

whose second part acts only on the algebraic, or tangent space indices. A Lorentz vector field ϕ^d, for example, is acted upon by the vector representation of the Lorentz generators, matrices S_{ab} with entries [18]

$$(S_{ab})^c{}_d = i\,(\delta^c_a\,\eta_{bd} - \delta^c_b\,\eta_{ad})\,. \tag{1.25}$$

The Fock–Ivanenko derivative is, in that case,

$$\mathcal{D}_\mu \phi^c = \partial_\mu \phi^c + A^c{}_{d\mu}\, \phi^d, \tag{1.26}$$

and so on for any tensor or spinor field.

In the case of soldered bundles, a tetrad field relates internal with external tensors. For example, if ϕ^a is an internal, or Lorentz vector, then

$$\phi^\rho = h_a{}^\rho\, \phi^a \tag{1.27}$$

will be a spacetime vector. Conversely, we can write

$$\phi^a = h^a{}_\rho\, \phi^\rho. \tag{1.28}$$

On the other hand, due to its non–tensorial character, a connection will acquire a vacuum, or non–homogeneous term, under the same transformation. For example, a general linear connection $\Gamma^\rho{}_{\nu\mu}$ is related to the corresponding spin connection $A^a{}_{b\mu}$ by

$$\Gamma^\rho{}_{\nu\mu} = h_a{}^\rho \partial_\mu h^a{}_\nu + h_a{}^\rho A^a{}_{b\mu} h^b{}_\nu \equiv h_a{}^\rho\, \mathcal{D}_\mu h^a{}_\nu. \tag{1.29}$$

The inverse relation is, consequently,

$$A^a{}_{b\mu} = h^a{}_\nu \partial_\mu h_b{}^\nu + h^a{}_\nu \Gamma^\nu{}_{\rho\mu} h_b{}^\rho \equiv h^a{}_\nu \nabla_\mu h_b{}^\nu, \tag{1.30}$$

where ∇_μ is the usual covariant derivative in the connection $\Gamma^\nu{}_{\rho\mu}$, which acts on *external* (spacetime) indices only. For a spacetime vector ϕ^ν, for example, it is given by

$$\nabla_\mu \phi^\nu = \partial_\mu \phi^\nu + \Gamma^\nu{}_{\rho\mu}\, \phi^\rho. \tag{1.31}$$

Using relations (1.27) and (1.28), it is easy to verify that

$$\mathcal{D}_\mu \phi^d = h^d{}_\rho \, \nabla_\mu \phi^\rho. \tag{1.32}$$

It is important to mention that, whereas the Fock-Ivanenko derivative $\mathcal{D}_\mu$ can be defined for all fields — tensorial and spinorial — the covariant derivative ∇_μ can be defined for tensorial fields only. In order to describe the interaction of spinor fields with gravitation the use of Fock–Ivanenko derivatives is, therefore, mandatory.

Equations (1.29) and (1.30) are simply different ways of expressing the property that the total — that is, with connection term for both indices — covariant derivative of the tetrad vanishes identically:

$$\partial_\mu h^a{}_\nu - \Gamma^\rho{}_{\nu\mu} h^a{}_\rho + A^a{}_{b\mu} h^b{}_\nu = 0. \tag{1.33}$$

On the other hand, a connection $\Gamma^\rho{}_{\lambda\mu}$ is said to be metric compatible if the so–called *metricity condition*

$$\nabla_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu} g_{\mu\rho} = 0 \tag{1.34}$$

is fulfilled. From the tetrad point of view, and using Eqs. (1.29) and (1.30), this equation can be rewritten in the form

$$\partial_\mu \eta_{ab} - A^d{}_{a\mu} \, \eta_{db} - A^d{}_{b\mu} \, \eta_{ad} = 0, \tag{1.35}$$

or equivalently

$$A_{ba\mu} = - \, A_{ab\mu}. \tag{1.36}$$

The underlying content of the metric preserving property, therefore, is that the spin connection is lorentzian. Conversely, when $\nabla_\lambda g_{\mu\nu} \neq 0$, the corresponding spin connection $A^a{}_{b\mu}$ cannot assume values in the Lie algebra of the Lorentz group — it is not a Lorentz connection.

1.3 Curvature and Torsion

Curvature and torsion are properties of connections [16], not of space itself. Strictly speaking, in the context of gauge interactions, there is no such a thing as curvature or torsion of spacetime, but only curvature or torsion of connections. This becomes evident if we note that many different connections are allowed to exist on the very same spacetime [19]. Of course, when restricted to the specific case of General Relativity, where only the Levi–Civita connection is present, universality of gravitation allows it to be interpreted — together with the metric — as part of the spacetime definition. However, in the presence of connections with different curvatures and torsions, it seems far wiser and convenient to follow the mathematicians and take spacetime simply as a manifold, and connections (with their curvatures and torsions) as additional structures.

The curvature and torsion tensors of the connection $A^a{}_{b\mu}$ are defined respectively by

$$R^a{}_{b\nu\mu} = \partial_\nu A^a{}_{b\mu} - \partial_\mu A^a{}_{b\nu} + A^a{}_{e\nu} A^e{}_{b\mu} - A^a{}_{e\mu} A^e{}_{b\nu} \tag{1.37}$$

and

$$T^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + A^a{}_{e\nu} h^e{}_\mu - A^a{}_{e\mu} h^e{}_\nu. \tag{1.38}$$

Whereas curvature is a 2-form assuming values in the Lie algebra of the Lorentz group,

$$R_{\nu\mu} = \tfrac{1}{2} R^a{}_{b\nu\mu} S_a{}^b,$$

torsion is a 2-form assuming values in the Lie algebra of the translation group,

$$T_{\nu\mu} = T^a{}_{\nu\mu} P_a,$$

with $P_a = \partial_a$ the translation generators. Using the relation (1.30), they can be expressed in purely spacetime forms:

$$R^\rho{}_{\lambda\nu\mu} \equiv h_a{}^\rho\, h^b{}_\lambda\, R^a{}_{b\nu\mu} = \partial_\nu \Gamma^\rho{}_{\lambda\mu} - \partial_\mu \Gamma^\rho{}_{\lambda\nu} + \Gamma^\rho{}_{\eta\nu} \Gamma^\eta{}_{\lambda\mu} - \Gamma^\rho{}_{\eta\mu} \Gamma^\eta{}_{\lambda\nu} \tag{1.39}$$

and

$$T^\rho{}_{\nu\mu} \equiv h_a{}^\rho\, T^a{}_{\nu\mu} = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}. \tag{1.40}$$

A general linear connection can be decomposed according to[2]

$$\Gamma^\rho{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu} + K^\rho{}_{\mu\nu}, \tag{1.41}$$

where

$$\overset{\circ}{\Gamma}{}^\sigma{}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \tag{1.42}$$

is the torsionless Christoffel, or Levi–Civita connection, and

$$K^\rho{}_{\mu\nu} = \tfrac{1}{2} \left(T_\nu{}^\rho{}_\mu + T_\mu{}^\rho{}_\nu - T^\rho{}_{\mu\nu} \right) \tag{1.43}$$

is the contortion tensor. In terms of the spin connection $A^a{}_{b\mu}$, this decomposition assumes the form

$$A^a{}_{b\mu} = \overset{\circ}{A}{}^a{}_{b\mu} + K^a{}_{b\mu}, \tag{1.44}$$

[2]The quantities related to General Relativity will be denoted with an over "∘".

where $\overset{\circ}{A}{}^a{}_{b\mu}$ is the spin connection of General Relativity.

Considering that the spin connection is a vector in the last index, we can write

$$A^a{}_{bc} = A^a{}_{b\mu}\, h_c{}^\mu. \tag{1.45}$$

It can thus be easily verified that, in the anholonomic basis $\{h_a\}$, the curvature and torsion components are given respectively by [20]

$$R^a{}_{bcd} = h_c\,(A^a{}_{bd}) - h_d\,(A^a{}_{bc}) + A^a{}_{ec}A^e{}_{bd} - A^a{}_{ed}A^e{}_{bc} - f^e{}_{cd}A^a{}_{be} \tag{1.46}$$

and

$$T^a{}_{bc} = A^a{}_{cb} - A^a{}_{bc} - f^a{}_{bc}. \tag{1.47}$$

Use of (1.47) for three combinations of the indices gives

$$A^a{}_{bc} = \tfrac{1}{2}(f_b{}^a{}_c + T_b{}^a{}_c + f_c{}^a{}_b + T_c{}^a{}_b - f^a{}_{bc} - T^a{}_{bc}), \tag{1.48}$$

or equivalently

$$A^a{}_{bc} = \overset{\circ}{A}{}^a{}_{bc} + K^a{}_{bc}, \tag{1.49}$$

with

$$\overset{\circ}{A}{}^a{}_{bc} = \tfrac{1}{2}\,(f_b{}^a{}_c + f_c{}^a{}_b - f^a{}_{bc}) \tag{1.50}$$

the usual expression of the General Relativity connection in terms of the coefficient of anholonomy.

Comment 1.6 The totally anti-symmetric Levi–Civita (or Kronecker) symbol is defined, in the 4–dimensional case, by

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} \;\;1 \text{ if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 \text{ if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ \;\;0 \text{ otherwise} \end{cases} \tag{1.51}$$

In applications, it may be necessary the contravariant version of this symbol, something like $\epsilon^{\mu\nu\rho\sigma}$ with the indices raised by the contravariant metric $g^{\mu\nu}$. This can be shown to be related to the covariant symbol by

$$\epsilon^{\mu\nu\rho\sigma} = -\frac{1}{h^2}\epsilon_{\mu\nu\rho\sigma}, \tag{1.52}$$

where, we recall, $h^2 = [\det(h^a{}_\mu)]^2 = -\det(g_{\alpha\beta}) = -g$. Using it, the following contractions can be worked out:

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\sigma} = -\frac{1}{h^2}\left(\delta^\mu_\alpha\delta^\nu_\beta\delta^\rho_\gamma - \delta^\mu_\beta\delta^\nu_\alpha\delta^\rho_\gamma + \delta^\mu_\beta\delta^\nu_\gamma\delta^\rho_\alpha - \delta^\mu_\gamma\delta^\nu_\beta\delta^\rho_\alpha + \delta^\mu_\gamma\delta^\nu_\alpha\delta^\rho_\beta - \delta^\mu_\alpha\delta^\nu_\gamma\delta^\rho_\beta\right) \tag{1.53}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\rho\sigma} = -\frac{2}{h^2}\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta\right) \tag{1.54}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\nu\rho\sigma} = -\frac{6}{h^2}\,\delta^\mu_\alpha \tag{1.55}$$

and

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -\frac{24}{h^2}. \tag{1.56}$$

1.4 Torsion Decomposition

The torsion tensor $T_{\lambda\mu\nu}$ can be decomposed into three components [21], irreducible under the global Lorentz group: there will be a vector part

$$\mathcal{V}_\mu = T^\nu{}_{\nu\mu}, \tag{1.57}$$

an axial part

$$\mathcal{A}^\mu = \tfrac{1}{6}\,\epsilon^{\mu\nu\rho\sigma}\,T_{\nu\rho\sigma}, \tag{1.58}$$

and a purely tensor part

$$\mathcal{T}_{\lambda\mu\nu} = \tfrac{1}{2}\,(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \tfrac{1}{6}\,(g_{\nu\lambda}\mathcal{V}_\mu + g_{\nu\mu}\mathcal{V}_\lambda) - \tfrac{1}{3}g_{\lambda\mu}\,\mathcal{V}_\nu, \tag{1.59}$$

that is, a tensor with vanishing vector and axial parts. These components are usually called “vector torsion”, “axial torsion” and “pure tensor torsion”. With them, the whole tensor can be written as

$$T_{\lambda\mu\nu} = \tfrac{2}{3}\,(\mathcal{T}_{\lambda\mu\nu} - \mathcal{T}_{\lambda\nu\mu}) + \tfrac{1}{3}\,(g_{\lambda\mu}\mathcal{V}_\nu - g_{\lambda\nu}\mathcal{V}_\mu) + \epsilon_{\lambda\mu\nu\rho}\,\mathcal{A}^\rho. \tag{1.60}$$

The names in the decomposition above come from their behavior under space (P) and time (T) reversals. We call “vector” and “axial-vector” objects with the following responses to $\mathcal{P}$ and $\mathcal{T}$ transformations:

$$\mathcal{V}^\mu = (\mathcal{V}^0, \vec{\mathcal{V}}) \quad \xrightarrow{P} \quad (\mathcal{V}^0, -\vec{\mathcal{V}}), \quad \mathcal{V} = (\mathcal{V}^0, \vec{\mathcal{V}}) \quad \xrightarrow{T} \quad (-\mathcal{V}^0, \vec{\mathcal{V}}) \tag{1.61}$$

$$\mathcal{A}^\mu = (\mathcal{A}^0, \vec{\mathcal{A}}) \quad \xrightarrow{P} \quad (-\mathcal{A}^0, \vec{\mathcal{A}}), \quad \mathcal{A} = (\mathcal{A}^0, \vec{\mathcal{A}}) \quad \xrightarrow{T} \quad (\mathcal{A}^0, -\vec{\mathcal{A}}) \tag{1.62}$$

The quantity $\mathcal{V}^\mu\mathcal{A}_\mu$, for example, is a pseudo-scalar under both P and T transformations, and a scalar under a combined PT transformation.

Comment 1.7 It will be seen in Section 5.3 that a gravitational field can be decomposed into a gravitoelectric and a gravitomagnetic parts. In Teleparallel Gravity, only the axial torsion appears in the gravitomagnetic component. For the gravitational interaction of spinors (see Section 10.3.2), the purely tensor piece is irrelevant: only the vector and the axial-vector torsions appear in the Dirac equation. This last fact will resurface in Section 11.4.

1.5 Bianchi Identities

Given a Lorentz connection $A^a{}_{b\mu}$, its torsion and curvature tensors satisfy two identities, called Bianchi identities. There is an identity for torsion,

$$\mathcal{D}_\nu T^a{}_{\rho\mu} + \mathcal{D}_\mu T^a{}_{\nu\rho} + \mathcal{D}_\rho T^a{}_{\mu\nu} = R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} + R^a{}_{\mu\nu\rho}, \tag{1.63}$$

usually called *first* Bianchi identity, and an identity for curvature,

$$\mathcal{D}_\nu R^a{}_{b\rho\mu} + \mathcal{D}_\mu R^a{}_{b\nu\rho} + \mathcal{D}_\rho R^a{}_{b\mu\nu} = 0, \tag{1.64}$$

usually called *second* Bianchi identity. Using relations (1.32) and (1.40), the Bianchi identity for torsion can be rewritten in the form

$$\begin{aligned} &\nabla_\nu T^\lambda{}_{\rho\mu} + \nabla_\mu T^\lambda{}_{\nu\rho} + \nabla_\rho T^\lambda{}_{\mu\nu} = \\ &\quad R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\nu\rho\mu} + R^\lambda{}_{\mu\nu\rho} + T^\lambda{}_{\rho\sigma} T^\sigma{}_{\mu\nu} + T^\lambda{}_{\nu\sigma} T^\sigma{}_{\rho\mu} + T^\lambda{}_{\mu\sigma} T^\sigma{}_{\nu\rho}. \end{aligned} \tag{1.65}$$

In a similar fashion, the Bianchi identity for curvature becomes

$$\begin{aligned} &\nabla_\nu R^\lambda{}_{\sigma\rho\mu} + \nabla_\mu R^\lambda{}_{\sigma\nu\rho} + \nabla_\rho R^\lambda{}_{\sigma\mu\nu} = \\ &\quad R^\lambda{}_{\sigma\mu\theta} T^\theta{}_{\nu\rho} + R^\lambda{}_{\sigma\nu\theta} T^\theta{}_{\rho\mu} + R^\lambda{}_{\sigma\rho\theta} T^\theta{}_{\mu\nu}. \end{aligned} \tag{1.66}$$

In the case of General Relativity, torsion vanishes for the relevant Levi-Civita connection $\overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu}$, and we obtain the usual Bianchi identities

$$\overset{\circ}{R}{}^\lambda{}_{\rho\mu\nu} + \overset{\circ}{R}{}^\lambda{}_{\nu\rho\mu} + \overset{\circ}{R}{}^\lambda{}_{\mu\nu\rho} = 0 \tag{1.67}$$

and

$$\overset{\circ}{\nabla}_\nu \overset{\circ}{R}{}^\lambda{}_{\sigma\rho\mu} + \overset{\circ}{\nabla}_\mu \overset{\circ}{R}{}^\lambda{}_{\sigma\nu\rho} + \overset{\circ}{\nabla}_\rho \overset{\circ}{R}{}^\lambda{}_{\sigma\mu\nu} = 0. \tag{1.68}$$

It is remarkable that, although torsion vanishes, Bianchi identity (1.67) for torsion remains in General Relativity — as the so–called cyclic identity.

Comment 1.8 Any tensor can be decomposed in its anti-symmetric and symmetric parts. The anti-symmetric part is defined by

$$\begin{aligned} C_{[ab]} &= \frac{1}{2!}\left(C_{ab} - C_{ba}\right) \\ D_{[abc]} &= \frac{1}{3!}\left(D_{abc} - D_{acb} + D_{cab} - D_{bac} + D_{bca} - D_{cba}\right), \end{aligned}$$

and so on. In the same way, the symmetric part is

$$\begin{aligned} C_{(ab)} &= \frac{1}{2!}\left(C_{ab} + C_{ba}\right) \\ D_{(abc)} &= \frac{1}{3!}\left(D_{abc} + D_{acb} + D_{cab} + D_{bac} + D_{bca} + D_{cba}\right), \end{aligned}$$

and so on. For example, equation (1.48) can be rewritten

$$A^a{}_{bc} = A^a{}_{(bc)} + A^a{}_{[bc]}, \tag{1.69}$$

with

$$A^a{}_{(bc)} = -\tfrac{1}{2}(f_{bc}{}^a + f_{cb}{}^a + T_{bc}{}^a + T_{cb}{}^a) \tag{1.70}$$

and

$$A^a{}_{[bc]} = -\tfrac{1}{2}(f^a{}_{bc} + T^a{}_{bc}). \tag{1.71}$$

From this expression we see that, given a tetrad, the connection is completely determined by its torsion — which is Ricci's theorem [24].

1.6 Lorentz Transformations

Vector base $\{h_a\}$ is far from unique. There exists actually a six-fold infinity of tetrad fields $\{h_a = h_a{}^\mu \, \partial_\mu\}$, each one relating $\mathbf{g}$ to the Lorentz metric η by Eqs. (1.9) and (1.11). This comes from the fact that, at each point of a riemannian spacetime, Eq. (1.11) only determines the tetrad field up to transformations of the six-parameter Lorentz group in the tangent space indices. Suppose in effect another tetrad $\{h'_a\}$ such that

$$g_{\mu\nu} = \eta_{cd} \; h'^c{}_\mu h'^d{}_\nu, \tag{1.72}$$

Contracting both sides with $h_a{}^\mu h_b{}^\nu$, we arrive at

$$\eta_{ab} = \eta_{cd} \; (h'^c{}_\mu h_a{}^\mu)(h'^d{}_\nu h_b{}^\nu). \tag{1.73}$$

This equation says that the matrix with entries

$$\Lambda^a{}_b = h'^a{}_\mu \, h_b{}^\mu, \tag{1.74}$$

which gives the transformation

$$h'^a{}_\mu = \Lambda^a{}_b \, h^b{}_\mu, \tag{1.75}$$

satisfies

$$\eta_{cd} \, \Lambda^c{}_a \, \Lambda^d{}_b = \eta_{ab}. \tag{1.76}$$

This is just the condition that a matrix Λ must satisfy in order to belong to (the vector representation of) the Lorentz group.

The converse reasoning will say that Lorentz transformations preserve the metric defined by $\{h_a\}$. This leads to a better characterization of the metrics defined by (1.11): (i) anholonomic base fields related by Lorentz transformations define one same metric; (ii) anholonomic base fields not related by Lorentz transformations define different metrics.

Under a local Lorentz transformation $\Lambda^a{}_b(x)$, the tetrad changes according to (1.75), whereas the spin connection undergoes the transformation

$$A'^a{}_{b\mu} = \Lambda^a{}_c(x) \, A^c{}_{d\mu} \, \Lambda_b{}^d(x) + \Lambda^a{}_c(x) \, \partial_\mu \Lambda_b{}^c(x). \tag{1.77}$$

In the same way, it is easy to verify that $T^a{}_{\nu\mu}$ and $R^a{}_{b\nu\mu}$ transform covariantly:

$$T'^a{}_{\nu\mu} = \Lambda^a{}_b(x) \, T^b{}_{\nu\mu} \tag{1.78}$$

and

$$R'^a{}_{b\nu\mu} = \Lambda^a{}_c(x) \, \Lambda_b{}^d(x) \, R^c{}_{d\nu\mu}. \tag{1.79}$$

This means that, just as $g_{\mu\nu}$, spacetime–indexed quantities $\Gamma^\rho{}_{\nu\mu}$, $T^\lambda{}_{\mu\nu}$ and $R^\rho{}_{\lambda\nu\mu}$ are invariant under a local Lorentz transformation.

Comment 1.9 Let us register one more point. Suppose the members of a tetrad base $\{h_a\}$ with commutation rule

$$[h_a, h_b] = f^c{}_{ab}\, h_c$$

are Lorentz-transformed: $h'^a{}_\mu = \Lambda^a{}_b\, h^b{}_\mu$. Then, in order to keep the commutation rule in the form

$$[h'_a, h'_b] = f'^c{}_{ab}\, h'_c, \tag{1.80}$$

the anholonomy coefficients must transform in a very special way:

$$f'^c{}_{ab} = \Lambda^c{}_d\, f^d{}_{ef}\, \Lambda^e{}_a\, \Lambda^f{}_b + \Lambda^c{}_d \left[\Lambda^e{}_a h_e(\Lambda^d{}_b) - \Lambda^e{}_b h_e(\Lambda^d{}_a)\right]. \tag{1.81}$$

The last two, zero-curvature terms, vanish for global $(\partial_\mu \Lambda^c{}_b = 0)$ Lorentz transformations. The anholonomy coefficients are, in that case, just covariant. Those derivative terms are, however, essential to compensate the behavior of the two connection terms in (1.47), keeping torsion covariant under local Lorentz transformations.

1.7 Dynamical Aspects

Dynamical considerations will in special require derivatives with respect to time, or better, to the proper time s defined by the metric tensor (1.11):

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \eta_{ab}\, h^a{}_\mu h^b{}_\nu\, dx^\mu dx^\nu = \eta_{ab}\, h^a h^b. \tag{1.82}$$

A curve γ (say, a particle trajectory) on spacetime, parametrized by proper time as $\gamma^\mu(s) = x^\mu(s)$, will have as four-velocity the vector of components

$$u^\mu = \frac{dx^\mu}{ds}\,. \tag{1.83}$$

The corresponding acceleration cannot be given a covariant meaning without a connection — and each different connection $\Gamma^\lambda{}_{\mu\nu}$ will define a different acceleration

$$a^\lambda = \frac{\nabla u^\lambda}{\nabla s} = u^\nu\, \nabla_\nu u^\lambda = \frac{du^\lambda}{ds} + \Gamma^\lambda{}_{\mu\nu} u^\mu u^\nu. \tag{1.84}$$

Comment 1.10 Observe that the definition of the four-velocity does not require a connection. In fact, even defined as an ordinary derivative, the four-velocity u^μ results to be a four-vector. The reason for this is that x^μ is not a four-vector, but a set of four scalar functions $\gamma^\mu(s)$ parametrizing the curve γ. As such, its ordinary derivative turns out to be covariant.

The Christoffel connection (1.42), for example, will define the acceleration

$$\overset{\circ}{a}{}^\lambda = u^\nu\, \overset{\circ}{\nabla}_\nu u^\lambda = \frac{du^\lambda}{ds} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu} u^\mu u^\nu. \tag{1.85}$$

As $\overset{\circ}{a}{}^{\lambda}$ is orthogonal to u^{λ}, its vanishing means that the u^{λ} keeps parallel to itself along the curve. This leads to the notion of parallel transport: we say that u^{λ} is parallel–transported along γ. Further, as every vector field is locally tangent to a curve (its local "integral curve"), a condition like

$$v^{\mu}\,\overset{\circ}{\nabla}_{\mu}u^{\lambda} \equiv \overset{\circ}{\nabla}_{v}u^{\lambda} = 0$$

says that u^{λ} is parallel–transported along the integral curve of v^{λ}. The metric compatibility condition (1.34) implies that

$$v^{\lambda}\,\overset{\circ}{\nabla}_{\lambda}g_{\mu\nu} = 0$$

for any vector field v^{λ}, which is equivalent to say that the metric $g_{\mu\nu}$ is parallel–transported everywhere on spacetime.

Let us now go a step further, and consider an observer attached to a particle moving along curve γ. An observer is abstractly conceived as a timelike worldline [25, 26]. We can do more: notice that the four members of a tetrad are (pseudo) orthogonal to each other. This means that one of them is timelike, and the other three are spacelike. As

$$\eta_{ab} = g^{\mu\nu}h_{a}{}^{\mu}h_{b}{}^{\nu} = h_{a\nu}h_{b}{}^{\nu}, \tag{1.86}$$

then

$$h_{0\nu}h_{0}{}^{\nu} = \eta_{00} = +1,$$

so that, in our convention with $\eta = \mathrm{diag}(+1,-1,-1,-1)$, h_0 is timelike and has unit modulus. The remaining h_k, for $k = 1,2,3$, are spacelike. We then "attach" h_0 to the observer by identifying

$$u = h_0 = \frac{d}{ds}, \tag{1.87}$$

with components $u^{\mu} = h_{0}{}^{\mu}$. Of course, h_0 will be the observer velocity. The tetrad field is, in this way, made into a *reference frame*, with an observer attached to it. If the observer is inertial, then

$$u^{\nu}\,\frac{\partial u^{\lambda}}{\partial x^{\nu}} + \Gamma^{\lambda}{}_{\mu\nu}u^{\mu}u^{\nu} = 0. \tag{1.88}$$

Take now a general connection Γ and examine the corresponding frame acceleration

$$a^{a}_{(f,\Gamma)} = h^{a}{}_{\lambda}\,a^{\lambda}_{(f,\Gamma)} = h^{a}{}_{\lambda}\,h_0(h_0^{\lambda}) + h^{a}{}_{\lambda}\Gamma^{\lambda}{}_{\mu\nu}h_0{}^{\mu}h_0{}^{\nu}. \tag{1.89}$$

The connection components, seen from the frame, are

$$A^a{}_{bc} = h^a{}_\lambda \Gamma^\lambda{}_{\mu\nu} h_b{}^\mu h_c{}^\nu + h^a{}_\lambda \, h_c(h_b{}^\lambda), \tag{1.90}$$

so that

$$a^a_{(f,\Gamma)} = A^a{}_{00} = \overset{\circ}{A}{}^a{}_{00} + K^a{}_{00} \tag{1.91}$$

for whatever connection. This is the recipe: each connection Γ, which seen from the observer tetrad is A, will attribute to the observer an acceleration $a^c_f = A^c{}_{00}$, seen by that very observer. Notice that Eq. (1.90) can be rewritten in the form

$$A^a{}_{bc} = h^a{}_\lambda \nabla_{h_c} h_b{}^\lambda. \tag{1.92}$$

For the Levi-Civita connection, we have simply

$$\overset{\circ}{a}{}^a_{(f)} = \overset{\circ}{A}{}^a{}_{00}. \tag{1.93}$$

It follows from Eq. (1.50) that

$$\overset{\circ}{a}{}^c_{(f)} = f_0{}^c{}_0. \tag{1.94}$$

A holonomic tetrad, therefore, has zero Christoffel acceleration. The point is that in the generic case, as

$$\overset{\circ}{\nabla}_{h_a} h_a{}^\lambda = h_c{}^\lambda \overset{\circ}{A}{}^c{}_{aa} \neq 0, \tag{1.95}$$

a tetrad member is not parallel-transported along its own integral curve. In particular, it has a Christoffel acceleration.

Equation (1.92) provides a general interpretation for $A^c{}_{ab}$. In particular, $\overset{\circ}{A}{}^i{}_{j0}$ is the time rate of change of h_j projected along h_i, with the covariant derivative given by the Levi-Civita connection. The space tetrad members rotate with angular velocity

$$\overset{\circ}{\omega}{}^k_{(f)} = \tfrac{1}{2}\, \epsilon^{kij}\, \overset{\circ}{A}_{ij0}, \tag{1.96}$$

which shows $\overset{\circ}{A}{}^a{}_{bc}$ in its historical role of Ricci's coefficient of rotation. The general meaning of (1.92) is clear: $A^a{}_{bc}$ is a "generalized frame proper acceleration", the covariant derivative of h_b along h_c, projected along h_a.

The above considerations give a new perception of the acceleration

$$\overset{\circ}{a}{}^k_{(f)} = \frac{du^k}{ds} + \overset{\circ}{A}{}^k{}_{bc}\, u^b u^c, \tag{1.97}$$

as seen from an accelerated frame. Besides the first, kinetic term, it includes contributions from the frame itself:

$$\overset{\circ}{a}{}^k_{(f)} = \frac{du^k}{ds} + \overset{\circ}{a}{}^k_{(f)} u^0 u^0 + 2\,(u \times \overset{\circ}{\omega}_{(f)})^k\, u^0 + \overset{\circ}{A}{}^k{}_{ij}\; u^i u^j. \tag{1.98}$$

Some of the terms turning up can be easily recognized. The piece

$$\overset{\circ}{a}{}^k_{(l)} = \overset{\circ}{a}{}^k_{(f)} u^0 u^0$$

represents the frame linear acceleration, whereas the piece

$$\overset{\circ}{a}{}^k_{(C)} = 2\,(u \times \overset{\circ}{\omega}_{(f)})^k\, u^0$$

represents the Coriolis force. The last piece represent additional inertial effects present in the frame [27].

Each connection Γ will attribute to a curve γ a different acceleration, given by Eq. (1.84). But acceleration must remain a measure of the velocity variation with *time*, and time appears in that formula as the proper time (1.82) defined by metric $g_{\mu\nu}$. If acceleration is to keep a meaning, it is necessary that the same metric be considered all along the curve. In other words, the acceleration–defining connection must parallel–transport $g_{\mu\nu}$, satisfying the metricity condition (1.34). Furthermore, an infinity of connections are metric compatible with a given metric, one (and only one) for each value of torsion. In this way, compatible connections are classified by their torsions [by the Ricci theorem mentioned in Comment 1.8]. The Levi-Civita connection is, consequently, the only one with vanishing torsion.

Comment 1.11 Another transport, distinct from parallel transport, can be introduced which absorbs the acceleration. It is given by the Fermi-Walker derivative:

$$\overset{\circ}{\nabla}{}^{(FW)}_u v^\lambda = \overset{\circ}{\nabla}_u v^\lambda + \overset{\circ}{a}_\nu v^\nu u^\lambda - u_\nu \overset{\circ}{a}{}^\lambda v^\nu.$$

With this derivative,

$$\overset{\circ}{\nabla}{}^{(FW)}_u u^\lambda = \overset{\circ}{\nabla}_u u^\lambda + \overset{\circ}{a}_\nu u^\nu u^\lambda - u_\nu \overset{\circ}{a}{}^\lambda u^\nu = 0$$

and now

$$\overset{\circ}{\nabla}{}^{(FW)}_u h_a{}^\lambda = \overset{\circ}{\nabla}_u h_a{}^\lambda + \overset{\circ}{a}_a u^\lambda - u_a \overset{\circ}{a}{}^\lambda.$$

In particular,

$$\overset{\circ}{\nabla}{}^{(FW)}_{h_0} h_0{}^\lambda = \overset{\circ}{\nabla}_{h_0} h_0{}^\lambda - \overset{\circ}{a}{}^\lambda = 0$$

implies that h_0 — by this Fermi-Walker transport — is kept tangent along its own integral curve.

Chapter 2

Gauge Theories and Gravitation

We give here short and rough *résumés* of gauge models — which describe three of the four fundamental interactions of Nature — and of General Relativity — the standard theory for gravitation. No more than a "cast of characters", with the main protagonists in each case. At the end, we discuss how to apply the gauge paradigm to gravitation.

2.1 The Gauge Tenets

The gauge theories which successfully describe electromagnetic, electroweak and strong interactions are all concerned with point-dependent transformations occurring in "internal" spaces, that is, spaces unrelated to the "external" spacetime differentiable manifold. The "point-dependence" means merely that different transformations in internal space take place at different points of the external manifold.

The forerunner of these theories is the Yang-Mills model, with the unitary group $SU(2)$ as gauge group acting on isotopic spin (isospin) spaces of varied dimensions, each one carrying a different linear representation of $SU(2)$. The proton-neutron pair stays in a 2-dimensional space of doublets (p, n), the pions in a 3-dimensional space of triplets (π^1, π^2, π^3), and so on. Particles are represented by these "multiplet" fields. More precisely, once the fields are quantized, the particles turn up as their quanta: protons are the quanta of what we call "the proton field", pions are the quanta of "the pion (complex scalar) field", and so on. A particle which is insensitive to a certain gauge field is assigned to a singlet representation of the group, in which it will not respond to any gauge transformation.

Fiber bundles are composite manifolds which encapsulate all the geometric aspects of these theories.[1] They are a combination of a base manifold (here, spacetime) and another space of interest (the gauge group, or any other space carrying one of its representations), built up with the strong proviso that the overall set of points constitute also a differentiable manifold. Given a point p on the base space, the bundle is locally (in a neighborhood of p) a direct product of both involved spaces.

Comment 2.1 With the circumference S^1 and the interval $(-1, +1)$ two simple but quite different bundles can be built: a cylinder, which is a global direct product, and a Möbius band, which is only locally a direct product.

A first bundle is constructed by attaching a copy of the gauge group at each spacetime point p. Each fiber — space attached at each point of the base spacetime manifold — is itself a group, and in this case the bundle is said to be "principal" (see Figure 2.1, where we have taken for base the Minkowski space $\mathbf{M}$). A "bundle projection" π takes all the points of the "fiber over p" into its corresponding base–space point p. And a converse "section" σ takes points on a neighborhood of p into a domain of the bundle manifold. Other bundles, called "associated", can be obtained by replacing

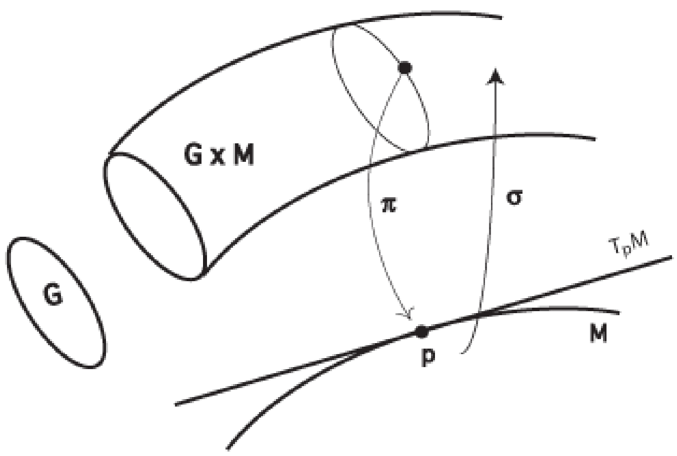

Figure 2.1: Local view of a principal bundle.

the group by one of its linear representations. "Source" fields inhabit (as sections) precisely the carrier vector spaces of such representations. They

[1] Detailed accounts of the geometries both of gauge theories and General Relativity can be found in Ref. [19].

experience gauge transformations of the form

$$\psi'^i(x^\mu) = U^i{}_j(x^\mu)\, \psi^j(x^\mu), \tag{2.1}$$

where $U^i{}_j(x^\mu)$ are the entries of the matrix $U(x^\mu)$ — the group element — representing the gauge transformation at the point p of coordinates x^μ, with $i, j, k = 1, 2, 3, ...d$, being d the dimension of the representation. This is actually the way sections transform. Notice that to each principal bundle corresponds an infinity of associated bundles, one for each group representation. The principal bundle has not that name for nothing: theorems proved for it can afterwards be transferred to each one of its infinite associates.

For an associated source field ψ belonging to a given representation, the group element assumes the form

$$U^i{}_j(x^\mu) = \exp[\alpha^B\, T_B]^i{}_j, \tag{2.2}$$

where $(T_B)^i{}_j$ $(A, B, C = 1, 2, 3, ..., n)$ are the generators, with n the dimension of the group. Dropping the matrix indices, the gauge transformation (2.1) will be given by

$$\psi'(x) = \exp[\alpha^B\, T_B]\, \psi(x), \tag{2.3}$$

where $\alpha^B = \alpha^B(x^\mu)$ are the parameters fixing the gauge transformation. The corresponding infinitesimal transformation is

$$\delta\psi(x) \equiv \psi'(x) - \psi(x) = \alpha^B\, T_B\, \psi(x). \tag{2.4}$$

The generators T_B satisfy the commutation relation

$$[T_B, T_C] = f^A{}_{BC} T_A, \tag{2.5}$$

where $f^A{}_{BC}$ are the structure constants of the group Lie algebra. We recall that the Lie algebra of a Lie group has this very special property, that the structure coefficients for the generator commutators are constants. The adjoint representation, which we denote by J_B, is given by matrices whose dimension d coincides with the group dimension n. It is given by the matrix with entries

$$(J_B)^A{}_C = f^A{}_{BC}.$$

The gauge boson field, which mediates the interactions, belong to the adjoint representation. It is a 1-form assuming values in the Lie algebra of the gauge group:

$$A = J_C\, A^C{}_\mu\, dx^\mu. \tag{2.6}$$

From the general definition of covariant derivative [19]

$$D_\mu\psi = \partial_\mu\psi - A^B{}_\mu\, \frac{\delta\psi}{\delta\alpha^B}, \tag{2.7}$$

together with the infinitesimal transformation (2.4), we found that the covariant derivative of the matter field ψ is

$$D_\mu \psi = \partial_\mu \psi - A^B{}_\mu T_B \psi, \tag{2.8}$$

where the 1-form $A^B{}_\mu$ now takes values on the appropriated representation of the Lie algebra: the covariant derivative depends on the representation of the field on which it acts. The covariant derivative of an object has the same behavior as the object itself. For example, since the field ψ transforms according to $\psi' = U\psi$, its covariant derivative must transform in the same way, that is,

$$D'_\mu \psi' = U(D_\mu \psi). \tag{2.9}$$

From this condition we obtain the transformation law of the gauge potential

$$A'_\mu = U A_\mu U^{-1} + U \partial_\mu U^{-1}, \tag{2.10}$$

which shows that A does not transform covariantly. This is so because A is actually a connection: the last, derivative term is just the non–covariance necessary to compensate the non–covariance of the ordinary derivative — defining in this way covariant derivatives. As said in the previous chapter, usual derivatives are not covariant under point–dependent transformations. Notice that the compensating, derivative term in (2.10) is independent of the connection. It disappears if we take the difference between two connections. That difference is, consequently, a covariant object, a tensor.

The infinitesimal version of (2.10) is

$$\delta A^C{}_\mu \equiv A'^C{}_\mu - A^C{}_\mu = - \left(\partial_\mu \alpha^C + f^C{}_{BD} A^B{}_\mu \, \alpha^D \right), \tag{2.11}$$

which can be written as

$$\delta A^C{}_\mu = - D_\mu \alpha^C, \tag{2.12}$$

where use has been made of the fact that the transformation parameter α^C belongs to the adjoint representation.

On the other hand, the covariant derivative of A itself is its curvature,

$$F = \tfrac{1}{2} \, J_A \, F^A{}_{\mu\nu} \, dx^\mu \wedge dx^\nu, \tag{2.13}$$

which has for components along generators J_A just the components of the field strength

$$F^A{}_{\mu\nu} = \partial_\mu A^A{}_\nu - \partial_\nu A^A{}_\mu + f^A{}_{BC} A^B{}_\mu A^C{}_\nu. \tag{2.14}$$

Under the gauge transformation (2.10), the field strength changes according to

$$F'_{\mu\nu} = U F_{\mu\nu} U^{-1}. \tag{2.15}$$

This means that F transforms covariantly. The corresponding infinitesimal transformation is

$$\delta F^A{}_{\mu\nu} \equiv F'^A{}_{\mu\nu} - F^A{}_{\mu\nu} = f^A{}_{BC}\, \alpha^B\, F^C{}_{\mu\nu}. \tag{2.16}$$

It is worth mentioning finally that, seen from a non-holonomous tetrad, instead of (2.14), a gauge field strength will assume the form

$$F^C{}_{ab} = h_a(A^C{}_b) - h_b(A^C{}_a) + f^C{}_{DE} A^D{}_a A^E{}_b - f^d{}_{ab} A^C{}_d. \tag{2.17}$$

The last term comes from the tetrad non-holonomicity,

$$[h_c, h_d] = f^e{}_{cd}\, h_e,$$

and is linear in the connection.

Comment 2.2 It is a good point to have in mind the well–known case of electromagnetism, which is a gauge theory with the gauge group $U(1)$. As a manifold, $U(1)$ is the 1-dimensional sphere, just the circumference S^1. In this abelian case the $f^A{}_{BC}$'s in (2.14) are all zero. Furthermore, because the group dimension is $n = 1$, the algebraic indices are usually omitted. Equation (2.14) gives then the usual expression of the Maxwell tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.18}$$

The observable, measurable field is $F_{\mu\nu}$, though the Aharonov-Bohm effect shows that some effects of A_λ can be eventually measurable at the quantum level. The potential A_μ is, however, the fundamental field: the photon is the quantum of field A_μ. Electromagnetism has been the historical prototype of a gauge theory, even though its simplicity left many aspects unnoticed. It has, for example, inspired the minimal coupling prescription (2.8) — though in its abelian, 1-dimensional simplicity T_B is single and can be taken as the unity.

From (2.14) it follows the identity

$$D_\rho F^A{}_{\mu\nu} + D_\nu F^A{}_{\rho\mu} + D_\mu F^A{}_{\nu\rho} = 0, \tag{2.19}$$

the indices being cyclically exchanged from term to term. This *Bianchi identity* generalizes to the non-abelian case the first pair of Maxwell's equations. Recall that those equations do not follow from the electromagnetic lagrangian, and in this sense are not dynamical.

The dynamical equations, which are those that generalize the second pair of Maxwell's equations, follow from the gauge lagrangian

$$\mathcal{L} = -\tfrac{1}{4}\, \gamma_{AB}\, F^A{}_{\mu\nu} F^{B\mu\nu}, \tag{2.20}$$

where

$$\gamma_{AB} = \mathrm{tr}(J_A J_B) = f^C{}_{AD} f^D{}_{BC} \tag{2.21}$$

is the Killing-Cartan metric, which is used to raise and lower internal indices. Since this metric can be defined only for semisimple groups (those with no invariant abelian subgroup), lagrangians for gauge theories can be constructed only for these groups [28, 29]. For non-semisimple groups, such as the Poincaré group, the Killing-Cartan bilinear form γ_{AB} is not a metric — it is not invertible.

Comment 2.3 It should be remarked that some abelian groups admit the construction of a gauge lagrangian. The most prominent example is electromagnetism, a gauge theory for the abelian $U(1)$ group. Its gauge lagrangian is constructed using, not the Killing-Cartan metric (which is degenerate), but a different gauge–invariant metric. Another example is Teleparallel Gravity, which corresponds to a gauge theory for the abelian translation group. In this case, the Minkowski metric replaces the Killing-Cartan metric. A more detailed discussion of this point will be presented at Chapter 7.

Let us then consider the lagrangian

$$\mathcal{L} = \mathcal{L}_s[\psi, D_\mu \psi] - \tfrac{1}{4}\, F^A{}_{\mu\nu} F_A{}^{\mu\nu}, \tag{2.22}$$

where the source lagrangian $\mathcal{L}_s[\psi, D_\mu \psi]$ is obtained from the free lagrangian for the source multiplet field $\psi = \{\psi_j\}$ by the *minimal coupling prescription*: usual derivatives ∂_μ are replaced by the covariant derivatives D_μ given in (2.8). Making use of the cyclic property

$$f_{ABC} = f_{CAB} = f_{BCA}, \tag{2.23}$$

characteristic of semisimple groups, the field equation that follows from the lagrangian (2.22) is

$$\partial_\mu F^{A\mu\nu} + f^A{}_{BC} A^B{}_\mu F^{C\mu\nu} = J^{A\nu}, \tag{2.24}$$

where

$$J^{A\nu} = -\ \frac{\partial \mathcal{L}_s}{\partial A_{A\nu}} \tag{2.25}$$

is the source current. This is the equation that generalizes the second, dynamical pair of Maxwell's equations

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{2.26}$$

Notice that knowledge of the group structure constants is enough to write down the Yang-Mills equations (2.24), which can also be written in the form

$$D_\mu F^{A\mu\nu} = J^{A\nu}. \tag{2.27}$$

The non-abelian character brings forward non-linearity: the gauge field interacts with itself. The abelian photon does not carry electromagnetic charge, but the gluon of strong interactions described by Chromodynamics carries the strong $SU(3)_{\text{color}}$ charge. The second term in the left-hand side of the field equation (2.24) is just the "self–current" $j^{A\nu}$, the current carried by the gauge fields themselves:

$$j^{A\nu} = -\, f^{A}{}_{BC}\, A^{B}{}_{\mu}\, F^{C\mu\nu}. \tag{2.28}$$

From Noether's theorem it can be verified that it is covariantly conserved:

$$D_\mu J^{A\nu} = 0. \tag{2.29}$$

Actually, this vanishing covariant divergence leads to no real conservation law — it is only a constraint on the source currents, usually called Noether identity. On the other hand, the field equation does lead to a true conservation: due to the anti-symmetry of $F^{A\mu\nu}$ in the spacetime indices, we see that

$$\partial_\nu \left(J^{A\nu} + j^{A\nu} \right) = 0. \tag{2.30}$$

This equation says that the *total* current — source plus gauge field — is conserved. Notice, however, that the self–current $j^{A\nu}$ is not, by itself, covariant under gauge transformations.

Comment 2.4 It is important to remark that this is a matter of consistency: since the derivative is not covariant, the conserved current cannot be covariant either in such a way that the conservation law as a whole is covariant, and is consequently physically meaningful. The gravitational version of this conservation law will be discussed in Chapter 8.

Recall from electromagnetism that, in the presence of an electromagnetic field, the Minkowski space equation of motion of a test particle of mass m and electric charge q is described by the Lorentz force law

$$\frac{du^\lambda}{ds} = \frac{q}{mc^2}\, F^{\lambda}{}_{\nu} u^\nu. \tag{2.31}$$

In this case the particle current $J^\nu = q u^\nu$, which is 1-dimensional in the internal space, obeys

$$\frac{dJ^\lambda}{ds} = \frac{q^2}{mc^2}\, F^{\lambda}{}_{\nu}\, J^\nu. \tag{2.32}$$

Generalization to a particle with a gauge charge (say, isospin) q^A leads to

$$\frac{du^\lambda}{ds} = \frac{q^A}{mc^2}\, F^{A\lambda}{}_{\nu} u^\nu. \tag{2.33}$$

The charge itself will obey Wong's equation [30]

$$D_u q^A \equiv \frac{dq^A}{ds} + A^{B}{}_{\nu}\, u^\nu\, (T_B)^{A}{}_{C}\, q^C = 0. \tag{2.34}$$

This describes an "internal precession": it leads to $q_A D_u q^A = 0$, so that

$$\mathbf{q}^2 = \gamma_{AB}\, q^A q^B$$

is a characteristic invariant of the representation which is kept (covariantly) constant.

Gauge theories present a special, fundamental property: the dynamic equation (2.24) is, in the sourceless case, just the geometrical identity (2.19) written for the dual

$$\star\, F^A{}_{\mu\nu} = \tfrac{1}{2}\, \epsilon_{\mu\nu\rho\sigma} F^{A\rho\sigma}, \tag{2.35}$$

here taken on Minkowski spacetime. In this sense, pure–gauge–field dynamics is dual to pure geometry. This *duality symmetry* of the gauge field is very important for its quantization — it is, together with conformal symmetry, one of the attributes which make gauge theories *renormalizable*. Renormalizability is, in almost all cases, checked in perturbation theory, the perturbation parameter being the coupling constant. The present text uses a common practice of field theory, hiding the coupling constant in the potential $A^C{}_\mu$. To obtain expressions with the explicit coupling constant g, just replace each $A^C{}_\mu$ by $g\, A^C{}_\mu$ [31]. Another important merit of gauge theories is that their coupling constants are dimensionless (the fine-structure constant $\alpha = q^2/\hbar c$ is the coupling constant of the electromagnetic interaction). This means that, order by order, they multiply (Feynman) integrals of the same dimension. In gravitation as described by General Relativity, the coupling constant is $8\pi G/c^4$, with dimension $M^{-1}L^{-1}T^2$. If we try to quantize it, the Feynman integrals have, at each other, to compensate for these dimensions. It turns out that they become more and more divergent [32].

Comment 2.5 The origin of this difference is that generic Noether currents have dimension MT^{-2}. The energy-momentum density, the source current for gravitation, has "abnormal" dimension $ML^{-1}T^{-2}$. The reason for this difference is that the energy–momentum current is related to translations on spacetime and, unlike most transformations, whose generators are dimensionless, the translation generator has dimension L^{-1}.

We have said that elementary particles appear in field theory as the quanta of the fundamental fields. Relativistic fields are actually sets of infinitely many degrees of freedom, one for each point of spacetime. Usually in modern phenomenology, particles come before: they are first detected in Nature, and then a field is attributed to each of them. When a field is found beforehand (as the electromagnetic field), a particle is identified to its quantum — provided the theory in quantizable. Models are built up by attributing particles to multiplets, and then collecting such multiplets into fields. These are, roughly speaking, the *source fields*. Interactions are then ascribed to *mediating fields* — here, just the gauge potentials. To the mediating fields are attributed new particles, the gauge bosons: photons for electromagnetism, gluons for the strong interactions regulated by Chromodynamics, the Z^0 and the $W^{(\pm)}$ for the weak sector of the electroweak interactions ruled by the group $SU(2) \otimes U(1)$. Non-renormalizability of a theory (as seems to be the case of General Relativity) jeopardises its quantization — which becomes meaningless. We do speak of a "graviton" but, as long as General Relativity remains non-renormalizable, this is a mere convenience of language.

Comment 2.6 Gauge fields (or the particles they represent) are massless; there is no mass term in lagrangian (2.20). Actually, a mass term would violate gauge symmetry. Nevertheless, the afore mentioned mediating bosons Z^0 and $W^{(\pm)}$ have masses, actually large masses as far as elementary particles are concerned. Such masses are theoretically obtained by the only known process which breaks the gauge symmetry while preserving renormalizability, the so–called "spontaneous symmetry breaking" (a name stemming from its original inspiration in superconductivity): a complex scalar field ϕ, which interacts with itself via a $\lambda\phi^4$ potential, is added to the lagrangian. The ensuing Hamiltonian exhibits a minimum value ("vacuum") which stands below zero, and corresponds to a state which is degenerate. In other words, this vacuum state is multiple — one can pass from one to another state of minimum energy by a gauge transformation. It is necessary to choose one of these states as a fundamental state in order to build the higher energy states. This choice breaks the symmetry and induces a change of field variables. The original $SU(2) \otimes U(1)$ gauge fields are no more the physical fields. The two components of the added complex field compose with the original gauge potentials to produce the (zero mass) photon, the Z^0 field, the $W^{(\pm)}$ bosons and a residual "Higgs boson". With this special mechanism, the gauge symmetry is preserved as a "hidden" symmetry: it remains behind the scene, but holds for non-physical fields. Thus the massive, physically observed mediating fields come from a redefinition of the degrees of freedom.

2.2 General Relativity

Consider the set of all linear bases exchanged by the linear group $GL(4, \mathbb{R})$, or the set of tetrad frames exchanged by its Lorentz subgroup. Take a particular point p on the manifold $\mathbf{R}$ and choose one particular base on the vector space $T_p\mathbf{R}$ to start with. Change then to any other: every other base can be attained by a transformation which is a group member; and to each group element will correspond one base obtained from the initial one. In consequence, the bundle of bases is the same as the bundle with one copy of the group at each point p — it is a principal bundle. It is one of the many miracles of tetrads that, being invertible, they determine the Lorentz transformation relating them [see Eq.(1.74)].

The principal bundle of frames puts all the geometry of General Relativity in a nutshell. A diagram is given in Figure 2.2. The tangent bundle, with a tangent space $T_p\mathbf{R}$ attached to each point p of the riemannian spacetime $\mathbf{R}$, is only one of its associates. Tensor bundles, and spinor bundles, are others. In

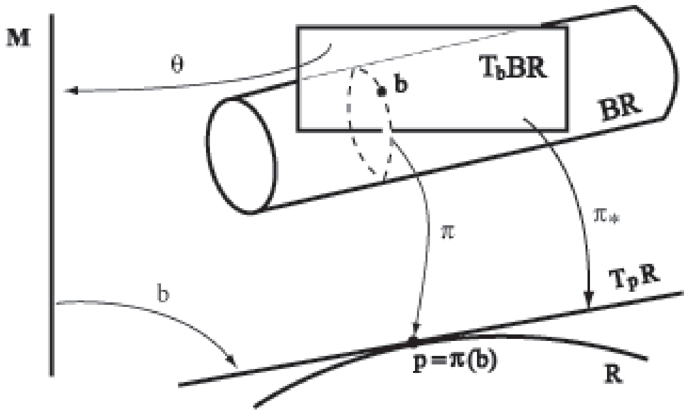

Figure 2.2: Diagram of the frame bundle.

the bundle **BR** of bases on **R**, the whole set of frames on $T_p\mathbf{R}$ is "attached" to point $p \in \mathbf{R}$. Its more intimate relation to the spacetime differentiable manifold makes it different from the corresponding gauge principal bundle pictured in Figure 2.1. As repeatedly said, the main difference is the presence of soldering — with the ensuing appearance of tetrads and torsion.

Comment 2.7 The parallelizable manifolds mentioned in Comment 1.2 have a simple definition in terms of bundles: their tangent bundle is trivial, that is, globally a direct product of the typical tangent space (say, Minkowski space) by the base manifold (say, a riemannian spacetime). This is a special case of a very general property: a bundle is a global direct product (fiber $\times$ base) *iff* there exists a global section.

Consider a particular base b on $T_p\mathbf{R}$: it will be a point on **BR**. This base, and all its companions obtained from it by a base transformation, are taken into point p by the bundle projection π. The solder 1-form θ relates each tangent space $T_b\mathbf{BR}$ *of the bundle of frames* to the Minkowski space **M** and — this is the main property — has, seen from each tetrad, just the components of that same tetrad. Form θ acts in a circuitous way: the mapping b — so called because it just represents the homonimous base b — in the diagram, which is a vector-space isomorphism, takes **M** into $T_p\mathbf{R}$ *and makes of* $T_p\mathbf{R}$ *a Minkowski space.* The torsion tensor of a given connection is just the covariant derivative of the solder 1-form or, due to the mentioned property, the covariant derivative of the tetrad field [see Eq. (1.38)]. Torsion is simply non-existent in internal (non-soldered) gauge theories.

General Relativity conceives the gravitational interaction — which it describes with paramount success at the classical level — as a change in the geometry of spacetime itself [33]. Specifically, as a change from the Lorentz metric η_{ab} of Minkowski space into a riemannian metric $g_{\mu\nu}$. This new metric plays the role of basic field, and is in principe defined everywhere. Derivatives compatible with this overall presence of the same metric must preserve it, must parallel–transport it everywhere [see Section 1.7]. Of all the connections preserving $g_{\mu\nu}$, there is only one which has vanishing torsion, the Christoffel or Levi-Civita connection $\overset{\circ}{\Gamma}{}^{\lambda}{}_{\mu\nu}$. This torsionless connection was chosen by Einstein to define all covariant derivatives, and its curvature, the Riemann tensor

$$\overset{\circ}{R}{}^{a}{}_{b\mu\nu} = \partial_\mu \overset{\circ}{A}{}^{a}{}_{b\nu} - \partial_\nu \overset{\circ}{A}{}^{a}{}_{b\mu} + \overset{\circ}{A}{}^{a}{}_{c\mu} \overset{\circ}{A}{}^{c}{}_{b\nu} - \overset{\circ}{A}{}^{a}{}_{c\nu} \overset{\circ}{A}{}^{c}{}_{b\mu} \,, \tag{2.36}$$

represents the field: gravitation is present whenever at least one component of $\overset{\circ}{R}{}^{a}{}_{b\mu\nu}$ is non-vanishing. The Ricci tensor is a symmetric second-order tensor defined as

$$\overset{\circ}{R}_{\mu\nu} = h_a{}^{\rho} h^b{}_{\mu} \overset{\circ}{R}{}^{a}{}_{b\rho\nu}, \tag{2.37}$$

and the scalar curvature is

$$\overset{\circ}{R} = g^{\mu\nu} \overset{\circ}{R}_{\mu\nu}. \tag{2.38}$$

It turns out that the Einstein tensor

$$G^{\mu\nu} \equiv \overset{\circ}{R}{}^{\mu\nu} - \tfrac{1}{2}\, \overset{\circ}{R}\, g^{\mu\nu} \tag{2.39}$$

is the only symmetric second-order tensor with vanishing covariant derivative. This result comes, actually, from the second Bianchi identity (1.68) with convenient contractions. Concerning the source fields, the energy-momentum tensor $\Theta^{\mu\nu}$ is the only symmetric second-order tensor with vanishing covariant derivative — as determmined by Noether's theorem. The source, in Newtonian gravitation, is the mass, whose concept is broadened into energy by Special Relativity. Energy, which in field theory is represented by the energy-momentum tensor, is to be the source of gravitation. It is thus natural to write

$$\overset{\circ}{R}{}^{\mu\nu} - \tfrac{1}{2}\, \overset{\circ}{R}\, g^{\mu\nu} = k\, \Theta^{\mu\nu}, \tag{2.40}$$

where k is some constant. Making the correspondence with Newton's law in the static weak-field limit, one determines $k = 8\pi G/c^4$ and the field equation, Einstein's equation, comes out as

$$\overset{\circ}{R}{}^{\mu\nu} - \tfrac{1}{2}\, \overset{\circ}{R}\, g^{\mu\nu} = \frac{8\pi G}{c^4}\, \Theta^{\mu\nu}. \tag{2.41}$$

This equation can be obtained from the lagrangian

$$\mathcal{L} = \overset{\circ}{\mathcal{L}} + \mathcal{L}_m, \tag{2.42}$$

where

$$\overset{\circ}{\mathcal{L}} = \frac{c^4}{16\pi G}\sqrt{-g}\,\overset{\circ}{R} \tag{2.43}$$

is the Einstein–Hilbert lagrangian, and $\mathcal{L}_m$ is the matter, or source field lagrangian. The right-hand side of Einstein equation (2.40) is the symmetric energy-momentum tensor

$$\Theta^{\mu\nu} = -\frac{1}{2\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}}. \tag{2.44}$$

It is actually the energy-momentum tensor modified by the presence of gravitation.

How does gravitation couple to other fields? Let us repeat that the metric $g_{\mu\nu}$ is related to the flat Lorentz tangent metric through the tetrad fields,

$$g_{\mu\nu} = \eta_{ab}\, h^a{}_\mu h^b{}_\nu. \tag{2.45}$$

Suppose for a moment that the $h^a{}_\mu$'s are trivial four-legs, mere coordinate changes. We can calculate the corresponding Christoffel symbol and the curvature. We find then that $\overset{\circ}{R}{}^a{}_{b\mu\nu} = 0$. This is a matter of course, as trivial tetrads will only lead to other representations, in terms of non-cartesian coordinates, of the flat Lorentz metric. We pass from η_{ab} to a different, riemannian metric only through a non-trivial tetrad field. This leads to an intuitive view of the *equivalence principle*. We have seen in Section 1.7 that non-trivial tetrads are actually accelerated frames. The equivalence principle will say that a gravitational field is, in some sense [34], locally equivalent to an accelerated frame. The presence of tetrads enforces also the passage of simple derivatives to covariant derivatives, so that usual derivatives are replaced by covariant ones. A rule turns up, which is reminiscent of the gauge prescription: to obtain the effect of gravitation on sources in general (particles or fields), (i) write all the usual equations they obey in Minkowski space in general coordinates, represented by trivial tetrads, and (ii) keep the same formulae, but with the trivial tetrads replaced by general tetrads, related to the metric by Eq. (2.45). The resulting equations will hold in General Relativity. Notice however that, as tetrad fields are only locally defined, gravitation is *only locally* equivalent to an accelerated frame.

The simplest case is the usual geodesic equation

$$\frac{du^\lambda}{ds} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu} u^\mu u^\nu = 0, \tag{2.46}$$

which describes the motion of a structureless, point-like test particle in the presence of a gravitational field. It says that $\overset{\circ}{a}{}^\lambda = 0$, which means that in

General Relativity there is no the concept of gravitational force. In this theory, the gravitational interaction is geometrized: the presence of gravitation produces a curvature in spacetime, and the gravitational interaction is described by letting the particles to follow the spacetime curvature. Any other interaction would contribute with a force term to the right-hand side of the geodesic equation.

For example, the motion of a test particle of mass m, electric charge q and four-velocity u^λ in the presence of both an electromagnetic and a gravitational field is described by a Lorentz force law which generalizes (2.31):

$$\frac{du^\lambda}{ds} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\rho}\, u^\mu u^\rho = \frac{q}{mc^2}\, F^\lambda{}_\nu u^\nu. \tag{2.47}$$

The different roles played by the two kinds of interaction would be summarized in the generalized Lorentz force equation for a particle with a gauge charge q^A, generalizing (2.33):

$$\frac{du^\lambda}{ds} + \overset{\circ}{\Gamma}{}^\lambda{}_{\mu\rho}\, u^\mu u^\rho = \frac{q_A}{mc^2}\, F^{A\lambda}{}_\nu\, u^\nu. \tag{2.48}$$

While the gauge interaction engenders the force in the right-hand side, gravitation is a left-hand–side geometric effect.

2.3 Gravitation and the Gauge Paradigm

General Relativity is not a gauge theory. It differs from gauge theories in many ways. The most relevant points are the following.

- The basic field of gauge theories is a connection, the gauge potential. In General Relativity, it is the metric, with respect to which variations are taken.

- There is a connection in General Relativity, but it is not a fundamental field: given a metric, the Levi-Civita connection is immediately known. Palatini approach is an attempt to circumvent this problem, but does not solve it.

- The gauge lagrangians are quadratic in the curvature. In General Relativity the Einstein-Hilbert lagrangian is linear in the curvature.

- Gauge interactions always appear as a force, while in General Relativity gravitation appears as a geometric effect.

- General Relativity does not have a gauge group. Sometimes, diffeomorphism is considered as the gauge group of gravitation. However, as is well known, any theory can be written in a covariant form, which means that diffeomorphism is empty of dynamical meaning. Furthermore, it takes place in spacetime, not in in the fiber.

In order to explore the possibility of describing gravitation as a gauge theory, let us put the gauge paradigm to work. To get some insight on how to proceed, let us first review the well known case of electromagnetism, a gauge theory for the $U(1)$ group. The source of the electromagnetic field is the electric four-current. According to Noether theorem [40], this current is conserved due to invariance of the source lagrangian under *global* transformations of the group $U(1)$. In order to recover this symmetry for the case of *local* transformations of the same group, it is necessary to introduce a connection assuming values in the Lie algebra of the $U(1)$ group. This connection represents the electromagnetic field, which emerges as a gauge theory for the unitary group $U(1)$.

Let us then play the same game for the gravitational case. As is well known, the source of gravitation is the energy-momentum tensor. From Noether theorem, this tensor is conserved provided the source lagrangian is invariant under spacetime translations. If gravity has a gauge description, therefore, it might be a gauge theory for the translation group. This theory is just the Teleparallel Equivalent of General Relativity — or Teleparallel Gravity — and is the theory that we will study in this book.

Chapter 3

Fundamentals of Teleparallel Gravity

The foundations of Teleparallel Gravity, a gauge theory for the translation group, are presented. Gauge transformations are introduced, and the coupling of matter fields to gravitation discussed. The field strength of the theory is shown to be the torsion tensor.

3.1 Geometrical Setting

The geometrical setting of Teleparallel Gravity — a gauge theory for the translation group — is the tangent bundle: at each point p of coordinates x^μ of a general, riemannian spacetime $\mathbf{R}$ — the base space — there is "attached" a Minkowski tangent-space $\mathbf{M} = T_p\mathbf{R} = T_{x^\mu}\mathbf{R}$ — the fiber — on which the gauge transformations take place (see Fig. 3.1, in which the tangent space at $p = \{x^\mu\}$ is indicated normally). A gauge transformation will be a point-dependent translation of the $T_{x^\mu}\mathbf{R}$ coordinates x^a,

$$x'^a = x^a + \epsilon^a, \tag{3.1}$$

with $\epsilon^a \equiv \epsilon^a(x^\mu)$ the transformation parameters.

The generators of infinitesimal translations are here differential operators,

$$P_a = \frac{\partial}{\partial x^a} \equiv \partial_a, \tag{3.2}$$

which satisfy the commutation relations

$$[P_a, P_b] = 0. \tag{3.3}$$

The corresponding infinitesimal transformation can be written in the form

$$\delta x^a = \epsilon^b P_b \, x^a. \tag{3.4}$$

Let us insist on the fact that, due to the peculiar character of translations,

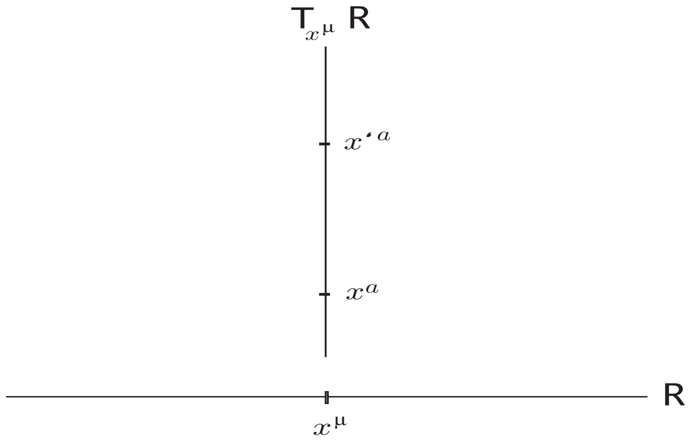

Figure 3.1: *Spacetime with the Minkowski tangent space at x^μ.*

any gauge theory including them will differ from the usual internal — Yang-Mills type — gauge models in many ways, the most significant being the presence of a tetrad field. The gauge bundle will always present the soldering property, and the internal and external sectors of the theory will be linked to each other. Teleparallelism will be necessarily a non-standard gauge theory.

3.2 Gauge Transformations of Source Fields

Let us consider now a general source field ψ. If V is an open set in spacetime, ψ is represented by a *local section* ψ_V of the fiber bundle (see Figure 2.2), which is given by a differentiable application of the form

$$\psi_V : V \to \pi^{-1}(V), \tag{3.5}$$

where π is the bundle projection from the fiber into spacetime [19]. Now, a fiber bundle is always locally trivial, that is, $\pi^{-1}(V)$ is always diffeomorphic to $V \times F$, with F the fiber on the gauge group acts:

$$\pi^{-1}(V) \sim V \times F. \tag{3.6}$$

This diffeomorphism, usually called a "local trivialization", is given by

$$f_V : \pi^{-1}(V) \to V \times F. \tag{3.7}$$

For a fiber of dimension d, a local section ψ_V is an application

$$x^\mu \to f_V^{-1}(x^\mu, x^A), \tag{3.8}$$

where the coordinate set $\{x^\mu\}$ indicates a point in spacetime and $\{x^A(x^\mu)\}$ indicates a point in the fiber over x^μ, with $A = 1, \ldots, d$. Therefore, a general source field ψ, defined as a section of the bundle, must depend on both coordinates x^μ and x^A:

$$\psi = \psi(x^\mu, x^A). \tag{3.9}$$

In the case of *internal*, usual Yang–Mills gauge theories, the source field ψ has a discrete set of components, it is a vector (a multiplet) in a fiber which is the carrier space of the representation to which it belongs (see Section 2.1)). For quantum–mechanical reasons, these representations must be unitary, and only compact groups have finite unitary representations. Thus, the fiber is a vector space of finite dimension, source fields are finite multiplets, whose internal coordinates are the components. These components are distorted by a multi–component phase α^A by a gauge transformation as in Eq. (2.3).

For the case of the non-compact translation group, unitarity is anyhow jeopardized. Each fiber is a copy of the whole Minkowski spacetime. The continuum of coordinates, now indicated $x^a(x^\mu)$, takes on the role of component indices in the above multiplets. The dependence of ψ on $x^a(x^\mu)$ is written simply as

$$\psi = \psi(x^a(x^\mu)). \tag{3.10}$$

Under an infinitesimal tangent space translation, it transforms according to

$$\delta\psi(x^a(x^\mu)) = -\,\epsilon^a \partial_a \psi(x^a(x^\mu)). \tag{3.11}$$

It gives the functional change of ψ at a fixed x^a and, of course, at fixed spacetime point x^μ — the typical transformation of gauge theories.

3.3 On the Coupling Prescriptions

In ordinary gauge theories, the coupling prescription amounts to replace ordinary derivatives by covariant derivatives involving a connection. As an example, let us consider the case of electromagnetism, a gauge theory for the unitary group $U(1)$. Under an infinitesimal gauge transformation with parameter $\alpha = \alpha(x)$, a (let us say) spinor field ψ changes according to

$$\delta\psi = i\alpha\psi. \tag{3.12}$$

Its ordinary derivative, however, does not transform covariantly:

$$\delta(\partial_\mu \psi) = i\alpha(\partial_\mu \psi) + i(\partial_\mu \alpha)\psi. \tag{3.13}$$

In order to recover the covariance it is necessary to introduce a gauge potential A_μ, which is a connection taking values at the Lie algebra of the gauge group $U(1)$. It is then easy to see that

$$D_\mu \psi = \partial_\mu \psi + iA_\mu \psi \tag{3.14}$$

transforms covariantly,

$$\delta(D_\mu \psi) = i\alpha(D_\mu \psi), \tag{3.15}$$

provided the gauge potential transforms according to

$$\delta A_\mu = -\partial_\mu \alpha. \tag{3.16}$$

The ensuing coupling prescription is consequently

$$\partial_\mu \psi \to D_\mu \psi. \tag{3.17}$$

Now, due to the fact that gravitation is not a background-independent theory, the gravitational coupling prescription has two distinct parts. The first is the replacement of the Minkowski metric $\eta_{\mu\nu}$ by a general pseudo-riemannian metric $g_{\mu\nu}$ representing a gravitational field:

$$\eta_{\mu\nu} \to g_{\mu\nu}. \tag{3.18}$$

This part is universal in the sense that it affects equally all matter fields. As we are going to see, it follows naturally from the requirement of covariance under spacetime translations. The second part is related to the coupling of the *spins* of matter fields to gravitation, and is related to the requirement of covariance under Lorentz transformations. It appears in the form

$$\partial_\mu \to \mathcal{D}_\mu, \tag{3.19}$$

with $\mathcal{D}_\mu$ a Lorentz covariant derivative, still to be determined. Of course, this part of the coupling is not universal in the sense that it depends on the spin content of each field. Let us then explore separately these two coupling prescriptions.

3.4 Translational Coupling Prescription

3.4.1 Translational Gauge Potential

Let us consider a general source field $\psi = \psi(x^a(x^\mu))$. As we have seen, under an infinitesimal gauge translation it transforms according to

$$\delta\psi = \epsilon^a \partial_a \psi. \tag{3.20}$$

Like in the electromagnetic case discussed in the previous section, its ordinary derivative does not transform covariantly:

$$\delta(\partial_\mu \psi) = \epsilon^a \partial_a (\partial_\mu \psi) + (\partial_\mu \epsilon^a) \partial_a \psi. \tag{3.21}$$

In order to recover the covariance, it is necessary to introduce a gauge potential B_μ, a 1-form assuming values in the Lie algebra of the translation group:

$$B_\mu = B^a{}_\mu \, P_a \,. \tag{3.22}$$

In fact, it is easy to verify that

$$h_\mu \psi = \partial_\mu \psi + B^a{}_\mu \, \partial_a \psi \tag{3.23}$$

transforms covariantly,

$$\delta(h_\mu \psi) = \epsilon^a \partial_a (h_\mu \psi), \tag{3.24}$$

provided the gauge potential transforms according to

$$\delta B^a{}_\mu = -\partial_\mu \epsilon^a. \tag{3.25}$$

The translational coupling prescription is then given by

$$\partial_\mu \psi \;\; \rightarrow \;\; h_\mu \psi = \partial_\mu \psi + B_\mu \psi. \tag{3.26}$$

Comment 3.1 The covariant derivative (3.23) can also be obtained from the general definition of covariant derivative [19]

$$h_\mu = \partial_\mu + B^a{}_\mu \, \frac{\delta}{\delta \epsilon^a}, \tag{3.27}$$

where

$$\frac{\delta}{\delta \epsilon^a} = \frac{\partial}{\partial \epsilon^a} - \partial_\rho \, \frac{\partial}{\partial(\partial_\rho \epsilon^a)} + \cdots \tag{3.28}$$

is the Lagrange derivative with respect to the parameter ϵ^a. Using the transformation (3.20), the covariant derivative of ψ is found to be

$$h_\mu \psi = (\partial_\mu + B^a{}_\mu \, \partial_a) \psi, \tag{3.29}$$

which is the same as (3.23).

3.4.2 Implications for the Metric

The translational covariant derivative (3.23) can be rewritten in the form

$$h_\mu \psi = (\partial_\mu x^a + B^a{}_\mu)\partial_a \psi. \tag{3.30}$$

Defining the tetrad field by

$$h^a{}_\mu = \partial_\mu x^a + B^a{}_\mu, \tag{3.31}$$

it can be rewritten as

$$h_\mu \psi = h^a{}_\mu \, \partial_a \psi. \tag{3.32}$$

Notice that the tetrad is gauge invariant, as can be easily seen by using the gauge transformations $\delta x^a = \epsilon^a$ and $\delta B^a{}_\mu = -\,\partial_\mu \epsilon^a$.

Comment 3.2 It is important to remark that, since the generators $P_a = \partial_a$ are derivatives which act on matter fields $\psi(x^a(x^\mu))$ through their tangent–space arguments x^a, which are the same for all, every source field in Nature will respond equally to their action, and consequently will couple equally to the translational gauge potentials. All of them, therefore, will feel gravitation the same. This is the origin of the concept of *universality* according to Teleparallel Gravity.

When the translational covariant derivative is written in the form (3.32), the translational coupling prescription acquires a very simple form. In fact, if we write the ordinary derivative $\partial_\mu \psi$ as

$$\partial_\mu \psi = e^a{}_\mu \partial_a \psi, \tag{3.33}$$

with $e^a{}_\mu = \partial_\mu x^a$, the translational coupling prescription assumes the form

$$e^a{}_\mu \, \partial_a \psi \;\to\; h^a{}_\mu \, \partial_a \psi. \tag{3.34}$$

That is to say,

$$e^a{}_\mu \;\to\; h^a{}_\mu. \tag{3.35}$$

Concomitantly with this replacement, the spacetime metric changes according to

$$\eta_{\mu\nu} = \eta_{ab} \, e^a{}_\mu e^b{}_\nu \;\to\; g_{\mu\nu} = \eta_{ab} \, h^a{}_\mu h^b{}_\nu. \tag{3.36}$$

The change in spacetime metric is, therefore, a direct consequence of the translational coupling prescription.

3.4.3 Translational Coupling in a General Frame

Up to now, we have used a class of Lorentz frames where no inertial effects are present. The equivalent expressions valid in a general Lorentz frame can be obtained by performing a local Lorentz transformation $x^a \to \Lambda_b{}^a \, x^b$, under which

$$\psi \;\to\; U(\Lambda)\,\psi, \tag{3.37}$$

with $U(\Lambda)$ an element of the Lorentz group in the representation appropriate for the source field ψ. Considering that $B^a{}_\mu \to \Lambda_b{}^a \, B^b{}_\mu$, it is then immediate to see that the translational covariant derivative (3.32) transforms covariantly,

$$h_\mu \psi \;\to\; U(\Lambda)\, h_\mu \psi, \tag{3.38}$$

where now

$$h^a{}_\mu = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} \, x^b + B^a{}_\mu, \tag{3.39}$$

with

$$\overset{\bullet}{A}{}^b{}_{c\mu} = \Lambda^b{}_d \, \partial_\mu \Lambda_c{}^d \tag{3.40}$$

a purely inertial Lorentz connection. In this class of frames, therefore, the translational coupling prescription assumes the form

$$e^a{}_\mu \partial_a \, \psi \;\to\; h^a{}_\mu \, \partial_a = (\partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} \, x^b + B^a{}_\mu)\, \partial_a \psi. \tag{3.41}$$

Introducing the Lorentz covariant derivative

$$\overset{\bullet}{\mathcal{D}}_\mu x^a = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} \, x^b, \tag{3.42}$$

the tetrad becomes

$$h^a{}_\mu = \overset{\bullet}{\mathcal{D}}_\mu x^a + B^a{}_\mu. \tag{3.43}$$

In this new class of frames, the gauge potential $B^a{}_\mu$ transforms according to

$$\delta B^a{}_\mu = -\,\overset{\bullet}{\mathcal{D}}_\mu \epsilon^a \tag{3.44}$$

under a gauge translation $\delta x^a = \epsilon^a$. We see in this way that the tetrad remains gauge invariant.

3.5 Spin Coupling Prescription

3.5.1 General Covariance Principle

Let us now obtain the spin part of the gravitational coupling prescription. This coupling prescription can be obtained from the so-called *principle of* *general covariance* [35]. This principle states that an equation valid in Special Relativity can be made to hold in the presence of gravitation if it is generally covariant, that is, if it preserves its form under general coordinate transformations. Now, in order to make an equation generally covariant, it is always necessary to introduce a connection, which is in principle concerned only with the *inertial* properties of the coordinate system under consideration. As long as only coordinate transformations are involved, just a vacuum connection (with zero curvature and torsion) is needed. Then, by using the equivalence between inertial and gravitational effects, that connection is replaced by a connection representing a *true gravitational field*. The crucial point is a formal property: the equations have the same forms for vacuum and non-vacuum connections. In this way, equations valid in the presence of gravitation are obtained from the corresponding equations holding in Special Relativity.

The principle of general covariance can be seen as an *active* version of the equivalence principle in the sense that, by making a special-relativistic equation covariant and using the strong equivalence principle, it is possible to obtain its form in the presence of gravitation. The usual form of the equivalence principle, on the other hand, can be interpreted as its *passive* version: the special-relativistic equation must be recovered in a locally inertial frame. It should be emphasized that general covariance by itself is empty of physical content, as any equation can be *made* generally covariant. Only when use is made of the local equivalence between inertial and gravitational effects, and the compensating term is re-interpreted as representing a true gravitational field, can the principle of general covariance be seen as an active version of the strong equivalence principle [36].

The above description of the general covariance principle refers to its usual *holonomic* version. An alternative, more general version of the principle can be obtained by using non-holonomic frames. The basic difference between these two versions is that, instead of requiring that an equation be covariant under a general coordinate transformation, in the anholonomic-frame version the equation is required to be covariant under a *local* Lorentz transformation of the frame. In spite of the different nature of the involved transformations, the physical content of both approaches is the same [37]. The frame version is, however, more general: unlike the coordinate version, it holds for integer as well as for half-integer spin fields.

An important point of the general covariance principle is that it defines in a natural way a Lorentz-covariant derivative, and consequently also a gravitational coupling prescription. The process of obtaining this coupling prescription comprises then two steps. The first is to pass to a general anholonomic frame, where inertial effects — which appear in the form of a compensating term, or vacuum Lorentz connection — are present. Then, by using the strong equivalence principle, instead of inertial effects, the compensating term can be replaced by a connection representing a true gravitational field, yielding in this way a gravitational coupling prescription.

3.5.2 Passage to an anholonomic frame

The first step to obtain the spin coupling prescription is to pass to a general anholonomic frame. Let us then consider a vector field v^c on Minkowski spacetime. Its ordinary derivative in frame is

$$\partial_a v^c = \delta_a^\mu \, \partial_\mu v^c. \tag{3.45}$$

In this expression,

$$\partial_a = \delta_a^\mu \, \partial_\mu \tag{3.46}$$

represents a trivial (holonomous) frame, with components δ_a^μ. Under a local Lorentz transformation $\Lambda^d{}_c(x) \equiv \Lambda^d{}_c$, a vector field transforms according to

$$v^c = \Lambda_d{}^c \, v^d. \tag{3.47}$$

The original and the Lorentz–transformed derivatives are related by

$$\partial_a v^c = \Lambda^b{}_a \, \Lambda_d{}^c \, \mathcal{D}_b V^d, \tag{3.48}$$

where

$$\mathcal{D}_b V^d = h_b V^d + \Lambda^d{}_e \, h_b(\Lambda_g{}^e) \, V^g, \tag{3.49}$$

and

$$h_b = \Lambda_b{}^a \, \partial_a \tag{3.50}$$

is the transformed frame. Due to the locality of the Lorentz transformation, it is anholonomic:

$$[h_b, h_c] = f^a{}_{bc} \, h_a. \tag{3.51}$$

Comment 3.3 Notice that the connection appearing in the covariant derivative (3.49) is exactly the inertial connection (3.40):

$$\Lambda^d{}_e \, h_b(\Lambda_g{}^e) \equiv \stackrel{\bullet}{A}{}^d{}_{gb} = \stackrel{\bullet}{A}{}^d{}_{g\mu} h_g{}^\mu. \tag{3.52}$$

Making use of the orthogonality property of the tetrads, we see from Eq. (3.50) that the Lorentz group element can be written in the form

$$\Lambda_b{}^d = h_b{}^\rho \, \delta_\rho^d. \tag{3.53}$$

From this expression, it follows that

$$\Lambda^c{}_d \, h_a(\Lambda_b{}^d) = \tfrac{1}{2} \left(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba} \right). \tag{3.54}$$

Substituting in the covariant derivative (3.49), it becomes

$$\mathcal{D}_a V^c = h_a V^c + \tfrac{1}{2} \left(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba} \right) V^b. \tag{3.55}$$

The freedom to choose any tetrad $\{h_a\}$ as a moving frame on Minkowski spacetime introduces the compensating term $\frac{1}{2}(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba})$ in the derivative of the vector field. This term is, of course, concerned only with the inertial properties of that frame. In other words, it represents the inertial effects inherent to the chosen frame.

3.5.3 Identifying inertia with gravitation

Let us begin by rewriting relation (1.47) in the form

$$A^c{}_{ba} - A^c{}_{ab} = T^c{}_{ab} + f^c{}_{ab}, \tag{3.56}$$

where $A^c{}_{ba}$ is a general Lorentz connection, with $T^c{}_{ba}$ its torsion. Use of this equation for three different combination of indices gives

$$\tfrac{1}{2} \left(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ba} \right) = A^c{}_{ba} - K^c{}_{ba}, \tag{3.57}$$

where

$$K^c{}_{ba} = \tfrac{1}{2} \left(T_b{}^c{}_a + T_a{}^c{}_b - T^c{}_{ba} \right) \tag{3.58}$$

is the contortion tensor in the tetrad frame. Considering that the covariant derivative (3.55) is written in Minkowski spacetime, the coefficient of anholonomy $f^c{}_{ba}$ represents inertial effects only. The left-hand side of expression (3.57) represents, in this case, inertial effects present in the frame. The right-hand side represents a gravitational field which is locally equivalent to those inertial effects.

Comment 3.4 It is important to remark that, in the presence of gravitation the coefficient of anholonomy $f^c{}_{ba}$ represents both inertia and gravitation. In Chapter 4 this point will be discussed in more details.

According to the general covariance principle, therefore, substituting (3.57) in the covariant derivative (3.55), which is a flat spacetime covariant derivative as seen in a general frame, we obtain the covariant derivative valid in the presence of gravitation:

$$\mathcal{D}_a V^c = h_a V^c + \left(A^c{}_{ba} - K^c{}_{ba} \right) V^b. \tag{3.59}$$

In terms of the vector representation

$$(S_{eb})^c{}_d = i\,(\delta^c_e\,\eta_{bd} - \delta^c_b\,\eta_{ed}) \tag{3.60}$$

of the Lorentz generators, it assumes the form

$$\mathcal{D}_a V^c = h_a V^c - \tfrac{i}{2}\left(A^{eb}{}_a - K^{eb}{}_a\right)(S_{eb})^c{}_d\, V^d. \tag{3.61}$$

Although obtained in the specific case of a Lorentz vector field, the compensating term (3.54) can be easily verified to be the same for any representation. In fact, considering a general source field ψ carrying an arbitrary representation of the Lorentz group, it Lorentz transformation will be

$$\psi' = U(\Lambda)\,\psi, \tag{3.62}$$

where

$$U(\Lambda) = \exp\left(\tfrac{i}{2}\epsilon_{bc}S^{bc}\right)$$

is the element of the Lorentz group in the arbitrary representation S^{bc}. As a simple calculation shows, also in this case we obtain that [37]

$$U(\Lambda)h_a U^{-1}(\Lambda) = \tfrac{i}{4}\left(f_{bca} + f_{acb} - f_{cba}\right)\, S^{bc}. \tag{3.63}$$

In this case, the covariant derivative (3.61) reads

$$\mathcal{D}_a\psi = h_a\psi - \tfrac{i}{2}\left(A^{bc}{}_a - K^{bc}{}_a\right) S_{bc}\,\psi. \tag{3.64}$$

3.6 Full Gravitational Coupling Prescription

The full gravitational coupling prescription is then composed of two parts: one, corresponding to the (universal) translational coupling prescription, which is represented by

$$e^a{}_\mu\,\partial_a\psi \;\rightarrow\; h^a{}_\mu\,\partial_a\psi, \tag{3.65}$$

and another, corresponding to the (non-universal) spin coupling prescription, represented by

$$\partial_a\psi \;\rightarrow\; \mathcal{D}_a\psi. \tag{3.66}$$

Put together, they yield the *full gravitational coupling prescription*,

$$e^a{}_\mu\,\partial_a\psi \;\rightarrow\; h^a{}_\mu\,\mathcal{D}_a\psi = h^a{}_\mu\left[h_a\psi - \tfrac{i}{2}\left(A^{bc}{}_a - K^{bc}{}_a\right) S_{bc}\,\psi\right]. \tag{3.67}$$

Equivalently, we can write

$$\partial_\mu\psi \;\rightarrow\; \mathcal{D}_\mu\psi = \partial_\mu\psi - \tfrac{i}{2}\left(A^{ab}{}_\mu - K^{ab}{}_\mu\right) S_{ab}\,\psi. \tag{3.68}$$

In this case, however, it is understood that, after the application of the coupling prescription, the spacetime indices $\mu, \nu, \rho \ldots$ are raised and lowered with the metric

$$g_{\mu\nu} = \eta_{ab}\, h^a{}_\mu h^b{}_\nu. \tag{3.69}$$

It is important to emphasize once more that this is the gravitational coupling prescription that follows from the general covariance principle, that is to say, from the strong equivalence principle. Any other form of the coupling prescription will be in contradiction with the equivalence principle.

3.7 Possible Connections

Then comes the crucial point: the general covariance principle does not determine uniquely the Lorentz connection $A^{bc}{}_\mu$. In fact, from the point of view of the coupling prescription, the connection can be chosen freely among the infinitely many possibilities, each one characterized by a connection with different values of curvature and torsion. Due to the identity

$$A^{bc}{}_\mu - K^{bc}{}_\mu = \overset{\circ}{A}{}^{bc}{}_\mu, \tag{3.70}$$

with $\overset{\circ}{A}{}^{bc}{}_\mu$ the (torsionless) spin connection of General Relativity, any one of the choices will give rise to a coupling prescription that is ultimately equivalent to the coupling prescription of General Relativity.

However, there is a strong constraint that must be taken into account: considering that the source of gravitation is the symmetric energy-momentum tensor, which has ten independent components, the gravitational field equations will be constituted by a set of ten independent differential equations. The choice of the connection, therefore, is restricted not to exceed ten independent components, otherwise the field equations will be unable to determine it univocally. As we are going to see, there are only two choices that respect the above constraint.

3.7.1 General Relativity Connection

The first possibility is to follow Einstein and choose the torsionless connection

$$A^{bc}{}_\mu = \overset{\circ}{A}{}^{bc}{}_\mu, \tag{3.71}$$

in which case the coupling prescription reads

$$e^a{}_\mu\, \partial_a \psi \;\rightarrow\; h^a{}_\mu\, \overset{\circ}{\mathcal{D}}_a \psi = h^a{}_\mu \left[h_a \psi - \tfrac{i}{2} \overset{\circ}{A}{}^{bc}{}_a\, S_{bc}\, \psi \right]. \tag{3.72}$$

Equivalently, we can write

$$\partial_\mu \psi \;\to\; \overset{\circ}{\mathcal{D}}_\mu \psi = \partial_\mu \psi - \tfrac{i}{2}\, \overset{\circ}{A}{}^{bc}{}_\mu\, S_{bc}\, \psi, \tag{3.73}$$

with the spacetime indices now raised and lowered with the spacetime metric $g_{\mu\nu}$. Since the connection $\overset{\circ}{A}{}^{bc}{}_\mu$ is completely determined by the spacetime metric, no additional degrees of freedom is introduced by this choice. In fact, the linear connection $\overset{\circ}{\Gamma}{}^\rho{}_{\nu\mu}$ corresponding to $\overset{\circ}{A}{}^{bc}{}_\mu$ is just the Christoffel connection of the metric $g_{\mu\nu}$:

$$\overset{\circ}{\Gamma}{}^\rho{}_{\nu\mu} = \tfrac{1}{2}\, g^{\rho\lambda} \left(\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\nu\mu}\right). \tag{3.74}$$

The gravitational theory based on such connection is General Relativity, whose main properties were described in Chapter 2.

3.7.2 Teleparallel Connection

A second possible choice that, like in the case of General Relativity, *does not introduce any additional degrees of freedom into the theory*, is to assume that the Lorentz connection $A^{ab}{}_\mu$ does not represent gravitation at all, but only inertial effects. This means to choose $A^b{}_{c\mu}$ as the inertial connection[1]

$$\overset{\bullet}{A}{}^b{}_{c\mu} = \Lambda^b{}_d\, \partial_\mu \Lambda_c{}^d. \tag{3.75}$$

The gravitational theory corresponding to this choice is just Teleparallel Gravity. In this theory, the gravitational field is fully represented by the the gauge potential $B^a{}_\mu$, which appears as the non-trivial part of the tetrad field [see Section 3.4]. In Teleparallel Gravity, therefore, Lorentz connections keep their special relativity role of representing inertial effects only.

The teleparallel coupling prescription is then given by [39]

$$e^a{}_\mu\, \partial_a \psi \;\to\; h^a{}_\mu\, \overset{\bullet\bullet}{\mathcal{D}}_a \psi = h^a{}_\mu \left[h_a \psi - \tfrac{i}{2}\left(\overset{\bullet}{A}{}^{bc}{}_a - \overset{\bullet}{K}{}^{bc}{}_a\right) S_{bc}\, \psi\right], \tag{3.76}$$

with $\overset{\bullet}{K}{}^{bc}{}_\mu$ the contortion of the connection $\overset{\bullet}{A}{}^{bc}{}_\mu$. Alternatively, one can write

$$\partial_\mu \psi \;\to\; \overset{\bullet\bullet}{\mathcal{D}}_\mu \psi = \partial_\mu \psi - \tfrac{i}{2}(\overset{\bullet}{A}{}^{bc}{}_\mu - \overset{\bullet}{K}{}^{bc}{}_\mu)\, S_{bc}\, \psi, \tag{3.77}$$

with the spacetime indices now raised and lowered with the spacetime metric $g_{\mu\nu}$. In the specific case of a Lorentz vector field V^b, for which S_{bc} is given by Eq. (3.60), the coupling prescription assumes the form

$$\partial_\mu V^b \;\to\; \overset{\bullet\bullet}{\mathcal{D}}_\mu V^b = \partial_\mu V^b + \left(\overset{\bullet}{A}{}^b{}_{c\mu} - \overset{\bullet}{K}{}^b{}_{c\mu}\right) V^c. \tag{3.78}$$

[1]All quantities related to Teleparallel Gravity will be denoted with an over "•".

The corresponding expression for the spacetime vector $V^\rho = h_c{}^\rho\, V^c$ is

$$\partial_\mu V^\rho \;\rightarrow\; \ddot{\nabla}_\mu V^\rho = \partial_\mu V^\rho + \left(\dot{\Gamma}^\rho{}_{\lambda\mu} - \dot{K}^\rho{}_{\lambda\mu}\right) V^\lambda. \tag{3.79}$$

These two derivatives are related by

$$\ddot{\mathcal{D}}_\mu V^b = h^b{}_\rho\, \ddot{\nabla}_\mu V^\rho.$$

Of course, due to the identity

$$\dot{A}^{bc}{}_\mu - \dot{K}^{bc}{}_\mu = \mathring{A}^{bc}{}_\mu, \tag{3.80}$$

the above coupling prescription is equivalent to the coupling prescription of General Relativity. However, the gravitational theory based on the spin connection (3.75), although physically equivalent to General Relativity is, conceptually speaking, completely different. In particular, since that connection represents inertial effects only, the gravitational field in this theory turns out to be fully represented by the translational gauge potential $B^a{}_\mu$, as it should be for a gauge theory for the translation group. This is the theory known as Teleparallel Gravity.

3.8 Curvature versus Torsion

The curvature of the teleparallel connection (3.75) vanishes identically:

$$\dot{R}^a{}_{b\nu\mu} = \partial_\nu \dot{A}^a{}_{b\mu} - \partial_\mu \dot{A}^a{}_{b\nu} + \dot{A}^a{}_{e\nu}\dot{A}^e{}_{b\mu} - \dot{A}^a{}_{e\mu}\dot{A}^e{}_{b\nu} = 0. \tag{3.81}$$

On the other hand, for a tetrad involving a non-trivial translational gauge potential $B^a{}_\mu$, that is to say, for

$$h^a{}_\mu = \dot{\mathcal{D}}_\mu x^a + B^a{}_\mu \tag{3.82}$$

with $B^a{}_\mu \neq \dot{\mathcal{D}}_\mu \epsilon^a$, torsion is non-vanishing:

$$\dot{T}^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + \dot{A}^a{}_{e\nu} h^e{}_\mu - \dot{A}^a{}_{e\mu} h^e{}_\nu \neq 0. \tag{3.83}$$

This connection can be considered a kind of "dual" of the General Relativity connection, which is a connection with vanishing torsion,

$$\mathring{T}^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + \mathring{A}^a{}_{e\nu} h^e{}_\mu - \mathring{A}^a{}_{e\mu} h^e{}_\nu = 0 \tag{3.84}$$

but non-vanishing curvature,

$$\overset{\circ}{R}{}^{a}{}_{b\nu\mu} = \partial_\nu \overset{\circ}{A}{}^{a}{}_{b\mu} - \partial_\mu \overset{\circ}{A}{}^{a}{}_{b\nu} + \overset{\circ}{A}{}^{a}{}_{e\nu} \overset{\circ}{A}{}^{e}{}_{b\mu} - \overset{\circ}{A}{}^{a}{}_{e\mu} \overset{\circ}{A}{}^{e}{}_{b\nu} \neq 0. \tag{3.85}$$

We see in this way that, whereas in General Relativity torsion vanishes, in Teleparallel Gravity it is curvature that vanishes. It is opportune to reinforce here that, from the gauge point of view, curvature and torsion are properties of connections, not of spacetime. Note, for example, that many different connections, each one with different curvature and torsion, can be defined on the very same metric spacetime.

Comment 3.5 As we are going to see later, General Relativity is completely equivalent to Teleparallel Gravity. This means essentially that torsion shows up simply as an alternative to curvature in the description of the gravitational interaction. In other words, torsion and curvature are related to the same degrees of freedom of gravity. In Chapter 14 we will present and discuss a gravitational model in which curvature and torsion represent independent degrees of freedom.

The linear connection corresponding to the spin connection $\overset{\bullet}{A}{}^{a}{}_{b\mu}$ is

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} = h_a{}^{\rho} \left(\partial_\mu h^a{}_\nu + \overset{\bullet}{A}{}^{a}{}_{b\mu} h^b{}_\nu \right) \equiv h_a{}^{\rho} \overset{\bullet}{\mathcal{D}}_\mu h^a{}_\nu. \tag{3.86}$$

This is the so-called Weitzenböck connection. Its definintion is equivalent to the identity

$$\partial_\mu h^a{}_\nu + \overset{\bullet}{A}{}^{a}{}_{b\mu} h^b{}_\nu - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} \, h^a{}_\rho = 0. \tag{3.87}$$

In the class of frames in which the spin connection $\overset{\bullet}{A}{}^{a}{}_{b\mu}$ vanishes, it becomes

$$\partial_\mu h^a{}_\nu - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} \, h^a{}_\rho = 0, \tag{3.88}$$

which is the absolute, or distant parallelism condition, from where Teleparallel Gravity got its name.

The Weitzenböck connection $\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu}$ is related to the Levi-Civita connection $\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu}$ of General Relativity by

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + \overset{\bullet}{K}{}^{\rho}{}_{\mu\nu}. \tag{3.89}$$

In terms of it, the Weitzenböck torsion is written as

$$\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu}, \tag{3.90}$$

whereas the (vanishing) Weitzenböck curvature is

$$\overset{\bullet}{R}{}^{\lambda}{}_{\rho\nu\mu} = \partial_\nu \overset{\bullet}{\Gamma}{}^{\lambda}{}_{\rho\mu} - \partial_\mu \overset{\bullet}{\Gamma}{}^{\lambda}{}_{\rho\nu} + \overset{\bullet}{\Gamma}{}^{\lambda}{}_{\sigma\nu} \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\rho\mu} - \overset{\bullet}{\Gamma}{}^{\lambda}{}_{\sigma\mu} \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\rho\nu} = 0. \tag{3.91}$$

Comment 3.6 It should be remarked that R. Weitzenböck does not seem to have ever written expression (3.86). In spite of this, the name "Weitzenböck connection" is commonly used to denote this particular case of a Cartan connection.

3.9 Translational Field Strength

Like in ordinary gauge theories, the field strength of Teleparallel Gravity is given by the covariant rotational of the gauge potential:

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu B^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu B^a{}_\mu. \tag{3.92}$$

Adding to it the vanishing torsion

$$\overset{\bullet}{\mathcal{D}}_\mu e^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu e^a{}_\mu \equiv [\overset{\bullet}{\mathcal{D}}_\mu, \overset{\bullet}{\mathcal{D}}_\nu] x^a = 0$$

of the inertial tetrad $e^a{}_\mu = \overset{\bullet}{\mathcal{D}}_\mu x^a$, we get

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu (\overset{\bullet}{\mathcal{D}}_\nu x^a + B^a{}_\nu) - \overset{\bullet}{\mathcal{D}}_\nu (\overset{\bullet}{\mathcal{D}}_\mu x^a + B^a{}_\mu). \tag{3.93}$$

Since, according to (3.43),

$$\overset{\bullet}{\mathcal{D}}_\mu x^a + B^a{}_\mu = h^a{}_\mu \tag{3.94}$$

is the tetrad field, we see that the field strength of Teleparallel Gravity coincides with torsion:

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu h^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu h^a{}_\mu. \tag{3.95}$$

Comment 3.7 As in any gauge theory, the field strength can also be obtained from the commutation relation of gauge covariant derivatives. Using the translational covariant derivative (3.23), one can easily verify that

$$[h_\mu, h_\nu] = \overset{\bullet}{T}{}^a{}_{\mu\nu} P_a. \tag{3.96}$$

Furthermore, due to the soldered character of the tangent bundle, torsion shows up also as the anholonomy of the translational covariant derivative:

$$[h_\mu, h_\nu] = \overset{\bullet}{T}{}^\rho{}_{\mu\nu} h_\rho. \tag{3.97}$$

Since the tetrad is gauge invariant, the field strength $\overset{\bullet}{T}{}^a{}_{\mu\nu}$ is also invariant under gauge transformations:

$$\overset{\bullet}{T}{}'^a{}_{\mu\nu} = \overset{\bullet}{T}{}^a{}_{\mu\nu}. \tag{3.98}$$

This is an expected result because, being the gauge group abelian, any field belonging to the adjoint representation, like for example the field strength, must be gauge invariant.

Comment 3.8 Remember that the generators of infinitesimal gauge transformations of fields belonging to the adjoint representation are the coefficient of structure of the group, taken as matrices. Since these coefficients vanish for abelian groups, fields belonging to the adjoint representations of abelian gauge theories will consequently be invariant. The gauge transformation of source fields, as we have seen in Section 3.3, are generated by $P_a = \partial/\partial x^a$.

Chapter 4

Particle Mechanics

The teleparallel equation of motion of test particles is obtained, its equivalence with the geodesic equation discussed, and the roles played by torsion and curvature in the description of the gravitational interaction clarified. It is then shown that, because its spin connection represents inertial effects only, Teleparallel Gravity produces a separation between inertia and gravitation. The gravitational field in this theory is represented by a translational gauge potential, whose time components coincide with the gravitational potential in the Newtonian limit.

4.1 Free Particles Revisited

4.1.1 Basic Notions

Let us start with Minkowski spacetime, whose quadratic interval is

$$d\sigma^2 = \eta_{ab}\, dx'^a\, dx'^b. \tag{4.1}$$

Consider now the trivial, holonomous (inertial) tetrad

$$e'^a{}_\mu = \partial_\mu x'^a. \tag{4.2}$$

Under a local Lorentz transformation $x^b = \Lambda^b{}_a\, x'^a$, the tetrad transforms according to

$$e^b{}_\mu = \Lambda^b{}_a\, e'^a{}_\mu, \tag{4.3}$$

where

$$e^a{}_\mu = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu}\, x^b \equiv \overset{\bullet}{\mathcal{D}}_\mu x^a \tag{4.4}$$

is the new tetrad, with $\overset{\bullet}{A}{}^a{}_{b\mu}$ the inertial connection (3.75). In the new frame

$$e^a = dx^a + \overset{\bullet}{A}{}^a{}_b\, x^b, \tag{4.5}$$

with $e^a = e^a{}_\mu \, dx^\mu$, the quadratic interval (4.1) reads

$$d\sigma^2 = \eta_{ab} \, e^a \, e^b \tag{4.6}$$

Since the metric is Lorentz invariant, the quadratic interval can also be written in the form

$$d\sigma^2 = \eta_{\mu\nu} \, dx^\mu dx^\nu \tag{4.7}$$

where

$$\eta_{\mu\nu} = \eta_{ab} \, e^a{}_\mu \, e^b{}_\nu \tag{4.8}$$

is the Minkowski spacetime metric.

The particle four-velocity is defined by

$$u^\mu = \frac{dx^\mu}{d\sigma}. \tag{4.9}$$

Hence, along the trajectory of the particle, we can write

$$d\sigma = u_\mu \, dx^\mu. \tag{4.10}$$

Using the tetrad (4.4), we define the anholonomic four-velocity:

$$u^a = e^a{}_\mu \, u^\mu = e^a \left(\frac{1}{d\sigma} \right). \tag{4.11}$$

In terms of the anholonomic base e^a, the spacetime interval along the particle trajectory can be written in the form

$$d\sigma = u_a \, e^a. \tag{4.12}$$

4.1.2 Free Equation of Motion

A free particle of mass m is represented by the action integral

$$\mathcal{S} = -\, mc \int_a^b d\sigma = -\, mc \int_a^b u_a \, e^a. \tag{4.13}$$

Substituting

$$e^a = dx^a + \overset{\bullet}{A}{}^a{}_{b\mu} \, dx^\mu, \tag{4.14}$$

it assumes the form

$$\mathcal{S} = -\, mc \int_a^b u_a (dx^a + \overset{\bullet}{A}{}^a{}_{b\mu} \, x^b \, dx^\mu). \tag{4.15}$$

Under the general spacetime variation $x^\mu \to x^\mu + \delta x^\mu$, action (4.13) changes according to

$$\delta \mathcal{S} = -\,m\,c \int_a^b \Big[e^a\,\delta u_a + u_a\,d\delta x^a + u_a\,\delta(\overset{\bullet}{A}{}^a{}_{b\mu} x^b) dx^\mu \\ + u_a\,\overset{\bullet}{A}{}^a{}_{b\mu} x^b\,d(\delta x^\mu)\Big]. \tag{4.16}$$

where we have used that $[\delta, d] = 0$. Writing the quadratic spacetime interval in the form

$$ds^2 = \eta_{ab}\,e^a\,e^b, \tag{4.17}$$

a direct calculation shows that

$$\delta(ds) = u_a\,\delta e^a. \tag{4.18}$$

On the other hand, writing the interval in the form

$$ds = u_a\,e^a, \tag{4.19}$$

we get

$$\delta(ds) = u_a\,\delta e^a + e^a\,\delta u_a. \tag{4.20}$$

From Eqs. (4.18) and (4.20), we see immediately that

$$e^a\,\delta u_a = 0. \tag{4.21}$$

Substituting in (4.16), it becomes

$$\delta \mathcal{S} = -\,m\,c \int_a^b \Big[u_a\,d\delta x^a + u_a\,\delta(\overset{\bullet}{A}{}^a{}_{b\mu} x^b) dx^\mu +\, u_a\,\overset{\bullet}{A}{}^a{}_{b\mu} x^b\,d(\delta x^\mu)\Big]. \tag{4.22}$$

Integrating by parts the first and the third terms and neglecting the surface terms, we obtain

$$\delta \mathcal{S} = mc \int_a^b \Big[du_a\,\delta x^a - u_a\,\delta(\overset{\bullet}{A}{}^a{}_{b\mu} x^b)\,dx^\mu + d(u_a\,\overset{\bullet}{A}{}^a{}_{b\mu} x^b)\,\delta x^\mu\Big]. \tag{4.23}$$

Performing the variations and differentials, and using the expressions

$$\delta x^a = \partial_\mu x^a \delta x^\mu \quad \text{and} \quad \delta \overset{\bullet}{A}{}^a{}_{b\mu} = \partial_\rho \overset{\bullet}{A}{}^a{}_{b\mu} \delta x^\rho,$$

after a straightforward algebra, we get

$$\delta \mathcal{S} = mc \int_a^b \Big[e^a{}_\mu\Big(\frac{du_a}{d\sigma} - \overset{\bullet}{A}{}^b{}_{a\rho}\,u_b\,u^\rho\Big) - \overset{\bullet}{R}{}^a{}_{b\mu\rho}\,x^b\,u_a\,u^\rho\Big] d\sigma\,\delta x^\mu. \tag{4.24}$$

Since the curvature $\stackrel{\bullet}{R}{}^a{}_{b\mu\rho}$ of the inertial connection $\stackrel{\bullet}{A}{}^a{}_{b\mu}$ vanishes identically, we are left with

$$\delta\mathcal{S} = mc \int_a^b \left[e^a{}_\mu \left(\frac{du_a}{d\sigma} - \stackrel{\bullet}{A}{}^b{}_{a\rho}\, u_b\, u^\rho \right) \right] d\sigma\, \delta x^\mu. \tag{4.25}$$

Considering then the invariance of the action, $\delta\mathcal{S} = 0$, and the arbitrariness of δx^μ, we obtain the equation of motion

$$u^\rho\, \stackrel{\bullet}{\mathcal{D}}_\rho u_a \equiv \frac{du_a}{d\sigma} - \stackrel{\bullet}{A}{}^b{}_{a\rho}\, u_b\, u^\rho = 0, \tag{4.26}$$

whose contravariant form is

$$u^\rho\, \stackrel{\bullet}{\mathcal{D}}_\rho u^a \equiv \frac{du^a}{d\sigma} + \stackrel{\bullet}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = 0. \tag{4.27}$$

This is the equation of motion of a free particle as seen from a general Lorentz frame. Of course, in the class of inertial frames e'^a, the inertial connection $\stackrel{\bullet}{A}{}'^a{}_{b\rho}$ vanishes and the equation of motion reduces to

$$\frac{du'^a}{d\sigma} = 0. \tag{4.28}$$

Comment 4.1 We have said at the previous chapter that the spin connection of Teleparallel Gravity keeps the special relativity role of describing inertial effects only. This is the reason why the same spin connection appears in the free-particle equation of motion when described from an anholonomic frame.

4.2 Gravitationally Coupled Particles

4.2.1 Coupling Prescription

In Classical Mechanics, where particles are not represented by fields, the gravitational coupling prescription is carried out by replacing a trivial tetrad on Minkowski space by a non-trivial tetrad representing a gravitational field:

$$e^a{}_\mu \to h^a{}_\mu. \tag{4.29}$$

In the specific case of Teleparallel Gravity, this is achieved by replacing

$$e^a{}_\mu = \stackrel{\bullet}{\mathcal{D}}_\mu x^a \to h^a{}_\mu = \stackrel{\bullet}{\mathcal{D}}_\mu x^a + B^a{}_\mu, \tag{4.30}$$

with $B^a{}_\mu$ the translational gauge potential. In consonance with this replacement, the spacetime metric changes according to

$$\eta_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu \to g_{\mu\nu} = \eta_{ab}\, h^a{}_\mu h^b{}_\nu. \tag{4.31}$$

As a consequence, the quadratic interval becomes

$$d\sigma^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu \to ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{4.32}$$

or equivalently, in terms of anholonomic bases,

$$d\sigma^2 = \eta_{ab}\, e^a\, e^b \to ds^2 = \eta_{ab}\, h^a\, h^b. \tag{4.33}$$

Along the particle trajectory, therefore, the spacetime interval can be written formally as

$$ds = g_{\mu\rho}\, u^\mu\, dx^\rho = \eta_{ab}\, u^a\, h^b. \tag{4.34}$$

The holonomic and the anholonomic particle four-velocities will now satisfy the relation

$$u^\rho \equiv \frac{dx^\rho}{ds} = u^a\, h_a{}^\rho. \tag{4.35}$$

4.2.2 Coupled Equation of Motion

As previously said, the action representing a free particle of mass m can be written as

$$\mathcal{S} = -\, mc \int_a^b u_a\, e^a. \tag{4.36}$$

In the presence of gravitation, the corresponding action is obtained by applying the coupling prescription (4.30), in which case it becomes

$$\mathcal{S} = -\, m\, c \int_a^b u_a\, \left[dx^a + \overset{\bullet}{A}{}^a{}_{b\mu}\, x^b\, dx^\mu + B^a{}_\mu\, dx^\mu \right]. \tag{4.37}$$

This is the teleparallel version of the action, as described from a general Lorentz frame. In the class of frames e'^a in which the inertial connection $\overset{\bullet}{A}{}'^a{}_b$ vanishes, it reduces to

$$\mathcal{S} = -\, m\, c \int_a^b u'_a\, \left[dx'^a + B'^a{}_\mu\, dx^\mu \right]. \tag{4.38}$$

Comment 4.2 It is interesting to observe that, due to the gauge structure of Teleparallel Gravity, the action has a form similar to the action of a charged particle in an electromagnetic field. In fact, if the particle has additionally an electric charge q and is in the presence of an electromagnetic potential A'_μ, the total action has the form

$$\mathcal{S} = -\,mc \int_a^b \left[u'_a \, dx'^a + u'_a \, B'^a{}_\mu \, dx^\mu + \frac{q}{mc^2} \, A'_\mu \, dx^\mu \right]. \tag{4.39}$$

Notice that, whereas the electromagnetic interaction depends on the relation q/m of the particle, the gravitational interaction has already been assumed to be universal in the sense that it does not depend on any property of the particle. In Chapter 9 we will present a more detailed discussion of this point.

Under the general spacetime variation $x^\mu \rightarrow x^\mu + \delta x^\mu$, action (4.37) changes according to

$$\delta\mathcal{S} = -\,m\,c \int_a^b \Big[h^a \, \delta u_a + u_a \, d\delta x^a + u^a \, \delta \overset{\bullet}{A}{}^a{}_{b\mu} x^b \, dx^\mu + u_a \, \overset{\bullet}{A}{}^a{}_{b\mu} \delta x^b \, dx^\mu$$
$$+ u^a \, \overset{\bullet}{A}{}^a{}_{b\mu} x^b \, d\delta x^\mu + u_a \, \delta B^a{}_\mu \, dx^\mu + u_a \, B^a{}_\mu \, d\delta x^\mu \Big]. \tag{4.40}$$

where

$$h^a = dx^a + \overset{\bullet}{A}{}^a{}_{b\mu} x^b dx^\mu + B^a{}_\mu dx^\mu, \tag{4.41}$$

and where we have already used that $[\delta, d] = 0$. Writing the quadratic spacetime interval in the form

$$ds^2 = \eta_{ab} \, h^a \, h^b, \tag{4.42}$$

a direct calculation shows that

$$\delta(ds) = u_a \, \delta h^a. \tag{4.43}$$

On the other hand, writing the interval in the form

$$ds = u_a \, h^a, \tag{4.44}$$

we get

$$\delta(ds) = u_a \, \delta h^a + h^a \, \delta u_a. \tag{4.45}$$

From Eqs. (4.43) and (4.45), we see immediately that

$$h^a \, \delta u_a = 0. \tag{4.46}$$

Substituting in (4.40), it becomes

$$\delta\mathcal{S} = -\,m\,c \int_a^b \Big[u_a \, d\delta x^a + u_a \, \delta \overset{\bullet}{A}{}^a{}_{b\mu} x^b \, dx^\mu + u_a \, \overset{\bullet}{A}{}^a{}_{b\mu} \delta x^b \, dx^\mu$$
$$+ u_a \, \overset{\bullet}{A}{}^a{}_{b\mu} x^b \, d\delta x^\mu + u_a \, \delta B^a{}_\mu \, dx^\mu + u_a \, B^a{}_\mu \, d\delta x^\mu \Big]. \tag{4.47}$$

Integrating by parts the terms containing differentials and neglecting the surface terms, we get

$$\delta\mathcal{S} = m\,c \int_a^b \Big[du_a \delta x^a - u_a \delta \dot{A}^a{}_{b\mu} x^b dx^\mu - u_a \dot{A}^a{}_{b\mu} \delta x^b dx^\mu + d(u_a \dot{A}^a{}_{b\mu} x^b) \delta x^\mu - u_a \delta B^a{}_\mu dx^\mu + d(u_a B^a{}_\mu) \delta x^\mu \Big]. \tag{4.48}$$

Performing the differentials and variations, substituting the expressions

$$\delta x^a = \partial_\mu x^a \delta x^\mu, \quad \delta \dot{A}^a{}_{b\mu} = \partial_\rho \dot{A}^a{}_{b\mu} \delta x^\rho, \quad \delta B^a{}_\mu = \partial_\rho B^a{}_\mu \delta x^\rho, \tag{4.49}$$

and considering that the curvature $\dot{R}^a{}_{b\mu\rho}$ of the teleparallel spin connection $\dot{A}^a{}_{b\rho}$ vanishes identically, we get finally

$$\delta\mathcal{S} = m\,c \int_a^b \left[h^a{}_\mu \left(\frac{du_a}{ds} - \dot{A}^b{}_{a\rho}\, u_b\, u^\rho\right) - \dot{T}^b{}_{\mu\rho}\, u_b\, u^\rho\right] \delta x^\mu\, ds, \tag{4.50}$$

where

$$\dot{T}^a{}_{\mu\rho} = \dot{\mathcal{D}}_\mu B^a{}_\rho - \dot{\mathcal{D}}_\rho B^a{}_\mu \tag{4.51}$$

is the translational field strength, or torsion. From the invariance of the action, $\delta\mathcal{S} = 0$, and taking into account the arbitrariness of δx^μ, the equation of motion is found to be

$$\frac{du_a}{ds} - \dot{A}^b{}_{a\rho}\, u_b\, u^\rho = \dot{T}^b{}_{a\rho}\, u_b\, u^\rho. \tag{4.52}$$

Its contravariant version is

$$\frac{du^a}{ds} + \dot{A}^a{}_{b\rho}\, u^b\, u^\rho = \dot{T}_b{}^a{}_\rho\, u^b\, u^\rho. \tag{4.53}$$

Using the identity

$$\dot{T}^b{}_{a\rho}\, u_b\, u^\rho = -\,\dot{K}^b{}_{a\rho}\, u_b\, u^\rho, \tag{4.54}$$

they can be rewritten, respectively, in the forms

$$\frac{du_a}{ds} - \dot{A}^b{}_{a\rho}\, u_b\, u^\rho = -\,\dot{K}^b{}_{a\rho}\, u_b\, u^\rho. \tag{4.55}$$

and

$$\frac{du^a}{ds} + \dot{A}^a{}_{b\rho}\, u^b\, u^\rho = \dot{K}^a{}_{b\rho}\, u^b\, u^\rho. \tag{4.56}$$

This is the teleparallel equation of motion of a particle of mass m in a gravitational field — as seen from a general Lorentz frame. It is a *force equation*, with torsion (or contortion) playing the role of force. It is interesting to observe that, since the conserved charge in teleparallel gravity is just the four-momentum p^a, the Wong equation [see Section 2.1] in this case coincides with the particle equation of motion.

Comment 4.3 By contraction with tetrads, and using identity (3.87), the equation of motion (4.52) can be written in a purely spacetime form

$$\frac{du_\mu}{ds} - \overset{\bullet}{\Gamma}{}^\theta{}_{\mu\nu}\, u_\theta\, u^\nu = \overset{\bullet}{T}{}^\theta{}_{\mu\nu}\, u_\theta\, u^\nu, \tag{4.57}$$

where $\overset{\bullet}{\Gamma}{}^\theta{}_{\mu\nu}$ is the Weitzenböck connection. Substituting

$$\overset{\bullet}{T}{}^\theta{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^\theta{}_{\nu\mu} - \overset{\bullet}{\Gamma}{}^\theta{}_{\mu\nu}, \tag{4.58}$$

it reduces to

$$\frac{du_\mu}{ds} - \overset{\bullet}{\Gamma}{}^\theta{}_{\nu\mu}\, u_\theta\, u^\nu = 0. \tag{4.59}$$

Due to the wrong positions of the indices, however, as well as to the fact that the Weitzenböck connection is not symmetric in the last two indices, the left-hand side of the equation (4.59) is not the covariant derivative of the four-velocity u_μ. This means that test particles do not follow the geodesics (or the auto-parallels) of the "torsioned spacetime". In Teleparallel Gravity, therefore, the gravitational interaction is not geometrized, but described by a force — with torsion playing the role of force.

4.2.3 Equivalence with the Geodesic Equation

Let us rewrite the force equation (4.56) in the form

$$\frac{du^a}{ds} + (\overset{\bullet}{A}{}^a{}_{b\rho} - \overset{\bullet}{K}{}^a{}_{b\rho})\, u^b\, u^\rho = 0. \tag{4.60}$$

Remembering that

$$\overset{\bullet}{A}{}^a{}_{b\rho} - \overset{\bullet}{K}{}^a{}_{b\rho} = \overset{\circ}{A}{}^a{}_{b\rho}, \tag{4.61}$$

with $\overset{\circ}{A}{}^a{}_{b\rho}$ the spin connection of General Relativity, the teleparallel force equation (4.56) is found to coincide with the geodesic equation

$$\frac{du^a}{ds} + \overset{\circ}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = 0 \tag{4.62}$$

of General Relativity. We see in this way that the teleparallel description of the gravitational interaction is completely equivalent to the description of General Relativity.

There are conceptual differences, though. In General Relativity, a theory fundamentally based on the weak equivalence principle, curvature is used to *geometrize* the gravitational interaction. The gravitational interaction in this case is described by letting (spinless) particles to follow the curvature of spacetime. Geometry replaces the concept of force, and the trajectories are determined, not by force equations, but by geodesics. Teleparallel Gravity, on the other hand, attributes gravitation to torsion. Torsion, however, accounts for gravitation not by geometrizing the interaction, but by acting as a force. In consequence, there are no geodesics in Teleparallel Gravity, only force equations quite analogous to the Lorentz force equation of electrodynamics [14]. This is an expected result because, like electrodynamics, Teleparallel Gravity is also a gauge theory.

Comment 4.4 This equivalence should not be surprising. In fact, let us take again the teleparallel action (4.37):

$$\mathcal{S} = -\,m\,c \int_a^b u_a \left[dx^a + \dot{A}^a{}_b\, x^b + B^a\right]. \tag{4.63}$$

If we use the identity

$$u_a(dx^a + \dot{A}^a{}_b\, x^b + B^a) = u_a\, h^a \equiv ds, \tag{4.64}$$

we see that the above action reduces to

$$\mathcal{S} = -\,m\,c \int_a^b ds, \tag{4.65}$$

which is the usual general-relativistic form of the action, from where the geodesic equation is usually obtained.

4.3 Separating Inertia from Gravitation

To begin with, let us consider again the tetrad field:

$$h^a{}_\mu = \dot{\mathcal{D}}_\mu x^a + B^a{}_\mu. \tag{4.66}$$

Whereas the first term is purely inertial, the second is purely gravitational. This means that both inertia and gravitation are included in $h^a{}_\mu$. As a consequence, the coefficient of anholonomy of a given frame h_a, which is given by

$$f^c{}_{ab} = h_a{}^\mu h_b{}^\nu (\partial_\nu h^c{}_\mu - \partial_\mu h^c{}_\nu), \tag{4.67}$$

will also include both inertia and gravitation.

Now, as discussed in Chapter 3, the spin connection of General Relativity is

$$\overset{\circ}{A}{}^a{}_{bc} = \tfrac{1}{2}(f_b{}^a{}_c + f_c{}^a{}_b - f^a{}_{bc}). \tag{4.68}$$

As a consequence, it represents both inertia and gravitation. To see that this is in fact the case, let us recall that, in its standard formulation, the strong equivalence principle says that it is always possible to find a frame in which inertia compensates gravitation *locally*, that is, in a point or along a world-line. In that local frame, the spin connection of general relativity vanishes:

$$\overset{\circ}{A}{}^a{}_{bc} = 0. \tag{4.69}$$

This vanishing is possible just because both inertia and gravitation are included in $\overset{\circ}{A}{}^a{}_{b\rho}$. When they locally compensate each other, the spin connection vanishes. In this local frame, the geodesic equation

$$\frac{du^a}{ds} + \overset{\circ}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = 0, \tag{4.70}$$

reduces to the equation of motion of a free particle in a local inertial frame:

$$\frac{du^a}{ds} = 0. \tag{4.71}$$

On the other hand, substituting the identity

$$\overset{\circ}{A}{}^a{}_{bc} = \overset{\bullet}{A}{}^a{}_{bc} - \overset{\bullet}{K}{}^a{}_{bc}, \tag{4.72}$$

the geodesic equation (4.70) becomes the force equation of Teleparallel Gravity:

$$\frac{du^a}{ds} + \overset{\bullet}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = \overset{\bullet}{K}{}^a{}_{b\rho}\, u^b\, u^\rho. \tag{4.73}$$

Since the teleparallel spin connection $\overset{\bullet}{A}{}^a{}_{b\rho}$ represents inertial effects only, the splitting (4.72) corresponds actually to a separation between inertia and gravitation [44]. Accordingly, the right-hand side of the equation of motion (4.73) represents the purely gravitational force, which transforms covariantly under local Lorentz transformations. The inertial forces coming from the frame non-inertiality are represented by the connection of the left-hand side, which is non-covariant by its very nature. In teleparallel gravity, therefore, whereas the gravitational effects are described by a covariant force, the non-inertial effects of the frame remain *geometrized* in the sense of general relativity, and are represented by an inertial-related connection. Notice that in the geodesic equation (4.70), both inertial and gravitational effects are described by the connection term of the left-hand side.

Comment 4.5 Although the inertial part of $\overset{\circ}{A}{}^a{}_{b\rho}$ does not contribute to some physical quantities, like curvature and torsion, it does contribute to others. One example is the energy-momentum tensor of gravitation, whose expression in General Relativity always include, in addition the energy-momentum of gravity itself, also the energy-momentum of inertia. This is the reason, by the way, why this density always shows up as a pseudotensor. Chapter 8 deals with this question in more details.

Let us consider again the local frame in which the spin connection $\overset{\circ}{A}{}^a{}_{b\rho}$ vanishes. On account of the identity (4.72), the teleparallel version of this local condition is

$$\overset{\bullet}{A}{}^a{}_{b\rho} = \overset{\bullet}{K}{}^a{}_{b\rho}. \tag{4.74}$$

This expression shows explicitly that, in such local frame, inertia (left-hand side) exactly compensates gravitation (right-hand side). On the other hand, owing to the non-tensorial character of the inertial effects, in teleparallel gravity it is also possible to choose a *global* frame h'_b in which only the inertial effects vanish. In this frame, $\overset{\bullet}{A}{}'^a{}_{b\rho} = 0$ and the equation of motion (4.73) becomes purely gravitational:

$$\frac{du'^a}{ds} = \overset{\bullet}{K}{}'^a{}_{bc}\, u'^b\, u'^c. \tag{4.75}$$

Since no inertial effects are present in this formulation, we can conclude that teleparallel gravity does not need to use the equivalence between inertia and gravitation to describe the gravitational interaction [45]. In Chapter 9 we will explore further this point.

Comment 4.6 In the form (4.75), the gravitational force becomes quite similar to the Lorentz force of electrodynamics. There is a difference, though: in contrast to the Lorentz force of electrodynamics, which is linear in the four-velocity, the gravitational force is quadratic in the four-velocity. This difference could be attributed, in principle, to the strict attractive character of gravitation. However, by decomposing the force on the right-hand side in time and space components, we obtain (dropping the primes)

$$\frac{du^a}{ds} = \gamma^2\, \overset{\bullet}{K}{}^a{}_{00} + \frac{\gamma^2}{c}\left(\overset{\bullet}{K}{}^a{}_{0j} + \overset{\bullet}{K}{}^a{}_{j0}\right) v^j + \frac{\gamma^2}{c^2}\, \overset{\bullet}{K}{}^a{}_{ij}\, v^i\, v^j, \tag{4.76}$$

where we have used that

$$u^a = \left(\gamma, \gamma\frac{v^i}{c}\right) \quad \text{with} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{4.77}$$

We see from Eq. (4.76) that the gravitational force includes, in additional to terms independent and quadratic in the particle velocity, also terms linear in the velocity. These terms are somewhat puzzling in the sense that the gravitational interaction will be attractive or repulsive depending on the sign of the particle velocity. They give rise to the gravitomagnetic interaction in the weak-field limit.

4.4 Newtonian Limit

By contraction with tetrads, and using the identity (1.30), the equation of motion (4.53) can be rewritten in the purely spacetime form

$$\frac{du^\rho}{ds} + \overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu}\, u^\mu\, u^\nu = \overset{\bullet}{T}{}_\mu{}^\rho{}_\nu\, u^\mu\, u^\nu, \tag{4.78}$$

with $\overset{\bullet}{T}_\mu{}^\rho{}_\nu$ the torsion of the Weitzenböck connection $\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu}$. The Newtonian limit is obtained by assuming that the gravitational field is stationary and weak. This means respectively that the time derivative of $B^a{}_\mu$ vanishes, and that $|B^a{}_\mu| \ll 1$. Accordingly, all particles are supposed to move with a sufficient small velocity so that u^i can be neglected in relation to u^0. Let us then rewrite the force equation (4.78) in the form

$$\frac{du^\rho}{ds} + \overset{\bullet}{\Gamma}{}^\rho{}_{00}\, u^0 u^0 = \overset{\bullet}{T}_0{}^\rho{}_0\, u^0 u^0. \tag{4.79}$$

In the class of frames in which the teleparallel spin connection $\overset{\bullet}{A}{}^a{}_{b\mu}$ vanishes, and choosing a (translational) gauge in which $\partial_\mu x^a = \delta^a_\mu$, the tetrad assumes the form

$$h^a{}_\mu = \delta^a{}_\mu + B^a{}_\mu. \tag{4.80}$$

Up to first order in $B^a{}_\mu$, therefore, we get

$$\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} \equiv \partial_\nu B^\rho{}_\mu, \tag{4.81}$$

where $B^\rho{}_\mu = \delta^\rho_a\, B^a{}_\mu$. In this case, Eq. (4.82) reduces to

$$\frac{d^2 x^\rho}{ds^2} = \partial^\rho B_{00}\; u^0 u^0. \tag{4.82}$$

Substituting

$$u^0 = c\, \frac{dt}{ds}, \tag{4.83}$$

it reduces to

$$\frac{d^2 x^\rho}{ds^2} = \partial^\rho B_{00}\, c^2\, \frac{dt^2}{ds^2}. \tag{4.84}$$

The time component of this equation reads

$$\frac{d^2 x^0}{ds^2} \equiv c^2\, \frac{d^2 t}{ds^2} = 0, \tag{4.85}$$

whose solution says that dt/ds equals a constant. Using this fact, the space component of Eq. (4.84) is found to be

$$\frac{dv^j}{dt} = c^2\, \partial^j B_{00}, \tag{4.86}$$

with $v^j = dx^j/dt$ the particle four-velocity. If we identify

$$c^2 B_{00} = \Phi, \tag{4.87}$$

with

$$\Phi = -\,\frac{GM}{|\vec{x}|} \tag{4.88}$$

the Newtonian gravitational potential, we get

$$\frac{d\vec{v}}{dt} = -\,\vec{\nabla}\Phi, \tag{4.89}$$

where we have used that the components of $\vec{x}$ are given by $x^i = -\,x_i$, so that $\partial^i = -\,\partial_i$. As expected, due to the equivalence with the geodesic equation, the force equation of Teleparallel Gravity also has the correct Newtonian limit.

Comment 4.7 It is interesting to observe that, since both teleparallel and Newtonian gravity describe the gravitational interaction through a force, the newtonian limit follows much more naturally from Teleparallel Gravity than from General Relativity, where no gravitational force exists.

4.5 Gravitomagnetic Field

Let us consider the same limit of the previous section, but now keeping also terms linear in u^i. In this case, the force equation (4.78) assumes the form

$$\frac{du^\rho}{ds} + \overset{\bullet}{\Gamma}{}^\rho{}_{00}\, u^0 u^0 + \Big(\overset{\bullet}{\Gamma}{}^\rho{}_{0i} + \overset{\bullet}{\Gamma}{}^\rho{}_{i0}\Big) u^0 u^i = \overset{\bullet}{T}{}_0{}^\rho{}_0\, u^0 u^0 + \Big(\overset{\bullet}{T}{}_0{}^\rho{}_i + \overset{\bullet}{T}{}_i{}^\rho{}_0\Big) u^0 u^i. \tag{4.90}$$

Substituting

$$\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} \equiv \partial_\nu B^\rho{}_\mu, \tag{4.91}$$

with $B^\rho{}_\mu = \delta^\rho_a\, B^a{}_\mu$, and discarding terms containing time derivatives of the potential, Eq. (4.90) reduces to

$$\frac{d^2x^\rho}{ds^2} = \partial^\rho B_{00}\, u^0 u^0 + \left[\partial^\rho (B_{0i} + B_{i0}) - \partial_i (B_0{}^\rho + B^\rho{}_0)\right] u^0 u^i. \tag{4.92}$$

We see from this expression that, in the linear approximation, only the symmetric part of $B^\rho{}_\mu$ contributes to the equation of motion. We can then assume that in this limit $B^\rho{}_\mu$ is symmetric [47]. In this case, Eq. (4.92) becomes

$$\frac{d^2x^\rho}{ds^2} = \partial^\rho B_{00}\, u^0 u^0 + 2\left(\partial^\rho B_{0i} - \partial_i B_0{}^\rho\right) u^0 u^i. \tag{4.93}$$

Substituting

$$u^0 = c\,\frac{dt}{ds} \qquad \text{and} \qquad u^i \equiv \frac{dx^i}{ds} = \frac{dx^i}{dt}\frac{dt}{ds} = v^i\,\frac{dt}{ds}, \tag{4.94}$$

and using the identification (4.87), it reduces to

$$\frac{d^2x^\rho}{ds^2} = \partial^\rho\Phi\,\frac{dt^2}{ds^2} + 2c\,(\partial^\rho B_{0i} - \partial_i B_0{}^\rho)\,\frac{dt^2}{ds^2}\,v^i. \tag{4.95}$$

Up to order v/c, the time component of this equation reads

$$\frac{d^2x^0}{ds^2} \equiv c\,\frac{d^2t}{ds^2} = 0, \tag{4.96}$$

whose solution implies that dt/ds equals a constant. Using this fact, the space component of Eq. (4.95) can be written in the form

$$\frac{d^2x^j}{dt^2} = -\,\partial_j\Phi + 2\left(\partial^j B_{0i} - \partial_i B_0{}^j\right) c\,v^i. \tag{4.97}$$

where we have used that $\partial^j = -\,\partial_j$. Now, if we identify

$$c^2\left(\partial^j B_{0i} - \partial_i B_0{}^j\right) = \epsilon^j{}_{ik}\,H^k, \tag{4.98}$$

with H^k the gravitomagnetic component of the gravitational field, we get

$$\frac{d\vec{v}}{dt} = -\,\vec{\nabla}\Phi + 2\,\frac{\vec{v}}{c}\times\vec{H}. \tag{4.99}$$

Comment 4.8 Since the torsion tensor

$$\overset{\bullet}{T}_{0ji} = \partial_j B_{0i} - \partial_i B_{0j} \tag{4.100}$$

has vanishing vector torsion, its decomposition in terms of the irreducible components under the global Lorentz group (see Section 1.4) is given by

$$\overset{\bullet}{T}_{0ji} = \epsilon_{0jik}\,\overset{\bullet}{\mathcal{A}}{}^k + \tfrac{2}{3}\left(\overset{\bullet}{\mathcal{T}}_{0ji} - \overset{\bullet}{\mathcal{T}}_{0ij}\right). \tag{4.101}$$

Comparing with Eq. (4.98), we see that the gravitomagnetic component of the gravitational field coincides with the space component of the axial torsion:

$$H^k = c^2\overset{\bullet}{\mathcal{A}}{}^k. \tag{4.102}$$

It is important to notice that, although the axial torsion

$$\overset{\bullet}{T}_\mu{}^\rho{}_\nu = \epsilon_\mu{}^\rho{}_{\nu\lambda}\,\overset{\bullet}{\mathcal{A}}{}^\lambda \tag{4.103}$$

does not contribute to the relativistic force appearing in the right-hand side of the equation of motion (4.78), in the non-relativistic limit it can give rise to gravitational effects.

4.6 The Spinning Particle

Let us consider now the motion of a classical particle of mass m and spin $\mathbf{s}$ in a gravitational field. In the context of Teleparallel Gravity, the action integral describing such a particle minimally coupled to gravitation is

$$\mathcal{S} = \int_a^b \left[-(\partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu}\, x^b + B^a{}_\mu)\, p_a + \tfrac{1}{2}\, (\overset{\bullet}{A}{}^{ab}{}_\mu - \overset{\bullet}{K}{}^{ab}{}_\mu) s_{ab} \right] dx^\mu, \quad (4.104)$$

where $p_a = mcu_a$ is the Noether charge associated with the invariance of $\mathcal{S}$ under spacetime translations, and $s_{ab} = -s_{ba}$ is the Noether charge associated with the invariance of $\mathcal{S}$ under Lorentz transformations [30, 148]. In other words, p_a is the momentum, and s_{ab} is the spin angular momentum density, which satisfies the Poisson relation

$$\{s_{ab}, s_{cd}\} = \eta_{ac}\, s_{bd} + \eta_{bd}\, s_{ac} - \eta_{ad}\, s_{bc} - \eta_{bc}\, s_{ad}. \quad (4.105)$$

Notice that, according to this prescription, the particle momentum couples minimally to the translational gauge potential $B^a{}_\mu$, whereas the spin of the particle, as implied by the general covariance principle, couples minimally to the dynamical spin connection

$$\overset{\bullet}{A}{}^{ab}{}_\mu - \overset{\bullet}{K}{}^{ab}{}_\mu \equiv \overset{\circ}{A}{}^{ab}{}_\mu. \quad (4.106)$$

A quite convenient way to get the equations of motion is by using the Routhian formalism, according to which the equation of motion for the particle trajectory comes from the Lagrange formulation, and the spin equation of motion comes from the Hamilton formulation. The Routhian arising from action (4.104) is

$$\mathcal{R}_0 = -\, (\partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} x^b + B^a{}_\mu)\, p_a\, u^\mu + \tfrac{1}{2}\, (\overset{\bullet}{A}{}^{ab}{}_\mu - \overset{\bullet}{K}{}^{ab}{}_\mu)\, s_{ab}\, u^\mu. \quad (4.107)$$

The equation of motion for the particle trajectory is obtained from

$$\frac{\delta}{\delta x^\mu} \int \mathcal{R}_0\, ds = 0, \quad (4.108)$$

whereas the equation of motion for the spin tensor follows from

$$\frac{ds_{ab}}{ds} = \{\mathcal{R}_0, s_{ab}\}. \quad (4.109)$$

The four-velocity and the spin angular momentum density must satisfy the constraints

$$s_{ab}\, s^{ab} \;=\; 2\, \mathbf{s}^2 \quad (4.110)$$

$$s_{ab}\, u^a \;=\; 0, \quad (4.111)$$

with $\mathbf{s}$ the particle spin vector. However, since the equations of motions that are obtained from the Routhian $\mathcal{R}_0$ do not satisfy the above constraints, it is necessary to include those constraints in the Routhian. The simplest way to achieve this amounts to the following [149]. First, a new expression for the spin is introduced:

$$\tilde{s}_{ab} = s_{ab} - \frac{s_{ac}u^c u_b}{u^2} - \frac{s_{cb}u^c u_a}{u^2}. \tag{4.112}$$

This new tensor satisfies the Poisson relation (4.105) with the metric

$$\eta_{ab} - u_a u_b/u^2.$$

A new Routhian that incorporates the above constraints is obtained by replacing all s_{ab} in $\mathcal{R}_0$ by $\tilde{s}_{ab}$, and by subtracting from it the term

$$\frac{du^a}{ds}\frac{s_{ab}u^b}{u^2}.$$

It is then given by

$$\mathcal{R} = -\left(\partial_\mu x^a + \stackrel{\bullet}{A}{}^a{}_{b\mu} x^b + B^a{}_\mu\right) p_a\, u^\mu + \tfrac{1}{2}\left(\stackrel{\bullet}{A}{}^{ab}{}_\mu - \stackrel{\bullet}{K}{}^{ab}{}_\mu\right) s_{ab}\, u^\mu - \frac{\stackrel{\bullet\bullet}{\mathcal{D}} u^a}{\mathcal{D}s}\frac{s_{ab}u^b}{u^2},$$

where

$$\frac{\stackrel{\bullet\bullet}{\mathcal{D}} u^a}{\mathcal{D}s} = u^\mu \stackrel{\bullet\bullet}{\mathcal{D}}_\mu u^a,$$

with $\stackrel{\bullet\bullet}{\mathcal{D}}_\mu$ the covariant derivative (3.78). Using this Routhian, the equation of motion for the spin is found to be

$$\frac{\stackrel{\bullet\bullet}{\mathcal{D}} s_{ab}}{\mathcal{D}s} = \left(u_a\, s_{bc} - u_b\, s_{ac}\right)\frac{\stackrel{\bullet\bullet}{\mathcal{D}} u^c}{\mathcal{D}s}, \tag{4.113}$$

which, on account of the equivalence (4.106), coincides with the corresponding result of General Relativity.

Making use of the lagrangian formalism, the next step is to obtain the equation of motion for the trajectory of the particle. Through a tedious but straightforward calculation, it is found to be

$$\frac{\stackrel{\bullet\bullet}{\mathcal{D}}}{\mathcal{D}s}\left(m\, c\, u_c\right) + \frac{\stackrel{\bullet\bullet}{\mathcal{D}}}{\mathcal{D}s}\left(\frac{\mathcal{D}u^a}{\mathcal{D}s}\frac{s_{ac}}{u^2}\right) = -\frac{1}{2}\stackrel{\bullet}{Q}{}^{ab}{}_{\mu\nu}\, s_{ab}\, u^\nu\, h_c{}^\mu, \tag{4.114}$$

where [see Eq. (7.23)]

$$\stackrel{\bullet}{Q}{}^a{}_{b\mu\nu} = \stackrel{\bullet}{\mathcal{D}}_\mu \stackrel{\bullet}{K}{}^a{}_{b\nu} - \stackrel{\bullet}{\mathcal{D}}_\nu \stackrel{\bullet}{K}{}^a{}_{b\mu} + \stackrel{\bullet}{K}{}^a{}_{d\mu}\, \stackrel{\bullet}{K}{}^d{}_{b\nu} - \stackrel{\bullet}{K}{}^a{}_{d\nu}\, \stackrel{\bullet}{K}{}^d{}_{b\mu} \tag{4.115}$$

is a curvature-like tensor, but which depends on torsion only. Using constraints (4.110-4.111), it is easy to verify that

$$\frac{\overset{\bullet\bullet}{\mathcal{D}} u^a}{\mathcal{D}s} \frac{s_{ac}}{u^2} = u^a \, \frac{\overset{\bullet\bullet}{\mathcal{D}} s_{ca}}{\mathcal{D}s}. \tag{4.116}$$

As a consequence, Eq. (4.114) acquires the form

$$\frac{\overset{\bullet\bullet}{\mathcal{D}}}{\mathcal{D}s} \left(mcu_c + u^a \frac{\overset{\bullet\bullet}{\mathcal{D}} s_{ca}}{\mathcal{D}s} \right) = -\,\frac{1}{2}\,\overset{\bullet}{Q}{}^{ab}{}_{\mu\nu}\, s_{ab}\, u^\nu \, h_c{}^\mu. \tag{4.117}$$

Defining the generalized four-momentum

$$\mathcal{P}_c = h_c{}^\mu \, \mathcal{P}_\mu \equiv m\, c\, u_c + u^a \, \frac{\overset{\bullet\bullet}{\mathcal{D}} s_{ca}}{\mathcal{D}s}, \tag{4.118}$$

we get

$$\frac{\overset{\bullet\bullet}{\mathcal{D}} \mathcal{P}_\mu}{\mathcal{D}s} = -\,\frac{1}{2}\,\overset{\bullet}{Q}{}^{ab}{}_{\mu\nu}\, s_{ab}\, u^\nu. \tag{4.119}$$

This is the teleparallel version of the Papapetrou equation. Notice that the particle spin, similarly to the electromagnetic field [see the teleparallel Maxwell equation (10.43)], couples to a curvature-like tensor, which is however a tensor written in terms of torsion only.

When the spin vanishes, it reduces to

$$\frac{\overset{\bullet\bullet}{\mathcal{D}}}{\mathcal{D}s} (mcu_c) = 0, \tag{4.120}$$

which is just the teleparallel force equation for spinless particles

$$\frac{du_c}{ds} - \overset{\bullet}{A}{}^b{}_{c\rho}\, u_b \, u^\rho = \overset{\bullet}{K}{}^b{}_{c\rho}\, u_b\, u^\rho. \tag{4.121}$$

When rewritten in terms of the spin connection of General Relativity, the teleparallel equation of motion (4.119) reduces to the ordinary Papapetrou equation [150]

$$\frac{\overset{\circ}{\mathcal{D}} \mathcal{P}_\mu}{\mathcal{D}s} = -\,\frac{1}{2}\,\overset{\circ}{R}{}^{ab}{}_{\mu\nu}\, s_{ab}\, u^\nu. \tag{4.122}$$

Comment 4.9 When the spin vanishes and the mass is constant, the Papapetrou equation (4.122) becomes to the geodesic equation

$$\frac{\overset{\circ}{\mathcal{D}}}{\mathcal{D}s} (mcu_c) = 0. \tag{4.123}$$

If, however, the mass changes along the trajectory, the above equation gives

$$\frac{\overset{\circ}{\mathcal{D}} u^{\mu}}{\mathcal{D}s} = (\delta^{\mu}_{\nu} - u^{\mu}u_{\nu})\,\overset{\circ}{\mathcal{D}}{}^{\nu}(\ln m) \equiv \partial^{\mu}(\ln m) - u^{\mu}\,\frac{d}{ds}(\ln m). \tag{4.124}$$

The last term, an acceleration which is proportional to the time–variation of the mass logarithm, is reminiscent of the classical velocity–propellant equation for rockets [126].

Chapter 5

Global Formulation for Gravity

Due to their shared abelian gauge structure, Teleparallel Gravity and electromagnetism are similar in several aspects. By analogy to the phase-factor approach to Maxwell's theory, a teleparallel non-integrable phase-factor formalism for gravitation can be developed. It represents the quantum mechanical version of the classical gravitational Lorentz force, and leads to simple descriptions of the Colella-Overhauser-Werner experiment and of the gravitational Aharonov-Bohm effect. In the classical (non-quantum) limit, it reduces to the force equation of Teleparallel Gravity.

5.1 Phase Factor Approach

As is widely known, in addition to the usual *differential* formalism, electromagnetism (as gauge theories in general) has a *global* formulation in terms of a non-integrable phase factor [48, 49]. According to that approach, electromagnetism can be considered as the gauge invariant effect of a non-integrable (path-dependent) phase factor. For a particle with electric charge q traveling from an initial point P to a final point Q, the phase factor is given by

$$\Phi_e(\mathsf{P}|\mathsf{Q}) = \exp\left[\frac{iq}{\hbar c}\int_{\mathsf{P}}^{\mathsf{Q}} A_\mu \, dx^\mu\right], \tag{5.1}$$

where A_μ is the electromagnetic gauge potential. In the classical (non-quantum) limit, the non-integrable phase factor approach yields the same results as those obtained from the Lorentz force equation

$$\frac{du^a}{ds} = \frac{q}{mc^2}\, F^a{}_b\, u^b. \tag{5.2}$$

In this sense, the phase-factor approach can be considered the *quantum* generalization of the *classical* Lorentz force equation. It is actually more general, as it can be used both on simply-connected and on multiply-connected domains. Its use is mandatory, for example, to describe the Aharonov-Bohm effect, a quantum phenomenon taking place in a multiply-connected space. While a differential equation as (5.2) is strictly local, an integrated object as the phase in (5.1) is global, and consequently able to take some topological effects into account.

By its similarity to A_μ, the teleparallel gauge potential $B^a{}_\mu$ can be used to construct a global formulation for gravitation [50]. To start with, let us notice that the electromagnetic phase factor has the form

$$\Phi_e(\mathsf{P}|\mathsf{Q}) = \exp\left[-\frac{i}{\hbar}\,\mathcal{S}_e\right], \tag{5.3}$$

where

$$\mathcal{S}_e = -\,\frac{q}{c}\int_{\mathsf{P}}^{\mathsf{Q}} A_\mu\, dx^\mu \tag{5.4}$$

is the *interaction part* of the action integral for a charged particle within an electromagnetic field. We can similarly write the gravitational phase factor as

$$\Phi_g(\mathsf{P}|\mathsf{Q}) = \exp\left[-\frac{i}{\hbar}\,\mathcal{S}_g\right], \tag{5.5}$$

where $\mathcal{S}_g$ is the *interaction part* of the action integral for a particle of mass m in a gravitational field. From Eq. (4.37) we see that the interaction part of the action is

$$\mathcal{S}_g = -\,mc\int_{\mathsf{P}}^{\mathsf{Q}} B^a{}_\mu\, u_a\, dx^\mu. \tag{5.6}$$

The gravitational phase factor is then written in the form

$$\Phi_g(\mathsf{P}|\mathsf{Q}) = \exp\left[\frac{i\,mc}{\hbar}\int_{\mathsf{P}}^{\mathsf{Q}} B^a{}_\mu\, u_a\, dx^\mu\right]. \tag{5.7}$$

As for the electromagnetic phase factor, it represents the *quantum* mechanical law that replaces the *classical* gravitational Lorentz force equation (4.75).

Observe that, since both the tetrad $h^a{}_\rho$ and the tangent space metric η_{ab} are gauge invariant, both the spacetime metric

$$g_{\mu\nu} = \eta_{ab}\, h^a{}_\mu\, h^a{}_\nu \tag{5.8}$$

and the corresponding interval

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{5.9}$$

are also gauge invariant. In consequence, the four-velocity

$$u^a = h^a\left(\frac{d}{ds}\right) = h^a{}_\mu \, u^\mu \tag{5.10}$$

is found to be gauge invariant.

Considering that any laboratory, even on Earth, can be considered as inertial, we assume the class of frames where the inertial spin connection $\overset{\bullet}{A}{}^a{}_{b\mu}$ vanishes. In this case, under a gauge transformation,

$$B'^a{}_\mu = B^a{}_\mu - \partial_\mu \epsilon^a, \tag{5.11}$$

analogously to the electromagnetic case, the integral in the phase factor (5.7) changes by a surface term. In the remaining of this chapter we are going to assume this specific class of frames.

5.2 Colella-Overhauser-Werner Experiment

As a first application of the gravitational non-integrable phase factor (5.7), we consider the Colella-Overhauser-Werner (COW) experiment [51]. It consists of using a neutron interferometer to observe the quantum mechanical phase shift of neutrons caused by their interaction with Earth's gravitational field, assumed to be Newtonian. As seen in Section 4.4, a Newtonian gravitational field is characterized by the condition that only $B_{00} \neq 0$. Furthermore, as the experience is performed with thermal neutrons, it is possible to use the small velocity approximation, according to which

$$u^0 = \gamma \equiv [1 - (v^2/c^2)]^{-1/2} \simeq 1. \tag{5.12}$$

When acting on the wave function of such particles, therefore, the gravitational phase factor (5.7) assumes the form

$$\Phi_g(\mathsf{P}|\mathsf{Q}) = \exp\left[\frac{im}{\hbar}\int_{\mathsf{P}}^{\mathsf{Q}} c^2 \, B_{00} \, dt\right]. \tag{5.13}$$

According to Eq. (**??**), in the Newtonian approximation the term $c^2 \, B_{00}$ can be identified with the (assumed homogeneous) Earth Newtonian potential, which is here given by

$$c^2 \, B_{00} = g \, z. \tag{5.14}$$

In this expression, g is the gravitational acceleration, assumed not to change significantly in the domain of the experience, and z is the distance from

Earth taken from some reference point. Consequently, the phase factor can be rewritten in the form

$$\Phi_g(\mathsf{P}|\mathsf{Q}) = \exp\left[\frac{img}{\hbar}\int_{\mathsf{P}}^{\mathsf{Q}} z(t)\,dt\right] \equiv \exp i\varphi. \tag{5.15}$$

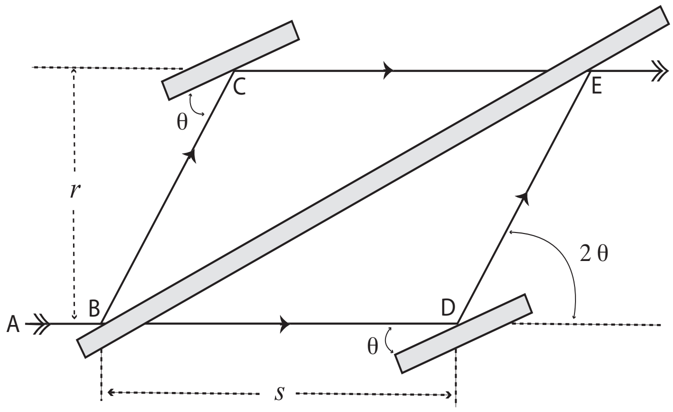

Figure 5.1: *Schematic illustration of the COW neutron interferometer.*

Let us now compute the phase φ through the two trajectories of Fig. 5.1. First, we consider the trajectory BDE. Assuming that the segment BD is at $z = 0$, we obtain

$$\varphi_{\mathsf{BDE}} = \frac{mg}{\hbar}\int_{\mathsf{D}}^{\mathsf{E}} z(t)\,dt. \tag{5.16}$$

For the trajectory BCE, we have

$$\varphi_{\mathsf{BCE}} = \frac{mg}{\hbar}\int_{\mathsf{B}}^{\mathsf{C}} z(t)\,dt + \frac{mgr}{\hbar}\int_{\mathsf{C}}^{\mathsf{E}} dt. \tag{5.17}$$

As the phase contribution along the segments DE and BC are equal, they cancel out from the phase difference

$$\Delta\varphi \equiv \varphi_{\mathsf{BCE}} - \varphi_{\mathsf{BDE}} = \frac{mgr}{\hbar}\int_{\mathsf{C}}^{\mathsf{E}} dt. \tag{5.18}$$

Since the neutron velocity is constant along the segment CE, the integral is

$$\int_{\mathsf{C}}^{\mathsf{E}} dt \equiv \frac{s}{v} = \frac{sm\lambda}{h}, \tag{5.19}$$

where s is the length of the segment CE, and $\lambda = h/(mv)$ is the de Broglie wavelength associated with the neutron. We thus obtain

$$\Delta\varphi = s\,\frac{2\pi g r\lambda m^2}{h^2}, \tag{5.20}$$

which is exactly the gravitationally-induced phase difference of the COW experiment [51].

5.3 Gravitational Aharonov-Bohm Effect

As a second application we use the phase factor (5.7) to study the gravitational analog of the Aharonov-Bohm effect [52]. The usual (electromagnetic) Aharonov-Bohm effect consists in a shift, by a constant amount, of the electron interferometry wave pattern, in a region where there is no magnetic field, but there is a nontrivial electromagnetic gauge potential A_i. Analogously, the gravitational Aharonov-Bohm effect will consist in a similar shift of the same wave pattern, but produced by the presence of a gravitational gauge potential B_{0i}.

Comment 5.1 As the phase shift of the COW experiment is produced by the coupling of the neutron mass with the component B_{00} of the translational gauge potential, it can be considered as a gravitoelectric Aharonov-Bohm effect. A similar denomination is used in the electromagnetic case [53].

Phenomenologically, this kind of effect might be present near a massive, rapidly rotating source, like a neutron star for example. Of course, differently from an ideal apparatus, in a real situation the gravitational field cannot be eliminated, and consequently the gravitational Aharonov-Bohm effect should be added to the other effects also causing a phase change.

Let us consider first the case in which there is no external field at all. If the electrons are emitted with a characteristic momentum p, then its wave-function has the de Broglie wavelength $\lambda = h/p$. Denoting by L the distance between the slit and the screen (see Fig. 5.2), and by d the distance between the two slits, when the conditions $L \gg \lambda$, $L \gg x$ and $L \gg d$ are satisfied the phase difference at a distance x from the central point of the screen is given by

$$\delta^0\varphi(x) = \frac{2\pi x d}{L\lambda}. \tag{5.21}$$

This expression defines the wave pattern on the screen.

We consider now the case in which a kind of infinite *gravitational solenoid* produces a purely static gravitomagnetic field flux concentrated in its interior. In the ideal situation, the gravitational field outside the solenoid vanishes completely, but there is a nontrivial gauge potential B_{0i}. When we let the electrons to move outside the solenoid, phase factors corresponding to paths lying on one side of the solenoid will interfere with phase factors corresponding to paths lying on the other side, which will produce an additional phase shift at the screen. Let us then calculate this additional phase

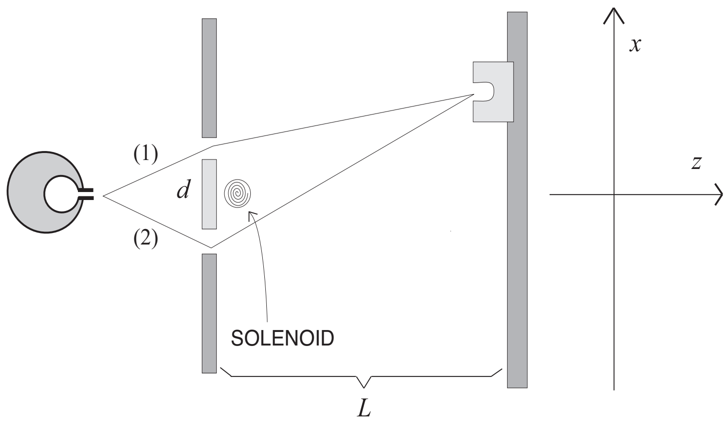

Figure 5.2: *Scheme of the Aharonov-Bohm electron interferometer.*

shift. The gravitational phase factor (5.7) for the physical situation above described is

$$\Phi_g(\mathsf{P}|\mathsf{Q}) = \exp\left[-\frac{imc}{\hbar}\int_{\mathsf{P}}^{\mathsf{Q}} u^0 \vec{B}_0 \cdot d\vec{r}\right], \tag{5.22}$$

where $\vec{B_0}$ is the vector with components ${B_0}^i = -B_{0i}$. Since

$$u^0 = \gamma \equiv [1-(v^2/c^2)]^{-1/2},$$

and considering a constant electron velocity v, we can write

$$\Phi_g(\mathsf{P}|\mathsf{Q}) = \exp\left[-\frac{i\gamma mc}{\hbar}\int_{\mathsf{P}}^{\mathsf{Q}} \vec{B}_0 \cdot d\vec{r}\right]. \tag{5.23}$$

Now, denoting by φ_1 the phase corresponding to a path lying on one side of the solenoid, and by φ_2 the phase corresponding to a path lying on the other side, the phase difference at the screen will be

$$\delta\varphi \equiv \varphi_2 - \varphi_1 = \frac{\gamma mc}{\hbar}\oint \vec{B}_0 \cdot d\vec{r}. \tag{5.24}$$

This can be rewritten in the form

$$\delta\varphi = \frac{\mathcal{E}\,\Omega}{\hbar\, c}, \tag{5.25}$$

where $\mathcal{E} = \gamma mc^2$ is the electron kinetic energy, and

$$\Omega = \oint \vec{B}_0 \cdot d\vec{r} = \oint (\vec{\nabla} \times \vec{B}_0) \cdot d\vec{\sigma} \equiv \oint \frac{\vec{H}}{c^2} \cdot d\vec{\sigma} \tag{5.26}$$

is the flux of gravitomagnetic field $\vec{H}$ inside the solenoid. In components, the gravitomagnetic field $\vec{H}$ is written as

$$\frac{H^i}{c^2} = \tfrac{1}{2}\,\epsilon^{ijk}\left(\partial_j B_{0k} - \partial_k B_{0j}\right) = \tfrac{1}{2}\,\epsilon^{ijk}\,\overset{\bullet}{T}_{0jk}, \tag{5.27}$$

with $\overset{\bullet}{T}_{0jk}$ the torsion of the Weitzenböck connection — that is, the teleparallel field strength. Recalling from Section 1.4 that the axial torsion is defined by

$$\overset{\bullet}{\mathcal{A}}{}^{\mu} = \tfrac{1}{6}\,\epsilon^{\mu\nu\rho\sigma}\,\overset{\bullet}{T}_{\nu\rho\sigma}, \tag{5.28}$$

we see that the gravitomagnetic field coincides with the component

$$\frac{H^i}{c^2} = \overset{\bullet}{\mathcal{A}}{}^{i} \equiv \tfrac{1}{2}\,\epsilon^{i0jk}\,\overset{\bullet}{T}_{0jk} \tag{5.29}$$

of the axial torsion [46].

Expression (5.25) gives the phase difference produced by the interaction of the particle kinetic energy with a gauge potential, which gives rise to the gravitational Aharonov-Bohm effect. As this phase difference depends on the energy, it applies equally to massive and massless particles. There is a difference, though: whereas for massive particles it is a genuine quantum effect, for massless particles, due to their intrinsic wave character, it can be considered as a classical effect. In fact, for $\mathcal{E} = h\omega$, the phase difference becomes

$$\delta\varphi = \frac{\omega\,\Omega}{c}, \tag{5.30}$$

which is seen not to depend on Planck's constant.

Like the electromagnetic Aharonov-Bohm effect, the phase difference is independent of the position x on the screen, and consequently the wave pattern defined by (5.25) will be wholly shifted by a constant amount. Differently from the electromagnetic case, however, the phase difference in the gravitational case depends on the particle kinetic energy, which in turn depends on the particle's mass and velocity. As a consequence, the requirement of invariance of the gravitational phase factor under the translational gauge transformation (5.11) implies that

$$\Omega\mathcal{E} = nhc, \tag{5.31}$$

with n an integer number. We see from this expression that, in contrast to the electromagnetic case, where it is possible to define a *quantum* of magnetic flux, in the gravitational case, due to the presence of the energy, it is not possible to define a particle-independent *quantum* of gravitomagnetic flux [54].

5.4 Quantum Versus Classical Approaches

We proceed now to show that, in the classical limit, the non-integrable phase factor approach reduces to the usual approach provided by the gravitational Lorentz force equation. In electromagnetism, the standard argument is well-known: the phase turning up in the quantum case is exactly the classical action, which leads to the Lorentz force. We intend, however, to illustrate the result directly, and for that we consider again the electron interferometry slit experiment, this time with a homogeneous static gravitomagnetic field $\vec{H}$ permeating the whole region between the slits and the screen (see Fig. 5.3). This field is supposed to point in the negative y-direction, and will produce a phase shift which is to be added to the phase

$$\delta^0\varphi(x) = \frac{2\pi x d}{L\lambda}\,, \tag{5.32}$$

extant in the absence of gravitomagnetic field. This shift, according to Eq. (5.25), is given by $\mathcal{E}\Omega/\hbar c$, with Ω the flux through the surface S circumscribed by the two trajectories. It is easily seen that

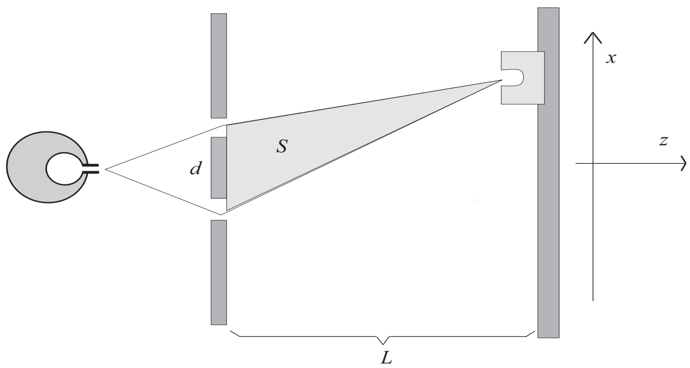

Figure 5.3: *Schematic illustration of the electron interference experiment in the presence of a gravitomagnetic field. The only contribution to the phase–shift comes from the flux inside the surface S delimited by the two trajectories.*

$$S = \frac{Ld}{2} \tag{5.33}$$

for any value of x. The flux is, consequently,

$$\Omega = -\,\frac{H_y L d}{2c^2}, \tag{5.34}$$

where $\vec{H} = -\, H_y \, \hat{e}_y$, with $\hat{e}_y$ a unity vector in the y direction. Therefore, the total phase difference will be

$$\delta\varphi = \frac{2\pi x d}{L\lambda} - \frac{\mathcal{E} H_y L d}{2\hbar c^3}. \tag{5.35}$$

This is the total phase–shift yielded by the phase–factor approach.

In the classical limit [55], the slit experiment can be interpreted in the following way. The electrons traveling through the gravitomagnetic field have their movement direction changed. This means that they are subjected to a force in the x-direction. For $x \ll L$, we can approximately write the electrons velocity as $v \simeq v_z$. In this case, they will be transversally accelerated by the gravitomagnetic field during the time interval

$$\Delta t = \frac{L}{v_z}. \tag{5.36}$$

This transversal x-acceleration is given by

$$a_x = \frac{2x}{(\Delta t)^2}. \tag{5.37}$$

Since the attained acceleration is constant, we can choose a specific point to calculate it. Let us then consider the point of maximum intensity on the screen, which is determined by the condition $\delta\varphi = 0$. This yields

$$x = \frac{H_y \, \lambda \, L^2 \, \mathcal{E}}{2 \, h \, c^3}. \tag{5.38}$$

The acceleration is then found to be

$$a_x = \frac{H_y \, \lambda \, \mathcal{E} \, v_z^2}{h \, c^3}. \tag{5.39}$$

From the classical point of view, therefore, we can say that the electrons experience a force in the x-direction given by

$$\mathcal{F}_x \equiv \gamma \, m \, a_x = \frac{\mathcal{E} \, v_z \, H_y}{c} \, \frac{\lambda \, p}{h}, \tag{5.40}$$

with $p = \gamma m v_z$ the electron momentum. Using the de Broglie relation $\lambda = h/p$, the Planck constant is eliminated, and we get the classical result

$$\mathcal{F}_x = \frac{\mathcal{E}}{c^3} \, v_z H_y = \frac{\mathcal{E}}{c^3} \, (\vec{v} \times \vec{H})_x. \tag{5.41}$$

The corresponding equation of motion is

$$\frac{\overset{\bullet}{\nabla} p_x}{\nabla t} = \frac{\mathcal{E}}{c^3} \, (\vec{v} \times \vec{H})_x, \tag{5.42}$$

where $\overset{\bullet}{\nabla}/\nabla t$ represents a time covariant derivative in the Weitzenböck connection.

Comment 5.2 The force equation (5.42) is quite similar to the electromagnetic Lorentz force, with the kinetic energy $\mathcal{E} = \gamma m c^2$ replacing the electric charge, and the gravitomagnetic vector field $\vec{H}$ replacing the usual magnetic field.

Let us show now that equation of motion (5.42) coincides with the x-component of the gravitational Lorentz force equation (4.57). Using $p^i = \gamma m v^i$, we can write

$$\frac{\overset{\bullet}{\nabla} v_i}{\nabla t} = \frac{1}{c} \, (\vec{v} \times \vec{H})_i \equiv \frac{1}{c} \, \epsilon_{ijk} \, v^j \, H^k. \tag{5.43}$$

From Eq. (5.27), however, we see that $\epsilon_{ijk} \, H^k = c^2 \, \overset{\bullet}{T}_{0ij}$. In terms of torsion components, therefore, we have

$$\frac{\overset{\bullet}{\nabla} v_i}{\nabla t} = c \, \overset{\bullet}{T}_{0ij} \, v^j. \tag{5.44}$$

Furthermore, a gravitomagnetic field ${B^0}_i$ does not change the time components of the spacetime metric g_{00}, which remains that of a Minkowski spacetime. This means that $dt = (\gamma/c) ds$, and consequently we can write

$$\frac{\overset{\bullet}{\nabla} v_i}{\nabla s} = \gamma \, \overset{\bullet}{T}_{0ij} \, v^j. \tag{5.45}$$

Using the relations $u^0 = \gamma$ and $u^j = (\gamma v^j)/c$, as well as the fact that the gravitomagnetic field does not change the absolute value of the particle velocity, and consequently $d\gamma/ds = 0$, we obtain

$$\frac{\overset{\bullet}{\nabla} u_i}{\nabla s} = \overset{\bullet}{T}_{0ij} \, u^0 \, u^j. \tag{5.46}$$

This force equation is a particular case of the gravitational Lorentz force equation [see Eq. (4.57)],

$$\frac{\overset{\bullet}{\nabla} u_\mu}{\nabla s} \equiv \frac{du_\mu}{ds} - {\overset{\bullet}{\Gamma}{}^\lambda}_{\mu\rho} \, u_\lambda \, u^\rho = \overset{\bullet}{T}_{\lambda\mu\rho} \, u^\lambda \, u^\rho, \tag{5.47}$$

for the case in which only the axial component of $\overset{\bullet}{T}_{0ij}$ is non-vanishing. We see in this way that the phase factor approach reduces, in the classical limit, to the gravitational Lorentz force equation, which is equivalent to the geodesic equation of General Relativity (as discussed in Section 4.2.3).

Chapter 6

Hodge Dual for Soldered Bundles

To account for all possible contractions allowed by the presence of the solder form, a generalized Hodge dual must be defined in the case of soldered bundles. Although for curvature the generalized dual coincides with the usual one, for torsion it gives a completely new dual definition.

6.1 Why a New Dual

Let Ω^p be the space of p-forms on an n-dimensional manifold $\mathbf{R}$ with metric $g_{\mu\nu}$. Since vector spaces Ω^p and Ω^{n-p} have the same finite dimension, they are isomorphic. The presence of a metric renders it possible to single out an unique isomorphism, called Hodge dual. Using a coordinate basis, the Hodge dual of a p-form $\alpha^p \in \Omega^p$,

$$\alpha^p = \frac{1}{p!}\, \alpha_{\mu_1 \cdots \mu_p}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}, \tag{6.1}$$

is the $(n-p)$-form $\star\alpha^p \in \Omega^{n-p}$ defined by

$$\star\, \alpha^p = \frac{h}{(n-p)!p!}\, \epsilon_{\mu_1 \mu_2 \cdots \mu_n}\, \alpha^{\mu_1 \cdots \mu_p}\, dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}, \tag{6.2}$$

where $h = \det(h^a{}_\mu) = \sqrt{-g}$, with $g = \det(g_{\mu\nu})$, and $\epsilon_{\mu_1 \mu_2 \cdots \mu_n}$ is the totally anti–symmetric Levi-Civita symbol discussed in Comment 1.6. The operator $\star$ satisfies the property

$$\star\star\, \alpha^p = (-1)^{p(n-p)+(n-s)/2}\, \alpha^p, \tag{6.3}$$

where s is the metric signature (see Section 1.1). Its inverse is, consequently,

$$\star^{-1} = (-1)^{p(n-p)+(n-s)/2} \star . \tag{6.4}$$

This dual operator can be used to define an inner product on Ω^p, given by

$$\alpha^p \wedge \star \beta^p = < \alpha^p, \beta^p > vol, \tag{6.5}$$

where *vol* is the volume element. Conversely, given an inner product, Eq. (6.5) can be used to define the dual operator.

For non-soldered bundles, the dual operator can be generalized in a straightforward way to act on vector-valued p-forms. Let β be a vector-valued p-form on the n-dimensional base space $\mathbf{R}$, taking values on a vector space $\mathbf{V}$. Its dual is the vector-valued $(n-p)$-form

$$\star \beta^p = \frac{h}{(n-p)!p!} \epsilon_{\mu_1\mu_2\ldots\mu_n} J_A \beta^{A\,\mu_1\ldots\mu_p} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}, \tag{6.6}$$

where the set $\{J_A\}$ is a basis for the vector space $\mathbf{V}$. In this case, the components $\beta^{A\,\mu_1\ldots\mu_p}$ have also an internal space index, which is not related to the external indices μ_i. As an example, let us consider the Yang–Mills field strength 2-form $F^A{}_{\mu\nu}$ in a four dimensional spacetime. Its dual is

$$\star F^A{}_{\mu\nu} = \frac{h}{2} \epsilon_{\mu\nu\rho\sigma} F^{A\rho\sigma}. \tag{6.7}$$

For soldered bundles, on the other hand, the situation is completely different. *Due to the presence of the solder form, internal and external indices can be transformed into each other, and this feature leads to the possibility of defining new dual operators* [56], each one of them related to an inner product on Ω^p. The main requirement of these new definitions is that, since (6.3) is still valid for p-forms on soldered bundles, we want to make it true also for vector-valued p-forms. As usual, the inner product $< \alpha^p, \beta^p >$ of two vector-valued p-forms α^p and β^b will be defined by

$$\mathrm{tr}\,(\alpha^p \wedge \star \beta^p) = < \alpha^p, \beta^p > vol. \tag{6.8}$$

We consider next, in a four dimensional spacetime, the specific cases of torsion and curvature, quantities related to connections living in soldered bundles.

6.2 Dual Torsion

Differently from internal (non-soldered) gauge theories, whose dual is defined by equation (6.6), in soldered bundles algebraic and spacetime indices can be transformed into each other through the use of the tetrad field. This property opens up the possibility of new contractions in relation to the usual definition (6.7). For the case of torsion, a general dual definition involving all possible index contractions is of the form

$$\star T^{\lambda}{}_{\mu\nu} = h\, \epsilon_{\mu\nu\rho\sigma} \left(a\, T^{\lambda\rho\sigma} + b\, T^{\rho\lambda\sigma} + c\, T^{\theta\rho}{}_{\theta}\, g^{\lambda\sigma} \right), \tag{6.9}$$

with a, b, c constant coefficients. It can be shown, however, that two coefficients suffice to define the generalized dual torsion. One way to see this is to observe that the first and second terms of (6.9) differ just by a permutation of indices. Since $T^{\lambda\rho\sigma}$ is anti-symmetric in the last two indices, whereas $T^{\lambda\rho\sigma}$ has no definite symmetry in the first and third indices, and considering that both are contracted with $\epsilon_{\mu\nu\rho\sigma}$, the first term contributes with half the number of independent terms in relation to the second term. This means that, in order to eliminate equivalent contractions, it is necessary that the coefficient a be half the value of b. We then set

$$b = 2a, \tag{6.10}$$

which yields

$$\star T^{\lambda}{}_{\mu\nu} = h\, \epsilon_{\mu\nu\rho\sigma} \left(a\, T^{\lambda\rho\sigma} + 2a\, T^{\rho\lambda\sigma} + c\, T^{\theta\rho}{}_{\theta}\, g^{\lambda\sigma} \right). \tag{6.11}$$

Comment 6.1 If we had chosen instead the alternative condition $b = -2a$, we would end up with an inconsistent algebraic system of equations for the coefficients a and c.

Now, in a four–dimensional spacetime with metric signature $s = 2$ the dual torsion must, as any 2-form, satisfy the relation

$$\star\star\, T^{\rho}{}_{\mu\nu} = -\, T^{\rho}{}_{\mu\nu}. \tag{6.12}$$

This condition yields the following algebraic system:

$$\begin{aligned} 8a^2 - 2ac &= 1 && (6.13) \\ 8a^2 + 2ac &= 0. && (6.14) \end{aligned}$$

There are two solutions, which differ by a global sign:

$$a = 1/4 \qquad c = -1 \tag{6.15}$$

and

$$a = -1/4 \qquad c = 1 \tag{6.16}$$

Since we are looking for a generalization of the usual expression (6.7), we choose the solution with $a > 0$. In this case, the generalized dual torsion is found to be

$$\star\, T^{\rho}{}_{\mu\nu} = h\, \epsilon_{\mu\nu\alpha\beta} \left(\tfrac{1}{4}\, T^{\rho\alpha\beta} + \tfrac{1}{2}\, T^{\alpha\rho\beta} - \, T^{\lambda\alpha}{}_{\lambda}\, g^{\rho\beta} \right). \tag{6.17}$$

Defining the tensor

$$S^{\rho\mu\nu} = -\, S^{\rho\nu\mu} := K^{\mu\nu\rho} - g^{\rho\nu} T^{\sigma\mu}{}_{\sigma} + g^{\rho\mu} T^{\sigma\nu}{}_{\sigma}, \tag{6.18}$$

the dual torsion can be rewritten in the form [56]

$$\star\, T^{\rho}{}_{\mu\nu} = \frac{h}{2}\, \epsilon_{\mu\nu\alpha\beta}\, S^{\rho\alpha\beta}. \tag{6.19}$$

Comment 6.2 Another way to see that two coefficients suffice to define the generalized dual torsion is to use the fact that torsion can be decomposed into irreducible components under the global Lorentz group (see Section 1.4). In terms of these components, the dual torsion reads

$$T_{\lambda\mu\nu} = \tfrac{2}{3}\, (\mathcal{T}_{\lambda\mu\nu} - \mathcal{T}_{\lambda\nu\mu}) + \tfrac{1}{3}\, (g_{\lambda\mu} \mathcal{V}_{\nu} - g_{\lambda\nu} \mathcal{V}_{\mu}) + \epsilon_{\lambda\mu\nu\rho}\, \mathcal{A}^{\rho}, \tag{6.20}$$

where

$$\mathcal{V}_{\mu} = T^{\nu}{}_{\nu\mu} \quad \text{and} \quad \mathcal{A}^{\mu} = \tfrac{1}{6}\, \epsilon^{\mu\nu\rho\sigma}\, T_{\nu\rho\sigma} \tag{6.21}$$

are respectively the vector and axial vector parts, and

$$\mathcal{T}_{\lambda\mu\nu} = \tfrac{1}{2}\, (T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \tfrac{1}{6}\, (g_{\nu\lambda} \mathcal{V}_{\mu} + g_{\nu\mu} \mathcal{V}_{\lambda}) - \tfrac{1}{3} g_{\lambda\mu}\, \mathcal{V}_{\nu}, \tag{6.22}$$

is a purely tensor part. Using the generalized dual definition (6.9), a simple calculation shows that

$$\star\, \mathcal{V}_{\mu} = 6ha\, \mathcal{A}_{\mu} \equiv A\, h\, \mathcal{A}_{\mu} \tag{6.23}$$

and

$$\star\, \mathcal{A}_{\mu} = \frac{1}{3h} (4a + 3c)\, \mathcal{V}_{\mu} \equiv \frac{B}{h}\, \mathcal{V}_{\mu}, \tag{6.24}$$

where A and B are two new parameters which, on account of the property (6.12), satisfy the relation $A\, B = -1$. In terms of the irreducible components, the generalized dual torsion is easily seen to be

$$\star\, T^{\lambda}{}_{\mu\nu} = h \Big[\pm \frac{2}{3}\, \epsilon_{\mu\nu\alpha\beta}\, \mathcal{T}^{\lambda\alpha\beta} + \frac{A}{3} (\delta^{\lambda}_{\mu} \mathcal{A}_{\nu} - \delta^{\lambda}_{\nu} \mathcal{A}_{\mu}) + \frac{B}{h^2}\, \epsilon^{\lambda}{}_{\mu\nu\rho}\, \mathcal{V}^{\rho} \Big]. \tag{6.25}$$

We see from this expression that two parameters suffice to define the generalized dual.

6.3 Dual Curvature

Let us consider now the curvature tensor. In a way analogous to the torsion case, we define its generalized dual by taking into account all possible contractions,

$$\star\, R^{\alpha\beta}{}_{\mu\nu} = h\, \epsilon_{\mu\nu\rho\sigma} \Big[a\, R^{\alpha\beta\rho\sigma} + b(R^{\alpha\rho\beta\sigma} - R^{\beta\rho\alpha\sigma}) + c(g^{\alpha\rho}\, R^{\beta\sigma} - g^{\beta\rho}\, R^{\alpha\sigma}) + d\, g^{\alpha\rho}\, g^{\beta\sigma}\, R \Big], \tag{6.26}$$

with a, b, c, d constants coefficients. Of course, since the curvature 2-form takes values on the Lie algebra of the Lorentz group, the above definition is anti–symmetric in α and β. By requiring, as in (6.12), that

$$\star\star\, R^{\alpha\beta}{}_{\mu\nu} = -\, R^{\alpha\beta}{}_{\mu\nu}, \tag{6.27}$$

we obtain a system of algebraic equations for a, b, c, d, whose solution with is

$$a = 1/2 \quad \text{and} \quad b = c = d = 0. \tag{6.28}$$

For the curvature, therefore, the generalized dual coincides with the usual expression, that is,

$$\star\, R^{\alpha\beta}{}_{\mu\nu} = \frac{h}{2}\, \epsilon_{\mu\nu\rho\sigma}\, R^{\alpha\beta\rho\sigma}. \tag{6.29}$$

This means that the additional index contractions related to soldering do not generate any additional contributions to the dual of curvature.

Chapter 7

Lagrangian and Field Equations

The lagrangian of Teleparallel Gravity is, like those of gauge theories, written in terms of contractions of its field strength — here represented by the torsion of the Weitzenböck connection. Nevertheless, due to the interchangeable character of algebraic and spacetime indices, additional contractions are possible and lead to a higher number of terms. These additional terms naturally yield the lagrangian of the teleparallel equivalent of General Relativity. The ensuing field equations are obtained.

7.1 Lagrangian of Teleparallel Gravity

As a gauge theory for the translation group, the gravitational action of Teleparallel Gravity can be written as

$$\overset{\bullet}{\mathcal{S}} = \frac{c^3}{16\pi G} \int \mathrm{tr} \left(\overset{\bullet}{T} \wedge \star \overset{\bullet}{T} \right), \tag{7.1}$$

where

$$\overset{\bullet}{T} = \tfrac{1}{2} \, \overset{\bullet}{T}{}^a{}_{\mu\nu} \, P_a \, dx^\mu \wedge dx^\nu \tag{7.2}$$

is the torsion 2-form, and

$$\star \, \overset{\bullet}{T} = \tfrac{1}{2} \left(\star \overset{\bullet}{T}{}^a{}_{\rho\sigma} \right) \, P_a \, dx^\rho \wedge dx^\sigma \tag{7.3}$$

is the corresponding dual form. There is, however, a problem: the translation group is abelian, its Cartan–Killing bilinear form is degenerate and cannot be used as a metric. It is necessary to use another invariant metric. To find out which one, we recall that the group manifold of translations is just the Minkowski spacetime $\mathbf{M}$, the quotient space between the Poincaré ($\mathcal{P}$) and the Lorentz ($\mathcal{L}$) groups:

$$\mathbf{M} = \mathcal{P}/\mathcal{L}.$$

In other words, $\mathbf{M}$ is homogeneous — or transitive — under spacetime translations. The role of the Cartan–Killing metric comes, when it exists, from its being invariant under the group action. Here it does not exist, but we can use the invariant Lorentz metric η_{ab} of $\mathbf{M}$ in its stead.

Comment 7.1 This is quite similar to electromagnetism, a gauge theory for the abelian $U(1)$ group. Also in this case the Cartan–Killing metric is degenerate and must be replaced by a different invariant metric. In general the trivial one-dimensional metric $\eta = +1$ is chosen, though the choice $\eta = -1$ would also be possible.

Action (7.1) can then be written as

$$\stackrel{\bullet}{\mathcal{S}} = \frac{c^3}{16\pi G} \int \eta_{ab} \stackrel{\bullet}{T}{}^a \wedge \star \stackrel{\bullet}{T}{}^b, \tag{7.4}$$

or, equivalently,

$$\stackrel{\bullet}{\mathcal{S}} = \frac{c^3}{64\pi G} \int \eta_{ab} \stackrel{\bullet}{T}{}^a{}_{\mu\nu} \left(\star \stackrel{\bullet}{T}{}^b{}_{\rho\sigma} \right) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma. \tag{7.5}$$

Taking into account that

$$dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = - \epsilon^{\mu\nu\rho\sigma} h^2 d^4x, \tag{7.6}$$

with $h = \sqrt{-g}$, the action functional reduces to

$$\stackrel{\bullet}{\mathcal{S}} = - \frac{c^3}{64\pi G} \int \stackrel{\bullet}{T}{}_{\alpha\mu\nu} \left(\star \stackrel{\bullet}{T}{}^\alpha{}_{\rho\sigma} \right) \epsilon^{\mu\nu\rho\sigma} h^2 d^4x, \tag{7.7}$$

where we have used the identity

$$\stackrel{\bullet}{T}{}^\alpha{}_{\rho\sigma} = h_a{}^\alpha \stackrel{\bullet}{T}{}^a{}_{\rho\sigma}. \tag{7.8}$$

Using then the generalized dual (6.19), as well as the identity (1.54), the action assumes the form [57]

$$\stackrel{\bullet}{\mathcal{S}} = \frac{c^3}{32\pi G} \int \stackrel{\bullet}{T}{}_{\rho\mu\nu} \stackrel{\bullet}{S}{}^{\rho\mu\nu} h\, d^4x, \tag{7.9}$$

where

$$\stackrel{\bullet}{S}{}^{\rho\mu\nu} = - \stackrel{\bullet}{S}{}^{\rho\nu\mu} = \stackrel{\bullet}{K}{}^{\mu\nu\rho} - g^{\rho\nu} \stackrel{\bullet}{T}{}^{\sigma\mu}{}_\sigma + g^{\rho\mu} \stackrel{\bullet}{T}{}^{\sigma\nu}{}_\sigma \tag{7.10}$$

is the tensor introduced in Section 6.2, which is usually called *superpotential*, and

$$\stackrel{\bullet}{K}{}^\nu{}_{\rho\mu} = \tfrac{1}{2} \left(\stackrel{\bullet}{T}{}_\rho{}^\nu{}_\mu + \stackrel{\bullet}{T}{}_\mu{}^\nu{}_\rho - \stackrel{\bullet}{T}{}^\nu{}_{\rho\mu} \right) \tag{7.11}$$

is the contortion of the Weitzenböck connection. The corresponding lagrangian density is

$$\overset{\bullet}{\mathcal{L}} = \frac{c^4 h}{32\pi G} \, \overset{\bullet}{S}{}^{\rho\mu\nu} \, \overset{\bullet}{T}{}_{\rho\mu\nu}. \tag{7.12}$$

Making use of the identity

$$\overset{\bullet}{T}{}^{\mu}{}_{\mu\rho} = \overset{\bullet}{K}{}^{\mu}{}_{\rho\mu}, \tag{7.13}$$

it can alternatively be written in the form

$$\overset{\bullet}{\mathcal{L}} = \frac{c^4 h}{16\pi G} \left(\overset{\bullet}{K}{}^{\mu\nu\rho} \overset{\bullet}{K}{}_{\rho\nu\mu} - \overset{\bullet}{K}{}^{\mu\rho}{}_{\mu} \overset{\bullet}{K}{}^{\nu}{}_{\rho\nu} \right). \tag{7.14}$$

Substituting $\overset{\bullet}{K}{}^{\rho\mu\nu}$, the teleparallel lagrangian becomes

$$\overset{\bullet}{\mathcal{L}} = \frac{c^4 h}{16\pi G} \left(\tfrac{1}{4} \, \overset{\bullet}{T}{}^{\rho}{}_{\mu\nu} \, \overset{\bullet}{T}{}_{\rho}{}^{\mu\nu} + \tfrac{1}{2} \, \overset{\bullet}{T}{}^{\rho}{}_{\mu\nu} \, \overset{\bullet}{T}{}^{\nu\mu}{}_{\rho} - \overset{\bullet}{T}{}^{\rho}{}_{\mu\rho} \, \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu} \right). \tag{7.15}$$

The first term corresponds to the usual lagrangian of internal gauge theories. The existence of the other two terms is related to presence of a tetrad field, which leads to new possible contractions — as discussed in Chapter 6.

Comment 7.2 If we rewrite a similar lagrangian with three free parameters, but with the coefficient of anholonomy replacing torsion, we get

$$\overset{\bullet}{\mathcal{L}} = \frac{c^4 h}{16\pi G} \left(\alpha \, f^{a}{}_{bc} f_{a}^{bc} + \beta \, f^{a}{}_{bc} \, f^{cb}{}_{a} - \gamma \, f^{a}{}_{ba} \, f^{cb}{}_{\nu} \right), \tag{7.16}$$

where α, β, γ are constants. Up to a surface term, this lagrangian is invariant under local Lorentz transformation only if the parameters satisfy the relation

$$\beta = 2\alpha \quad \text{and} \quad \gamma = -4\alpha. \tag{7.17}$$

For $\alpha = 1/4$, we get exactly the same coefficients of (7.15) [62].

7.2 Equivalence with Einstein–Hilbert

Substituting the relation

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + \overset{\bullet}{K}{}^{\rho}{}_{\mu\nu} \tag{7.18}$$

in expression (3.91) for the curvature of the Weitzenböck connection, we find

$$\overset{\bullet}{R}{}^{\rho}{}_{\theta\mu\nu} = \overset{\circ}{R}{}^{\rho}{}_{\theta\mu\nu} + \overset{\bullet}{Q}{}^{\rho}{}_{\theta\mu\nu} \equiv 0, \tag{7.19}$$

where

$$\overset{\circ}{R}{}^{\rho}{}_{\theta\mu\nu} = \partial_\mu \overset{\circ}{\Gamma}{}^{\rho}{}_{\theta\nu} - \partial_\nu \overset{\circ}{\Gamma}{}^{\rho}{}_{\theta\mu} + \overset{\circ}{\Gamma}{}^{\rho}{}_{\sigma\mu} \, \overset{\circ}{\Gamma}{}^{\sigma}{}_{\theta\nu} - \overset{\circ}{\Gamma}{}^{\rho}{}_{\sigma\nu} \, \overset{\circ}{\Gamma}{}^{\sigma}{}_{\theta\mu} \tag{7.20}$$

is the curvature of the Levi–Civita connection, and

$$\begin{aligned} \overset{\bullet}{Q}{}^{\rho}{}_{\theta\mu\nu} = \; & \partial_\mu \overset{\bullet}{K}{}^{\rho}{}_{\theta\nu} - \partial_\nu \overset{\bullet}{K}{}^{\rho}{}_{\theta\mu} + \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\mu} \, \overset{\bullet}{K}{}^{\sigma}{}_{\theta\nu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\nu} \, \overset{\bullet}{K}{}^{\sigma}{}_{\theta\mu} \\ & - \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\theta\mu} \, \overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu} + \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\theta\nu} \, \overset{\bullet}{K}{}^{\rho}{}_{\sigma\mu} + \overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu} \, \overset{\bullet}{K}{}^{\sigma}{}_{\theta\mu} - \overset{\bullet}{K}{}^{\rho}{}_{\sigma\mu} \, \overset{\bullet}{K}{}^{\sigma}{}_{\theta\nu} \end{aligned} \tag{7.21}$$

is a tensor written in terms of the Weitzenböck connection only. Like the Riemann curvature tensor, it is a 2-form assuming values in the Lie algebra of the Lorentz group,

$$\overset{\bullet}{Q} = \tfrac{1}{2} \, S_a{}^b \, \overset{\bullet}{Q}{}^a{}_{b\mu\nu} \, dx^\mu \wedge dx^\nu, \tag{7.22}$$

with the components given by

$$\overset{\bullet}{Q}{}^a{}_{b\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu \overset{\bullet}{K}{}^a{}_{b\nu} - \overset{\bullet}{\mathcal{D}}_\nu \overset{\bullet}{K}{}^a{}_{b\mu} + \overset{\bullet}{K}{}^a{}_{c\nu} \, \overset{\bullet}{K}{}^c{}_{b\mu} - \overset{\bullet}{K}{}^a{}_{c\mu} \, \overset{\bullet}{K}{}^c{}_{b\nu}. \tag{7.23}$$

By taking appropriate contractions, it is easy to show that the scalar version of Eq. (7.19) is

$$- \overset{\circ}{R} = \overset{\bullet}{Q} \equiv \left(\overset{\bullet}{K}{}^{\mu\nu\rho} \overset{\bullet}{K}{}_{\rho\nu\mu} - \overset{\bullet}{K}{}^{\mu\rho}{}_{\mu} \overset{\bullet}{K}{}^{\nu}{}_{\rho\nu} \right) + \frac{2}{h} \partial_\mu (h \, \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}). \tag{7.24}$$

Comparing with Eq. (7.14), we see that

$$\overset{\bullet}{\mathcal{L}} = \overset{\circ}{\mathcal{L}} - \partial_\mu \left(\frac{c^4 h}{8\pi G} \, \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu} \right), \tag{7.25}$$

where

$$\overset{\circ}{\mathcal{L}} = - \frac{c^4}{16\pi G} \, \sqrt{-g} \, \overset{\circ}{R} \tag{7.26}$$

is the Einstein–Hilbert lagrangian of General Relativity. Up to a divergence, therefore, the lagrangian of Teleparallel Gravity is equivalent to the lagrangian of General Relativity.

The Einstein–Hilbert lagrangian (7.26) depends on the metric, as well as on the first and second derivatives of the metric. The terms containing second derivatives, however, form a surface, or divergence term [58]. In consequence, it is possible to rewrite Einstein–Hilbert lagrangian in the form

$$\overset{\circ}{\mathcal{L}} = \overset{\circ}{\mathcal{L}}_1 + \partial_\mu (\sqrt{-g} \, w^\mu), \tag{7.27}$$

where $\overset{\circ}{\mathcal{L}}_1$ is a first-order lagrangian — it depends only on the metric and its first derivatives. Of course, $\overset{\circ}{\mathcal{L}}_1$ is not by itself a scalar. However, since it changes by a divergence under general coordinate transformations, there are actually infinitely many different lagrangians $\overset{\circ}{\mathcal{L}}_1$, each one with its particular surface term $\sqrt{-g}\, w^\mu$. One specific first-order lagrangian for General Relativity is the Møller lagrangian [8]

$$\overset{\circ}{\mathcal{L}}_{\rm M} = \frac{c^4 h}{16\pi G} \left(\overset{\circ}{\nabla}_\mu h^{a\nu}\, \overset{\circ}{\nabla}_\nu h_a{}^\mu - \overset{\circ}{\nabla}_\mu h_a{}^\mu\, \overset{\circ}{\nabla}_\nu h^{a\nu} \right). \tag{7.28}$$

The interesting point of this lagrangian is that, in the class of frames h'_a in which the teleparallel spin connection vanishes, it is found to coincide exactly — that is, without any surface term — with the teleparallel lagrangian:

$$\overset{\circ}{\mathcal{L}}_{\rm M} = \overset{\bullet}{\mathcal{L}}{}'. \tag{7.29}$$

7.3 Matter Energy-Momentum Density

The action integral of a general matter field is

$$\mathcal{S}_m = \frac{1}{c} \int \mathcal{L}_m\, d^4x. \tag{7.30}$$

The lagrangian $\mathcal{L}_m$ is assumed to depend only on the fields and on their first derivatives. Thus, under an arbitrary variation $\delta h_a{}^\mu$ of the tetrad field, the action variation is written in the form

$$\delta\mathcal{S}_m = \frac{1}{c} \int \Theta^a{}_\mu\, \delta h_a{}^\mu\, h\, d^4x, \tag{7.31}$$

where

$$h\, \Theta^a{}_\mu = \frac{\delta\mathcal{L}_m}{\delta B_a{}^\mu} \equiv \frac{\delta\mathcal{L}_m}{\delta h_a{}^\mu} = \frac{\partial\mathcal{L}_m}{\partial h_a{}^\mu} - \partial_\lambda \frac{\partial\mathcal{L}_m}{\partial_\lambda \partial h_a{}^\mu} \tag{7.32}$$

is the matter energy-momentum tensor.

Let us consider first an infinitesimal Lorentz transformation

$$\Lambda_a{}^b = \delta_a{}^b + \epsilon_a{}^b,$$

with $\epsilon_a{}^b = -\,\epsilon^b{}_a$, the transformation parameter. Under such transformation, the tetrad changes according to

$$\delta h_a{}^\mu = \epsilon_a{}^b\, h_b{}^\mu. \tag{7.33}$$

The requirement of invariance of the action under local Lorentz transformations, therefore, yields

$$\int \Theta^a{}_b\, \epsilon_a{}^b\, h\, d^4x = 0. \tag{7.34}$$

Since $\epsilon_a{}^b$ is anti–symmetric, the energy–momentum tensor must be symmetric to yield a vanishing result [35].

Let us consider now a general transformation of the spacetime coordinates,

$$x'^{\rho} = x^{\rho} + \xi^{\rho}. \tag{7.35}$$

Under such transformation, the tetrad changes according to

$$\delta h_a{}^{\mu} \equiv h'_a{}^{\mu}(x) - h_a{}^{\mu}(x) = h_a{}^{\rho}\, \partial_{\rho}\xi^{\mu} - \xi^{\rho}\, \partial_{\rho} h_a{}^{\mu}. \tag{7.36}$$

Substituting in (7.31), we obtain

$$\delta \mathcal{S}_m = \frac{1}{c} \int \Theta^a{}_{\mu}\, \left[h_a{}^{\rho}\, \partial_{\rho}\xi^{\mu} - \xi^{\rho}\, \partial_{\rho} h_a{}^{\mu}\right] h\, d^4x, \tag{7.37}$$

or equivalently

$$\delta \mathcal{S}_m = \frac{1}{c} \int \left[\Theta^{\rho}{}_c\, \partial_{\rho}\xi^c + \xi^c\, \Theta^{\rho}{}_{\mu}\, \partial_{\rho} h_c{}^{\mu} - \xi^{\rho}\, \partial_{\rho} h_a{}^{\mu}\right] h\, d^4x, \tag{7.38}$$

Substituting the identity

$$\partial_{\rho} h_a{}^{\mu} = \overset{\bullet}{A}{}^{b}{}_{a\rho}\, h_b{}^{\mu} - \overset{\bullet}{\Gamma}{}^{\mu}{}_{\lambda\rho}\, h_a{}^{\lambda}, \tag{7.39}$$

which follows from Eq. (3.87) and making use of the symmetry of the energy-momentum tensor, the action variation assumes the form

$$\delta \mathcal{S}_m = \frac{1}{c} \int \Theta_c{}^{\rho} \left[\partial_{\rho}\xi^c + (\overset{\bullet}{A}{}^{c}{}_{b\rho} - \overset{\bullet}{K}{}^{c}{}_{b\rho})\, \xi^b\right] h\, d^4x. \tag{7.40}$$

Integrating the first term by parts, neglecting the surface term, the invariance of the action yields

$$\int \left[\partial_{\mu}(h\Theta_a{}^{\mu}) - (\overset{\bullet}{A}{}^{b}{}_{a\mu} - \overset{\bullet}{K}{}^{\cdot b}{}_{a\mu})\, (h\, \Theta_b{}^{\mu})\right] \xi^a\, h\, d^4x = 0. \tag{7.41}$$

From the arbitrariness of ξ^c, it follows that

$$\overset{\bullet\bullet}{\mathcal{D}}_{\mu}(h\Theta_a{}^{\mu}) \equiv \partial_{\mu}(h\Theta_a{}^{\mu}) - (\overset{\bullet}{A}{}^{b}{}_{a\mu} - \overset{\bullet}{K}{}^{\cdot b}{}_{a\mu})\, (h\, \Theta_b{}^{\mu}) = 0. \tag{7.42}$$

Making use of the identity

$$\partial_{\rho} h = h\, \overset{\circ}{\Gamma}{}^{\nu}{}_{\nu\rho} \equiv h \left(\overset{\bullet}{\Gamma}{}^{\nu}{}_{\rho\nu} - \overset{\bullet}{K}{}^{\nu}{}_{\rho\nu}\right), \tag{7.43}$$

the above conservation law becomes

$$\partial_{\mu}\Theta_a{}^{\mu} + (\overset{\bullet}{\Gamma}{}^{\mu}{}_{\rho\mu} - \overset{\bullet}{K}{}^{\mu}{}_{\rho\mu})\, \Theta_a{}^{\rho} - (\overset{\bullet}{A}{}^{b}{}_{a\mu} - \overset{\bullet}{K}{}^{b}{}_{a\mu})\, \Theta_b{}^{\mu} = 0. \tag{7.44}$$

In a purely spacetime form, it reads

$$\overset{\bullet\bullet}{\nabla}_\mu \Theta_\lambda{}^\mu \equiv \partial_\mu \Theta_\lambda{}^\mu + (\overset{\bullet}{\Gamma}{}^\mu{}_{\rho\mu} - \overset{\bullet}{K}{}^\mu{}_{\rho\mu})\, \Theta_\lambda{}^\rho - (\overset{\bullet}{\Gamma}{}^\rho{}_{\lambda\mu} - \overset{\bullet}{K}{}^\rho{}_{\lambda\mu})\, \Theta_\rho{}^\mu = 0. \quad (7.45)$$

This is the conservation law of the source energy-momentum tensor in Teleparallel Gravity. Of course, due to the relation (7.18), it coincides with the corresponding conservation law of General Relativity:

$$\overset{\circ}{\nabla}_\mu \Theta_\lambda{}^\mu \equiv \partial_\mu \Theta_\lambda{}^\mu + \overset{\circ}{\Gamma}{}^\mu{}_{\rho\mu} \Theta_\lambda{}^\rho - \overset{\circ}{\Gamma}{}^\rho{}_{\lambda\mu} \Theta_\rho{}^\mu = 0. \quad (7.46)$$

Comment 7.3 It is important to remark that "covariant conservation laws" are not, strictly speaking, true conservation laws because they fail to yield a conserved "charge". They are actually identities, called Noether identities, which rule the exchange of energy and momentum between the source and the gravitational fields [40].

7.4 Field Equations

Consider now the lagrangian

$$\mathcal{L} = \overset{\bullet}{\mathcal{L}} + \mathcal{L}_m, \quad (7.47)$$

with $\mathcal{L}_m$ the lagrangian of a general matter field. Introducing the notation

$$k = \frac{8\pi G}{c^4}, \quad (7.48)$$

variation with respect to the gauge potential $B^a{}_\rho$ yields the teleparallel version of the gravitational field equation

$$\partial_\sigma (h \overset{\bullet}{S}_a{}^{\rho\sigma}) - k\, h\, \overset{\bullet}{\jmath}_a{}^\rho = k\, h\, \Theta_a{}^\rho. \quad (7.49)$$

In this equation,

$$h \overset{\bullet}{S}_a{}^{\rho\sigma} = h h_a{}^\lambda \overset{\bullet}{S}_\lambda{}^{\rho\sigma} \equiv -\,k\, \frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial(\partial_\sigma h^a{}_\rho)} \quad (7.50)$$

is the superpotential,

$$h\, \overset{\bullet}{\jmath}_a{}^\rho = -\,\frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial B^a{}_\rho} \equiv -\,\frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\rho} \quad (7.51)$$

stands for the gauge current, which in this case represents the Noether energy-momentum density of gravitation [59], and

$$h\,\Theta_a{}^\rho = -\frac{\delta\mathcal{L}_m}{\delta B^a{}_\rho} = -\frac{\delta\mathcal{L}_m}{\delta h^a{}_\rho} \equiv -\left(\frac{\partial\mathcal{L}_m}{\partial h^a{}_\rho} - \partial_\lambda \frac{\partial\mathcal{L}_m}{\partial_\lambda \partial h^a{}_\rho}\right) \tag{7.52}$$

is the matter (or source) energy-momentum tensor. In Appendix A we present a detailed computation leading to the results

$$\overset{\bullet}{S}_a{}^{\rho\sigma} = \overset{\bullet}{K}{}^{\rho\sigma}{}_a - h_a{}^\sigma\,\overset{\bullet}{T}{}^{\theta\rho}{}_\theta + h_a{}^\rho\,\overset{\bullet}{T}{}^{\theta\sigma}{}_\theta \tag{7.53}$$

and

$$\overset{\bullet}{\jmath}_a{}^\rho = \frac{1}{k}\,h_a{}^\lambda\,\overset{\bullet}{S}_c{}^{\nu\rho}\,\overset{\bullet}{T}{}^c{}_{\nu\lambda} - \frac{h_a{}^\rho}{h}\,\overset{\bullet}{\mathcal{L}} + \frac{1}{k}\,\overset{\bullet}{A}{}^c{}_{a\sigma}\overset{\bullet}{S}_c{}^{\rho\sigma}. \tag{7.54}$$

The gravitational field equation (7.49) depends on torsion only. Using the identity (7.18), it can be rewritten in terms of the curvature only. In fact, through a lengthy but straightforward calculation, the left-hand side of (7.49) can be shown to satisfy

$$\partial_\sigma(h\overset{\bullet}{S}_a{}^{\rho\sigma}) - k\,(h\overset{\bullet}{\jmath}_a{}^\rho) = h\Big(\overset{\circ}{R}_a{}^\rho - \tfrac{1}{2}\,h_a{}^\rho\,\overset{\circ}{R}\Big). \tag{7.55}$$

This means that, as expected due to the equivalence between the corresponding lagrangians, the teleparallel field equation (7.49) is equivalent to Einstein's field equation

$$\overset{\circ}{R}_a{}^\rho - \tfrac{1}{2}\,h_a{}^\rho\,\overset{\circ}{R} = k\,\Theta_a{}^\rho. \tag{7.56}$$

It is important to observe that the energy-momentum tensor appears as the source in both theories: as the source of curvature in General Relativity, and as the source of torsion in Teleparallel Gravity. It is not surprising, therefore, that its conservation law (7.45) in Teleparallel Gravity, coincides with its conservation law (7.46) in General Relativity. This is just a matter of consistency.

7.5 Bianchi Identities

The Bianchi identities of Teleparallel Gravity [60] can be obtained from the general Bianchi identities presented in Section 1.5, by replacing

$$A^a{}_{b\mu} \to \overset{\bullet}{A}{}^a{}_{b\mu}, \tag{7.57}$$

which implies the concomitant replacements

$$T^a{}_{\mu\nu} \rightarrow \overset{\bullet}{T}{}^a{}_{\mu\nu} \tag{7.58}$$

and

$$R^a{}_{b\mu\nu} \rightarrow \overset{\bullet}{R}{}^a{}_{b\mu\nu} \equiv \overset{\circ}{R}{}^a{}_{b\mu\nu} + \overset{\bullet}{Q}{}^a{}_{b\mu\nu} = 0. \tag{7.59}$$

Thus, the Bianchi identity for torsion, which in the general case is given by [see Eq. (1.63)]

$$\mathcal{D}_\nu T^a{}_{\rho\mu} + \mathcal{D}_\mu T^a{}_{\nu\rho} + \mathcal{D}_\rho T^a{}_{\mu\nu} = R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} + R^a{}_{\mu\nu\rho}, \tag{7.60}$$

assumes the teleparallel form

$$\overset{\bullet}{\mathcal{D}}_\nu \overset{\bullet}{T}{}^a{}_{\rho\mu} + \overset{\bullet}{\mathcal{D}}_\mu \overset{\bullet}{T}{}^a{}_{\nu\rho} + \overset{\bullet}{\mathcal{D}}_\rho \overset{\bullet}{T}{}^a{}_{\mu\nu} = 0. \tag{7.61}$$

Through a tedious, but straightforward calculation, it can be rewritten in the form,

$$\overset{\bullet}{Q}{}^\rho{}_{\theta\mu\nu} + \overset{\bullet}{Q}{}^\rho{}_{\nu\theta\mu} + \overset{\bullet}{Q}{}^\rho{}_{\mu\nu\theta} = 0, \tag{7.62}$$

which is equivalent to the first Bianchi identity of General Relativity

$$\overset{\circ}{R}{}^\rho{}_{\theta\mu\nu} + \overset{\circ}{R}{}^\rho{}_{\nu\theta\mu} + \overset{\circ}{R}{}^\rho{}_{\mu\nu\theta} = 0. \tag{7.63}$$

Comment 7.4 In the class of frames h'_a in which $\overset{\bullet}{A}{}'^a{}_{b\mu} = 0$, the Bianchi identity (7.61) reduces to

$$\partial_\nu \overset{\bullet}{T}{}'^a{}_{\rho\mu} + \partial_\mu \overset{\bullet}{T}{}'^a{}_{\nu\rho} + \partial_\rho \overset{\bullet}{T}{}'^a{}_{\mu\nu} = 0. \tag{7.64}$$

In this form it becomes similar to the Bianchi identity of electromagnetism,

$$\partial_\nu F'_{\rho\mu} + \partial_\mu F'_{\nu\rho} + \partial_\rho F'_{\mu\nu} = 0. \tag{7.65}$$

On the other hand, the Bianchi identity for curvature, which in the general case is given by [see Eq. (1.64)]

$$\mathcal{D}_\nu R^a{}_{b\rho\mu} + \mathcal{D}_\mu R^a{}_{b\nu\rho} + \mathcal{D}_\rho R^a{}_{b\mu\nu} = 0, \tag{7.66}$$

in Teleparallel Gravity acquires the form

$$\overset{\bullet}{\mathcal{D}}_\nu \overset{\bullet}{Q}{}^a{}_{b\rho\mu} + \overset{\bullet}{\mathcal{D}}_\mu \overset{\bullet}{Q}{}^a{}_{b\nu\rho} + \overset{\bullet}{\mathcal{D}}_\rho \overset{\bullet}{Q}{}^a{}_{b\mu\nu} = 0, \tag{7.67}$$

where use has been made of the General Relativity Bianchi identity

$$\overset{\circ}{\mathcal{D}}_\nu \overset{\circ}{R}{}^a{}_{b\rho\mu} + \overset{\circ}{\mathcal{D}}_\mu \overset{\circ}{R}{}^a{}_{b\nu\rho} + \overset{\circ}{\mathcal{D}}_\rho \overset{\circ}{R}{}^a{}_{b\mu\nu} = 0. \tag{7.68}$$

As is well known, the contracted form of this identity is

$$\overset{\circ}{\mathcal{D}}_\rho \left[h \left(\overset{\circ}{R}_a{}^\rho - \tfrac{1}{2} h_a{}^\rho \overset{\circ}{R} \right) \right] = 0. \tag{7.69}$$

Through a similar procedure, the contracted form of the teleparallel Bianchi identity (7.67) is found to be

$$\overset{\bullet\bullet}{\mathcal{D}}_\rho \left[\partial_\sigma (h \overset{\bullet}{S}_a{}^{\rho\sigma}) - k\, h\, \overset{\bullet}{\jmath}_a{}^\rho \right] = 0. \tag{7.70}$$

If we remember that, in the presence of a source field, the teleparallel field equation is given by

$$\partial_\sigma (h \overset{\bullet}{S}_a{}^{\rho\sigma}) - k\, h\, \overset{\bullet}{\jmath}_a{}^\rho = k\, h\, \Theta_a{}^\rho, \tag{7.71}$$

the Bianchi identity (7.70) is seen to be consistent with the conservation law

$$\overset{\bullet\bullet}{\mathcal{D}}_\rho (h \Theta_a{}^\rho) = 0, \tag{7.72}$$

as obtained from Noether's theorem [see Section 7.3].

7.6 A Glimpse on New General Relativity

New General Relativity corresponds to a generalized teleparallel model with three arbitrary parameters. Like in Teleparallel Gravity, the relevant connection is the Weitzenböck connection

$$\Gamma^\rho{}_{\mu\nu} = h_a{}^\rho\, \partial_\nu h^a{}_\mu, \tag{7.73}$$

but the lagrangian has the form

$$\mathcal{L}_{ngr} = \frac{c^4}{16\pi G} \left[a_1\, T^\rho{}_{\mu\nu}\, T_\rho{}^{\mu\nu} + a_2\, T^\rho{}_{\mu\nu}\, T^{\nu\mu}{}_\rho + a_3\, T_{\rho\mu}{}^\rho\, T^{\nu\mu}{}_\nu \right], \tag{7.74}$$

where a_1, a_2, a_3 are arbitrary coefficients, to be determined by experience. In terms of the irreducible components of torsion [see Section 1.4], it can be rewritten in the form [21]:

$$\mathcal{L}_{ngr} = \frac{c^4}{16\pi G} \left[b_1\, \mathcal{T}^\rho{}_{\mu\nu}\, \mathcal{T}_\rho{}^{\mu\nu} + b_2\, \mathcal{V}^\mu\, \mathcal{V}_\mu + b_3\, \mathcal{A}^\mu\, \mathcal{A}_\mu \right], \tag{7.75}$$

with b_1, b_2, b_3 new arbitrary coefficients. A straightforward calculation shows that

$$\tfrac{2}{3}\, \mathcal{T}^\rho{}_{\mu\nu}\, \mathcal{T}_\rho{}^{\mu\nu} + \tfrac{2}{3}\, \mathcal{V}^\mu\, \mathcal{V}_\mu - \tfrac{3}{2}\, \mathcal{A}^\mu\, \mathcal{A}_\mu = \overset{\circ}{R}\,, \tag{7.76}$$

with $\stackrel{\circ}{R}$ the scalar curvature of the Levi–Civita connection. Using this identity, lagrangian (7.74) can be recast in the form

$$\mathcal{L}_{ngr} = \frac{c^4}{16\pi G} \left[\stackrel{\circ}{R} + c_1 \, \mathcal{T}^{\rho}{}_{\mu\nu} \, \mathcal{T}_{\rho}{}^{\mu\nu} + c_2 \, \mathcal{V}^{\mu} \, \mathcal{V}_{\mu} + c_3 \, \mathcal{A}^{\mu} \, \mathcal{A}_{\mu} \right], \tag{7.77}$$

with the new coefficients given by

$$c_1 = b_1 - \tfrac{2}{3}, \quad c_2 = b_2 - \tfrac{2}{3}, \quad c_3 = b_3 + \tfrac{3}{2}. \tag{7.78}$$

The first term of (7.77) is the Einstein–Hilbert lagrangian. In this theory, therefore, torsion is assumed to produce deviations from the predictions of General Relativity — or equivalently, from the predictions of the teleparallel equivalent of General Relativity. This means that torsion represents additional degrees of freedom in relation to curvature. It should be remarked that solar system experiments restrict severely the existence of non-vanishing c_1 and c_2. Considering that all precision experiments have up to now found a reasonable explanation within General Relativity, we can say that there is no any experimental evidence for a non-vanishing c_3 either. Furthermore, it has already been shown that the Schwarzschild solution exists only for the case with [173]

$$c_1 = c_2 = c_3 = 0.$$

In principle, therefore, we can say that New General Relativity lacks experimental support.

Chapter 8

Gravitational Energy-Momentum Density

Using the possibility of separating gravitation and inertia allowed by teleparallel gravity, explicit expressions for the energy-momentum density of gravitation and inertia are obtained. The energy-momentum density of gravity turns out to be a true tensor which satisfies a covariant conservation law. The energy-momentum density of inertia, on the other hand, is neither conserved nor covariant. Together with the energy-momentum tensor of gravity, they form a pseudotensor conserved in the ordinary sense. The non-covariance of the usual expressions of the gravitational energy-momentum density is not an intrinsic property of gravity, but a consequence of the fact that they include also the energy-momentum of inertia.

As for any true field, it is natural to expect that gravitation have its own local energy-momentum density. In the preface to his classic textbook [25], Synge says that *in Einstein's theory, either there is a gravitational field or there is none, according to as the Riemann tensor does not or does vanish. This is an absolute property; it has nothing to do with any observer's world line.* Bondi [72] argues in that same direction, saying that *in relativity a non-localizable form of energy is inadmissible, because any form of energy contributes to gravitation and so its location can in principle be found.* In this line of argument, the energy of the gravitational field should be localizable independently of any observer.

On the other hand, it is usually argued that such a density cannot be locally defined because of the equivalence principle [61]. According to these arguments, any attempt to identify an energy-momentum density for the gravitational field leads to complexes that are not true tensors. The first of such attempts was made by Einstein, who proposed an expression which was nothing but the canonical expression obtained from Noether's theorem [63]. This quantity is, like many others, a pseudotensor, an object that depends on the coordinate system. Several other attempts have been made, leading to different expressions for this pseudotensor [64].

Comment 8.1 The problem of defining an energy-momentum density for the gravitational field has a close analogy with the problem of defining a gauge self-current for the Yang-Mills field, discussed in Section 2.1. The sourceless field equation is

$$\partial_\mu F^{A\mu\nu} + j^{A\nu} = 0, \tag{8.1}$$

with $F^{A\mu\nu}$ the curvature of the connection $A^A{}_\mu$. The piece

$$j^{A\nu} = - f^A{}_{BC}\, A^B{}_\mu\, F^{C\mu\nu} \tag{8.2}$$

represents the gauge pseudocurrent. Due to the explicit presence of the connection $A^B{}_\mu$, this current is clearly not gauge covariant, in analogy with the gravitational energy-momentum pseudotensor, which is not covariant under general coordinate transformations.

Despite the existence of some controversial points related to the equivalence principle [25, 65], it seems true that in the context of General Relativity no tensorial expression for the gravitational energy-momentum density exists. In the stream of this perception, a *quasilocal* approach [67] has been proposed which is highly clarifying [68]. According to this approach, for each gravitational energy-momentum pseudotensor there is an associated *superpotential* which is a hamiltonian boundary term. The energy-momentum defined by such a pseudotensor does not really depend on the local reference frame, but only on its values on the boundary of a region — from which its *quasilocal* character. As the relevant boundary conditions are physically acceptable, this approach is said to validate the pseudotensor approach to the gravitational energy-momentum problem. Independently of this and others attempts to circumvent the problem of the gravitational energy-momentum density, the question remains: is the impossibility of defining a tensorial expression for the gravitational energy–momentum density a fundamental property of Nature, or just a drawback of the particular geometrical description of General Relativity?

8.1 Field Equations and Conservation Laws

In the first order formalism, where the lagrangian depends on the field and on its first derivative only, the gravitational field equation can be obtained from the usual, first order Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial h^a{}_\rho} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\sigma h^a{}_\rho)} = 0. \tag{8.3}$$

It can be rewritten in the form

$$\partial_\sigma(hS_a{}^{\rho\sigma}) - k\, h\jmath_a{}^\rho = 0, \tag{8.4}$$

where k is defined in Eq. (7.48),

$$S_a{}^{\rho\sigma} = -\, S_a{}^{\sigma\rho} \equiv -\,\frac{k}{h}\,\frac{\partial\mathcal{L}}{\partial(\partial_\sigma h^a{}_\rho)} \tag{8.5}$$

is the so-called *superpotential*, and

$$\jmath_a{}^\rho \equiv -\,\frac{1}{h}\frac{\partial\mathcal{L}}{\partial h^a{}_\rho} \tag{8.6}$$

stands for the Noether gravitational energy-momentum current. Equation (8.4) is known as the *potential form* of the gravitational field equation [8]. It is, in structure, similar to the Yang-Mills equation. Its main virtue is to explicitly exhibit the complex defining the energy-momentum current of the gravitational field. Due to the anti-symmetry of $S_a{}^{\rho\sigma}$ in the last two indices, the field equation implies the conservation of the gravitational energy-momentum current:

$$\partial_\rho(h\jmath_a{}^\rho) = 0. \tag{8.7}$$

Any conservation law of this form, namely, written as a four-dimensional *ordinary* divergence, is a *true conservation law* in the sense that it yields a time-conserved "charge". On the other hand, in order to be physically meaningful, the equation expressing any conservation law must be covariant under both general coordinate and local Lorentz transformations.

Comment 8.2 Of course, in the case of gauge theories, the conservation laws must also be gauge covariant.

Although trivial in the absence of gravitation, this simple property has an important consequence in the gravitational case: *no tensorial quantity can be truly conserved*. In fact, since the derivative in this case is not covariant, in order to have a covariant conservation law, the conserved current cannot be covariant either. This means that the energy-momentum current $\jmath_a{}^\rho$ appearing in the gravitational field equation (8.4), which has the conservation law (8.7), cannot be a tensor. Conversely, due to the fact that it is a true tensor, the symmetric energy-momentum tensor $\Theta_a{}^\rho$ of a source field—that is, the energy-momentum tensor appearing in the right-hand side of the gravitational field equation—can only be conserved in the covariant sense,

$$\overset{\circ}{\mathcal{D}}_\rho(h\Theta_a{}^\rho) \equiv h(\partial_\rho\Theta_a{}^\rho - \overset{\circ}{A}{}^c{}_{a\rho}\,\Theta_c{}^\rho + \overset{\circ}{\Gamma}{}^\rho{}_{\lambda\rho}\,\Theta_a{}^\lambda) = 0, \tag{8.8}$$

otherwise the conservation law itself would not be covariant. We reinforce that this is not a true conservation law, but just an identity (the Noether identity) regulating the exchange of energy-momentum between matter and gravitation.

The discussion above makes it clear that, if a *tensorial* expression $t_a{}^\rho$ for the energy-momentum density of gravitation exists, it must be conserved in the covariant sense. On the other hand, inertial (or fictitious) effects related to the non-inertiality of a frame are non-covariant, as they are represented by Lorentz connections. The energy-momentum density associated to these fictitious effects, which we denote by $\imath_a{}^\rho$, cannot therefore be a true tensor. This means that, even if a tensorial expression for the gravitational energy-momentum density exists, the sum of the inertial and the purely gravitational densities, $\imath_a{}^\rho + t_a{}^\rho$, will necessarily be a pseudotensor. This is a matter of consistency: since the sum represents the total (in the absence of source fields) energy-momentum density, it has to be truly conserved,

$$\partial_\rho[h(\imath_a{}^\rho + t_a{}^\rho)] = 0, \tag{8.9}$$

and consequently cannot be a true tensor.

8.2 Teleparallel Gravity

In Special Relativity, the anholonomy of the frames is entirely related to the fictitious forces present in those frames. The preferred class of inertial frames is then characterized by the absence of fictitious forces. In the presence of gravitation, however, the anholonomy of the frames is related to both gravitational and inertial effects. Since the gravitational effects can never be eliminated globally, no holonomous frames can be defined in the presence of gravitation. Like in Special Relativity, however, also in the presence of gravitation there is a preferred class of frames: the class that reduces to the inertial class in the absence of gravitation. Seen from that preferred class of frames, which we denote by h'_b, the spin connection of Teleparallel Gravity vanishes everywhere:

$$\overset{\bullet}{A}{}'^a{}_{b\mu} = 0. \tag{8.10}$$

It is important to remark that this has nothing to do with the strong equivalence principle, which says that a connection can be made to vanish at a point, or locally along a trajectory. In a Lorentz rotated frame $h_a = \Lambda_a{}^b\, h'_b$, it assumes the form

$$\overset{\bullet}{A}{}^a{}_{b\mu} = \Lambda^a{}_e\, \partial_\mu \Lambda_b{}^e. \tag{8.11}$$

As seen in Chapter 7, the sourceless gravitational field equation is [59]

$$\partial_\sigma(h\overset{\bullet}{S}_a{}^{\rho\sigma}) - k\,h\,\overset{\bullet}{\jmath}_a{}^{\rho} = 0, \tag{8.12}$$

where

$$\overset{\bullet}{S}_a{}^{\rho\sigma} \equiv -\frac{k}{h}\,\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial(\partial_\sigma h^a{}_\rho)} = \overset{\bullet}{K}{}^{\rho\sigma}{}_a - h_a{}^\sigma\,\overset{\bullet}{T}{}^{\theta\rho}{}_\theta + h_a{}^\rho\,\overset{\bullet}{T}{}^{\theta\sigma}{}_\theta \tag{8.13}$$

is the superpotential, and

$$\overset{\bullet}{\jmath}_a{}^{\rho} \equiv -\frac{1}{h}\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\rho} = \frac{1}{k}\,h_a{}^\lambda\,\overset{\bullet}{S}_c{}^{\nu\rho}\,\overset{\bullet}{T}{}^c{}_{\nu\lambda} - \frac{h_a{}^\rho}{h}\,\overset{\bullet}{\mathcal{L}} + \frac{1}{k}\,\overset{\bullet}{A}{}^c{}_{a\sigma}\overset{\bullet}{S}_c{}^{\rho\sigma} \tag{8.14}$$

is the teleparallel energy-momentum current. Due to the anti-symmetry of the superpotential in the last two indices, it is conserved in the ordinary sense:

$$\partial_\rho(h\overset{\bullet}{\jmath}_a{}^{\rho}) = 0. \tag{8.15}$$

An important property of Teleparallel Gravity is that its spin connection, given by Eq. (8.11), is related only to the inertial properties of the frame, not to gravitation. In fact, it is possible to choose an appropriate frame in which it vanishes everywhere. This means that the last term of the gravitational current (8.14) will also vanish in that appropriate frame. Furthermore, put together with the potential term of field equation (8.12), they make up a Fock-Ivanenko covariant derivative of the superpotential:

$$\partial_\sigma(h\overset{\bullet}{S}_a{}^{\rho\sigma}) - \overset{\bullet}{A}{}^c{}_{a\sigma}(h\,\overset{\bullet}{S}_c{}^{\rho\sigma}) \equiv \overset{\bullet}{\mathcal{D}}_\sigma(h\overset{\bullet}{S}_a{}^{\rho\sigma}). \tag{8.16}$$

This allows us to rewrite that field equation in the form

$$\overset{\bullet}{\mathcal{D}}_\sigma(h\overset{\bullet}{S}_a{}^{\rho\sigma}) - k\,h\,\overset{\bullet}{t}_a{}^{\rho} = 0, \tag{8.17}$$

where

$$\overset{\bullet}{t}_a{}^{\rho} = \frac{1}{k}\,h_a{}^\lambda\,\overset{\bullet}{S}_c{}^{\nu\rho}\,\overset{\bullet}{T}{}^c{}_{\nu\lambda} - \frac{h_a{}^\rho}{h}\,\overset{\bullet}{\mathcal{L}} \tag{8.18}$$

is a tensorial current. The crucial point comes out now: since the teleparallel spin connection (8.11) has vanishing curvature, the Fock-Ivanenko derivative $\overset{\bullet}{\mathcal{D}}_\sigma$ is commutative:

$$[\overset{\bullet}{\mathcal{D}}_\rho, \overset{\bullet}{\mathcal{D}}_\sigma] = 0. \tag{8.19}$$

Taking into account the anti-symmetry of the superpotential in the last two indices, it follows from (8.17) that the tensorial current (8.18) is covariantly conserved:

$$\overset{\bullet}{\mathcal{D}}_\rho(h\overset{\bullet}{t}_a{}^{\rho}) = 0. \tag{8.20}$$

Due to these properties, $\overset{\bullet}{t}_a{}^\rho$ can be interpreted as the energy-momentum density of gravitation alone. Accordingly,

$$\overset{\bullet}{\imath}_a{}^\rho = \frac{1}{k}\, \overset{\bullet}{A}{}^c{}_{a\sigma} \overset{\bullet}{S}_c{}^{\rho\sigma} \tag{8.21}$$

can be interpreted as the energy-momentum density of the inertial (or fictitious) forces. The total (inertia plus gravitation) energy-momentum density is consequently

$$\overset{\bullet}{\jmath}_a{}^\rho = \overset{\bullet}{\imath}_a{}^\rho + \overset{\bullet}{t}_a{}^\rho, \tag{8.22}$$

which is non-covariant due to the inertial effects, and is conserved in the ordinary sense. We see in this way that the basic reason for the usual general relativistic expressions to be a pseudotensor is that they include, in addition to gravitation, a contribution from the inertial effects of the frame. If considered separately from inertia, however, the gravitational energy-momentum density is found to be a true tensor and conserved in the covariant sense. Of course, as it does not represent the total energy-momentum density, it does not need to be truly conserved.

Comment 8.3 An interesting property of the tensorial current (8.18) is that, as field theory would expect for a massless field, its trace vanishes identically:

$$\overset{\bullet}{t}_\rho{}^\rho \equiv h^a{}_\rho\, \overset{\bullet}{t}_a{}^\rho = 0. \tag{8.23}$$

On the other hand, the trace of the pseudocurrent (8.14) is found to be proportional to the lagrangian:

$$h\, \overset{\bullet}{\jmath}_\rho{}^\rho \equiv h\, h^a{}_\rho\, \overset{\bullet}{\jmath}_a{}^\rho = -\frac{h}{2k}\, \overset{\bullet}{S}_a{}^{\mu\nu}\, \overset{\bullet}{T}{}^a{}_{\mu\nu} = -\,2\, \overset{\bullet}{\mathcal{L}}. \tag{8.24}$$

Similar results hold for the symmetric and the canonical energy-momentum densities of the electromagnetic field, given respectively by [58]

$$\Theta_\lambda{}^\rho = -\, F_\lambda{}^\mu F^\rho{}_\mu + \tfrac{1}{4}\, \delta_\lambda{}^\rho\, F_{\mu\nu} F^{\mu\nu} \tag{8.25}$$

and

$$\theta_\lambda{}^\rho = -\, \partial_\lambda A^\mu F^\rho{}_\mu + \tfrac{1}{4}\, \delta_\lambda{}^\rho\, F_{\mu\nu} F^{\mu\nu}. \tag{8.26}$$

In fact, one can easily check that

$$\Theta_\rho{}^\rho = 0 \quad \text{and} \quad \theta_\rho{}^\rho = -\,2\mathcal{L}_{em}.$$

We can then say that $\overset{\bullet}{t}_a{}^\rho$ and $\overset{\bullet}{\jmath}_\mu{}^\rho$ play a role similar to the symmetric and the canonical energy–momentum tensors of massless source fields. This similarity becomes still more evident if we note that, whereas the symmetric energy-momentum tensor $\Theta_\lambda{}^\rho$ is gauge invariant, the canonical tensor $\theta_\lambda{}^\rho$ is not, as it depends explicitly on the electromagnetic potential A^μ.

8.3 General Relativity

It is well known that no first order *invariant* lagrangian for General Relativity exists. What does exist is the second order Einstein-Hilbert invariant lagrangian, in which the second-derivative terms reduce to a total divergence. This lagrangian is the of the form

$$\overset{\circ}{\mathcal{L}} = -\frac{h}{2k}\,\overset{\circ}{R} \equiv \overset{\circ}{\mathcal{L}}_1 + \partial_\mu(h\,w^\mu), \tag{8.27}$$

where $\overset{\circ}{\mathcal{L}}_1$ is a first-order lagrangian and w^μ is a contravariant four-vector. There are actually infinitely many different first order lagrangians, each one connected to a different surface term:

$$\overset{\circ}{\mathcal{L}} = \overset{\circ}{\mathcal{L}}_1 + \partial_\mu(h\,w^\mu) = \overset{\circ}{\mathcal{L}}{}'_1 + \partial_\mu(h\,w'^\mu) = \overset{\circ}{\mathcal{L}}{}''_1 + \partial_\mu(h\,w''^\mu) = \ldots\,. \tag{8.28}$$

Since the divergence term does not contribute to the field equation, any one of the first order lagrangians will lead to the same expression

$$\partial_\sigma(h\overset{\circ}{S}{}_a{}^{\rho\sigma}) - k\,h\overset{\circ}{\jmath}{}_a{}^\rho = 0 \tag{8.29}$$

for the potential form of Einstein's field equation [8]. In this equation, $\overset{\circ}{S}{}_a{}^{\rho\sigma}$ is the superpotential and $\overset{\circ}{\jmath}{}_a{}^\rho$ is the gravitational current, which is conserved in the ordinary sense:

$$\partial_\rho(h\overset{\circ}{\jmath}{}_a{}^\rho) = 0. \tag{8.30}$$

Now, as is well known, the general-relativistic approach to gravitation is strongly rooted on the equivalence between inertia and gravitation. As a consequence, inertial and gravitational effects are considered on the same footing: they are both embodied in the spin connection $\overset{\circ}{A}{}^a{}_{b\mu}$, and cannot be separated. Because of this inseparability, the energy-momentum current in General Relativity will always include, in addition to the purely gravitational density, also the energy-momentum density of inertia. Since the latter is a pseudotensor, so will the whole current be. In General Relativity, therefore, it is not possible to define a tensorial expression for the gravitational energy-momentum density. To see that, let us rewrite Einstein equation (8.29) in the form

$$\overset{\circ}{\mathcal{D}}_\sigma(h\overset{\circ}{S}{}_a{}^{\rho\sigma}) - kh\,\overset{\circ}{t}{}_a{}^\rho = 0, \tag{8.31}$$

where

$$\overset{\circ}{\mathcal{D}}_\sigma(h\overset{\circ}{S}{}_a{}^{\rho\sigma}) = \partial_\sigma(h\overset{\circ}{S}{}_a{}^{\rho\sigma}) - \overset{\circ}{A}{}^b{}_{a\sigma}\,(h\overset{\circ}{S}{}_b{}^{\rho\sigma}) \tag{8.32}$$

is the covariant derivative in the general relativity spin connection, and

$$\overset{\circ}{t}_a{}^{\rho} = \overset{\circ}{\jmath}_a{}^{\rho} - \frac{1}{k} \overset{\circ}{A}{}^b{}_{a\sigma} \, \overset{\circ}{S}_b{}^{\rho\sigma} \tag{8.33}$$

is a tensorial quantity that, at least in principle, could be interpreted as the energy-momentum density of gravitation. That this is a covariant quantity can be verified by noting that, since Einstein equation (8.31) is covariant, and considering that its first term is also covariant, the current term must necessarily be a tensor. However, there is a problem in this interpretation: due to the fact that $\overset{\circ}{\mathcal{D}}_\sigma$ is not commutative, this tensorial quantity is not covariantly conserved, and consequently cannot be physically meaningful. Conversely, we can say that the requirement of covariant conservation of $\overset{\circ}{t}_a{}^{\rho}$ would impose unphysical constraints on the spacetime geometry.

Comment 8.4 This last inconsistency is similar to the inconsistencies that appear in the theory of a fundamental spin-2 field coupled to gravitation in the context of general relativity [76]. In Section 10.5 we show that Teleparallel Gravity is able to solve also this problem.

8.4 Further Remarks

Using the fact that teleparallel gravity allows a separation between gravitation and inertia, we have shown that it is possible to obtain explicit expressions for the energy-momentum density of gravitation and inertia. The energy-momentum density of gravity turns out to be a true tensor which satisfies a covariant conservation law. Since it does not represent the total energy-momentum density—in the sense that the inertial part is not included—it does not need to be truly (or ordinarily) conserved. This is consistent with the property that a covariant quantity can only be conserved in the covariant sense, otherwise the conservation law itself will be physically meaningless. The energy-momentum density of inertia, on the other hand, is neither conserved nor covariant. It constitutes, furthermore, an example of an energy-momentum density that is not source of gravitation. Together with the energy-momentum tensor of gravity, they form a pseudotensor conserved in the ordinary sense. We can then say that the non-covariance of the gravitational energy-momentum density is not an intrinsic property of gravity, but a consequence of the fact that the usual expressions include also the energy-momentum of inertia. Considering that, in general relativity, gravitation and inertia are mixed up in the spin connection of the theory and cannot be separated, the energy-momentum tensor of gravity in this theory will alway include the the energy-momentum tensor of inertia, and will consequently be a pseudotensor. The existence of a purely gravitational energy-momentum tensor in teleparallel gravity allows one to compute unequivocally the energy and momentum of gravitation in a given frame, and through Lorentz transformations to obtain the value in any other frame without necessity of a regularization process [77].

Chapter 9

Gravitation Without the Equivalence Principle

General Relativity is fundamentally grounded on the equivalence principle. If universality is found to fail, even in a small quantity, the geometric description of General Relativity breaks down. On the other hand, due to its gauge structure, Teleparallel Gravity does not require the equivalence principle to describe the gravitational interaction. As a consequence, it remains a consistent theory in the absence of universality. Although this is not important at the classical level, where universality has passed all experimental tests, it may become important at the quantum level, where universality may fail to be true.

9.1 Introductory Remarks

Universality of gravitation means that everything in the universe feels gravity the same. Provided the initial conditions are the same, all particles, independently of their mass and constitution, will follow the same trajectory. In the study of a massive particle motion, universality of free fall is directly connected with the equality between inertial and gravitational masses: $m_i = m_g$. In fact, in order to be eliminated from the equation of motion — such that the motion becomes universal — they must necessarily coincide.

In Einstein's General Relativity, which is a theory fundamentally based on the universality of free fall or, equivalently, on the weak equivalence principle, geometry replaces the concept of force in the description of the gravitational interaction. Despite its success in all experimental tests at the classical level [78, 80], it is worth to keep in mind that a possible violation of the weak equivalence principle would lead, among other consequences, to the non-universality of free fall, and consequently to its conceptual breakdown. Of course, like Newtonian gravity, it could still to be used in most practical cases but, again like Newton theory, it could fail in describing some specific physical situation. A violation of the weak equivalence principle would constitute the discovery of a new force of Nature, not predicted by General Relativity.

The non-universal character of electromagnetism is the reason why there is no a geometric description, in the sense of General Relativity, for the electromagnetic interaction. On the other hand, as a gauge theory for the translation group, the teleparallel equivalent of General Relativity does not describe the gravitational interaction through a geometrization of spacetime, but as a gravitational force similar to the Lorentz force of electrodynamics. Since Maxwell theory, a gauge theory for the unitary group $U(1)$, is able to describe the non-universal electromagnetic interaction, the question then arises whether the gauge approach of Teleparallel Gravity would also be able to describe the gravitational interaction in the lack of universality, that is, in the absence of the weak equivalence principle. This is the issue we are going to address in this chapter.

9.2 The Electromagnetic Case as an Example

9.2.1 The Electromagnetic Coupling Prescription

The non-universal electromagnetic interaction is described by a gauge theory for the unitary group $U(1)$. An element of this group is written as

$$U = \exp[i\epsilon J], \tag{9.1}$$

where $\epsilon = \epsilon(x^\mu)$ is the transformation parameter, and J is the transformation generator. When applied to a general field ψ representing a particle of electric charge q, the generator in the ψ representation is simply $J = \sqrt{\alpha_e}$, with

$$\alpha_e = \frac{q^2}{\hbar c} \tag{9.2}$$

the *electromagnetic fine structure constant*. Considering units in which $\hbar = c = 1$, the gauge transformation $\psi' = U\psi$ is written as

$$\psi' = \exp[i\epsilon q]\psi, \tag{9.3}$$

whose infinitesimal form is

$$\delta\psi = i\epsilon q\psi. \tag{9.4}$$

In order to render the derivative

$$D_\mu\psi = (\partial_\mu + iqA_\mu)\psi \tag{9.5}$$

really covariant, the gauge potential must transform according to

$$A'_\mu = A_\mu - \partial_\mu \epsilon. \tag{9.6}$$

Comment 9.1 Sometimes, the $U(1)$ group element is written in the form

$$U = \exp[i\epsilon]. \tag{9.7}$$

The gauge transformation is then

$$\psi' = \exp[i\epsilon]\psi, \tag{9.8}$$

whose infinitesimal form is

$$\delta = i\epsilon\psi. \tag{9.9}$$

In this case, to make the derivative (9.5) really covariant, the gauge potential is required to transform according to

$$A'_\mu = A_\mu - \frac{1}{q}\,\partial_\mu \epsilon. \tag{9.10}$$

9.2.2 Lorentz Force Equation

Considering an inertial frame in Minkowski spacetime, the action describing a particle of mass m and electric charge q in the presence of an electromagnetic field A_μ, is

$$\mathcal{S} = -\,m\,c \int_a^b \left[u_a\,\partial_\mu x^a + \frac{q}{mc^2}\,A_\mu\right] dx^\mu. \tag{9.11}$$

The invariance of the action under a general spacetime transformation yields the equation of motion

$$\frac{du^a}{ds} = \frac{q}{mc^2}\,F^{ac}\,u_c, \tag{9.12}$$

where F^{ac} is the electromagnetic field strength. This is the so-called Lorentz force equation. This force is not universal in the sense that it depends on the relation q/m of the particle.

9.3 Managing without Universality

9.3.1 Non-Universal Coupling Prescription

To see how Teleparallel Gravity is able to manage without universality, let us imagine the existence of particles with different gravitational and inertial masses: $m_g \neq m_i$. Analogously to the electromagnetic case, whose gauge transformation involves the electromagnetic fine structure constant, the translational gauge transformation in the non-universal case must depend on some kind of *gravitational constant* α_g. Since it must reduce to $\alpha_g = 1$ in the universal case, we write

$$\alpha_g = \frac{m_g}{m_i}\,. \tag{9.13}$$

Let us then denote by ψ a field representing a particle with $m_g \neq m_i$. Its translational gauge transformation is

$$\psi' = \tilde{U}\,\psi, \tag{9.14}$$

with

$$\tilde{U} = \exp\left[\alpha_g\,\epsilon^a\,\partial_a\right]. \tag{9.15}$$

The corresponding infinitesimal transformation is

$$\tilde{\delta}\psi = \alpha_g\,\epsilon^a\,\partial_a\psi. \tag{9.16}$$

Since it must have the form

$$\tilde{\delta}\psi = \delta x^a\,\partial_a\psi, \tag{9.17}$$

the non-universal gauge transformation of the tangent space coordinates is

$$\tilde{\delta}x^a = \alpha_g\,\epsilon^a. \tag{9.18}$$

Using now the general definition (3.27) of covariant derivative, the translational gauge covariant derivative of ψ is found to be

$$\tilde{h}_\mu\psi = \partial_\mu\psi + \alpha_g\,B^a{}_\mu\,\partial_a\,\psi. \tag{9.19}$$

Comment 9.2 To see that this derivative is in fact covariant, let us take the transformed derivative

$$\tilde{h}'_\mu\psi' = \partial_\mu\psi' + \alpha_g\,B'^a{}_\mu\,\partial_a\psi', \tag{9.20}$$

where we have used the gauge invariance of the generators ∂_a. Substituting

$$\psi' = \psi + \alpha_g\,\epsilon^c\,\partial_c\psi, \tag{9.21}$$

as well as the gauge potential transformation

$$B'^a{}_\mu = B^a{}_\mu - \partial_\mu\epsilon^a, \tag{9.22}$$

we see that

$$\tilde{\delta}(\tilde{h}_\mu\psi) = \alpha_g\,\epsilon^a\partial_a(\tilde{h}_\mu\psi), \tag{9.23}$$

which shows that (9.19) is gauge covariant.

Similarly to the universal case, the covariant derivative (9.19) can be rewritten in the form

$$\tilde{h}_\mu \psi = (\tilde{h}_\mu x^a)\, \partial_a \, \psi, \tag{9.24}$$

where

$$\tilde{h}_\mu x^a = \partial_\mu x^a + \alpha_g B^a{}_\mu \tag{9.25}$$

is the translational covariant derivative of x^a. In a general Lorentz rotated frame, it assumes the form

$$\tilde{h}_\mu x^a = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu}\, x^b + \alpha_g B^a{}_\mu. \tag{9.26}$$

The non-universal gravitational coupling prescription is then achieved by replacing

$$\partial_\mu x^a \to \tilde{h}_\mu x^a. \tag{9.27}$$

It is interesting to observe that the break down of universality modifies the gravitational behavior of the particle, but does not change their inertial properties. This is the real meaning of breaking universality: particles couple differently to gravitation and to inertia.

9.3.2 Particle Equation of Motion

If the inertial and gravitational masses of a given particle are different, its free action is written as

$$\mathcal{S} = -\, m_i \, c \int_a^b u_a \, \partial_\mu x^a \, dx^\mu. \tag{9.28}$$

The corresponding action in the presence of gravitation is obtained by applying the coupling prescription

$$\partial_\mu x^a \to \tilde{h}_\mu x^a, \tag{9.29}$$

with $\tilde{h}_\mu x^a$ given by (9.25). The result is

$$\mathcal{S} = -\, m_i \, c \int_a^b u_a \left[\partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu}\, x^b + \alpha_g B^a{}_\mu \right] dx^\mu. \tag{9.30}$$

Comment 9.3 Notice that, due to the gauge structure of Teleparallel Gravity, the action assumes a form similar to the action of a charged particle in an electromagnetic field. In fact, if the particle has additionally an electric charge q and is in the presence of an electromagnetic field A_μ, the action becomes

$$\mathcal{S} = -\, m_i \, c \int_a^b \left[u_a \, \overset{\bullet}{\mathcal{D}}_\mu x^a + \frac{m_g}{m_i}\, B^a{}_\mu u_a + \frac{q}{m_i}\, \frac{A_\mu}{c^2} \right] dx^\mu. \tag{9.31}$$

We see from this expression that the gravitational mass m_g plays a role similar to the electric charge q.

Variation of the action integral (9.30) under general spacetime transformations $x^\mu \to x^\mu + \delta x^\mu$ yields

$$\delta\mathcal{S} = -m_i c \int_a^b \Big[\delta u_a \tilde{h}^a + u_a\Big(\delta dx^a + \delta\overset{\bullet}{A}{}^a{}_{b\mu} x^b dx^\mu + \overset{\bullet}{A}{}^a{}_{b\mu}\delta x^b dx^\mu + \overset{\bullet}{A}{}^a{}_{b\mu} x^b \delta dx^\mu + \alpha_g \delta B^a{}_\mu dx^\mu + \alpha_g B^a{}_\mu \delta dx^\mu\Big)\Big], \tag{9.32}$$

where we have introduced the notation

$$\tilde{h}^a \equiv \tilde{h}^a{}_\mu dx^\mu = (\partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} x^b + \alpha_g B^a{}_\mu) dx^\mu. \tag{9.33}$$

Comment 9.4 It is important to remark that $\tilde{h}^a{}_\mu$ is not a tetrad. It is related to the true tetrad $h^a{}_\mu$ through

$$\tilde{h}^a{}_\mu = h^a{}_\mu + \frac{m_g - m_i}{m_i}\, B^a{}_\mu.$$

When $m_i = m_g$ it becomes the tetrad.

Using the property $[\delta, d] = 0$, integrating by parts the terms containing $d\delta x^\mu$, and neglecting the surface terms, Eq. (9.32) becomes

$$\delta\mathcal{S} = -m_i c \int_a^b \Big[\delta u_a \tilde{h}^a + u_a\Big(- du_a \delta x^a + u_a \delta\overset{\bullet}{A}{}^a{}_{b\mu} x^b dx^\mu + u_a \overset{\bullet}{A}{}^a{}_{b\mu}\delta x^b dx^\mu + - d(u_a \overset{\bullet}{A}{}^a{}_{b\mu} x^b)\delta x^\mu + \alpha_g u_a \delta B^a{}_\mu dx^\mu - \alpha_g d(u_a B^a{}_\mu)\delta x^\mu\Big)\Big]. \tag{9.34}$$

The variation of the four-velocity u_a is

$$\delta u_a = \frac{du_a}{ds}\, \delta s. \tag{9.35}$$

From the relation $ds = g_{\mu\nu} u^\mu dx^\nu$, we can write

$$\delta s = g_{\mu\nu} u^\mu \delta x^\nu. \tag{9.36}$$

Substituting in (9.35), we obtain

$$\delta u_a = u_\mu \frac{du_a}{ds}\, \delta x^\mu. \tag{9.37}$$

Using this result, as well as the relations

$$\delta x^a = \partial_\mu x^a \delta x^\mu, \quad \delta\overset{\bullet}{A}{}^a{}_{b\mu} = \partial_\rho \overset{\bullet}{A}{}^a{}_{b\mu}\delta x^\rho, \quad \delta B^a{}_\mu = \partial_\rho B^a{}_\mu \delta x^\rho, \tag{9.38}$$

variation (9.34) assumes the form

$$\delta\mathcal{S} = -m_i c \int_a^b \Big[\tilde{h}^a \frac{du_a}{ds}\, u_\mu - du_a \tilde{h}^a{}_\mu + u_a \Big(\partial_\mu \overset{\bullet}{A}{}^a{}_{b\rho} - \partial_\rho \overset{\bullet}{A}{}^a{}_{b\mu}\Big) x^b dx^\rho$$
$$+ u_a \Big(\overset{\bullet}{A}{}^a{}_{b\rho} \partial_\mu x^b - \overset{\bullet}{A}{}^a{}_{b\mu} \partial_\rho x^b\Big) dx^\rho + \alpha_g u_a \Big(\partial_\mu B^a{}_\rho - \partial_\rho B^a{}_\mu\Big) dx^\rho \Big] \delta x^\mu. \quad (9.39)$$

Since the teleparallel spin connection $\overset{\bullet}{A}{}^a{}_{b\mu}$ has vanishing curvature,

$$\overset{\bullet}{R}{}^a{}_{b\mu\rho} = \partial_\mu \overset{\bullet}{A}{}^a{}_{b\rho} - \partial_\rho \overset{\bullet}{A}{}^a{}_{b\mu} + \overset{\bullet}{A}{}^a{}_{e\mu} \overset{\bullet}{A}{}^e{}_{b\rho} - \overset{\bullet}{A}{}^a{}_{e\rho} \overset{\bullet}{A}{}^e{}_{b\mu} = 0, \quad (9.40)$$

we arrive at

$$\delta\mathcal{S} = -m_i c \int_a^b \Big[- P^\rho{}_\mu \tilde{h}^a{}_\rho \frac{du_a}{ds} + u_a \overset{\bullet}{A}{}^a{}_{b\rho} \tilde{h}^b{}_\mu u^\rho$$
$$- u_a \overset{\bullet}{A}{}^a{}_{b\mu} \tilde{h}^b{}_\rho u^\rho + \alpha_g u_a \overset{\bullet}{T}{}^a{}_{\mu\rho} u^\rho \Big] \delta x^\mu ds, \quad (9.41)$$

where

$$P^\rho{}_\mu = \delta^\rho{}_\mu - u^\rho u_\mu \quad (9.42)$$

is a projection tensor, and

$$\overset{\bullet}{T}{}^a{}_{\mu\rho} = \partial_\mu B^a{}_\rho - \partial_\rho B^a{}_\mu + \overset{\bullet}{A}{}^a{}_{b\mu} B^b{}_\rho - \overset{\bullet}{A}{}^a{}_{b\rho} B^b{}_\mu \quad (9.43)$$

is the field strength, or torsion. Due to the invariance of the action, and the arbitrariness of δx^μ, we get finally

$$P^\rho{}_\mu \tilde{h}^a{}_\rho \frac{du_a}{ds} - \Big(\overset{\bullet}{A}{}^a{}_{b\rho} \tilde{h}^b{}_\mu - \overset{\bullet}{A}{}^a{}_{b\mu} \tilde{h}^b{}_\rho\Big) u_a\, u^\rho = \alpha_g \overset{\bullet}{T}{}^a{}_{\mu\rho} u_a u^\rho. \quad (9.44)$$

This is the equation of motion for particles with $m_g \neq m_i$ in the presence of gravitation. It can also be written in the alternative form

$$\tilde{h}^a{}_\rho \Big[P^\rho{}_\mu \frac{\overset{\bullet}{\mathcal{D}} u_a}{\mathcal{D}s} + P^\lambda{}_\mu\, \overset{\bullet}{A}{}^b{}_{a\lambda}\, u_b\, u^\rho \Big] = \alpha_g \overset{\bullet}{T}{}^a{}_{\mu\rho} u_a u^\rho. \quad (9.45)$$

Due to the fact that $u^\mu P^\rho{}_\mu = 0$, both sides of the equation of motion are clearly orthogonal to u^μ. A crucial point is that, though this equation of motion depends explicitly on the relation $\alpha_g = m_g/m_i$ of the particle, fields $B^a{}_\mu$ and $\overset{\bullet}{T}{}^a{}_{\rho\mu}$ do not. This means essentially that the teleparallel field equation (7.49) can be consistently solved for the gravitational potential $B^a{}_\mu$, independently of the validity or not of the weak equivalence principle. That is to say, Teleparallel Gravity is able to describe the motion of a particle even in its absence [82]. For $m_g = m_i$, the above equation of motion reduces to

$$\frac{du_a}{ds} - \overset{\bullet}{A}{}^b{}_{a\rho}\, u_b\, u^\rho = \overset{\bullet}{T}{}^b{}_{a\rho}\, u_b\, u^\rho, \tag{9.46}$$

which is the universal teleparallel force equation(4.52)

The equation of motion (9.45) is Lorentz covariant, as can be seen by rewriting it in the form

$$\tilde{h}^a{}_\mu \left(\frac{du_a}{ds} - u_b \overset{\bullet}{A}{}^b{}_{a\rho} u^\rho \right) + \tilde{h}^a{}_\rho u^\rho \left(u_\mu \frac{du_a}{ds} - u_b \overset{\bullet}{A}{}^b{}_{a\mu} \right) = \alpha_g u_a \overset{\bullet}{T}{}^a{}_{\mu\rho} u^\rho. \tag{9.47}$$

It is important to note that, under the gauge transformations

$$dx'^a = dx^a + \alpha_g\, d\epsilon^a \tag{9.48}$$

and

$$B'^a = B^a - d\epsilon^a, \tag{9.49}$$

both the action (9.30) and the equation of motion (9.45) are gauge invariant. Furthermore, in the Newtonian limit, Eq. (9.45) reduces to

$$m_i \, \frac{d^2 \vec{x}}{dt^2} = -\, m_g \, \nabla \Phi,$$

with

$$\Phi = c^2 B_{00},$$

which is just Newton equation for $m_i \neq m_g$. We recall that Newtonian gravity, like Teleparallel Gravity, can comply with the lack of universality.

9.4 Non-universality and General Relativity

Of course, the equation of motion (9.45) does not reduce to a geodesic equation. In order to comply with the foundations of General Relativity, and rewrite it as a geodesic equation, it would be necessary to incorporate the particle properties into the spacetime geometry. This can be achieved by assuming that

$$\tilde{h}^a{}_\mu = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu} x^b + \alpha_g B^a{}_\mu \tag{9.50}$$

which takes into account the relation $\alpha_g = m_g/m_i$ of the particle under consideration, is a true tetrad field. This tetrad defines a new spacetime metric tensor

$$\tilde{g}_{\mu\nu} = \eta_{ab}\, \tilde{h}^a{}_\mu\, \tilde{h}^b{}_\nu, \tag{9.51}$$

in terms of which the corresponding spacetime invariant interval is

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}\, dx^\mu dx^\nu. \tag{9.52}$$

It defines also a new Weitzenböck connection

$$\Gamma^\lambda{}_{\mu\rho} = \tilde{h}_a{}^\lambda \, \overset{\bullet}{\mathcal{D}}_\rho \tilde{h}^a{}_\mu, \tag{9.53}$$

whose torsion is

$$T^a{}_{\mu\rho} = \partial_\mu \tilde{h}^a{}_\rho - \partial_\rho \tilde{h}^a{}_\mu + \overset{\bullet}{A}{}^a{}_{b\mu} \tilde{h}^b{}_\rho - \overset{\bullet}{A}{}^a{}_{b\rho} \tilde{h}^b{}_\mu. \tag{9.54}$$

Then, by noticing that in this case the relation between the gravitational field strength

$$\overset{\bullet}{T}{}^a{}_{\mu\rho} = \partial_\mu B^a{}_\rho - \partial_\rho B^a{}_\mu + \overset{\bullet}{A}{}^a{}_{b\mu} B^b{}_\rho - \overset{\bullet}{A}{}^a{}_{b\rho} B^b{}_\mu \tag{9.55}$$

and torsion turns out to be

$$\alpha_g \, \overset{\bullet}{T}{}^a{}_{\mu\rho} = \tilde{h}^a{}_\lambda \, T^\lambda{}_{\mu\rho}, \tag{9.56}$$

it is an easy task to verify that, for a fixed relation $\alpha = m_g/m_i$, the equation of motion (9.45) is equivalent to the geodesic equation

$$\frac{d\tilde{u}_\mu}{d\tilde{s}} - \tilde{\Gamma}^\lambda{}_{\mu\rho}\, \tilde{u}_\lambda \, \tilde{u}^\rho = 0, \tag{9.57}$$

where

$$\tilde{u}^\mu \equiv \frac{dx^\mu}{d\tilde{s}} = u^a \, \tilde{h}_a{}^\mu$$

is the particle four-velocity, and $\tilde{\Gamma}^\rho{}_{\mu\nu}$ is the Christoffel connection of the metric $\tilde{g}_{\mu\nu}$. Notice that this equation can also be obtained from the action integral

$$\tilde{S} = -\, m_i c \int_a^b d\tilde{s}, \tag{9.58}$$

which has the usual form of the action in the context of General Relativity.

Nevertheless, the price for imposing a geodesic equation of motion to describe a non-universal interaction is that the theory becomes inconsistent. In fact, the solution of the corresponding Einstein's field equation

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\, \tilde{g}_{\mu\nu} \tilde{R} = \frac{8\pi G}{c^4}\, \tilde{\Theta}_{\mu\nu}, \tag{9.59}$$

would in this case depend on the relation m_g/m_i of the test particle, which renders the theory self-contradictory: test particles with different relations

m_g/m_i would require connections with different curvatures to keep all equations of motion given by geodesics. Of course the gravitational field, as a true field, cannot depend on any test particle properties. We can then conclude that, in the absence of the weak equivalence principle, the geometric description of General Relativity breaks down. Since the gauge potential $B^a{}_\mu$ can always be obtained independently of any property of the test particle, Teleparallel Gravity remains a consistent theory [82].

Chapter 10

Gravitational Coupling of the Fundamental Fields

The teleparallel coupling of some of the main fundamental relativistic fields — scalar, Dirac and electromagnetic fields — to gravitation are examined. The dominant aspect is that they all couple to gravitation through torsion.

10.1 Gravitational Coupling Revisited

We recall that a *relativistic field* is a field with a definite, covariant behavior under transformations of the Poincaré group $\mathcal{P}$ [See Appendix B, Section B.1]. This means, actually, that it belongs to one of the representations of $\mathcal{P}$. There is one of such representations for each value of mass and spin. With fixed masses, a scalar field belongs to the null representation, which means that it is invariant (spin 0); a Dirac field belongs to the bi-spinor representation (spin 1/2); a vector fields belongs to the vector representation (spin 1); and so on.

As discussed in Chapter 3, the teleparallel coupling of any field to gravitation is obtained by applying the coupling prescription

$$\partial_\mu \psi \ \rightarrow \ \ddot{\mathcal{D}}_\mu \psi, \tag{10.1}$$

where

$$\ddot{\mathcal{D}}_\mu \psi = \partial_\mu \psi - \tfrac{i}{2} \left(\dot{A}^{bc}{}_\mu - \dot{K}^{bc}{}_\mu \right) S_{bc}\, \psi, \tag{10.2}$$

with S_{bc} the Lorentz generators written in the representation appropriate for the field under consideration. In the case of a Lorentz vector field V^b, for which S_{bc} is given by Eq. (3.60), the coupling prescription assumes the form

$$\partial_\mu V^b \;\to\; \overset{\bullet\bullet}{\mathcal{D}}_\mu V^b = \partial_\mu V^b + \left(\overset{\bullet}{A}{}^b{}_{c\mu} - \overset{\bullet}{K}{}^b{}_{c\mu}\right) V^c. \tag{10.3}$$

The corresponding expression for the spacetime vector $V^\rho = h_c{}^\rho\, V^c$ is

$$\partial_\mu V^\rho \;\to\; \overset{\bullet\bullet}{\nabla}_\mu V^\rho = \partial_\mu V^\rho + \left(\overset{\bullet}{\Gamma}{}^\rho{}_{\lambda\mu} - \overset{\bullet}{K}{}^\rho{}_{\lambda\mu}\right) V^\lambda. \tag{10.4}$$

In this chapter we apply the teleparallel coupling prescription to some specific fundamental fields.

10.2 Scalar Field

Let us consider the lagrangian of a scalar field ϕ of mass m, which on Minkowski spacetime is written as

$$\mathcal{L}_\phi = \frac{e}{2}\left(\eta^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - \mu^2\phi^2\right), \tag{10.5}$$

where $e = \det(e^a{}_\mu) = 1$ and $\mu = mc/\hbar$. The corresponding field equation is the Klein–Gordon equation

$$\eta^{\mu\nu}\, \partial_\mu\partial_\nu\phi + \mu^2\phi = 0. \tag{10.6}$$

Since by definition a scalar field belongs to the null representation,

$$S_{ab}\phi = 0, \tag{10.7}$$

the coupling prescription (3.76 assumes the form

$$\partial_\mu \;\to\; \overset{\bullet\bullet}{\mathcal{D}}_\mu \equiv \partial_\mu. \tag{10.8}$$

Applying this prescription to the free lagrangian (10.5), it becomes

$$\mathcal{L}_\phi = \frac{h}{2}\left[g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - \mu^2\phi^2\right], \tag{10.9}$$

where we have concomitantly changed

$$e^a{}_\mu \;\to\; h^a{}_\mu \quad \text{and} \quad \eta^{\mu\nu} \;\to\; g^{\mu\nu}. \tag{10.10}$$

Variation of this lagrangian yields the field equation

$$\overset{\bullet}{\Box}\phi + \mu^2\phi = 0, \tag{10.11}$$

where

$$\overset{\bullet}{\Box}\,\phi = h^{-1}\,\partial_\rho\,(h\,\partial^\rho\phi) \tag{10.12}$$

is the Laplace–Beltrami operator. Using the identity

$$\partial_\mu h = h\,(\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\rho} - \overset{\bullet}{K}{}^\rho{}_{\mu\rho}), \tag{10.13}$$

it can be rewritten in the form

$$\overset{\bullet}{\Box}\phi = \left(\partial_\mu + \overset{\bullet}{\Gamma}{}^\rho{}_{\mu\rho} - \overset{\bullet}{K}{}^\rho{}_{\mu\rho}\right)\partial^\mu\phi \equiv \overset{\bullet\bullet}{\nabla}_\mu\partial^\mu\phi, \tag{10.14}$$

with $\overset{\bullet\bullet}{\nabla}_\mu$ the covariant derivative (10.4). This is the teleparallel version of the Laplace–Beltrami operator. The teleparallel version of the Klein–Gordon equation is, consequently [85],

$$\overset{\bullet\bullet}{\nabla}_\mu\partial^\mu\phi + \mu^2\phi = 0. \tag{10.15}$$

We see from this equation that, in Teleparallel Gravity, a scalar field couples to torsion. We see also that it results equivalent to apply the coupling prescription to the lagrangian or to the field equation.

Comment 10.1 It is important to remark that, in the Einstein–Cartan models (see Chapter 14), only a spin distribution could produce or feel torsion [83]. A scalar field, for example, should be able to feel only curvature [84]. However, as we have seen in Chapter 7, the gravitational interaction can be described alternatively in terms of curvature or torsion. Since a scalar field couples to curvature, it must also couple to torsion [85]. Actually, its own spin is zero, but its derivative is a vector field which "feels" torsion.

10.3 Dirac Spinor Field

10.3.1 The Dirac Equation

On Minkowski spacetime, the spinor field lagrangian is

$$\mathcal{L}_\psi = \frac{ic\hbar}{2}\,\left(\bar{\psi}\,\gamma^a\,e_a{}^\mu\,\partial_\mu\psi - e_a{}^\mu\,\partial_\mu\bar{\psi}\gamma^a\,\psi\right) - mc^2\,\bar{\psi}\psi. \tag{10.16}$$

The corresponding field equation is the free Dirac equation

$$i\hbar\gamma^a\,e_a{}^\mu\,\partial_\mu\psi - mc\,\psi = 0. \tag{10.17}$$

Comment 10.2 Due to the importance of spinor fields in the study of the gravitational interaction at the microscopic scale, in Appendix B we present a resumé on the Dirac equation.

The gravitationally coupled Dirac lagrangian is obtained by applying the teleparallel coupling prescription

$$e_a{}^\mu \, \partial_\mu \psi \;\; \rightarrow \;\; h_a{}^\mu \, \ddot{\mathcal{D}}_\mu = \partial_\mu \psi - \tfrac{i}{2} \left(\dot{A}^{bc}{}_\mu - \dot{K}^{bc}{}_\mu \right) S_{bc} \, \psi, \tag{10.18}$$

with S_{bc} the spinor representation

$$S_{bc} \equiv \tfrac{1}{2} \sigma_{bc} = \tfrac{i}{4} [\gamma_b, \gamma_c]. \tag{10.19}$$

The result is

$$\mathcal{L}_\psi = h \left[\frac{ic\hbar}{2} \left(\bar{\psi} \gamma^\mu \, \ddot{\mathcal{D}}_\mu \psi - \ddot{\mathcal{D}}_\mu \bar{\psi} \, \gamma^\mu \psi \right) - m \, c^2 \, \bar{\psi} \psi \right], \tag{10.20}$$

where we have used the notation $\gamma^\mu \equiv \gamma^\mu(x) = \gamma^a \, h_a{}^\mu$. Using the identity

$$\ddot{\mathcal{D}}_\mu (h \gamma^\mu) = 0, \tag{10.21}$$

the teleparallel version of the coupled Dirac equation is found to be

$$i\hbar \gamma^\mu \left[\partial_\mu \psi - \tfrac{i}{4} \left(\dot{A}^{bc}{}_\mu - \dot{K}^{bc}{}_\mu \right) \sigma_{bc} \right] \psi - mc \, \psi = 0. \tag{10.22}$$

In the class of frames in which the inertial connection $\dot{A}^{ab}{}_\mu$ vanishes, the Dirac equation becomes

$$i\hbar \gamma^\mu \left[\partial_\mu \psi + \tfrac{i}{4} \, \dot{K}^{bc}{}_\mu \, \sigma_{bc} \right] \psi - mc \, \psi = 0. \tag{10.23}$$

We see from this equation that, whereas in Teleparallel Gravity the Dirac spinor couples to the contortion tensor $\dot{K}^a{}_{b\mu}$, in General Relativity it couples to the spin connection $\mathring{A}^a{}_{b\mu}$.

Comment 10.3 Sometimes, the coupling prescription of Teleparallel Gravity is assumed to be defined by the covariant derivative

$$\dot{\mathcal{D}}_\mu \psi = \partial_\mu \psi - \frac{i}{2} \, \dot{A}^{ab}{}_\mu \, S_{ab} \psi. \tag{10.24}$$

Since $\dot{A}^{ab}{}_\mu$ vanishes in a specific class of frames, it is argued that, for spinor fields ψ, the Fock–Ivanenko derivative reduces to an ordinary derivative [12]:

$$\dot{\mathcal{D}}_\mu \psi = \partial_\mu \psi. \tag{10.25}$$

For this reason, it is usually asserted that, in the presence of spinor fields, the equivalence between General Relativity and Teleparallel Gravity is broken down. This argument is of course misleading as it is based on a coupling prescription that does not follow from the equivalence principle. Actually, the coupling prescription based on the covariant derivative (10.24) violates the principle, and is therefore physically unacceptable.

10.3.2 Torsion Decomposition and Spinors

As discussed in Section 1.4, torsion can be decomposed in irreducible components under the global Lorentz group:

$$T_{\lambda\mu\nu} = \tfrac{2}{3}\left(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}\right) + \tfrac{1}{3}\left(g_{\lambda\mu}v_\nu - g_{\lambda\nu}v_\mu\right) + \epsilon_{\lambda\mu\nu\rho}\, a^\rho, \tag{10.26}$$

where $t_{\lambda\mu\nu}$ represents the purely tensor torsion, and v_μ and a^ρ represent respectively its vector and axial parts. Let us consider the Dirac equation (10.22). As a simple calculation shows, the coupling term of the covariant derivative is

$$\tfrac{i}{4}\, \overset{\bullet}{K}{}^{ab}{}_\mu\, \gamma^\mu\, \sigma_{ab} = -\,\gamma^\mu \left(\tfrac{1}{2}\overset{\bullet}{\mathcal{V}}_\mu + \tfrac{3i}{4}\, \overset{\bullet}{\mathcal{A}}_\mu\, \gamma^5\right), \tag{10.27}$$

where $\gamma^5 = \gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$. As a consequence, in the class of frames in which the inertial connection $\overset{\bullet}{A}{}^{ab}{}_\mu$ vanishes, the teleparallel covariant derivative of a spinor field can be written as

$$\gamma^\mu\, \overset{\bullet\bullet}{\mathcal{D}}_\mu \psi = \gamma^\mu \left(\partial_\mu - \tfrac{1}{2}\, \overset{\bullet}{\mathcal{V}}_\mu - \tfrac{3i}{4}\, \overset{\bullet}{\mathcal{A}}_\mu\, \gamma^5\right) \psi. \tag{10.28}$$

The corresponding teleparallel Dirac equation then reads

$$i\,\hbar\,\gamma^\mu \left(\partial_\mu - \tfrac{1}{2}\, \overset{\bullet}{\mathcal{V}}_\mu - \tfrac{3i}{4}\, \overset{\bullet}{\mathcal{A}}_\mu \gamma^5\right) \psi = mc\,\psi. \tag{10.29}$$

It involves the vector $\overset{\bullet}{\mathcal{V}}_\mu$ and the axial $\overset{\bullet}{\mathcal{A}}_\mu$ torsions only [37]. This means essentially that the purely tensor piece $t_{\lambda\mu\nu}$ of torsion is irrelevant for the description of the gravitational interaction of fermions. Notice that the property above is necessary for invariance under temporal and space invertions: vector $\overset{\bullet}{\mathcal{V}}_\mu$ couples to the vector current γ^μ, axial $\overset{\bullet}{\mathcal{A}}_\mu$ couples to the axial current $\gamma^\mu\gamma^5$ [see Appendix B for a discussion on the different kind of currents allowed by the Grassmann algebra].

Comment 10.4 It is interesting to remark that, in General Relativity, where the covariant derivative is given by

$$\overset{\circ}{\mathcal{D}}_\mu \psi = \partial_\mu \psi - \tfrac{i}{2}\, \overset{\circ}{A}{}^{ab}{}_\mu\, S_{ab}\psi, \tag{10.30}$$

if the spin connection $\overset{\circ}{A}{}^{bc}{}_\mu$ is written in terms of the coefficient of nonholonomy $f^a{}_{bc}$ according to

$$\overset{\circ}{A}{}^a{}_{bc} = -\tfrac{1}{2}(f^a{}_{bc} + f_{bc}{}^a + f_{cb}{}^a), \tag{10.31}$$

a decomposition similar to (10.27) can be made, and the Dirac equation turns out to be written in terms of the trace and the pseudo-trace of $f^a{}_{bc}$ only. The purely tensor part of $f^a{}_{bc}$ is also irrelevant for spinors.

10.4 Electromagnetic Field

On Minkowski spacetime, the electromagnetic field is described by the Lagrangian density

$$\mathcal{L}_{em} = -\frac{e}{4}\, F_{\mu\nu} F^{\mu\nu}, \tag{10.32}$$

where $e = \det(e^a{}_\mu) = 1$, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{10.33}$$

is the electromagnetic field–strength. The corresponding field equation is

$$\partial_\mu F^{\mu\nu} = 0, \tag{10.34}$$

which along with the Bianchi identity

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0, \tag{10.35}$$

constitute the sourceless Maxwell's equations. In the Lorenz gauge $\partial_\mu A^\mu = 0$, field equation (10.34) becomes the wave equation

$$\eta^{\mu\nu}\, \partial_\mu \partial_\nu A^\rho = 0. \tag{10.36}$$

Let us obtain now, by applying the coupling prescription (10.4), Maxwell's equation in Teleparallel Gravity. In the specific case of the electromagnetic vector field A^ρ, the coupling prescription assumes the form

$$\partial_\mu V^\rho \;\; \rightarrow \;\; \ddot{\nabla}_\mu A^\rho = \partial_\mu A^\rho + \left(\dot{\Gamma}^\rho{}_{\nu\mu} - \dot{K}^\rho{}_{\nu\mu}\right) A^\nu. \tag{10.37}$$

As a consequence, the gravitationally coupled Maxwell lagrangian in Teleparallel Gravity can be written as

$$\mathcal{L}_{em} = -\frac{h}{4}\, F_{\mu\nu} F^{\mu\nu}, \tag{10.38}$$

where now

$$F_{\mu\nu} = \ddot{\nabla}_\mu A_\nu - \ddot{\nabla}_\nu A_\mu. \tag{10.39}$$

Using the explicit form of $\ddot{\nabla}_\mu$, and the definitions of torsion and contortion tensors, it is easy to verify that the field strength, like in General Relativity, does not change:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{10.40}$$

The corresponding field equation is

$$\ddot{\nabla}_\mu F^{\mu\nu} \equiv \partial_\mu (h F^{\mu\nu}) = 0, \tag{10.41}$$

which yields Maxwell's equation in Teleparallel Gravity. Assuming the teleparallel Lorenz gauge $\overset{\bullet\bullet}{\nabla}_\mu A^\mu = 0$, and using the commutation relation

$$\left[\overset{\bullet\bullet}{\nabla}_\mu, \overset{\bullet\bullet}{\nabla}_\nu\right] A^\mu = -\overset{\bullet}{Q}_{\mu\nu}\, A^\mu, \tag{10.42}$$

where $\overset{\bullet}{Q}_{\mu\nu} = \overset{\bullet}{Q}{}^\rho{}_{\mu\rho\nu}$, with $\overset{\bullet}{Q}{}^\rho{}_{\mu\sigma\nu}$ given by Eq. (7.21), we obtain

$$\overset{\bullet\bullet}{\nabla}_\mu \overset{\bullet\bullet}{\nabla}{}^\mu A_\nu + \overset{\bullet}{Q}{}^\mu{}_\nu A_\mu = 0. \tag{10.43}$$

This is the teleparallel version of Maxwell's equation in terms of the electromagnetic potential. Consider now the Bianchi identity (10.35). Applying the coupling prescription (10.37), it is an easy task to verify that it remains invariant:

$$\partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} = 0. \tag{10.44}$$

This is the teleparallel version of the second pair of Maxwell's equation.

In the context of the teleparallel equivalent of General Relativity, therefore, the electromagnetic field is able to couple to torsion, and this coupling does not violate the gauge invariance of Maxwell's theory. Furthermore, using relation (7.18), it is easy to verify that the teleparallel version of Maxwell's equations, which are equations written in terms of the Weitzenböck connection only, are completely equivalent with the usual Maxwell's equations in the context of General Relativity, which are equations written in terms of the Levi–Civita connection only. The basic lesson is that Teleparallel Gravity is able to provide a consistent description of the interaction of torsion with the electromagnetic field [96].

10.5 Spin-2 Field

10.5.1 Defining a Spin-2 Field

Due to the fact that linearized gravity represents a spin-2 field, the dynamics of a fundamental spin-2 field in Minkowski space, according to the usual approach [103], is expected to coincide with the dynamics of a linear perturbation of the metric around flat spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + \psi_{\mu\nu}. \tag{10.45}$$

For this reason, a fundamental spin-2 field is usually assumed to be described by a rank-two, symmetric tensor $\psi_{\mu\nu} = \psi_{\nu\mu}$. However, conceptually speaking, this is not the most fundamental notion of a spin-2 field. As is well known, although the gravitational interaction of scalar and vector fields can be described in the metric formalism, the gravitational interaction of spinor fields requires a tetrad formalism [104]. The tetrad formalism can then be considered to be more fundamental than the metric formulation in the sense that it is able to describe the gravitational interaction of both tensor and spinor fields. Accordingly, the tetrad field can be said to be more fundamental than the metric. Relying on this property, instead of similar to a linear perturbation of the metric, a fundamental spin-2 field should be considered a linear perturbation of the tetrad field [111].

Denoting by $e^a{}_\mu$ a trivial tetrad representing the Minkowski metric, a fundamental spin-2 field $\phi^a{}_\mu$ should, therefore, be defined by

$$h^a{}_\mu = e^a{}_\mu + \phi^a{}_\mu. \tag{10.46}$$

Since the tetrad is a translational-valued vector field, $\phi^a{}_\mu$ will also be a translational-valued vector field,

$$\phi_\mu = \phi^a{}_\mu \, P_a, \tag{10.47}$$

with $P_a = \partial_a$ the translation generators. Observe that, in the usual *metric* formulation of gravity, the symmetry of the metric tensor eliminates six degrees of freedom of the sixteen original degrees of freedom of $g_{\mu\nu}$. In the *tetrad* formulation, on the other hand, local Lorentz invariance is responsible for eliminating six degrees of freedom of the sixteen original degrees of freedom of $h^a{}_\mu$, yielding the same number of independent components of $g_{\mu\nu}$. Of course, the same equivalence must hold in relation to the fields $\psi_{\mu\nu}$ and $\phi^a{}_\mu$.

Now, in teleparallel gravity the gravitational field is represented by a translational gauge potential $B_\mu = B^a{}_\mu P_a$, which appears as the nontrivial part of the tetrad field [106]. This means essentially that $\phi^a{}_\mu$ is similar to the gauge potential of teleparallel gravity. Accordingly, its dynamics must coincide with the dynamics of linearized teleparallel gravity.

10.5.2 The Flat Spacetime Case

Gauge transformations

In the inertial frame e'^a, the tetrad describing the flat Minkowski spacetime is of the form

$$e'^a{}_\mu = \partial_\mu x'^a. \tag{10.48}$$

A spin-2 field $\phi'^a{}_\mu$ corresponds to a linear perturbation of this tetrad:

$$h'^a{}_\mu = \partial_\mu x'^a + \phi'^a{}_\mu. \tag{10.49}$$

In this class of frames, therefore, the vacuum is represented by

$$\phi'^a{}_\mu = \partial_\mu \xi^a(x), \tag{10.50}$$

with $\xi^a(x)$ an arbitrary function of the spacetime coordinates x^ρ. In fact, such $\phi'^a{}_\mu$ represents simply a gauge translation

$$x'^a \to x'^a + \xi^a(x) \tag{10.51}$$

in the fiber fiber, or tangent space. This means that the gauge transformation associated to the spin-2 field $\phi^a{}_\mu$ is

$$\phi'^a{}_\mu \to \phi'^a{}_\mu - \partial_\mu \xi^a(x). \tag{10.52}$$

Of course, $h'^a{}_\mu$ is invariant under such transformations.

Field strength and Bianchi identity

The next step towards the construction of a field theory for $\phi'^a{}_\mu$ is to define the analogous of torsion:

$$F'^a{}_{\mu\nu} = \partial_\mu \phi'^a{}_\nu - \partial_\nu \phi'^a{}_\mu. \tag{10.53}$$

This tensor is actually the spin-2 field-strength. As can be easily verified, $F'^a{}_{\mu\nu}$ is gauge invariant. Furthermore, it satisfies the Bianchi identity

$$\partial_\rho F'^a{}_{\mu\nu} + \partial_\nu F'^a{}_{\rho\mu} + \partial_\mu F'^a{}_{\nu\rho} = 0, \tag{10.54}$$

which can equivalently be written in the form

$$\partial_\rho (\varepsilon^{\lambda\rho\mu\nu} \, F'^a{}_{\mu\nu}) = 0, \tag{10.55}$$

with $\varepsilon^{\lambda\rho\mu\nu}$ the totally anti-symmetric, flat spacetime Levi-Civita tensor.

Lagrangian and field equation

Considering that the dynamics of a spin-2 field must coincide with the dynamics of linear gravity, its Lagrangian will be similar to the Lagrangian (7.12) of teleparallel gravity. One has just to replace the teleparallel torsion $\dot{T}^a{}_{\mu\nu}$ by the spin-2 field strength $\sqrt{k}\, F'^a{}_{\mu\nu}$. It is interesting to notice that when we do that, the spin-2 analogous of the teleparallel superpotential $\dot{S}^a{}_{\mu\nu}$ is the Fierz tensor [97]

$$\mathcal{F}'^{\,\mu\nu}_a = e'_a{}^\rho \, \mathcal{K}'^{\mu\nu}{}_\rho - e'_a{}^\nu \, e'_b{}^\rho \, F'^{b\mu}{}_\rho + e'_a{}^\mu \, e'_b{}^\rho \, F'^{b\nu}{}_\rho, \tag{10.56}$$

with

$$\mathcal{K}'^{\mu\nu}{}_{\rho} = \tfrac{1}{2}\left(e'_a{}^{\nu}\, F'^{a\mu}{}_{\rho} + e'^{a}{}_{\rho}\, F'^{\mu\nu}_{a} - e'_a{}^{\mu}\, F'^{a\nu}{}_{\rho}\right) \tag{10.57}$$

the spin-2 analogous of the teleparallel contortion. Considering that in the absence of gravitation $\det(e'^{a}{}_{\mu}) = 1$, the Lagrangian for a massless spin-2 field is

$$\mathcal{L}' = \frac{1}{4}\, \mathcal{F}'^{\mu\nu}_{a}\, F'^{a}{}_{\mu\nu}. \tag{10.58}$$

By performing variations with respect to $\phi^{a}{}_{\rho}$, we obtain

$$\partial_\mu \mathcal{F}'^{\rho\mu}_{a} = 0. \tag{10.59}$$

This is the field equation satisfied by a massless spin-2 field in Minkowski spacetime, as seen from the inertial frame $e'^{a}{}_{\mu}$.

Comment 10.5 It is interesting to notice that teleparallel gravity naturally yields the Fierz formulation for a spin-2 field [107].

Duality symmetry

The spin-2 field can be viewed as an Abelian gauge field with the *internal* index replaced by an *external* Lorentz index. Due to the presence of the tetrad, Lorentz and spacetime indices can be transformed into each other. As a consequence, its Hodge dual will necessarily include additional index contractions in relation to the usual dual. Taking into account all possible contractions, its dual turns out to be given by [108]

$$^{\star}F'^{a}{}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\, \mathcal{F}'^{a\rho\sigma}. \tag{10.60}$$

Substituting the Fierz tensor (10.56), we find

$$^{\star}F'^{a}{}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left(F'^{a\rho\sigma} - e'^{a\sigma} e'_b{}^{\lambda}\, F'^{b\rho}{}_{\lambda} + e'^{a\rho} e'_b{}^{\lambda}\, F'^{b\sigma}{}_{\lambda}\right). \tag{10.61}$$

Let us now consider the Bianchi identity (10.55). Written for the dual of $F'^{a}{}_{\mu\nu}$, it reads

$$\partial_\rho(\varepsilon^{\lambda\rho\mu\nu}\, {}^{\star}F'^{a}{}_{\mu\nu}) = 0. \tag{10.62}$$

Substituting ${}^{\star}F'^{a}{}_{\mu\nu}$ as given by Eq. (10.60), we get

$$\partial_\rho \mathcal{F}'^{a\mu\rho} = 0, \tag{10.63}$$

which is the field equation (10.59). We see in this way that, provided the generalized Hodge dual (10.60) for soldered bundles is used, the spin-2 field has duality symmetry. This is actually an expected result because the dynamics of a spin-2 field must coincide with the dynamics of linear gravity, which has already been shown to present duality symmetry [?]. We remark finally that, in the usual Fierz formulation of a spin-2 field, identity (10.55) has to be put by hand in order to get the number of equations consistent with the number of degrees of freedom [109]. In the present formulation it appears as a consequence of considering the spin-2 field as a perturbation of the tetrad instead of the metric.

Passage to a general frame

In a Lorentz rotated frame $e^a = \Lambda^a{}_b(x)\, e'^a$, the tetrad assumes the form

$$e^a{}_\mu \equiv \overset{\bullet}{\mathcal{D}}_\mu x^a = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu}\, x^b. \tag{10.64}$$

In this frame, the vacuum of $\phi^a{}_\mu$ turns out to be represented by

$$\phi^a{}_\mu = \overset{\bullet}{\mathcal{D}}_\mu \xi^a(x), \tag{10.65}$$

whereas the gauge transformations assume the form

$$\phi^a{}_\mu \to \phi^a{}_\mu - \overset{\bullet}{\mathcal{D}}_\mu \xi^a(x). \tag{10.66}$$

The field strength (10.53), on the other hand, becomes

$$F^a{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu \phi^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu \phi^a{}_\mu. \tag{10.67}$$

Accordingly, the Bianchi identity reads

$$\overset{\bullet}{\mathcal{D}}_\rho F^a{}_{\mu\nu} + \overset{\bullet}{\mathcal{D}}_\nu F^a{}_{\rho\mu} + \overset{\bullet}{\mathcal{D}}_\mu F^a{}_{\nu\rho} = 0, \tag{10.68}$$

which is equivalent to

$$\overset{\bullet}{\mathcal{D}}_\rho (\varepsilon^{\lambda\rho\mu\nu}\, F^a{}_{\mu\nu}) = 0. \tag{10.69}$$

Analogously, the Fierz tensor turns out to be

$$\mathcal{F}_a{}^{\mu\nu} = e_a{}^\rho\, \mathcal{K}^{\mu\nu}{}_\rho - e_a{}^\nu\, e_b{}^\rho\, F^{b\mu}{}_\rho + e_a{}^\mu\, e_b{}^\rho\, F^{b\nu}{}_\rho. \tag{10.70}$$

Now, as a simple inspection shows, the Lagrangian (10.58) is invariant under local Lorentz transformations, that is,

$$\mathcal{L}' \equiv \mathcal{L} = \frac{1}{4}\, \mathcal{F}_a{}^{\mu\nu}\, F^a{}_{\mu\nu}. \tag{10.71}$$

The corresponding field equation,

$$\overset{\bullet}{\mathcal{D}}_\mu \mathcal{F}_a{}^{\rho\mu} \equiv \partial_\mu \mathcal{F}_a{}^{\rho\mu} - \overset{\bullet}{A}{}^b{}_{a\mu}\, \mathcal{F}_b{}^{\rho\mu} = 0, \tag{10.72}$$

represents the field equation satisfied by a massless spin-2 field in Minkowski spacetime, as seen from the general frame $e^a{}_\mu$. It is clearly invariant under the gauge transformation (10.66). It is important to observe that the theory has twenty two constraints: sixteen of the invariance under the gauge transformations (10.52), and six from the invariance of the Lagrangian $\mathcal{L}$ under local Lorentz transformations. The twenty four original components of the Fierz tensor are then reduced to only two, as appropriate for a massless spin-2 field.

Relation to the metric approach

Let us consider the inertial frame e'_a endowed with a cartesian coordinate system. In this case all connections vanish, and according to Eq. (1.33) the tetrad $e'^a{}_\mu$ satisfies the condition $\partial_\rho e'^a{}_\mu = 0$. Using this tetrad, we can define $\phi'^\rho{}_\mu := e'_a{}^\rho \, \phi'^a{}_\mu$. Since $\phi'^\rho{}_\mu$ is not in principle symmetric, the perturbation of the metric — which is usually supposed to represent a fundamental spin-2 field — is to be identified with the symmetric part of $\phi'^\rho{}_\mu$:

$$\psi^\rho{}_\mu = \phi'^\rho{}_\mu + \phi'_\mu{}^\rho. \tag{10.73}$$

It is then easy to see that, in terms of $\psi^\rho{}_\mu$, the gauge transformation (10.52) acquires the form,

$$\psi^\rho{}_\mu \to \psi^\rho{}_\mu - \partial^\rho \xi_\mu(x) - \partial_\mu \xi^\rho(x), \tag{10.74}$$

where $\xi^\mu(x) = \xi^a(x) \, e'_a{}^\mu$. The Bianchi identity (10.54), on the other hand, is seen to be trivially satisfied, whereas the field equation (10.59) assumes the form

$$\Box \, (\delta^\mu{}_\lambda \, \psi - \psi^\mu{}_\lambda) - \partial_\lambda \partial^\mu \, \psi - \delta^\mu{}_\lambda \, \partial_\nu \partial_\rho \, \psi^{\nu\rho} + \partial_\nu \partial^\mu \, \psi^\nu{}_\lambda + \partial_\lambda \partial_\rho \, \psi^{\rho\mu} = 0, \tag{10.75}$$

with $\psi = \psi^\alpha{}_\alpha$. This is precisely the linearized Einstein equation [112], which means that in absence of gravity the teleparallel-based approach is totally equivalent to the usual general relativity-based approach to the spin-2 field. Namely, $\phi^a{}_\rho$ and $\psi_{\mu\nu}$ are the same physical field, which in the massless case is well known to represent waves with helicity 2. The teleparallel approach, however, is much more elegant and simple in the sense that it is similar to the spin-1 electromagnetic theory — an heritage of the (abelian) gauge structure of teleparallel gravity. In addition, it allows a precise distinction between gauge transformations — local translations in the tangent space — and spacetime coordinate transformations. Furthermore, in contrast to the usual metric approach, the field $\phi^a{}_\rho$, as well as its gauge transformations, are not restricted to be infinitesimal.

10.5.3 Coupling with Gravitation

Gravitational coupling prescription

In absence of gravitation, as we have seen in the previous section, the algebraic index of ${\phi^a}_\rho$ can be transformed into a spacetime index through contraction with the tetrad field, and vice-versa. In the presence of gravitation, this index transformation can lead to problems with the coupling prescription. In fact, as is well known, higher ($s > 1$) spin fields, and in particular a spin-2 field, present consistency problems when coupled to gravitation [112]. The problem is that the divergence identities satisfied by the field equations of a spin-2 field in Minkowski spacetime are no longer valid when it is coupled to gravitation. In addition, the coupled equations are no longer gauge invariant. The basic underlying difficulty is related to the fact that the covariant derivative of general relativity — which defines the gravitational coupling prescription — is non-commutative, and this introduces unphysical constraints on the spacetime curvature. As we are going to see here, if Teleparallel Gravity is used as paradigm, all inconsistencies disappear.

To begin with we note that, because ${\phi^a}_\rho$ is a vector field assuming values in the Lie algebra of the translation group, $\phi_\rho = {\phi^a}_\rho P_a$, the algebraic index "$a$" is not an ordinary vector index. It is actually a gauge index which, due to the "external" character of translations, happens to be similar to the usual, true vector index "ρ". To understand better this difference, let us consider a scalar field ϕ. As is well known, its gravitational coupling prescription is trivial:

$$\partial_\mu \phi \rightarrow \partial_\mu \phi. \tag{10.76}$$

Its interaction with gravitation comes solely from the tetrad replacement

$${e^a}_\mu \rightarrow {h^a}_\mu = {e^a}_\mu + {B^a}_\mu, \tag{10.77}$$

or equivalently, from the metric replacement

$$\eta_{\mu\nu} = {e^a}_\mu {e^b}_\nu \, \eta_{ab} \rightarrow g_{\mu\nu} = {h^a}_\mu {h^b}_\nu \, \eta_{ab}. \tag{10.78}$$

The crucial point is to note that this is true independently of whether the scalar field is or is not a translational-valued field $\phi = \phi^a P_a$. We see in this way that translational gauge indices are irrelevant for the gravitational coupling prescription.

Based on the above considerations, in the class of frames h'_a in which the inertial connection $\overset{\bullet}{A}{}^a{}_{b\mu}$ vanishes, the gravitational coupling prescription of the spin-2 field $\phi_\rho = \phi^a{}_\rho P_a$ is written in the form

$$\partial_\mu \phi'_\rho \to \partial_\mu \phi'_\rho - \left(\overset{\bullet}{\Gamma}{}^\lambda{}_{\rho\mu} - \overset{\bullet}{K}{}^\lambda{}_{\rho\mu} \right) \phi'_\lambda. \tag{10.79}$$

In components, it reads

$$\partial_\mu \phi'^a{}_\rho \to \partial_\mu \phi'^a{}_\rho - \left(\overset{\bullet}{\Gamma}{}^\lambda{}_{\rho\mu} - \overset{\bullet}{K}{}^\lambda{}_{\rho\mu} \right) \phi'^a{}_\lambda. \tag{10.80}$$

Of course, because of the identity

$$\overset{\bullet}{\Gamma}{}^\lambda{}_{\rho\mu} - \overset{\bullet}{K}{}^\lambda{}_{\rho\mu} = \overset{\circ}{\Gamma}{}^\lambda{}_{\rho\mu}, \tag{10.81}$$

with $\overset{\circ}{\Gamma}{}^\lambda{}_{\rho\mu}$ the Levi-Civita connection of the metric $g_{\mu\nu}$, this coupling prescription coincides with the coupling prescription of general relativity. This is a key point of the equivalence between general relativity and teleparallel gravity [110].

Let us consider now the coupling prescription in an arbitrary frame. Under a local Lorentz transformation, the ordinary derivative of $\phi'^a{}_\rho$ transforms according to

$$\partial_\mu \phi'^a{}_\rho \to \overset{\bullet}{\mathcal{D}}_\mu \phi^a{}_\rho = \partial_\mu \phi^a{}_\rho + \overset{\bullet}{A}{}^a{}_{b\mu}\, \phi^b{}_\rho. \tag{10.82}$$

In a general Lorentz frame, therefore, the gravitational coupling prescription of a fundamental spin-2 field $\phi^a{}_\rho$ is written as

$$\partial_\mu \phi^a{}_\rho \to \partial_\mu \phi^a{}_\rho + \overset{\bullet}{A}{}^a{}_{b\mu}\, \phi^b{}_\rho - \left(\overset{\bullet}{\Gamma}{}^\lambda{}_{\rho\mu} - \overset{\bullet}{K}{}^\lambda{}_{\rho\mu} \right) \phi^a{}_\lambda. \tag{10.83}$$

This coupling prescription provides different connection-terms for each index of $\phi^a{}_\rho$: whereas the algebraic index is connected to inertial effects only, the spacetime index is connected to the gravitational coupling prescription. It constitutes one of the main differences of the teleparallel-based approach in relation to the usual metric approach based on general relativity. In fact, the latter considers both indices of the spin-2 variable $\psi_{\mu\nu}$ on an equal footing, leading to a coupling prescription that breaks the gauge invariance of the spin-2 theory.

Comment 10.6 It is important to note that, as a simple inspection shows, the coupling prescription (10.83) cannot be rewritten in terms of the connections $\overset{\circ}{A}{}^a{}_{b\mu}$ and $\overset{\circ}{\Gamma}{}^\lambda{}_{\rho\mu}$ of General Relativity. In spite of the equivalenece between Teleparallel Gravity and General Relativity, such spin-2 field theory has not a General Relativity counterpart.

Field strength and Bianchi identity

Let us now apply the gravitational coupling prescription (10.83) to the free theory. To begin with we notice that, because the gauge parameter ξ^a has an algebraic index only, the gauge transformation (10.66) does not change in the presence of gravitation:

$$\phi^a{}_\mu \rightarrow \phi^a{}_\mu - \overset{\bullet}{\mathcal{D}}_\mu \xi^a(x). \tag{10.84}$$

On the other hand, considering that the connection (10.81) is symmetric in the last two indices, we see that the field strength $F^a{}_{\mu\nu}$ does not change in the presence of gravitation:

$$F^a{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu \phi^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu \phi^a{}_\mu. \tag{10.85}$$

Due to the fact that the teleparallel Fock-Ivanenko derivative $\overset{\bullet}{\mathcal{D}}_\mu$ is commutative, the Bianchi identity also remains unchanged,

$$\overset{\bullet}{\mathcal{D}}_\rho F^a{}_{\mu\nu} + \overset{\bullet}{\mathcal{D}}_\nu F^a{}_{\rho\mu} + \overset{\bullet}{\mathcal{D}}_\mu F^a{}_{\nu\rho} = 0. \tag{10.86}$$

Since $\varepsilon^{\lambda\rho\mu\nu}$ is a density of weight $\omega = -1$, in the presence of gravitation the Levi-Civita *tensor* is given by $h\varepsilon^{\lambda\rho\mu\nu}$, and the Bianchi identity can be rewritten in the form

$$\overset{\bullet}{\mathcal{D}}_\rho (h\,\varepsilon^{\lambda\rho\mu\nu}\, F^a{}_{\mu\nu}) = 0. \tag{10.87}$$

This is similar to what happens to the electromagnetic field in the presence of gravitation.

Lagrangian and field equation

Analogously to the flat background case, the Lagrangian of the spin-2 field in the presence of gravitation can be obtained from the teleparallel Lagrangian (7.12) by replacing the teleparallel torsion $\overset{\bullet}{T}{}^a{}_{\mu\nu}$ by the spin-2 field strength $\sqrt{k}\, F^a{}_{\mu\nu}$. The result is

$$\mathcal{L} = \frac{h}{4}\, \mathcal{F}_a{}^{\mu\nu}\, F^a{}_{\mu\nu}, \tag{10.88}$$

where

$$\mathcal{F}_a{}^{\mu\nu} = h_a{}^\rho\, \mathcal{K}^{\mu\nu}{}_\rho - h_a{}^\mu\, h_b{}^\sigma F^{b\nu}{}_\sigma + h_a{}^\nu\, h_b{}^\sigma F^{b\mu}{}_\sigma \tag{10.89}$$

is the gravitationally-coupled Fierz tensor, with

$$\mathcal{K}^{\mu\nu}{}_\rho = \tfrac{1}{2}\,(h_a{}^\nu\, F^{a\mu}{}_\rho + h^a{}_\rho\, F_a{}^{\mu\nu} - h_a{}^\mu\, F^{a\nu}{}_\rho) \tag{10.90}$$

the corresponding spin-2 analogous of the coupled contortion. We notice in passing that this Lagrangian is invariant under the gauge transformation (10.66). It is furthermore invariant under local Lorentz transformation $h'^a{}_\mu = \Lambda^a{}_b(x)\, h^a{}_\mu$ of the frames.

Performing variations in relation to $\phi^a{}_\rho$, we get

$$\dot{\mathcal{D}}_\mu \mathcal{F}_a{}^{\rho\mu} + (\dot{\Gamma}^\mu{}_{\nu\mu} - \dot{K}^\mu{}_{\nu\mu})\, \mathcal{F}_a{}^{\rho\nu} = 0. \tag{10.91}$$

Using the identity

$$\partial_\mu h = h\, \mathring{\Gamma}^\mu{}_{\lambda\mu} \equiv h \left(\dot{\Gamma}^\mu{}_{\lambda\mu} - \dot{K}^\mu{}_{\lambda\mu} \right), \tag{10.92}$$

it can be rewritten in the form

$$\dot{\mathcal{D}}_\mu (h \mathcal{F}_a{}^{\rho\mu}) = 0. \tag{10.93}$$

This is the field equation of a fundamental spin-2 field in the presence of gravitation, as seen from the general frame $h^a{}_\mu$. It is important to remark that, on account of the commutativity of the covariant derivative $\dot{\mathcal{D}}_\mu$, the gravitationally-coupled theory is gauge invariant and, like the free theory, has the correct number of independent components.

Comment 10.7 It is important to observe that the coupled equation (10.93) can also be obtained from the free field equation (10.72) by applying the gravitational coupling prescription (10.83).

10.5.4 Spin-2 Field as Source of Gravitation

Let us consider now the total Lagrangian

$$\mathcal{L}_t = \dot{\mathcal{L}} + \mathcal{L}, \tag{10.94}$$

where $\dot{\mathcal{L}}$ is the teleparallel Lagrangian (7.12), and $\mathcal{L}$ is the Lagrangian (10.88) of a spin-2 field in the presence of gravitation. The corresponding field equation is

$$\partial_\sigma (h \dot{S}_a{}^{\rho\sigma}) - k\, h\, (\dot{t}_a{}^\rho + \dot{\imath}_a{}^\rho) = k\, h\, \theta_a{}^\rho, \tag{10.95}$$

where $\dot{t}_a{}^\rho$ is the gravitational energy-momentum tensor, $\dot{\imath}_a{}^\rho$ is the energy-momentum pseudotensor of inertia, and

$$\theta_a{}^\rho \equiv -\frac{1}{h} \frac{\delta \mathcal{L}}{\delta h^a{}_\rho} = h_a{}^\nu\, \mathcal{F}_c{}^{\mu\rho}\, F^c{}_{\mu\nu} - \frac{h_a{}^\rho}{h}\, \mathcal{L} \tag{10.96}$$

is the spin-2 field source energy-momentum tensor. Observe that

$$\theta_\rho{}^\rho \equiv h^a{}_\rho \, \theta_a{}^\rho = 0, \tag{10.97}$$

as it should be for a massless field. Furthermore, from the invariance of $\mathcal{L}$ under general coordinate transformation, it is found to satisfy the usual (general relativity) covariant conservation law [35]

$$\overset{\circ}{\mathcal{D}}_\rho(h\theta_a{}^\rho) \equiv \partial_\rho(h\theta_a{}^\rho) - \overset{\circ}{A}{}^b{}_{a\rho}\,(h\theta_b{}^\rho) = 0. \tag{10.98}$$

Due to the anti-symmetry of the superpotential in the last two indices, we see from the field equation (10.95) that the total energy-momentum density is conserved in the ordinary sense:

$$\partial_\rho[h(\overset{\bullet}{\imath}_a{}^\rho + \overset{\bullet}{t}_a{}^\rho + \theta_a{}^\rho)] = 0. \tag{10.99}$$

As we have seen in Chapter 7, the field equation (10.95) can be rewritten in the form

$$\overset{\bullet}{\mathcal{D}}_\sigma(h\overset{\bullet}{S}_a{}^{\rho\sigma}) = k\,h\,(\overset{\bullet}{t}_a{}^\rho + \theta_a{}^\rho), \tag{10.100}$$

where the right-hand side represents the true gravitational field source. We recall that the energy-momentum density of inertia, although entering the total energy-momentum conservation, is not source of gravitation, and accordingly must remain in the left-hand side of the field equation. Considering that the covariant derivative $\overset{\bullet}{\mathcal{D}}_\rho$ is commutative, the true source of gravitation is found to be conserved in the covariant sense:

$$\overset{\bullet}{\mathcal{D}}_\rho[h(\overset{\bullet}{t}_a{}^\rho + \theta_a{}^\rho)] = 0. \tag{10.101}$$

This property ensures the consistency of the theory in the sense that no constraints on the background spacetime geometry show up.

10.5.5 Further Remarks

Due to the fact that it describes the gravitational interaction through a geometrization of spacetime, General Relativity is not, strictly speaking, a field theory in the usual sense of classical fields. On the other hand, owing to its gauge structure, teleparallel gravity does not geometrize the gravitational interaction, and for this reason it is much more akin to a field theory than general relativity. When looking for a field theory for the spin-2 field, therefore, it seems far more reasonable to use teleparallel gravity as paradigm. Accordingly, instead of a symmetric second-rank tensor $\psi_{\mu\nu}$, the spin-2 field is assumed to be represented by a spacetime (world) vector field assuming values in the Lie algebra of the translation group. Its components $\phi^a{}_\mu$, like the gauge potential of teleparallel gravity (or the tetrad), represent a set of four spacetime vector fields.

Although in the absence of gravitation the teleparallel-based approach to the spin-2 field coincides with the usual metric approach based on General Relativity, in the presence of gravitation it differs substantially from the usual metric approach. The reason is that the index "a" of the translational-valued field $\phi^a{}_\rho$ is not an ordinary vector index, but a gauge index. As such, it is irrelevant for the gravitational coupling prescription, as discussed in section 10.5.3. This point is usually overlooked in the metric approach, which considers both indices of the spin-2 field $\psi_{\mu\nu}$ on an equal footing. As a result, the ensuing gravitational coupling prescription is found to break the gauge invariance of the theory. When the correct coupling prescription is used, a sound gravitationally-coupled spin-2 field theory emerges, which is quite similar to the gravitationally-coupled electromagnetic theory. Furthermore, it is both gauge and local Lorentz invariance, and it preserves the duality symmetry of the free theory.

Finally, it is important to remark that, because the teleparallel spin connection $\overset{\bullet}{A}{}^a{}_{b\mu}$ is purely inertial, the covariant derivative $\overset{\bullet}{\mathcal{D}}_\sigma$ is commutative. Taking into account the anti-symmetry of the superpotential in the spacetime indices, we obtain the divergence identity

$$\overset{\bullet}{\mathcal{D}}_\rho \overset{\bullet}{\mathcal{D}}_\sigma (h \overset{\bullet}{S}_a{}^{\rho\sigma}) = 0, \tag{10.102}$$

which is consistent with the covariant conservation law (10.101). This property, together with the gauge and local Lorentz invariance, render the gravitationally–coupled spin-2 theory fully consistent. In the context of general relativity, whose spin connection represents both inertia and gravitation, the inertial part of the gravitational energy-momentum pseudotensor cannot be separated, and consequently the gravitational field equation cannot be written in a form equivalent to (10.100). In the context of general relativity, therefore, no consistent gravitationally-coupled spin-2 field theory can be obtained.

Chapter 11

Duality Symmetry

Duality symmetry is an important property of sourceless gauge theories. It says that the field equation is just the Bianchi identity written for the dual of the field strength. This means that, if we know the geometrical background — that is, the Bianchi identity — we know the dynamics — that is, the field equations. Duality is not present in General Relativity nor in Teleparallel Gravity. It is, nevertheless, present in the linearized approximation to both. Taking advantage of the possibility of separating torsion into irreducible components under the global Lorentz group, a dual-symmetric sub-theory of Teleparallel Gravity is obtained.

General Relativity does not exhibit the remarkable duality symmetry described in Section 2.1, characteristic of *internal–space* gauge theories. Teleparallel Gravity, on the other hand, is a gauge theory, though for the translation group [115, 116]. It presents, anyhow, several characteristics distinguishing it from General Relativity and approaching the Yang–Mills paradigm. In particular, it does not describe the gravitational interaction by a geometrization of spacetime, but by a true force (see Chapter 4). Another important point refers to the lagrangian of the gravitational field: whereas in General Relativity it is linear in the curvature, in Teleparallel Gravity it is quadratic the field strength of the theory, the torsion tensor.

The main difference Teleparallel Gravity shows with respect to the internal gauge paradigm comes precisely from its external fingerprint: the presence of a solder form connecting the internal with the external sectors of the theory. This form, whose components are the tetrad field, gives rise to new types of contractions, absent in internal gauge theories. In consequence the gauge lagrangian, as well as the field equation, will include additional terms if compared to the internal theories. As discussed in Chapter 6, these additional terms can be taken into account through a generalization of the concept of *dual*, which holds in soldered bundles. We intend now to analyze whether, under some specific conditions, teleparallelism could present duality symmetry [117].

11.1 Duality Symmetry and Gravitation

Consider the first Bianchi identity of Teleparallel Gravity, given by Eq. (7.61):

$$\dot{\mathcal{D}}_\nu \dot{T}^a{}_{\rho\mu} + \dot{\mathcal{D}}_\mu \dot{T}^a{}_{\nu\rho} + \dot{\mathcal{D}}_\rho \dot{T}^a{}_{\mu\nu} = 0. \tag{11.1}$$

It can equivalently be written in the form

$$\dot{\mathcal{D}}_\rho (\epsilon^{\lambda\rho\mu\nu} \, \dot{T}^a{}_{\mu\nu}) = 0. \tag{11.2}$$

For the dual torsion, it reads

$$\dot{\mathcal{D}}_\rho (\epsilon^{\lambda\rho\mu\nu} \star \dot{T}^a{}_{\mu\nu}) = 0. \tag{11.3}$$

Substituting the dual definition [see Eq. (6.19)]

$$\star\, \dot{T}^a{}_{\mu\nu} = \tfrac{1}{2} \, h \, \epsilon_{\mu\nu\alpha\sigma} \, \dot{S}^{a\alpha\sigma}, \tag{11.4}$$

and using the relation [see Eq. (1.54)],

$$\epsilon^{\lambda\rho\mu\nu} \, \epsilon_{\mu\nu\alpha\sigma} = - \frac{2}{h^2} \left(\delta^\lambda_\alpha \, \delta^\rho_\sigma - \delta^\lambda_\sigma \, \delta^\rho_\alpha \right), \tag{11.5}$$

it reduces to

$$\dot{\mathcal{D}}_\sigma (h \dot{S}_a{}^{\rho\sigma}) = 0. \tag{11.6}$$

Comparing with the sourceless teleparallel gravitational field equation [see Eq. (8.17)]

$$\dot{\mathcal{D}}_\sigma (h \dot{S}_a{}^{\rho\sigma}) - k \, (h \, \dot{t}_a{}^\rho) = 0, \tag{11.7}$$

we see that the Bianchi identity written for the dual torsion does not yield the sourceless gravitational field equation. As a matter of fact, it yields the potential term of the field equation, but not the current term. This means essentially that gravitation is not dual–symmetric.

The condition for gravitation to present duality symmetry, therefore, is that the gravitational energy–momentum current [see Eq. (8.18)]

$$\dot{t}_a{}^\rho = \frac{1}{k} \, h_a{}^\lambda \, \dot{S}_c{}^{\nu\rho} \, \dot{T}^c{}_{\nu\lambda} - \frac{h_a{}^\rho}{h} \, \dot{\mathcal{L}} \tag{11.8}$$

vanishes. Using the expression (7.12) for the teleparallel lagrangian, we arrive at the condition

$$S_c{}^{\mu\rho} \, T^c{}_{\mu\lambda} = \tfrac{1}{4} \, \delta^\rho_\lambda \, S_c{}^{\mu\nu} \, T^c{}_{\mu\nu}. \tag{11.9}$$

This is quite a restrictive condition, which seems not to be realized, at least in the general case. Nevertheless, under some specific conditions, it is possible that gravitation may present duality symmetry. Let us begin by analyzing the case of linear gravity.

11.2 Linear Gravity

As discussed in Chapter 1, the trivial tetrad

$$e^a{}_\mu \equiv \overset{\bullet}{\mathcal{D}}_\mu x^a = \partial_\mu x^a + \overset{\bullet}{A}{}^a{}_{b\mu}\, x^b \tag{11.10}$$

describes the flat Minkowski geometry. In the class of frames in which the inertial connection $\overset{\bullet}{A}{}^a{}_{b\mu}$ vanishes, it becomes

$$e^a{}_\mu = \partial_\mu x^a. \tag{11.11}$$

For such tetrads, it is always possible to properly choose the translational gauge in such a way that

$$e^a{}_\mu = \delta^a_\mu. \tag{11.12}$$

In this case, the spacetime metric will be

$$\eta_{\mu\nu} = \delta^a_\mu\, \delta^b_\nu\, \eta_{ab} = \mathrm{diag}(+1, -1, -1, -1). \tag{11.13}$$

Next, we consider the gauge potential $B^a{}_\mu$ as a small perturbation in relation to the trivial tetrad, and write

$$B^a{}_\mu = \varepsilon B^a_{(1)\mu} + \varepsilon^2 B^a_{(2)\mu} + \dots\ , \tag{11.14}$$

where ε is a small dimensionless parameter introduced to label the successive orders of the perturbation. We obtain, in consequence, an expansion of the tetrad field around the flat background:

$$h^a{}_\mu = \delta^a_\mu + \varepsilon B^a_{(1)\mu} + \varepsilon^2 B^a_{(2)\mu} + \dots\ . \tag{11.15}$$

The corresponding expansion of the metric tensor is

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon\left(B_{(1)\mu\nu} + B_{(1)\nu\mu}\right) + \dots\ , \tag{11.16}$$

where

$$B^\rho_{(1)\nu} = \delta^\rho_a B^a_{(1)\nu}, \tag{11.17}$$

and the spacetime indices are now raised and lowered with the Minkowski metric $\eta_{\mu\nu}$:

$$B_{(1)\mu\nu} = \eta_{\mu\rho} B^\rho_{(1)\nu}. \tag{11.18}$$

Although the perturbation $B_{(1)\mu\nu}$ of the gauge potential is not in principle symmetric, it has already been shown that the anti-symmetric part of $B_{(1)\mu\nu}$ drops off completely from the first-order lagrangian and field equation [121].

For this reason, we are going to assume from now on that $B_{(1)\mu\nu}$ is symmetric. In this case, and up to first order, the Weitzenböck connection (3.86) reads

$$\overset{\bullet}{\Gamma}{}^{\rho}_{(1)\mu\nu} = \partial_\nu B^{\rho}_{(1)\mu}. \tag{11.19}$$

The torsion and the contortion tensors are found to be, respectively,

$$\overset{\bullet}{T}{}^{\rho}_{(1)\mu\nu} = \partial_\mu B^{\rho}_{(1)\nu} - \partial_\nu B^{\rho}_{(1)\mu} \tag{11.20}$$

and

$$\overset{\bullet}{K}{}^{\rho}_{(1)\mu\nu} = \partial^\rho B_{(1)\mu\nu} - \partial_\mu B^{\rho}_{(1)\nu}. \tag{11.21}$$

The corresponding first–order superpotential is

$$\begin{aligned}\overset{\bullet}{S}{}_{(1)\nu}{}^{\rho\mu} = {} & \partial^\rho B^{\mu}_{(1)\nu} - \partial^\mu B^{\rho}_{(1)\nu} \\ & -\delta^{\mu}_{\nu}\left(\partial^\rho B^{\sigma}_{(1)\sigma} - \partial_\sigma B^{\sigma\rho}_{(1)}\right) + \delta^{\rho}_{\nu}\left(\partial^\mu B^{\sigma}_{(1)\sigma} - \partial_\sigma B^{\sigma\mu}_{(1)}\right)\end{aligned} \tag{11.22}$$

Comment 11.1 We notice in passing that the superpotential (11.22) coincides with the Fierz tensor [97] for a symmetric spin-2 field [see Section 10.5].

The first–order sourceless gravitational field equation is consequently

$$\partial_\mu \overset{\bullet}{S}{}_{(1)a}{}^{\rho\mu} = 0, \tag{11.23}$$

where

$$\overset{\bullet}{S}{}_{(1)a}{}^{\rho\mu} = \delta^{\nu}_{a}\, \overset{\bullet}{S}{}_{(1)\nu}{}^{\rho\mu}. \tag{11.24}$$

On the other hand, we see from Eq. (11.6) that the first order Bianchi identity written for the dual torsion is

$$\partial_\mu \overset{\bullet}{S}{}_{(1)a}{}^{\rho\mu} = 0. \tag{11.25}$$

Comparing with the first–order field equation (11.23), we conclude that linear gravity does present duality symmetry. This is actually an expected result because, as already remarked in Chapter 10, linear General Relativity presents duality symmetry [113].

Comment 11.2 Let us consider the teleparallel action of the gravitational field, which according to Eq. (7.4) is given by

$$\overset{\bullet}{\mathcal{S}}[\overset{\bullet}{T}] = \frac{c^3}{16\pi G}\int \eta_{ab}\, \overset{\bullet}{T}{}^a \wedge \star \overset{\bullet}{T}{}^b. \tag{11.26}$$

Written for the dual, it reads

$$\dot{\mathcal{S}}[\star \dot{T}] = \frac{c^3}{16\pi G} \int \eta_{ab} \star \dot{T}^a \wedge \star\star \dot{T}^b. \tag{11.27}$$

Since

$$\star\star \dot{T}^b = -\dot{T}^b,$$

we see immediately that the action changes sign under the dual transformation:

$$\dot{\mathcal{S}}[\star \dot{T}] = -\dot{\mathcal{S}}[\dot{T}] \tag{11.28}$$

This is a known property of linear General Relativity [113], which emerges quite trivially in the context of Teleparallel Gravity.

11.3 In Search of a Dual Gravity

As discussed in Section 10.3.2, the gravitational interaction of a Dirac spinor in Teleparallel Gravity involves only the vector and the axial torsions [37]. In the class of frames in which the inertial connection $\dot{A}^a{}_{b\mu}$ vanishes, the teleparallel Dirac equation is

$$i\hbar \gamma^\mu \left(\partial_\mu - \tfrac{1}{2} \dot{\mathcal{V}}_\mu - \tfrac{3i}{4} \dot{\mathcal{A}}_\mu \gamma^5\right) \psi = mc\, \psi. \tag{11.29}$$

We then argue: since the pure tensor piece $\dot{t}_{\lambda\mu\nu}$ is irrelevant for the description of the gravitational interaction of spinor fields, if we restrict ourselves to the microscopic world of the fermions we can consider a gravitational theory in which the purely tensor piece of torsion vanishes. In this case, torsion reduces to

$$\dot{T}_{\lambda\mu\nu} = \tfrac{1}{3}\left(g_{\lambda\mu} \dot{\mathcal{V}}_\nu - g_{\lambda\nu} \dot{\mathcal{V}}_\mu\right) + \epsilon_{\lambda\mu\nu\rho}\, \dot{\mathcal{A}}^\rho. \tag{11.30}$$

The corresponding contortion tensor is

$$\dot{K}^{\rho\mu\nu} = \tfrac{1}{3}\left(g^{\nu\rho} \dot{\mathcal{V}}^\mu - g^{\nu\mu} \dot{\mathcal{V}}^\rho\right) - \tfrac{1}{2} \epsilon^{\nu\rho\mu\lambda}\, \dot{\mathcal{A}}_\lambda, \tag{11.31}$$

whereas the superpotential becomes

$$\dot{S}^{\rho\mu\nu} = -\tfrac{2}{3}\left(g^{\rho\mu} \dot{\mathcal{V}}^\nu - g^{\rho\nu} \dot{\mathcal{V}}^\mu\right) - \tfrac{1}{2} \epsilon^{\rho\mu\nu\lambda}\, \dot{\mathcal{A}}_\lambda. \tag{11.32}$$

Substituting these expressions in the teleparallel lagrangian

$$\dot{\mathcal{L}} = \frac{c^4 h}{32\pi G}\, \dot{S}^{\rho\mu\nu}\, \dot{T}_{\rho\mu\nu}, \tag{11.33}$$

we obtain

$$\dot{\mathcal{L}} = \frac{c^4 h}{16\pi G} \left(-\tfrac{2}{3}\, \dot{\mathcal{V}}_\mu\, \dot{\mathcal{V}}^\mu + \tfrac{3}{2}\, \dot{\mathcal{A}}_\mu\, \dot{\mathcal{A}}^\mu\right). \tag{11.34}$$

Comment 11.3 An alternative way to get this lagrangian is to note that the teleparallel lagrangian (11.33) can be written in the form [12]

$$\overset{\bullet}{\mathcal{L}} = \frac{c^4 h}{16\pi G} \left(\frac{2}{3} \overset{\bullet}{\mathcal{T}}{}^{\lambda\mu\nu} \overset{\bullet}{\mathcal{T}}{}_{\lambda\mu\nu} - \frac{2}{3} \overset{\bullet}{\mathcal{V}}{}^{\mu} \overset{\bullet}{\mathcal{V}}{}_{\mu} + \frac{3}{2} \overset{\bullet}{\mathcal{A}}{}^{\mu} \overset{\bullet}{\mathcal{A}}{}_{\mu} \right). \tag{11.35}$$

When $\overset{\bullet}{\mathcal{T}}{}^{\lambda\mu\nu}$ vanishes, it reduces to (11.34).

We can then ask: would this theory present duality symmetry? Or in other words, would the purely tensor torsion be responsible for spoiling the duality symmetry of gravitation? To answer this question, we rewrite the condition for a gravitational theory to present duality symmetry — given by Eq. (11.9) — in terms of the vector and the axial torsions. The result is

$$\begin{aligned} - \tfrac{4}{9} \overset{\bullet}{\mathcal{V}}{}_{\lambda} \overset{\bullet}{\mathcal{V}}{}^{\rho} - \tfrac{2}{9} \delta_{\lambda}{}^{\rho} \overset{\bullet}{\mathcal{V}}{}_{\mu} \overset{\bullet}{\mathcal{V}}{}^{\mu} - \tfrac{1}{2} \epsilon^{\rho}{}_{\lambda\nu\alpha} \overset{\bullet}{\mathcal{V}}{}^{\nu} \overset{\bullet}{\mathcal{A}}{}^{\alpha} + \delta_{\lambda}{}^{\rho} \overset{\bullet}{\mathcal{A}}{}_{\mu} \overset{\bullet}{\mathcal{A}}{}^{\mu} - \overset{\bullet}{\mathcal{A}}{}_{\lambda} \overset{\bullet}{\mathcal{A}}{}^{\rho} \\ = \tfrac{1}{2} \delta^{\rho}_{\lambda} \left(- \tfrac{2}{3} \overset{\bullet}{\mathcal{V}}{}_{\mu} \overset{\bullet}{\mathcal{V}}{}^{\mu} + \tfrac{3}{2} \overset{\bullet}{\mathcal{A}}{}_{\mu} \overset{\bullet}{\mathcal{A}}{}^{\mu} \right). \end{aligned} \tag{11.36}$$

Now, as a simple inspection shows, no solution exists if torsion is real. However, in the complex domain, if the axial and vector parts of torsion are related by

$$\overset{\bullet}{\mathcal{A}}{}_{\mu} = \pm \tfrac{2i}{3} \overset{\bullet}{\mathcal{V}}{}_{\mu}, \tag{11.37}$$

the above condition is fulfilled, and the resulting gravitational theory turns out to present duality symmetry.

Applying the generalized dual definition (6.17) to the axial and vector torsions, we obtain

$$\star \overset{\bullet}{\mathcal{A}}{}_{\mu} = - \tfrac{2}{3} \overset{\bullet}{\mathcal{V}}{}_{\mu} \quad \text{and} \quad \star \overset{\bullet}{\mathcal{V}}{}_{\mu} = \tfrac{3}{2} \overset{\bullet}{\mathcal{A}}{}_{\mu}. \tag{11.38}$$

On account of the relation (11.37), we see that in this theory torsion turns out to be self dual (upper sign) or anti-self dual (lower sign):

$$\star \overset{\bullet}{\mathcal{A}}{}_{\mu} = \pm i \overset{\bullet}{\mathcal{A}}{}_{\mu} \quad \text{and} \quad \star \overset{\bullet}{\mathcal{V}}{}_{\mu} = \pm i \overset{\bullet}{\mathcal{V}}{}_{\mu}. \tag{11.39}$$

Notice that we are using here the extended notions of self duality and anti-self duality for complex fields [119]. In that context we can say that we have obtained a sub-theory of Teleparallel Gravity which is able to describe the gravitational interaction of fermions and presents duality symmetry.

11.4 A Self-Dual Gravitational Theory

Let us briefly explore some properties of the self–dual gravitational theory obtained in the previous Section. We begin by noting that, according to

Eq. (11.37), torsion becomes a complex tensor. In fact, in terms of the vector torsion, it reads

$$\overset{\bullet}{T}_{\lambda\mu\nu} = \tfrac{1}{3}\left(g_{\lambda\mu}\overset{\bullet}{\mathcal{V}}_{\nu} - g_{\lambda\nu}\overset{\bullet}{\mathcal{V}}_{\mu}\right) \pm \tfrac{2i}{3}\,\epsilon_{\lambda\mu\nu\rho}\,\overset{\bullet}{\mathcal{V}}{}^{\rho}. \tag{11.40}$$

The corresponding contortion tensor is

$$\overset{\bullet}{K}{}^{\rho\mu\nu} = \tfrac{1}{3}\left(g^{\nu\rho}\overset{\bullet}{\mathcal{V}}{}^{\mu} - g^{\nu\mu}\overset{\bullet}{\mathcal{V}}{}^{\rho}\right) \mp \tfrac{i}{3}\epsilon^{\nu\rho\mu\lambda}\,\overset{\bullet}{\mathcal{V}}_{\lambda}, \tag{11.41}$$

whereas the superpotential acquires the form

$$\overset{\bullet}{S}{}^{\lambda\rho\sigma} = -\tfrac{2}{3}\left(g^{\lambda\rho}\overset{\bullet}{\mathcal{V}}{}^{\sigma} - g^{\lambda\sigma}\overset{\bullet}{\mathcal{V}}{}^{\rho}\right) \mp \tfrac{i}{3}\,\epsilon^{\lambda\rho\sigma\theta}\,\overset{\bullet}{\mathcal{V}}_{\theta}. \tag{11.42}$$

Even though torsion is complex, the gravitational lagrangian is real:

$$\overset{\bullet}{\mathcal{L}} = -\frac{c^4 h}{12\pi G}\,\overset{\bullet}{\mathcal{V}}_{\mu}\overset{\bullet}{\mathcal{V}}{}^{\mu}. \tag{11.43}$$

The corresponding field equation is

$$\partial_\sigma(h\,\overset{\bullet}{S}_{a}{}^{\rho\sigma}) = 0. \tag{11.44}$$

Of course, as the theory is dual symmetric, this field equation coincides with the Bianchi identity written for the dual torsion, which is given by Eq. (11.6). Using the relation

$$\overset{\bullet}{S}_{a}{}^{\rho\sigma} = h_{a\lambda}\,\overset{\bullet}{S}{}^{\lambda\rho\sigma},$$

with $\overset{\bullet}{S}{}^{\lambda\rho\sigma}$ given by Eq. (11.42), its explicit form is the complex equation

$$-\tfrac{2}{3}\,\partial_\sigma\left[h\left(h_a{}^{\rho}\,\overset{\bullet}{\mathcal{V}}{}^{\sigma} - h_a{}^{\sigma}\,\overset{\bullet}{\mathcal{V}}{}^{\rho}\right)\right] + \tfrac{i}{3}\,\partial_\sigma\left(hh_{a\nu}\,\epsilon^{\nu\rho\sigma\lambda}\,\overset{\bullet}{\mathcal{V}}_{\lambda}\right) = 0. \tag{11.45}$$

The imaginary part is

$$\partial_\sigma\left(hh_{a\nu}\,\epsilon^{\nu\rho\sigma\lambda}\,\overset{\bullet}{\mathcal{V}}_{\lambda}\right) = 0, \tag{11.46}$$

or equivalently,

$$\partial_\sigma\left(h\,h_{a\nu}\overset{\bullet}{\mathcal{V}}_{\lambda}\right) + \partial_\lambda\left(h\,h_{a\sigma}\overset{\bullet}{\mathcal{V}}_{\nu}\right) + \partial_\nu\left(h\,h_{a\lambda}\overset{\bullet}{\mathcal{V}}_{\sigma}\right) = 0. \tag{11.47}$$

This is the Bianchi identity of the theory. The real part of Eq. (11.45), on the other hand, is

$$-\tfrac{2}{3}\,\partial_\sigma\left[h\left(h_a{}^{\rho}\,\overset{\bullet}{\mathcal{V}}{}^{\sigma} - h_a{}^{\sigma}\,\overset{\bullet}{\mathcal{V}}{}^{\rho}\right)\right] = 0. \tag{11.48}$$

This is the dynamical field equation of the theory. We then have the following property: whereas the Bianchi identity shows up as the imaginary part, the dynamical field equation is obtained as the real part of the complex field equation (11.45). A similar mechanism holds in the Palatini formulation of self dual General Relativity [see Ref. [124], page 1511].

In the presence of a source field represented by $\mathcal{L}_m$, since the corresponding energy-momentum tensor

$$\Theta_a{}^\rho \equiv -\frac{1}{h}\frac{\delta \mathcal{L}_m}{\delta B^a{}_\rho} = -\frac{1}{h}\frac{\delta \mathcal{L}_m}{\delta h^a{}_\rho} \tag{11.49}$$

is real, it only contributes to the dynamical equation, which acquires then the form

$$\partial_\sigma \left[h \left(h_a{}^\sigma \, \overset{\bullet}{\mathcal{V}}{}^\rho - h_a{}^\rho \, \overset{\bullet}{\mathcal{V}}{}^\sigma \right) \right] = \frac{12\pi G}{c^4} \, (h \, \Theta_a{}^\rho). \tag{11.50}$$

This is the equation governing the dynamics of the self dual gravitational field.

Comment 11.4 In the weak field limit of (macroscopic) Teleparallel Gravity, the axial torsion is found not to contribute to the newtonian potential. In fact, only the vector and purely tensor parts of torsion contribute to the Newton potential [46]. On the other hand, in the (microscopic) limit of the gravitational interaction of fermions, it is the purely tensor part of torsion that does not contribute. As a consequence, the dual gravitation will not present a newtonian limit. Of course, this is not a problem as this theory might be valid only at the microscopic level, where the newtonian limit is not required to hold.

It is interesting to note that, when the vector and axial–vector parts of torsion are related by Eq. (11.37), the Dirac equation (11.29) assumes the form

$$i \, \hbar \, \gamma^\mu \left[\partial_\mu - \tfrac{1}{2} \, \overset{\bullet}{\mathcal{V}}{}_\mu \, (1 \mp \gamma^5) \right] \psi = mc \, \psi, \tag{11.51}$$

with the upper (lower) sign refering to the self dual (anti-self dual) case. We see from this equation that a self dual (anti-self dual) torsion couples only to the left-hand (right-hand) component of the spinor field. In other words, gravitation becomes a chiral interaction at the microscopic level of the gravitational interaction of fermions. Observe furthermore that the Dirac equation (11.51) is invariant under a chiral transformation

$$\psi \to \gamma^5 \, \psi,$$

except for a change of sign of the spinor mass term. A similar property holds in Electrodynamics [see, for example, Ref. [125], page 520].

It should be remarked finally that, although this self dual gravitational theory is able to describe the gravitational interaction of fermions, its physical meaning is still quite obscure. In particular, since its energy–momentum current vanishes, it seem unable to transport energy and momentum. Anyway, because it presents duality symmetry, this sub-theory has a natural intrinsic interest and may deserve further analysis.

Comment 11.5 Even though it has no meaning at the microscopic level of the gravitational interaction of fermions, it is instructive to obtain the (classical) equation of motion of a spinless particle in the presence of gravitation. In the context of Teleparallel Gravity, this equation of motion is given by [see Eq. 4.56]

$$\frac{du^a}{ds} + \overset{\bullet}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = \overset{\bullet}{K}{}^a{}_{b\rho}\, u^b\, u^\rho. \tag{11.52}$$

Substituting $\overset{\bullet}{K}{}^a{}_{b\rho}$ as given by Eq. (11.41), the imaginary part of torsion drops out, and we get

$$\frac{du^a}{ds} + \overset{\bullet}{A}{}^a{}_{b\rho}\, u^b\, u^\rho = -\frac{1}{3}\,(h^a{}_\rho - u^a\, u_\rho)\, \overset{\bullet}{\mathcal{V}}{}^\rho. \tag{11.53}$$

We see from this equation that the gravitational force in this case has the form of a projector, and is orthogonal to the particle four-velocity.

Chapter 12

Teleparallel Kaluza-Klein Theory

In gauge theories the total space (or the bundle) is locally a direct product of spacetime — described by General Relativity — and the internal space of gauge variables. In Kaluza-Klein theories the bundle is taken as a *global* direct product. The Einstein–Hilbert lagrangian written on the total space leads, when the coordinates are convenient separated, to the 4–dimensional lagrangian of General Relativity plus a lagrangian of Yang–Mills type — the Kaluza-Klein "miracle". Replacing General Relativity by its teleparallel equivalent leads to versions of Kaluza-Klein models which are closer to the formal unification which is their ultimate goal.

In ordinary Kaluza-Klein theories [127], the geometrical approach of General Relativity is used as the paradigm for the description of all other interactions of Nature. In the original Kaluza-Klein theory, for example, gravitational and electromagnetic fields are described by a Einstein-Hilbert type lagrangian in a five-dimensional spacetime. On the other hand, as discussed in previous chapters, Teleparallel Gravity, a gauge theory for the translation group, is equivalent to General Relativity. This equivalence opens up new roads for the study of unified theories. In fact, instead of using the general–relativistic geometrical description, we can adopt the gauge description as the basic paradigm, and in this way construct what we call the *teleparallel equivalent of Kaluza-Klein theory* [128]. According to this approach, instead of a Einstein-Hilbert type lagrangian, both gravitational and electromagnetic fields are described by a gauge, or Maxwell type lagrangian.

12.1 Kaluza-Klein Theory: a Brief Review

In its simplest form, the Kaluza-Klein theory [129, 130] is an extension of General Relativity to a five dimensional pseudo-Riemannian spacetime $\mathbf{R}^5$ with a topology given by the product between the usual four dimensional spacetime $\mathbf{R}^4$ and the circumference S^1:

$$\mathbf{R}^5 = \mathbf{R}^4 \otimes S^1.$$

Denoting the five dimensional indices by capital Latin letters $(A, B, C, \ldots = 0, 1, 2, 3, 5)$, the metric γ_{AB} of $\mathbf{R}^5$ is, in principle, a function of the coordinates x^μ of $\mathbf{R}^4$ and of the coordinates x^5 of S^1:

$$\gamma_{AB} = \gamma_{AB}(x^\mu, x^5).$$

Since x^5 is a periodic coordinate, we can write

$$x^5 = \rho\,\theta,$$

with θ the angular coordinate and ρ the radius of S^1. Using the metric γ_{AB}, we define now the five dimensional Levi-Civita connection [131]

$$\Gamma^C{}_{AB} = \tfrac{1}{2}\,\gamma^{CD}(\partial_A \gamma_{DB} + \partial_B \gamma_{DA} - \partial_D \gamma_{AB}). \tag{12.1}$$

Its curvature tensor is

$$R^A{}_{BCD} = \partial_C \Gamma^A{}_{BD} - \partial_D \Gamma^A{}_{BC} + \Gamma^A{}_{CE} \Gamma^E{}_{BD} - \Gamma^A{}_{DE} \Gamma^E{}_{BC}. \tag{12.2}$$

The action integral of the theory is written as a five dimensional Einstein-Hilbert Lagrangian

$$\mathcal{S}_5 = -\,\frac{c^3}{16\pi G_5} \int d^5x \, \sqrt{-\gamma}\; R, \tag{12.3}$$

where $\gamma = \det(\gamma_{AB})$, $R = \gamma^{BD} R^A{}_{BAD}$ is the five dimensional scalar curvature, and G_5 is the five dimensional version of Newton constant.

Comment 12.1 Around 1912, G. Nordström developed a scalar theory for gravitation [132]. He was the first to use a five dimensional spacetime in an attempt to unify gravitation and electromagnetism [133]. For a historical account, as well as for a comprehensive list of references, see Ref. [127]

The fifteen components of the five dimensional metric γ_{AB} can be represented by the four dimensional spacetime metric $g_{\mu\nu}$, a vector field A_μ, and a scalar field ϕ. In terms of these variables, it can be conveniently parametrized in the form

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\mu\nu} + \beta^2 \phi A_\mu A_\nu & \beta \phi A_\mu \\ \hline \beta \phi A_\nu & \phi \end{array} \right), \tag{12.4}$$

with β a parameter to be determined by the unification process. On the other hand, the assumed topology for the five dimensional space allows us to expand any field quantity, and in particular each component of the metric γ_{AB}, in a Fourier series of the form

$$\gamma_{AB} = \sum_{n=-\infty}^{\infty} \gamma_{AB}^{(n)}(x^\mu) \exp\left[inx^5/r\right]. \tag{12.5}$$

In order to obtain the four dimensional theory, Kaluza originally imposed the so-called *cylindric condition*

$$\frac{\partial \gamma_{AB}}{\partial x^5} = 0, \tag{12.6}$$

which corresponds to taking only the $n = 0$ mode in the Fourier expansion (12.5). Substituting the $n = 0$ piece of γ_{AB} into the action (12.3), integrating over x^5 and choosing

$$\beta^2 = \frac{16\pi G}{c^4}, \tag{12.7}$$

with

$$G = \frac{G_5}{2\pi\rho} \tag{12.8}$$

the ordinary Newton gravitational constant, we get (dropping the label $^{(0)}$ of the Fourier expansions)

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(-\frac{c^3}{16\pi G} \overset{\circ}{R} + \frac{\phi}{4} F_{\mu\nu} F^{\mu\nu} + \frac{c^3}{24\pi G} \frac{\partial_\mu \phi \, \partial^\mu \phi}{\phi^2} \right), \tag{12.9}$$

where $g = \det(g_{\mu\nu})$, $\overset{\circ}{R}$ is the Ricci scalar curvature of the four dimensional spacetime, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{12.10}$$

If we take $\phi = -1$, we can then identify A_μ as the electromagnetic potential, and action (12.9) reduces to the Einstein-Maxwell action

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(-\frac{c^3}{16\pi G} \overset{\circ}{R} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right). \tag{12.11}$$

This is usually considered the “miracle” of the Kaluza-Klein theories: gauge theories emerge naturally from geometry in the dimensional reduction process. It is particularly interesting to observe how the action (12.3), which is invariant under five dimensional general coordinate transformations, reduces to the action (12.9), which is invariant under both four dimensional general coordinate transformations and $U(1)$ gauge transformations.

Comment 12.2 In order to get the proper relative sign between Einstein and Maxwell lagrangians, so that energy is positive, it is necessary that

$$\phi \equiv \gamma_{55} < 0.$$

According to our metric convention, this means that the fifth dimension must be space-like. This is consistent with causality as more than one time-like dimension would lead to closed time-like curves.

An important point of the abelian Kaluza-Klein theory is that the five dimensional space $\mathbf{R}^5$ is a solution of the five dimensional Einstein equation

$$R_{AB} - \frac{1}{2}\gamma_{AB} R = 0, \tag{12.12}$$

which follows from action (12.3). Using the decomposition (12.4), it yields respectively for $AB = \mu\nu$, $AB = \mu 5$ and $AB = 55$,

$$\overset{\circ}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{R} = -\frac{\beta^2 \phi}{2}\, \Theta_{\mu\nu} - \frac{1}{\phi}\left[\overset{\circ}{\nabla}_\mu (\partial_\nu \phi) - g_{\mu\nu}\, \Box \phi\right], \tag{12.13}$$

$$\overset{\circ}{\nabla}_\nu F_\mu{}^\nu = -\,3\, \frac{\partial^\nu \phi}{\phi}\, F_{\mu\nu}, \tag{12.14}$$

and

$$\Box\, \phi = -\,\frac{\beta^2 \phi}{4} F_{\mu\nu} F^{\mu\nu}, \tag{12.15}$$

with $\overset{\circ}{R}_{\mu\nu}$ the four dimensional Ricci tensor, and $\Theta_{\mu\nu}$ the symmetric energy-momentum tensor of the electromagnetic field. If we set again $\phi = -1$, and use Eq. (12.7), we obtain the Einstein-Maxwell system of equations,

$$\overset{\circ}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{R} = \frac{8\pi G}{c^4}\, \Theta_{\mu\nu} \tag{12.16}$$

and

$$\overset{\circ}{\nabla}_\nu F_\mu{}^\nu = 0. \tag{12.17}$$

The five dimensional *sourceless* field equation (12.12), therefore, reduces to the usual four dimensional Einstein equation with the electromagnetic energy-momentum tensor as source. This means that matter in four dimensions comes from the geometry of a five dimensional spacetime.

Although used by Kaluza in his original paper, the condition $\phi = constant$ leads to inconsistencies. In fact, we see from Eq. (12.15) that it implies

$$F_{\mu\nu} F^{\mu\nu} = 0, \tag{12.18}$$

as first noticed by Jordan [134], Bergmann [135] and Thiry [136]. This problem can be solved by taking into account all harmonics of the Fourier expansion (12.5). In this case, however, the scalar field ϕ will remain in the theory. Initially, this fact was considered a drawback of the theory because it leads actually to a scalar-tensor theory for gravitation. Afterwards, because scalar fields are useful to explain several phenomena of modern physics, like for example inflation and spontaneous symmetry breaking, this point became well accepted. Furthermore, quantum corrections would provide a mass for ϕ, thereby removing its long range gravitational effects [137, 138]. Since these fields remain in the theory, they imply the existence of a huge family of new particles which emerge as a byproduct of the unification process.

Comment 12.3 The generalization of the Kaluza-Klein theory for non-abelian groups [139] can be made by assuming a spacetime manifold of the form

$$\mathbf{R}^{4+n} = \mathbf{R}^4 \otimes S^n,$$

with S^n a compact manifold. In complete analogy with the five dimensional case, the dimensional reduction of a $(4+n)$-dimensional Einstein lagrangian yields four–dimensional Einstein's plus a Yang-Mills type lagrangians. It should be remarked however that, differently from the five dimensional abelian theory, the non-abelian case is plagued by conceptual difficulties. One of the main problems is that the space $\mathbf{R}^{4+n}$ cannot be a solution of the $(4+n)$-dimensional Einstein equation [137]. This is a serious drawback which cannot be overlooked when considering these models.

12.2 Teleparallel Kaluza-Klein

12.2.1 Five-Vector Potential

In the framework of Teleparallel Gravity, the action describing a particle of mass m and charge q in the presence of both an electromagnetic field A_μ and a gravitational field $B^a{}_\mu$ is [14]

$$\mathcal{S} = -\, mc \int_a^b \left[u_a \, \overset{\bullet}{\mathcal{D}}_\mu x^a + B^a{}_\mu \, u^b \, \eta_{ab} + \frac{q}{mc^2} \, A_\mu \right] dx^\mu. \tag{12.19}$$

The corresponding equation of motion is [see Section 4.2.2]

$$h^a{}_\rho \, \frac{\overset{\bullet}{\mathcal{D}} u_a}{\mathcal{D}s} = \overset{\bullet}{T}{}^a{}_{\rho\mu} \, u^b u^\mu \, \eta_{ab} + \frac{q}{mc^2} \, F_{\rho\mu} \, u^\mu, \tag{12.20}$$

where

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu B^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu B^a{}_\mu \tag{12.21}$$

is the gravitational field strength — that is, torsion — and

$$F_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu A_\nu - \overset{\bullet}{\mathcal{D}}_\nu A_\mu \tag{12.22}$$

is the electromagnetic field strength.

We see from the equation of motion (12.20) that torsion acts on particles in the very same way the electromagnetic field acts on charges, that is, as a force. This similarity allows a unified description of the gravitational and electromagnetic interactions. In order to get such description, we begin by choosing the $U(1)$ gauge index of the electromagnetic theory as the *fifth* component of the gauge potential. Accordingly, we define a five-vector gauge potential in the form $(A, B, \ldots = 0, 1, 2, 3, 5)$

$$\mathcal{A}^A{}_\mu = \left(B^a{}_\mu, A^5{}_\mu \right), \tag{12.23}$$

where

$$A^5{}_\mu = \frac{q}{\kappa mc^2} \, A_\mu, \tag{12.24}$$

with κ a dimensionless parameter to be determined in the unification procedure. Accordingly, we define a unified field strength

$$\mathcal{F}^A{}_{\mu\nu} = \left(\overset{\bullet}{T}{}^a{}_{\mu\nu}, F^5{}_{\mu\nu} \right),$$

with

$$F^5{}_{\mu\nu} = \frac{q}{\kappa mc^2} \, F_{\mu\nu}. \tag{12.25}$$

In terms of the potential $\mathcal{A}^A{}_\mu$, therefore, the unified field strength is

$$\mathcal{F}^A{}_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu \mathcal{A}^A{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu \mathcal{A}^A{}_\mu. \tag{12.26}$$

Implicit in the above definitions is the introduction of a five-dimensional space $\mathcal{M}^5$, given by the cartesian product between the Minkowski space M^4 and the circle S^1:

$$\mathcal{M}^5 = M^4 \otimes S^1.$$

A point in this space is determined by the coordinates

$$x^A = (x^a, x^5),$$

where x^a are the coordinates of M^4, and x^5 is a coordinate on S^1. The corresponding metric tensor is

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{55} \end{pmatrix}, \tag{12.27}$$

We introduce now a velocity five-vector

$$u^A = (u^a, u^5).$$

The components u^a form the usual anholonomic four-velocity, and u^5 is a strictly internal component. In this case, by choosing

$$u^5 = -\,\kappa \quad \text{and} \quad \eta_{55} = -1, \tag{12.28}$$

action (12.19) can be rewritten in the form

$$\mathcal{S} = -\,mc \int_a^b \left[u_a \, \overset{\bullet}{\mathcal{D}}_\mu x^a + \mathcal{A}^A{}_\mu \, u^B \, \eta_{AB} \right] dx^\mu. \tag{12.29}$$

The corresponding equation of motion is

$$h^a{}_\rho \, \frac{\overset{\bullet}{\mathcal{D}} u_a}{\mathcal{D}s} = \mathcal{F}^A{}_{\rho\mu} \, u^B \; u^\mu \, \eta_{AB}. \tag{12.30}$$

Due to the fact that torsion acts on particles in the very same way the electromagnetic field acts on charges, the trajectory of a charged particle submitted to both an electromagnetic and a gravitational field can be described by a unified Lorentz-type force equation.

Comment 12.4 Alternatively, we could have chosen

$$u^5 = \kappa \quad \text{and} \quad \eta_{55} = 1, \tag{12.31}$$

which would lead to the same action integral, and consequently to the same equation of motion. This choice corresponds to another metric convention for the internal space. In principle, both conventions are possible. However, as we are going to see, the unification process will introduce a constraint according to which the choice of η_{55} will depend on the metric convention adopted for the tangent Minkowski space.

12.2.2 Unified Lagrangian and Field Equations

In a gauge theory for the translation group, the gauge transformation is defined as a local translation of the tangent-space coordinates,

$$\delta x^a = \alpha^b P_b \, x^a, \tag{12.32}$$

with $P_b = \partial/\partial x^b$ the generators, and α^b the corresponding infinitesimal parameters. In a unified teleparallel Kaluza-Klein model, a general gauge transformation is represented by a translation of the five-dimensional space coordinates x^A,

$$\delta x^A = \alpha^B \, P_B \, x^A, \tag{12.33}$$

where $P_B = \partial/\partial x^B$ are the generators, and

$$\alpha^B = \left(\alpha^a, \alpha^5\right) \tag{12.34}$$

are the transformation parameters. Analogously to the definitions used for the gauge potentials, we take

$$\alpha^5 = \frac{q}{\kappa m c^2} \, \alpha. \tag{12.35}$$

Furthermore, in the same way as in ordinary Kaluza-Klein models, we assume that the gauge potentials $\mathcal{A}^A{}_\mu$, and consequently the tetrad $h^a{}_\mu$ and the metric tensor $g_{\mu\nu}$, do not depend on the coordinate x^5.

As discussed in Chapter 7, the gravitational action of Teleparallel Gravity is

$$\overset{\bullet}{\mathcal{S}} = \frac{c^3}{16\pi G} \int \eta_{ab} \, \overset{\bullet}{T}{}^a \wedge \star \overset{\bullet}{T}{}^b. \tag{12.36}$$

The action of the electromagnetic field, on the other hand, is

$$\mathcal{S}_{em} = -\,\frac{1}{4} \int F \wedge \star F, \tag{12.37}$$

with F the electromagnetic field 2-form. Owing to the gauge structure of both actions, and using the unified field strength (12.26), we can write a unified action in the form

$$\mathcal{S} = \frac{c^3}{16\pi G} \int \eta_{AB} \, \mathcal{F}^A \wedge \star \mathcal{F}^B, \tag{12.38}$$

with

$$\mathcal{F}^A = \tfrac{1}{2} \, \mathcal{F}^A{}_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

the unified 2-form field strength (12.26).

Separating the spacetime and the electromagnetic components, the corresponding lagrangian is

$$\mathcal{L} = \frac{hc^4}{32\pi G}\, \dot{T}^{\rho}{}_{\mu\nu}\, \dot{S}_{\rho}{}^{\mu\nu} + \eta_{55}\, \frac{\kappa^{-2}q^2}{16\pi G m^2}\, \frac{h}{4}\, F_{\mu\nu}F^{\mu\nu}. \tag{12.39}$$

The first term of $\mathcal{L}$ is the teleparallel gauge lagrangian (7.12), which is equivalent to the Einstein-Hilbert lagrangian of General Relativity. In order to get Maxwell's lagrangian from the second term, two conditions must be satisfied. First, it is necessary that

$$\kappa^2 = \frac{q^2}{16\pi G m^2}. \tag{12.40}$$

We see from this expression that κ^2 turns out to be proportional to the ratio between electric and gravitational forces [140]. Second, in order to have a positive-definite energy for the electromagnetic field, and to get the appropriate relative sign between the gravitational and electromagnetic lagrangians, it is necessary that $\eta_{55} = -1$. With these conditions, the lagrangian (12.39) becomes

$$\mathcal{L} \equiv \dot{\mathcal{L}} + \mathcal{L}_{em} = \frac{h}{4k}\, \dot{T}^{\rho}{}_{\mu\nu}\, \dot{S}_{\rho}{}^{\mu\nu} - \frac{h}{4}\, F_{\mu\nu}F^{\mu\nu}, \tag{12.41}$$

with $k = 8\pi G/c^4$. As said in Section 12.1, that the Maxwell lagrangian in four dimensions shows up from the Einstein-Hilbert lagrangian in five dimensions, is usually considered as a miracle of the standard Kaluza-Klein theory [137]. That the Einstein-Hilbert lagrangian of general relativity shows up from a Maxwell-type lagrangian for a five-dimensional translation gauge theory, can be considered as the other face of the same miracle.

The functional variation of $\mathcal{L}$ with respect to $B^a{}_\rho$ yields the teleparallel field equation

$$\partial_\sigma(h\dot{S}_a{}^{\rho\sigma}) - k\,(h\dot{\jmath}_a{}^{\rho}) = k\,(h\Theta_a{}^{\rho}), \tag{12.42}$$

where $\dot{\jmath}_a{}^{\rho}$ is the Noether energy-momentum current of the gravitational field, and

$$h\Theta_a{}^{\rho} \equiv -\frac{\delta\mathcal{L}_{em}}{\delta B^a{}_\rho} \equiv -\frac{\delta\mathcal{L}_{em}}{\delta h^a{}_\rho} = hF_{a\nu}F^{\rho\nu} - h_a{}^{\rho}\mathcal{L}_{em} \tag{12.43}$$

is the symmetric energy-momentum tensor of the electromagnetic field. On the other hand, the functional variation of $\mathcal{L}$ with respect to A_μ yields the teleparallel version of Maxwell's equation

$$\ddot{\nabla}_\mu F^{\mu\nu} = 0, \tag{12.44}$$

as obtained in Section 10.4.

12.2.3 Metric Constraint

An interesting feature of the teleparallel Kaluza-Klein model is that it imposes a constraint between the spacetime metric and the metric on the $U(1)$ group. In fact, when we choose the metric of the Minkowski space to be

$$\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1), \tag{12.45}$$

we have necessarily that $\eta_{55} = -1$, and the resulting metric of the five-dimensional "internal" space will be

$$\eta_{AB} = \mathrm{diag}(+1, -1, -1, -1, -1). \tag{12.46}$$

This means that the fifth dimension must necessarily be space-like, and the metric with signature $(3, 2)$ is excluded. On the other hand, if we had chosen instead

$$\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1) \tag{12.47}$$

for Minkowski spacetime, it is easy to verify that the same consistency arguments would require that $\eta_{55} = 1$. The resulting metric of the five-dimensional "internal" space would then be

$$\eta_{AB} = \mathrm{diag}(-1, +1, +1, +1, +1), \tag{12.48}$$

and the same conclusion would be obtained: the fifth dimension must necessarily be space-like, and the metric with signature $(3, 2)$ is excluded. The unification of the gravitational and electromagnetic lagrangians, therefore, imposes a constraint on the metric conventions for Minkowski and for the electromagnetic internal manifold S^1. In fact, the choice between $\eta_{55} = +1$ and $\eta_{55} = -1$ for the metric of the $U(1)$ group depends on the metric convention adopted for the Minkowski space. As a consequence, the metric of the five-dimensional internal space turns out to be restricted to either (12.46) or (12.48). Metrics with signature $(3, 2)$, which would imply a time-like fifth dimension, are excluded.

12.2.4 Matter Fields

Let us consider now a general matter field Ψ. In contrast to the gauge fields, it depends on the coordinate x^5:

$$\Psi(x^A) = \Psi(x^\mu, x^5).$$

Under an infinitesimal generalized gauge translation, it behaves as

$$\delta\Psi = \alpha^A \, P_A \, \Psi. \tag{12.49}$$

Its covariant derivative is consequently

$$\mathcal{D}_\mu \Psi = \partial_\mu \Psi + \mathcal{A}^A{}_\mu P_A \, \Psi, \tag{12.50}$$

with the infinitesimal gauge transformation of $\mathcal{A}^A{}_\mu$ given by

$$\delta \mathcal{A}^B{}_\mu = - \, \partial_\mu \alpha^B. \tag{12.51}$$

This transformation is consistent with the fact that both electromagnetism and Teleparallel Gravity are described by abelian gauge theories.

Now, as the internal manifold S^1 is compact, we assume that the dependence of $\Psi(x^\mu, x^5)$ on the coordinate x^5 is of the form

$$\Psi(x^\mu, x^5) = \exp\left[i \, 2\pi\theta\right] \, \psi(x^\mu), \tag{12.52}$$

where

$$\theta = \frac{\kappa \, x^5}{\lambda_C} \tag{12.53}$$

is the dimensionless coordinate (angle) of S^1, with $\lambda_C = (2\pi\hbar/mc)$ the Compton wavelength of the particle the wave function Ψ represents. That is to say,

$$\Psi(x^\mu, x^5) = \exp\left[i2\pi \, \frac{\kappa \, x^5}{\lambda_C}\right] \, \psi(x^\mu). \tag{12.54}$$

As a consequence, a translation in the coordinate x^5 turns out to be a $U(1)$ gauge transformation, and a translation in the coordinates x^a turns out to be a gauge transformation of the translation group. For a simultaneous infinitesimal translation in the five coordinates x^A, we see from transformation (12.49) that

$$\delta\Psi = \alpha^a \, \partial_a \Psi + \alpha \left(\frac{iq}{\hbar c}\right) \Psi, \tag{12.55}$$

where we have used Eq. (12.35).

On the other hand, the covariant derivative (12.50) assumes the form

$$\mathcal{D}_\mu \Psi = h_\mu \Psi + \frac{iq}{\hbar c} \, A_\mu \, \Psi, \tag{12.56}$$

with $h_\mu = h^a{}_\mu \, \partial_a$. Using the relation $A_a = h_a{}^\mu \, A_\mu$, we can rewrite (12.56) in the form

$$\mathcal{D}_\mu \Psi = h^a{}_\mu \, \mathcal{D}_a \Psi, \tag{12.57}$$

with

$$\mathcal{D}_a \Psi = \partial_a \Psi + \frac{iq}{\hbar c} \, A_a \, \Psi \tag{12.58}$$

the electromagnetic covariant derivative in Minkowski spacetime. As usual, the commutator of covariant derivatives yields the field strength

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\Psi = \mathcal{F}^A{}_{\mu\nu} P_A \Psi, \tag{12.59}$$

with

$$\mathcal{F}^A{}_{\mu\nu} P_A \Psi = \overset{\bullet}{T}{}^a{}_{\mu\nu} P_a \Psi + \frac{iq}{\hbar c} F_{\mu\nu} \Psi. \tag{12.60}$$

12.3 Further Remarks

The teleparallel equivalent of the standard Kaluza-Klein theory is a five–dimensional Maxwell-type translational gauge theory on a four-dimensional spacetime. In this theory, owing to the fact that both torsion and the electromagnetic field act on particles through a Lorentz-type force, the electromagnetic field strength can be considered as an extra, fifth component of torsion. For this reason, the unification in this approach can be considered to be much more natural than in the ordinary Kaluza-Klein theory.

An important feature of this model is that, in contrast to ordinary Kaluza–Klein models, no scalar field is generated by the unification process. Accordingly, no unphysical constraints appear, and the gravitational action can be naturally truncated at the zero mode. In other words, the cylindric condition can be naturally imposed for matter fields, which corresponds to keep only the $n = 0$ Fourier mode. The infinite spectrum of massive new particles is eliminated, strongly reducing the redundancy present in ordinary Kaluza-Klein theories. Furthermore, spacetime is kept four-dimensional, with no extra dimension. Only the "internal" space has additional dimensions.

Another important point concerns the relation between geometry and gauge theories. According to ordinary Kaluza-Klein models, gauge theories emerge from higher-dimensional geometric theories as a consequence of the dimensional reduction process. According to the teleparallel approach, however, gauge theories are the natural structures to be introduced, the four–dimensional geometry (gravitation) emerging from the noncompact sector of the "internal" space. In fact, only this sector can give rise to a tetrad field, which is the responsible for the geometrical structure (either metric or teleparallel) induced in spacetime. As the gauge theories are introduced in their original form — they do not come from geometry — the unification turns out to be much more natural and easier to be performed.

The generalization of the teleparallel Kaluza-Klein model to non-abelian gauge theories is straightforward, and can be realized by introducing a $(4+n)$-dimensional internal space formed by the cartesian product between Minkowski space and a compact riemannian manifold, where n is the dimension of the gauge group [141]. Like in the electromagnetic case, the gauge field-strength appears as extra gauge-components of torsion. It is worth mentioning also that most of the conceptual problems present in ordinary Kaluza-Kllein models do not appear in the teleparallel version. In particular, due to the fact that the gauge structure — and not geometry — forms the basic paradigm, the problem discussed in Comment 12.3 does not exist in non-abelian teleparallell Kaluza-Klein.

Chapter 13

The Connection Space

The space of connections, like the space of points, is an affine space. Different points of this space defines different connections, which are not related by Lorentz transformations. Each connection defines an acceleration, and provides different ways of describing the gravitational interaction. Whereas in General Relativity the spin connection defines a vanishing acceleration, and then the geodesic equations, in Teleparallel gravity the spin connection defines an acceleration which is related to a gravitational force.

13.1 Translations in the Connection Space

The space of connections has a very interesting character: it is an affine space [22, 23]. This means that the points on any straight line drawn through two connections is also a connection. Given two connections $A_{(0)}$ and $A_{(1)}$,

$$A_{(\alpha)} = \alpha\, A_{(1)} + (1-\alpha) A_{(0)} \tag{13.1}$$

will be a connection for any real value of α. These properties hold also for gauge potentials which, as said above, are connections on bundles with internal spaces. This is similar to the space of points, which is also an affine space. The difference between two points is not a point, but a vector. Points themselves are not covariant in any sense, but vectors are. In the same token, connections are not covariant (and in particular a connection which is zero in one base is not zero in another base), but the difference between two connections is a tensor — an object with well-defined behavior. The above property does not transfer to curvature, but transfers to torsion:

$$T^{\lambda}_{(\alpha)\mu\nu} = \alpha\, T^{\lambda}_{(1)\mu\nu} + (1-\alpha) T^{\lambda}_{(0)\mu\nu}. \tag{13.2}$$

A generic point in the space of Lorentz connections is specified by a connection 1-form

$$A_c = \tfrac{1}{2}\, A^{ab}{}_{c}\, S_{ab} \tag{13.3}$$

presenting curvature and torsion. Given two different connections A_c and $\bar{A}_c$, the difference

$$\bar{A}^a{}_{bc} - A^a{}_{bc} = k^a{}_{bc} \tag{13.4}$$

is also a 1-form assuming values in the Lorentz Lie algebra,

$$k_c = \tfrac{1}{2}\, k^a{}_{bc}\, S_a{}^b, \tag{13.5}$$

but transforming covariantly under local Lorentz transformations. As such, it is necessarily anti-symmetric in the *first two* indices. Separating $k^a{}_{bc}$ in the symmetric and anti-symmetric parts in the *last two* indices, one gets

$$k^a{}_{bc} = \tfrac{1}{2}(k^a{}_{bc} + k^a{}_{cb}) + \tfrac{1}{2}(k^a{}_{bc} - k^a{}_{cb}). \tag{13.6}$$

Defining a tensor $t^a{}_{cb} = -\,t^a{}_{bc}$ such that

$$k^a{}_{bc} - k^a{}_{cb} = t^a{}_{cb}, \tag{13.7}$$

and using Eq. (13.6) for three combination of indices, it is easy to verify that

$$k^a{}_{bc} = \tfrac{1}{2}(t_b{}^a{}_c + t_c{}^a{}_b - t^a{}_{bc}). \tag{13.8}$$

We see that the difference between two connections has the form of a contortion tensor. In other words, a translation in the space of Lorentz connections is achieved by adding to a given connection a contortion-type 1-form.

13.2 Curvature and Torsion

In the language of differential forms, curvature and torsion of a given connection 1-form A are defined respectively by

$$R = dA + AA \equiv \mathcal{D}_A A \tag{13.9}$$

and

$$T = dh + Ah \equiv \mathcal{D}_A h, \tag{13.10}$$

where h is a tetrad $\mathcal{D}_A$ denotes the covariant differential in the connection A. Given two connections A and $\bar{A}$, the difference

$$\bar{A} - A \equiv k \tag{13.11}$$

is also a 1-form assuming values in the Lorentz Lie algebra,

$$k_c = \tfrac{1}{2}\, k^{ab}{}_c\, S_{ab}, \tag{13.12}$$

but transforming covariantly under local Lorentz transformations. Its covariant derivative, therefore, is written as

$$\mathcal{D}_A k = dk + \{A, k\}. \tag{13.13}$$

It is then easy to verify that, given two connections such that $\bar{A} = A + k$, their curvature and torsion will be related by

$$\bar{R} = R + \mathcal{D}_A k + k\, k \tag{13.14}$$

and

$$\bar{T} = T + k\, h. \tag{13.15}$$

The effect of adding a covector k to a given connection A is then to change its curvature and torsion 2-forms. On account of the torsion transformation (13.15), the contortion tensor is found to transform according to

$$\bar{K} = K + k. \tag{13.16}$$

13.3 Equivalence under Connection Translations

Let us choose the point of the space of connections representing the vanishing connection

$$\overset{\bullet}{A}{}^a{}_{bc} = 0. \tag{13.17}$$

See from a general Lorentz frame, this connection has the form

$$\overset{\bullet}{A}{}^a{}_{bc} = \Lambda^a{}_e\, h_c \Lambda_b{}^e, \tag{13.18}$$

with $\Lambda^a{}_e$ the Lorentz transformation. As we have seen, this is just the spin connection of Teleparallel Gravity.

Performing now a translation with parameter $\overset{\bullet}{K}{}^a{}_{bc}$, we obtain the new connection

$$\overset{\bullet}{A}{}^a{}_{bc} - \overset{\bullet}{K}{}^a{}_{bc} = \overset{\circ}{A}{}^a{}_{bc}, \tag{13.19}$$

which is just the spin connection of General Relativity. We see in this way that Teleparallel Gravity and General Relativity are related by a translation in the space of connections. This is actually an universal property: given a general connection $A^a{}_{bc}$, if one performs a translation using the connection contortion $K^a{}_{bc}$, one ends up with the spin connection of General Relativity:

$$A^a{}_{bc} - K^a{}_{bc} = \overset{\circ}{A}{}^a{}_{bc}. \tag{13.20}$$

Now, as we have discussed in Section 1.7, each connection defines an acceleration. There are, thus, many different accelerations, one for each point of the connection space. Of course, they are independent in the sense that they are not related by any Lorentz transformations. The same holds for the connections $\overset{\bullet}{A}{}^a{}_{bc}$ and $\overset{\circ}{A}{}^a{}_{bc}$. In General Relativity, for example, the connection is such that all accelerations vanish identically. The geodesic equation

$$\frac{du^c}{ds} + \overset{\circ}{A}{}^c{}_{ab}\, u^a\, u^b = 0 \tag{13.21}$$

shows clearly this property. As a consequence, there is no the concept of force in General Relativity. In Teleparallel Gravity, on the other hand, the connection has no dynamical meaning, and the resulting acceleration is fully related to a gravitational force. In fact, the equation of motion in this case is the force equation

$$\frac{du^c}{ds} + \overset{\bullet}{A}{}^c{}_{ab}\, u^a\, u^b = \overset{\bullet}{K}{}^c{}_{ab}\, u^a\, u^b. \tag{13.22}$$

Of course, due to the relation (13.20), the acceleration defined by any connection will give rise to an equivalent equation of motion

$$\frac{du^c}{ds} + A^c{}_{ab}\, u^a\, u^b = K^c{}_{ab}\, u^a\, u^b. \tag{13.23}$$

However, since this general connection cannot be obtained from the metric, or the tetrad, which are the variables of the gravitational field equations, they cannot be determined by the field equations. In this sense, we can say that the space of connections has two special points: one, which defines, up to a local Lorentz transformation, a vanishing connection, and another, which defines the Levi-Civita connection, a connection fully determined by the spacetime metric, or tetrad.

Comment 13.1 Connections — including gauge potentials — belong to affine spaces. This allows a detailed examination of an interesting question, the Wu–Yang ambiguity or the *problem of copies* [23]. The fundamental field of a gauge theory is the potential $A^C{}_\lambda$, but the observable, measurable field is the field strength $F^C{}_{\mu\nu}$. It may come as a surprise that, in non–abelian theories, many quite non-equivalent potentials $A^C{}_\lambda$ ("copies") can have one same field strength $F^C{}_{\mu\nu}$. In what concerns torsion, there are no copies for lorentzian connections. Let us state in more detail the Ricci theorem mentioned in Comment 1.8 and at the end of Section 1.7: given a metric $g_{\mu\nu}$ and any tensor of type $T^\lambda{}_{\mu\nu}$, there exists one and only one linear connection Γ which preserves $g_{\mu\nu}$ and has torsion equal to $T^\lambda{}_{\mu\nu}$. In particular, the only lorentzian connection with $T^\lambda{}_{\mu\nu} = 0$ is the Levi-Civita connection, from which the others differ precisely by their torsions. Lorentzian connections are, in this way, *classified* by their torsions. General Relativity, by postulating $T^\lambda{}_{\mu\nu} = 0$ fixes the connection as $\overset{\circ}{\Gamma}$ and avoids the copies problem. Notice that this is still another consequence of the existence of torsion, even if vanishing.

Chapter 14

A Glimpse on Einstein-Cartan

Einstein-Cartan can be considered as a prototype of those theories in which curvature and torsion represent different gravitational degrees of freedom. For the sake of comparison with Telaparallel Gravity, in which curvature and torsion are related to the same degrees of freedom, a brief review of this theory is presented here. Some drawbacks are pointed out.

Alternative gravitational models, like Einstein-Cartan and gauge theories for the Poincaré and the affine groups, consider curvature and torsion as representing independent gravitational degrees of freedom. Torsion appears as intimately related to spin, and consequently turns out to be important mainly at the microscopic level, where spins become relevant. A fundamental difference between Einstein–Cartan and the mentioned gauge theories is that, whereas in the former torsion is a non-propagating field, in the latter both curvature and torsion are propagating fields. Notwithstanding this difference, as these models present all the same relationship between torsion and spin, the Einstein-Cartan model can be taken as representative of this class, and for this reason it will be the only one to be discussed here.

The basic motivation for the Einstein–Cartan construction [142] is the fact that, at a microscopic level, matter is represented by elementary particles, which are characterized by mass *and* spin. If one adopts the same *geometrical spirit of General Relativity*, not only mass but also spin should be source of gravitation at that level. According to this scheme, like in General Relativity, energy–momentum should appear as source of curvature, whereas spin should appear as source of torsion. The relevant connection of this theory, therefore, is a general Cartan connection $A^a{}_{b\mu}$ presenting both curvature and torsion. It can be decomposed as in Eq.(1.44), that is,

$$A^a{}_{b\mu} = \overset{\circ}{A}{}^a{}_{b\mu} + K^a{}_{b\mu}. \tag{14.1}$$

The corresponding spacetime linear connection is

$$\Gamma^\rho{}_{\lambda\mu} = h_a{}^\rho \partial_\mu h^a{}_\lambda + h_a{}^\rho A^a{}_{b\mu} h^b{}_\lambda \equiv h_a{}^\rho \mathcal{D}_\mu h^a{}_\lambda. \tag{14.2}$$

In terms of this connection, decomposition (14.1) assumes the form

$$\Gamma^\rho{}_{\lambda\mu} = \overset{\circ}{\Gamma}{}^\rho{}_{\lambda\mu} + K^\rho{}_{\lambda\mu}, \tag{14.3}$$

with $\overset{\circ}{\Gamma}{}^\rho{}_{\lambda\mu}$ the Levi–Civita connection.

14.1 Field Equations

The Einstein–Cartan gravitational lagrangian is

$$\mathcal{L}_{EC} = -\frac{c^4}{16\pi G} \sqrt{-g}\, R. \tag{14.4}$$

Although it formally coincides with the Einstein–Hilbert lagrangian of General Relativity, the scalar curvature

$$R = g^{\mu\nu} R^\rho{}_{\mu\rho\nu} \tag{14.5}$$

refers now to the curvature of the general Cartan connection:

$$R^\rho{}_{\lambda\nu\mu} = \partial_\nu \Gamma^\rho{}_{\lambda\mu} - \partial_\mu \Gamma^\rho{}_{\lambda\nu} + \Gamma^\rho{}_{\eta\nu}\Gamma^\eta{}_{\lambda\mu} - \Gamma^\rho{}_{\eta\mu}\Gamma^\eta{}_{\lambda\nu}. \tag{14.6}$$

Since the connection $\Gamma^\rho{}_{\lambda\mu}$ is not symmetric in the last two indices, the Ricci curvature tensor is not symmetric either: $R_{\mu\nu} \neq R_{\nu\mu}$.

Considering the total lagrangian

$$\mathcal{L} = \mathcal{L}_{EC} + \mathcal{L}_m, \tag{14.7}$$

with $\mathcal{L}_m$ the lagrangian of a matter source field ψ, the gravitational field equations are obtained by taking variations with respect to the metric $g^{\mu\nu}$ and the contortion tensor $K_\rho{}^{\mu\nu}$. The resulting field equations are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R = \frac{8\pi G}{c^4}\, \theta_{\mu\nu} \tag{14.8}$$

and

$$T^\rho{}_{\mu\nu} + \delta^\rho_\mu\, T^\alpha{}_{\nu\alpha} - \delta^\rho_\nu\, T^\alpha{}_{\mu\alpha} = \frac{8\pi G}{c^4}\, s^\rho{}_{\mu\nu}. \tag{14.9}$$

In these equations,

$$\sqrt{-g}\, \theta_\mu{}^\nu = \frac{\partial \mathcal{L}_m}{\partial(\mathcal{D}_\nu \psi)}\, h^a{}_\mu \partial_a \psi - \delta^\nu_\mu\, \mathcal{L}_m \tag{14.10}$$

is the canonical energy–momentum tensor and

$$\sqrt{-g}\, s^{\rho}{}_{\mu\nu} = \frac{1}{2}\,\frac{\partial \mathcal{L}_m}{\partial(\mathcal{D}_\rho \psi)}\; S_{\mu\nu}\psi \tag{14.11}$$

is the canonical spin tensor of the source, with

$$S_{\mu\nu} = h^a{}_\mu h^b{}_\nu S_{ab}$$

the spacetime version of the Lorentz generators. Equation (14.9) can be rewritten in the form

$$T^{\rho}{}_{\mu\nu} = \frac{8\pi G}{c^4}\left(s^{\rho}{}_{\mu\nu} + \tfrac{1}{2}\,\delta^{\rho}_{\mu}\, s^{\alpha}{}_{\nu\alpha} - \tfrac{1}{2}\,\delta^{\rho}_{\nu}\, s^{\alpha}{}_{\mu\alpha}\right). \tag{14.12}$$

We see from this equation that, for spinless matter torsion vanishes, the canonical energy–momentum tensor becomes symmetric, and the field equation (14.8) reduces to ordinary Einstein equation. In particular, in empty space there is no difference between the Einstein–Cartan and Einstein theories. In the presence of spinning matter, however, there will be a non-vanishing torsion, as given by Eq. (14.12). As this is a purely algebraic equation, torsion is a non-propagating field.

14.2 Coupling Prescription

The coupling prescription in Einstein–Cartan theory is achieved through the replacement

$$\partial_\mu \;\to\; \mathcal{D}_\mu = \partial_\mu - \tfrac{i}{2} A^{ab}{}_\mu\, S_{ab}. \tag{14.13}$$

Acting on a spinor field ψ, it assumes the form

$$\mathcal{D}_\mu \psi = \partial_\mu \psi - \tfrac{i}{2} A^{ab}{}_\mu\, S_{ab}\psi, \tag{14.14}$$

with S_{ab} the spinor representation (10.19). Substituting the decomposition (14.1), we obtain

$$\mathcal{D}_\mu \psi = \overset{\circ}{\mathcal{D}}_\mu \psi - \tfrac{i}{2}\, K^{ab}{}_\mu\, S_{ab}\psi, \tag{14.15}$$

with $\overset{\circ}{\mathcal{D}}_\mu \psi$ the covariant derivative of General Relativity. We see from this expression that new physical phenomena in relation to General Relativity (and consequently in relation to Teleparallel Gravity) are expected in the presence of spin, or equivalently, in the presence of torsion. These additional effects are related to the fact that, in this theory, curvature and torsion represent independent gravitational degrees of freedom.

In the case of a Lorentz vector V^a, for which the generators S_{ab} are given by Eq. (1.25), the covariant derivative (14.15) reads

$$\mathcal{D}_\mu V^a = \partial_\mu V^a + A^a{}_{b\mu}\, V^b. \tag{14.16}$$

The corresponding covariant derivative of a spacetime vector $V^\rho = h_a{}^\rho V^a$ has the form

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho{}_{\lambda\mu}\, V^\lambda. \tag{14.17}$$

14.3 Particle Equations of Motion

According to the Einstein–Cartan construction, torsion vanishes in absence of spin, and the theory reduces to General Relativity. As a consequence, a spinless particle must satisfy the geodesic equation

$$\frac{du^\rho}{ds} + \overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu}\, u^\mu\, u^\nu = 0. \tag{14.18}$$

Comment 14.1 It is worth mentioning that, considering the equation of motion of a free particle in an inertial frame with cartesian coordinates,

$$\frac{du^\rho}{ds} = 0, \tag{14.19}$$

and applying the coupling prescription (14.17), we obtain the auto-parallel equation

$$\frac{du^\rho}{ds} + \Gamma^\rho{}_{\mu\nu}\, u^\mu\, u^\nu = 0, \tag{14.20}$$

which is not the correct equation of motion for spinless particles in Einstein–Cartan theory. In fact, according to this equation, all particles are equally affected by torsion, independently of their spin content. This means that even spinless particles would follow a trajectory that deviates from the geodesic motion. There are actually more problems with the auto-parallel equation of motion. It has already been shown that they cannot be obtained from a lagrangian formalism [147], which means that a spinless particle following such a trajectory does not have a lagrangian. Taking into account that the energy-momentum density is defined as the functional derivative of the lagrangian with respect to the metric tensor (or equivalently, with respect to the tetrad field), it is not possible to define an energy-momentum density for such particle.

The geodesic equation (14.18) is obtained from the lagrangian

$$\mathcal{S} = -\int_a^b h^a{}_\mu\, p_a\, dx^\mu. \tag{14.21}$$

where $p_a = mcu_a$ is the is the particle four–momentum. According to the minimal coupling prescription (14.16), the action integral describing a spinning particle minimally coupled to the gauge potential $A^a{}_{b\mu}$ is

$$\mathcal{S} = \int_a^b \left[-h^a{}_\mu\, p_a + \tfrac{1}{2}\, A^{ab}{}_\mu\, s_{ab}\right] dx^\mu, \tag{14.22}$$

where s_{ab} is the particle spin angular momentum. Following the same procedure of Section 4.6, the Routhian describing a spinning particle in Einstein–Cartan theory is

$$\mathcal{R} = -\, h^a{}_\mu \, p_a \, u^\mu + \tfrac{1}{2} \, A^{ab}{}_\mu \, s_{ab} \, u^\mu - \frac{\mathcal{D}u^a}{\mathcal{D}s} \frac{s_{ab} u^b}{u^2}, \tag{14.23}$$

where

$$\frac{\mathcal{D}u^a}{\mathcal{D}s} = u^\mu \, \mathcal{D}_\mu u^a,$$

with $\mathcal{D}_\mu$ the covariant derivative (14.16). Using this Routhian, the equation of motion for the spin is found to be

$$\frac{\mathcal{D}s_{ab}}{\mathcal{D}s} = (u_a \, s_{bc} - u_b \, s_{ac}) \, \frac{\mathcal{D}u^c}{\mathcal{D}s}. \tag{14.24}$$

Making use of the lagrangian formalism, the equation of motion for the trajectory of the particle is found to be

$$\frac{\mathcal{D}\mathcal{P}_\mu}{\mathcal{D}s} = T^a{}_{\mu\nu} \, \mathcal{P}_a \, u^\nu - \frac{1}{2} \, R^{ab}{}_{\mu\nu} \, s_{ab} \, u^\nu, \tag{14.25}$$

where

$$\mathcal{P}_\mu = h_\mu{}^c \, \mathcal{P}_c, \tag{14.26}$$

with

$$\mathcal{P}_c = m \, c \, u_c + u^a \, \frac{\mathcal{D}s_{ca}}{\mathcal{D}s} \tag{14.27}$$

the generalized momentum. This is the Einstein–Cartan version of the Papapetrou equation [176]. In addition to the usual Papapetrou coupling between the particle spin and the Riemann tensor, there is also a coupling between torsion and the generalized momentum $\mathcal{P}_\rho$. Since curvature and torsion represent in this case independent degrees of freedom, new physics is associated to the coupling of torsion with $\mathcal{P}_\rho$. This equation should be compared with the equation of motion (4.119) and (4.122), in which curvature and torsion appears as alternative ways of describing the same gravitational field, and in which no new physics is associated to torsion.

14.4 Some Caveats

The Einstein–Cartan theory briefly described in this chapter presents a series of conceptual problems, some of them usually overlooked in the literature. The first problem refers to the coupling prescription (14.13). Although it complies with the *passive* strong equivalence principle, it violates the general covariance principle, an active version of the strong equivalence principle [see Section 3.5.1]. This problem can be circumvented by assuming a totally anti–symmetric torsion [see, for example, Ref. [176]], but this is totally unjustified from the conceptual point of view. Furthermore, when used to describe the interaction of the electromagnetic field with gravitation, the coupled Maxwell equation results to be not gauge invariant.

Another, quite important problem, refers to the gravitational field equation (14.8). As is well known, the canonical energy–momentum tensor of the electromagnetic field is not gauge covariant [see, for example, Ref. [58], page 81]. When the electromagnetic field is considered as the source field, therefore, the field equation (14.8) will not be gauge covariant. This is a serious drawback of the model.

These problems are usually circumvented by *postulating* that the electromagnetic field does not couple nor produce torsion [175]. In other words, torsion is assumed to be irrelevant to the Maxwell's equations [93]. This "solution", however, is far from reasonable. It is not Nature that has to comply with our theories, but the other way round.

Comment 14.2 It is important to remark that the above postulate lacks physical support. In fact, from a quantum point of view, one may always expect an interaction between photons and torsion [94]. The reason for this is that a photon, perturbatively speaking, can virtually disintegrate into an electron–positron pair. Considering that these particles are massive fermions that do couple to torsion, a photon will necessarily feel the presence of torsion. Since all macroscopic phenomena must have an interpretation based on an average of microscopic phenomena, and taking into account the strictly attractive character of gravitation which eliminates the possibility of a vanishing average, the photon field must interact with torsion through the virtual pair produced by the vacuum polarization.

Finally, it is worth mentioning that, although the equation of motion (14.25) is widely considered to be the Einstein–Cartan version of the Papapetrou equation [see, for example, Ref. [176]], it somehow contradicts the Einstein–Cartan paradigm in the sense that torsion does not couple to spin, but to the generalized four momentum. Like in General Relativity, spin couples to curvature. There has been some attempts to obtain equations of motions in which torsion couples somehow to spin [see, for example, Ref. [13], page 249], but these equations do not follow from a variational principle, nor makes use of the Einstein–Cartan coupling prescription (14.13).

Chapter 15

Epilogue

If Teleparallel Gravity is equivalent to General Relativity, why should one study it? A discussion of some of the reasons is presented here.

15.1 On the Gravitational Interaction

Gravitation has, at least at the classical level, a quite distinctive property: structureless particles with different masses and compositions experience it in such a way that all of them acquire the same acceleration and, given the same initial conditions, follow the same path. This universality of response — usually referred to as *universality of free fall*, and embodied in the *weak equivalence principle* — is its most peculiar characteristic. It is unique: no other fundamental interaction of Nature exhibits it. That said, effects equally felt by all bodies were known since long: they are the *inertial* effects, which show up in non-inertial frames. Examples on Earth are the centrifugal and the Coriolis forces.

Universality of both gravitational and inertial effects was a conceptual clue used by Einstein in building up General Relativity. Another ingredient was the notion of field, which provides the best approach to interactions consistent with Special Relativity: all known forces are mediated by fields on spacetime. If gravitation is to be represented by a field it should, by the considerations above, be a universal field, equally felt by every particle. A natural solution is then to assume that gravitation changes spacetime itself. The simplest way to change spacetime would be to change what appears as its most fundamental field — the metric. The presence of a gravitational field should be, therefore, represented by a change in the metric of Minkowski spacetime.

The metric tensor, however, defines neither curvature nor torsion by itself.

As a matter of fact, curvature and torsion are properties of connections, and many different connections, with different curvature and torsion tensors, can be defined on the very same metric spacetime. The question then arises: how can we determine the relevant connection to describe the gravitational field? To answer this question, we observe first that a general Lorentz connection has 24 independent components. However, any gravitational theory in which the source is the 10 components symmetric energy–momentum tensor, will not be able to determine uniquely the connection. There are only two possible options.

The first is to choose the Levi–Civita, or Christoffel connection, which is a connection completely specified by the 10 components of the metric tensor. The second option is to choose a Lorentz connection not related to gravitation, but to inertial effects only. In this case, the gravitational field turns out to be fully represented by the tetrad field — or more specifically, by the non–trivial part of the tetrad field. In the first case, which corresponds to Einstein's choice, we get General Relativity, a theory that assumes torsion to vanish from the very beginning. In the second case, we get Teleparallel Gravity, a gauge theory for the translation group, in which curvature is assumed to vanish from the very beginning.

Einstein's choice, it must be said, is the most intuitive from the point of view of universality. Gravitation can be easily understood by supposing that it produces a curvature in spacetime, in such a way that all (spinless, structureless) particles, independently of their masses and constitutions, will follow a geodesic of the curved spacetime. Universality of free fall is, in this way, naturally incorporated into gravitation. Geometry replaces the concept of force and the trajectories are solutions, not of a force equation, but of a geodesic equation. Nevertheless, because such a geometrization requires the weak equivalence principle, in the absence of universality the general–relativistic description of gravitation would simply break down.

This restriction apart, one may wonder whether there is any problem with Einstein's choice. Was Einstein wrong when he chose a torsionless connection to describe gravitation? This question is as old as General Relativity. In fact, since the early days of General Relativity, there have been theoretical speculations on the necessity of including torsion, *in addition to curvature*, in the description of the gravitational interaction. This comes from the fact that a Lorentz connection, as described in its generality by Cartan, presents naturally *both* curvature and torsion. Theories like the Einstein–Cartan model [142] and the gauge theories for the Poincaré [143, 28, 29] and the affine groups [144] consider curvature and torsion as representing separate degrees of freedom. In these theories, torsion is directly related to spin and should become relevant only when spins are important. According to these models, therefore, new physical phenomena are expected from the presence of torsion. From this point of view, therefore, Einstein would have made a mistake by neglecting torsion.

On the other hand, although conceptually different, General Relativity and Teleparallel Gravity are found to yield equivalent descriptions of the gravitational interaction. An immediate implication of this equivalence is that curvature and torsion turn out to be simply alternative ways of describing the gravitational field, and are consequently related to the same degrees of freedom of gravity. This is corroborated by the fact that the same matter energy-momentum tensor appears as source in both theories: of curvature in General Relativity, of torsion in Teleparallel Gravity. According to this interpretation, both General Relativity and Teleparallel Gravity are complete theories. From this point of view, therefore, Einstein did not make a mistake by not introducing torsion into gravitation. It should be emphasized that, as of today, there is no experimental evidence for *new physics* associated to torsion. Furthermore, *ordinary physics* related to all known gravitational phenomena, including the physics of the solar system, can be consistently reinterpreted in terms of teleparallel torsion. This means essentially that, though unnoticed by many, torsion has already been detected.

15.2 Why to Study Teleparallel Gravity

Then comes the inevitable question: if Teleparallel Gravity is equivalent to General Relativity, why should one study it? There are several points that can be used to justify this study. Here, we discuss some of them.

15.2.1 Matters of Consistency

Differently from the coupling prescriptions of other models involving torsion, the coupling prescription of Teleparallel Gravity, like that of General Relativity, is consistent with both the active and passive versions of the strong equivalence principle. In consequence, when applied to describe the gravitational interaction of the electromagnetic field, that prescription is found not to violate the $U(1)$ gauge invariance of Maxwell theory. As this invariance is of paramount importance for physics, the torsion interpretation provided by Teleparallel Gravity can be considered as the most natural in the sense that it is not in conflict with well established theories.

15.2.2 Gauge Structure and Universality

Although equivalent to General Relativity, Teleparallel Gravity gives of gravitation a completely different picture. Curvature is replaced by torsion, geometry by force. Behind this difference lies the gauge structure: teleparallelism shows up as a gauge theory for the group of translation on Minkowski space, the tangent space at every point of any spacetime — which, by the way, explains why gravitation has for source energy–momentum, just the Noether current for those translations. Soldering makes of it a non-standard gauge theory, keeping nevertheless a remarkable similarity to electromagnetism, also a gauge theory for an abelian group. Due to the gauge structure, it dispenses with the weak equivalence principle. It can comply with universality, but remains a consistent theory in its absence. And, because also Newtonian gravity can comply with non–universality, the Newtonian limit follows much more naturally from Teleparallel Gravity than from General Relativity.

15.2.3 Gravitational Energy-Momentum Density

All fundamental fields have a well-defined local energy-momentum density. It should be expected that the same happen to the gravitational field. It is true, however, that no tensorial expression for the gravitational energy-momentum density can be defined in the context of General Relativity. The basic reason for this impossibility is that both gravitational and inertial effects are mixed in the spin connection of the theory, and cannot be separated. Even though some quantities, like for example curvature, are not affected by inertia, some others turn out to depend on it. For example, the energy-momentum density of gravitation will necessarily include both the energy-momentum density of gravity and the energy–momentum density of inertia. Since the inertial effects are essentially non-tensorial — they depend on the frame — it is not surprising that in this theory the complex defining the energy–momentum density of the gravitational field shows up as a non-tensorial object.

On the other hand, although equivalent to general relativity, teleparallel gravity naturally separates gravitation from inertia. As a consequence, it is possible to write down an energy-momentum density for gravitation only, excluding the contribution from inertia. This object is a true tensor, which means that *gravitation alone, like any other field of nature, does have a tensorial energy-momentum definition*. Since the purely gravitational energy-momentum density does not represent the total energy-momentum density — in the sense that the inertial part is not included — it does not need to be truly conserved, but only covariantly conserved. Of course, the total energy-momentum density, which in the general case includes contributions from inertia, gravitation and matter, remains conserved in the ordinary sense. We can then say that the impossibility of defining a tensorial expression for the energy-momentum density of gravity is not a property of nature, but just a drawback of the geometrical picture of general relativity.

15.2.4 The Case of the Spin-2 Field

It is well known that higher spin fields, and in particular a spin-2 field, present consistency problems when coupled to gravitation [112, **?**]. The problem is that the divergence identities satisfied by the field equations of a spin-2 field in Minkowski spacetime are no longer valid when it is coupled to gravitation. In addition, the coupled equations are no longer gauge invariant. The basic underlying difficulty is related to the fact that the covariant derivative of general relativity — which defines the gravitational coupling prescription — is non-commutative, and this introduces unphysical constraints on the spacetime curvature.

Now, due to the fact that General Relativity describes the gravitational interaction through a geometrization of spacetime, it is not, strictly speaking, a field theory in the usual sense of classical fields. On the other hand, owing to its gauge structure, teleparallel gravity does not geometrize the gravitational interaction, and for this reason it is much more akin to a field theory than general relativity. When defining a spin-2 field, therefore, instead of using General Relativity, it seems far more reasonable to use Teleparallel Gravity as paradigm. Accordingly, instead of a symmetric second-rank tensor, a spin-2 field must be assumed to be represented by a spacetime (world) vector field assuming values in the Lie algebra of the translation group. Its components, like the gauge potential of teleparallel gravity, represent a set of four spacetime vector fields.

In absence of gravitation, the resulting spin-2 field theory naturally emerges in the Fierz formalism, and turns out to be structurally similar to electromagnetism, a gauge theory for the $U(1)$ group. In fact, in addition to satisfy a dynamic field equation, the spin-2 field is found to satisfy also a Bianchi identity, which is related to the dynamic field equation by duality transformation. Furthermore, the gauge and the local Lorentz invariance of the theory provide the correct number of independent components for a massless spin-2 field. Upon contraction with the tetrad, the translational-valued vector field and the symmetric second-rank tensor field represent the same physical field, and consequently both approaches are equivalent in absence of gravitation. The teleparallel-based construction, however, can be considered to be more elegant in the sense that it has a gauge structure, it presents duality symmetry, and it allows for a precise distinction between gauge transformations — local translations in the tangent space — and spacetime coordinate transformations.

In the presence of gravitation, if the teleparallel correct coupling prescription is used, a sound gravitationally–coupled spin-2 field theory emerges, which is quite similar to the gravitationally–coupled electromagnetic theory. Furthermore, it is both gauge and local Lorentz invariance, and it preserves the duality symmetry of the free theory. In addition, owing to the fact that the teleparallel spin connection is purely inertial, the corresponding covariant derivative is commutative, no unphysical constraints on the spacetime geometry shows up. This property, together with the gauge and local Lorentz invariance, render the teleparallel–based gravitationally–coupled spin-2 theory fully consistent. Namely, it does not present the consistency problems of the spin-2 theory constructed on the basis of general relativity.

15.2.5 Gauge Structure and Unification

An argument comes from the age–old unification dream: all other fundamental interactions of Nature are described by gauge theories and gravitation, with Teleparallel Gravity, comes to the fold. This is far from enough to attain unification, but it is a step towards it. Consider, for example, the unification approach of the Kaluza-Klein theories. In their ordinary versions, gauge theories emerge from higher–dimensional geometric theories as a consequence of the dimensional reduction process. According to the teleparallel approach, on the other hand, it is the gauge makeup the natural structure to be introduced. In this case, it is the four–dimensional geometry (or gravitation) that emerges from the soldered sector of the gauge theory. As the gauge theories are introduced in their original form — they do not come from geometry — the unification turns out to be much more simple and natural. In fact, in contrast to ordinary Kaluza–Klein models, in the teleparallel version studied in Chapter 12 no scalar field is generated by the unification process. Accordingly, no unphysical constraints appear, and the gravitational action can naturally be truncated at the zero mode. The infinite spectrum of new massive particles is eliminated, strongly reducing the redundancy present in ordinary Kaluza–Klein theories.

15.2.6 Gravity and the Quantum

General Relativity is fundamentally grounded on the universality of free fall — or, equivalently, on the weak equivalence principle. This has been confirmed by all experimental tests at the classical level [78], but not at the quantum level [79]. In fact, at this level, as discussed in Chapter 5, the phase of the particle wavefunction turns out to depend on the particle mass (in the COW experiment, obtained in the non-relativistic limit), or on the relativistic kinetic energy (in the gravitational Aharonov–Bohm effect). On the other hand, owing to its gauge structure, Teleparallel Gravity does not require the weak equivalence principle. Although classically equivalent to General Relativity, therefore, Teleparallel Gravity seems to be a more convenient theory to deal with gravitationally–related quantum phenomena.

The fact that teleparallelism can dispense with the weak equivalence principle can have some deeper consequences. In fact, as is well known, General Relativity and quantum mechanics are not consistent with each other. This conflict stems from the very principles on which these theories take their roots. General Relativity, on one hand, is based on the equivalence principle, whose strong version establishes the *local* equivalence between gravitation and inertia. The fundamental asset of quantum mechanics, on the other hand, is the uncertainty principle, which is essentially *nonlocal*: a test particle does not follow a given trajectory, but infinitely many trajectories, each one with a different probability. Is there a consistent way of reconciling the equivalence and the uncertainty principles?

First of all, observe that the strong version of the equivalence principle, which requires the weak one, presupposes an ideal observer [34], represented by a timelike curve which intersects the space-section *at a point*. In each space-section, it applies at that intersecting point. The conflict comes from that idealization and extends, clearly, also to Special Relativity. In the geodesic equation, gravitation only appears through the Levi–Civita connection, which can be made to vanish all along. An *ideal* observer can choose frames whose acceleration exactly compensate the effect of gravitation. A *real* observer, on the other hand, will be necessarily an object extended in space, consequently intersecting a congruence of curves. Such congruences are described by the geodesic deviation equation and, consequently, detect the true covariant object characterizing the gravitational field, the curvature tensor — which cannot be made to vanish. Quantum Mechanics requires real observers, pencils of ideal observers. The inconsistency with the strong principle, therefore, is a mathematical necessity which precludes the existence of a quantum version of the strong equivalence principle [177].

It seems, therefore, that the equivalence and the uncertainty principles cannot hold simultaneously. It then comes the question: is it possible to discard one of them? From the point of view of General Relativity, the answer is *no*, as this theory cannot survive without it. From the point of view of Teleparallel Gravity, however, the old Synge's injunction [25] to the effect that the *midwife be now buried with appropriate honours*, can finally take place. This may represent an important step towards a reconciliation between gravity and the quantum.

15.2.7 Quantizing Gravity

Due to the fact that General Relativity is deeply rooted on the equivalence principle, its spin connection involves both gravitation and inertia. As a consequence, any approach to quantum gravity based on this connection will necessarily include a quantization of the inertial forces — whatever this means. Considering furthermore the non–covariant character of inertia, as well as its divergent asymptotic behavior, such approach has great chances to face severe consistency problems. It is also important to mention that, as is well known [35], general covariance by itself is empty of dynamical content as any relativistic equation can be made generally covariant. As a consequence, Lorentz covariance is also empty of dynamical content as any relativistic equation can be made Lorentz covariant. The invariance of a physical system under Lorentz transformations has to do only with changes of frames. In fact, whereas a *local* Lorentz transformation relates different classes of frames, a *global* Lorentz transformation relates frames inside each one of those classes. Observe furthermore that the Lorentz connection of General Relativity is not an independent field: it is completely determined by the tetrad field. One should not expect, therefore, any dynamical effect coming from a "gaugefication" of the Lorentz group.

On the other hand, as a gauge theory for the translation group, the gravitational field in Teleparallel Gravity is not represented by a Lorentz connection, but by a translational–valued connection that appears as the non–trivial part of the tetrad. In this theory, the Lorentz connection keeps its special relativistic role of representing inertial effects only. Considering that the translational gauge potential does not represent inertia, but gravitation only, a quantization approach based on the teleparallel variables will probably appear much more natural and consistent. Furthermore, due to the abelian character of translations, such approach will certainly be much simpler than those based on the non–abelian Lorentz connections.

15.2.8 Matters of Concept

Due to the geometric interpretation provided by General Relativity, which makes use of the torsionless Levi–Civita connection, there is a widespread belief that gravitation produces a curvature in spacetime. In consequence, the universe as a whole must be curved. According to Teleparallel Gravity, however, the above perspective changes, producing deep implications on the way we see the universe. In fact, because of the equivalence between Teleparallel Gravity and General Relativity, it becomes a matter of convention to describe the gravitational interaction in terms of curvature or in terms of torsion. This means essentially that the attribution of curvature to spacetime is not an absolute statement, but a model–dependent conclusion. Of course, the cosmology based on General Relativity is not incorrect. However, an appraisal based on teleparallel gravity could provide a new way to look at the universe, and eventually unveil new perspectives not visible in the standard General Relativity approach.

Appendix A

Teleparallel Field Equation

We present in this appendix a detailed computation leading to the teleparallel field equation introduced in Section 7.4. The teleparallel lagrangian is given by [see Eq. (7.15)]

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{2k} \left(\tfrac{1}{4} \, \overset{\bullet}{T}{}^{\rho}{}_{\mu\nu} \, \overset{\bullet}{T}{}_{\rho}{}^{\mu\nu} + \tfrac{1}{2} \, \overset{\bullet}{T}{}^{\rho}{}_{\mu\nu} \, \overset{\bullet}{T}{}^{\nu\mu}{}_{\rho} - \overset{\bullet}{T}{}^{\rho}{}_{\mu\rho} \, \overset{\bullet}{T}{}^{\nu\mu}{}_{\nu} \right), \tag{A.1}$$

where $k = 8\pi G/c^4$, and $\overset{\bullet}{T}{}^{\rho}{}_{\nu\mu} = h_a{}^{\rho} \, \overset{\bullet}{T}{}^{a}{}_{\nu\mu}$ with

$$\overset{\bullet}{T}{}^{a}{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + \overset{\bullet}{A}{}^{a}{}_{e\nu} h^e{}_\mu - \overset{\bullet}{A}{}^{a}{}_{e\mu} h^e{}_\nu \tag{A.2}$$

the torsion tensor. The gravitational field equation is obtained from the Euler–Lagrange equation for the the gauge potential $B^a{}_\mu$ or, equivalently, for the tetrad field $h^a{}_\mu$:

$$\frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\rho} - \partial_\sigma \, \frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial(\partial_\sigma h^a{}_\rho)} = 0. \tag{A.3}$$

The corresponding field equation can be written in the form

$$\partial_\sigma (h \, \overset{\bullet}{S}{}_a{}^{\rho\sigma}) - k \, (h \, \overset{\bullet}{\jmath}{}_a{}^{\rho}) = 0, \tag{A.4}$$

where

$$\overset{\bullet}{S}{}_a{}^{\rho\sigma} = - \, \overset{\bullet}{S}{}_a{}^{\sigma\rho} \equiv - \, \frac{k}{h} \, \frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial(\partial_\sigma h^a{}_\rho)} \tag{A.5}$$

is the superpotential, and

$$\overset{\bullet}{\jmath}{}_a{}^{\rho} \equiv - \, \frac{1}{h} \frac{\partial \overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\rho} \tag{A.6}$$

stands for the Noether gravitational energy-momentum current.

A.1 The superpotential

Taking the functional derivative of the first term of Eq. (A.1) with respect to $\partial_\sigma h^a{}_\rho$, we obtain

$$
\begin{aligned}
\frac{1}{4}\,\frac{\partial}{\partial\left(\partial_\sigma h^a{}_\rho\right)}\left(\overset{\bullet}{T}_{\lambda\mu\nu}\overset{\bullet}{T}{}^{\lambda\mu\nu}\right) &= \frac{1}{2}\,\overset{\bullet}{T}{}_b{}^{\mu\nu}\,\frac{\partial \overset{\bullet}{T}{}^b{}_{\mu\nu}}{\partial\left(\partial_\sigma h^a{}_\rho\right)} \\
&= \frac{1}{2}\,\overset{\bullet}{T}{}_b{}^{\mu\nu}\delta^b_a\left(\delta^\sigma_\mu\delta^\rho_\nu - \delta^\sigma_\nu\delta^\rho_\mu\right) \\
&= \overset{\bullet}{T}{}_a{}^{\sigma\rho}.
\end{aligned} \tag{A.7}
$$

In the same way, the second term of Eq. (A.1) yields

$$
\begin{aligned}
\frac{1}{2}\,\frac{\partial}{\partial\left(\partial_\sigma h^a{}_\rho\right)}\left(\overset{\bullet}{T}_{\rho\mu\nu}\,\overset{\bullet}{T}{}^{\mu\rho\nu}\right) &= \frac{1}{2}\,\frac{\partial \overset{\bullet}{T}{}^c{}_{\mu\nu}}{\partial\left(\partial_\sigma h^a{}_\rho\right)}\,\overset{\bullet}{T}{}^{\nu\mu}{}_c + \frac{1}{2}\,\overset{\bullet}{T}{}^c{}_{\mu\nu}\,\frac{\partial \overset{\bullet}{T}{}^{\nu\mu}{}_c}{\partial\left(\partial_\sigma h^a{}_\rho\right)} \\
&= \frac{1}{2}\,\delta^c_a\left(\delta^\sigma_\mu\delta^\rho_\nu - \delta^\sigma_\nu\delta^\rho_\mu\right)\overset{\bullet}{T}{}^{\nu\mu}{}_c \\
&\qquad + \frac{1}{2}\,\overset{\bullet}{T}{}^{\beta\alpha}{}_a\left(\delta^\sigma_\alpha\delta^\rho_\beta - \delta^\sigma_\beta\delta^\rho_\alpha\right) \\
&= \overset{\bullet}{T}{}^{\rho\sigma}{}_a - \overset{\bullet}{T}{}^{\sigma\rho}{}_a,
\end{aligned} \tag{A.8}
$$

where we have used the fact that neither the metric nor the tetrad depends on the derivative of the tetrad. The functional derivative of the last term of the lagrangian yields

$$
\begin{aligned}
-\,\frac{\partial}{\partial\left(\partial_\sigma h^a{}_\rho\right)}\left(\overset{\bullet}{T}{}^\nu{}_{\mu\nu}\overset{\bullet}{T}{}^{\lambda\mu}{}_\lambda\right) &= -2\,\overset{\bullet}{T}{}^{\nu\mu}{}_\nu\,\frac{\partial \overset{\bullet}{T}{}^\lambda{}_{\mu\lambda}}{\partial\left(\partial_\sigma h^a{}_\rho\right)} \\
&= -2\,\overset{\bullet}{T}{}^{\nu\mu}{}_\nu\,h_b{}^\lambda\,\frac{\partial \overset{\bullet}{T}{}^b{}_{\mu\lambda}}{\partial\left(\partial_\sigma h^a{}_\rho\right)} \\
&= -2\,\overset{\bullet}{T}{}^{\nu\mu}{}_\nu\,h_b{}^\lambda\,\delta^b_a\left(\delta^\sigma_\mu\delta^\rho_\lambda - \delta^\sigma_\lambda\delta^\rho_\mu\right) \\
&= -2\,\overset{\bullet}{T}{}^{\nu\sigma}{}_\nu\,h_a{}^\rho + 2\,\overset{\bullet}{T}{}^{\nu\rho}{}_\nu\,h_a{}^\sigma.
\end{aligned} \tag{A.9}
$$

Combining these results, the superpotential (A.5) is found to be

$$
\overset{\bullet}{S}{}_a{}^{\sigma\rho} = -\frac{1}{2}\left(\overset{\bullet}{T}{}_a{}^{\sigma\rho} + \overset{\bullet}{T}{}^{\rho\sigma}{}_a - \overset{\bullet}{T}{}^{\sigma\rho}{}_a - 2\,\overset{\bullet}{T}{}^{\nu\sigma}{}_\nu h_a{}^\rho + 2\,\overset{\bullet}{T}{}^{\nu\rho}{}_\nu h_a{}^\sigma\right), \tag{A.10}
$$

or equivalently,

$$
\overset{\bullet}{S}{}_a{}^{\sigma\rho} = \overset{\bullet}{K}{}^{\rho\sigma}{}_a + \overset{\bullet}{T}{}^{\nu\sigma}{}_\nu h_a{}^\rho - \overset{\bullet}{T}{}^{\nu\rho}{}_\nu h_a{}^\sigma, \tag{A.11}
$$

with

$$
\overset{\bullet}{K}{}^{\rho\sigma}{}_a = \tfrac{1}{2}\left(\overset{\bullet}{T}{}_a{}^{\rho\sigma} + \overset{\bullet}{T}{}^{\sigma\rho}{}_a - \overset{\bullet}{T}{}^{\rho\sigma}{}_a\right) \tag{A.12}
$$

the contortion tensor.

A.2 The energy-momentum current

Using the identity

$$\frac{\partial h}{\partial h^a{}_\rho} = h_a{}^\rho \, h, \tag{A.13}$$

the functional derivative of Eq. (A.1) with respect to $h^a{}_\rho$ can bw written in the form

$$\frac{\partial \dot{\mathcal{L}}}{\partial h^a{}_\rho} = \frac{h}{2k}\left(\frac{1}{4}\,\frac{\partial \dot{T}^c{}_{\mu\nu}}{\partial h^a{}_\rho}\,\dot{T}_c{}^{\mu\nu} + \frac{1}{4}\,\dot{T}^c{}_{\mu\nu}\frac{\partial \dot{T}_c{}^{\mu\nu}}{\partial h^a{}_\rho} + \frac{1}{4}\,\dot{T}^c{}_{\mu\nu}\frac{\partial \dot{T}^{\nu\mu}{}_c}{\partial h^a{}_\rho} + \frac{1}{2}\,\frac{\partial \dot{T}^c{}_{\mu\nu}}{\partial h^a{}_\rho}\,\dot{T}^{\nu\mu}{}_c \right.$$
$$\left. - \dot{T}_{\lambda\mu}{}^\lambda\,\frac{\partial \dot{T}^{\nu\mu}{}_\nu}{\partial h^a{}_\rho} - \frac{\partial \dot{T}_{\lambda\mu}{}^\lambda}{\partial h^a{}_\rho}\,\dot{T}^{\nu\mu}{}_\nu\right) + h_a{}^\rho \dot{\mathcal{L}}\,. \tag{A.14}$$

In what follows, we are going to use the properties

$$\frac{\partial h_c{}^\nu}{\partial h^a{}_\rho} = -h_a{}^\nu h_c{}^\rho \tag{A.15}$$

and

$$\frac{\partial g^{\mu\nu}}{\partial h^c{}_\rho} = -g^{\rho\nu} h_c{}^\mu - g^{\rho\mu} h_c{}^\nu. \tag{A.16}$$

From the torsion definition (A.2), the first functional derivative that appears in (A.14) is trivially calculated:

$$\frac{\partial \dot{T}^c{}_{\mu\nu}}{\partial h^a{}_\rho} = \dot{A}^c{}_{a\mu}\,\delta^\rho_\nu - \dot{A}^c{}_{a\nu}\,\delta^\rho_\mu. \tag{A.17}$$

The second kind of functional derivative that appears in (A.14) is

$$\begin{aligned}
\frac{\partial \dot{T}_c{}^{\mu\nu}}{\partial h^a{}_\rho} &= \eta_{cb}\frac{\partial}{\partial h^a{}_\rho}\left(g^{\mu\alpha}g^{\nu\beta}\dot{T}^b{}_{\alpha\beta}\right) \\
&= \left(\frac{\partial g^{\mu\alpha}}{\partial h^a{}_\rho}g^{\nu\beta} + g^{\mu\alpha}\frac{\partial g^{\nu\beta}}{\partial h^a{}_\rho}\right)\dot{T}_{c\alpha\beta} + \eta_{cb}\,g^{\mu\alpha}g^{\nu\beta}\frac{\partial \dot{T}^b{}_{\alpha\beta}}{\partial h^a{}_\rho} \\
&= \left(-g^{\nu\beta}g^{\alpha\rho}h_a{}^\mu - g^{\nu\beta}g^{\mu\rho}h_a{}^\alpha - g^{\mu\alpha}g^{\nu\rho}h_a{}^\beta - g^{\mu\alpha}g^{\beta\rho}h_a{}^\nu\right)\dot{T}_{c\alpha\beta} \\
&\qquad + \eta_{cb}\,g^{\mu\alpha}g^{\nu\beta}\left(\dot{A}^b{}_{a\alpha}\delta^\rho{}_\beta - \dot{A}^b{}_{a\beta}\delta^\rho{}_\alpha\right) \\
&= -h_a{}^\mu\dot{T}_c{}^{\rho\nu} - g^{\mu\rho}\dot{T}_{ca}{}^\nu - g^{\nu\rho}\dot{T}_c{}^\mu{}_a - h_a{}^\nu\dot{T}_c{}^{\mu\rho} \\
&\qquad + \eta_{cb}\,g^{\mu\alpha}g^{\nu\rho}\dot{A}^b{}_{a\alpha} - \eta_{cb}\,g^{\mu\rho}g^{\nu\beta}\dot{A}^b{}_{a\beta}.
\end{aligned}$$

The third kind of functional derivative is

$$
\begin{aligned}
\frac{\partial \dot{T}^{\nu\mu}{}_{c}}{\partial h^{a}{}_{\rho}} &= \eta_{cb}\, \frac{\partial}{\partial h^{a}{}_{\rho}} \left(h^{b}{}_{\sigma} \dot{T}^{\nu\mu\sigma} \right) \\
&= \eta_{ca}\, \dot{T}^{\nu\mu\rho} + \eta_{cb}\, h^{b}{}_{\sigma}\, \frac{\partial \dot{T}^{\nu\mu\sigma}}{\partial h^{a}{}_{\rho}},
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial \dot{T}^{\nu\mu\sigma}}{\partial h^{a}{}_{\rho}} &= \frac{\partial}{\partial h^{a}{}_{\rho}} \left(\eta^{bc} h_{b}{}^{\nu} \dot{T}_{c}{}^{\mu\sigma} \right) \\
&= -\eta^{bc} h_{a}{}^{\nu} h_{b}{}^{\rho} \dot{T}_{c}{}^{\mu\sigma} + \eta^{bc} h_{b}{}^{\nu}\, \frac{\partial \dot{T}_{c}{}^{\mu\sigma}}{\partial h^{a}{}_{\rho}} \\
&= -\, h_{a}{}^{\nu} \dot{T}^{\rho\mu\sigma} - \eta^{bc} h_{b}{}^{\nu} \left(h_{a}{}^{\mu} \dot{T}_{c}{}^{\rho\sigma} + g^{\mu\rho} \dot{T}_{ca}{}^{\sigma} + g^{\sigma\rho} \dot{T}_{c}{}^{\mu}{}_{a} + h_{a}{}^{\sigma} \dot{T}_{c}{}^{\mu\rho} \right) \\
&\qquad +\, \eta^{bc} h_{b}{}^{\nu} \eta_{cd}\, g^{\mu\alpha} g^{\sigma\rho} \dot{A}^{d}{}_{a\alpha} - \eta^{bc} h_{b}{}^{\nu} \eta_{cd}\, g^{\mu\rho} g^{\sigma\beta} \dot{A}^{d}{}_{a\beta} \\
&= -\, h_{a}{}^{\nu} \dot{T}^{\rho\mu\sigma} - h_{a}{}^{\mu} \dot{T}^{\nu\rho\sigma} - g^{\mu\rho} \dot{T}^{\nu}{}_{a}{}^{\sigma} - g^{\sigma\rho} \dot{T}^{\nu\mu}{}_{a} - h_{a}{}^{\sigma} \dot{T}^{\nu\mu\rho} \\
&\qquad +\, h_{b}{}^{\nu} g^{\mu\alpha} g^{\sigma\rho} \dot{A}^{b}{}_{a\alpha} - h_{b}{}^{\nu} g^{\mu\rho} g^{\sigma\beta} \dot{A}^{b}{}_{a\beta}.
\end{aligned}
\tag{A.18}
$$

The fourth kind of derivative that appears in (A.14) can be calculated with help of (A.18). It is given by

$$
\begin{aligned}
\frac{\partial \dot{T}^{\nu\mu}{}_{\nu}}{\partial h^{a}{}_{\rho}} &= \frac{\partial}{\partial h^{a}{}_{\rho}} \left(g_{\nu\sigma} \dot{T}^{\nu\mu\sigma} \right) \\
&= \frac{\partial g_{\nu\sigma}}{\partial h^{a}{}_{\rho}}\, \dot{T}^{\nu\mu\sigma} + g_{\nu\sigma} \frac{\partial \dot{T}^{\nu\mu\sigma}}{\partial h^{a}{}_{\rho}} \\
&= -\dot{T}^{\rho\mu}{}_{a} - h_{a}{}^{\mu} \dot{T}^{\nu\rho}{}_{\nu} - g^{\mu\rho} \dot{T}^{\nu}{}_{a\nu} + h_{b}{}^{\rho} g^{\mu\nu} \dot{A}^{b}{}_{a\nu} - h_{b}{}^{\nu} g^{\mu\rho} \dot{A}^{b}{}_{a\nu}.
\end{aligned}
$$

Finally, the fifth and last functional derivative that appears in (A.14) is

$$
\begin{aligned}
\frac{\partial \dot{T}_{\lambda\mu}{}^{\lambda}}{\partial h^{a}{}_{\rho}} &= \frac{\partial}{\partial h^{a}{}_{\rho}} \left(h_{c}{}^{\lambda} \dot{T}^{c}{}_{\mu\lambda} \right) \\
&= -h_{a}{}^{\lambda} h_{c}{}^{\rho} \dot{T}^{c}{}_{\mu\lambda} + h_{c}{}^{\lambda} (\dot{A}^{c}{}_{a\mu} \delta^{\rho}{}_{\lambda} - \dot{A}^{c}{}_{a\lambda} \delta^{\rho}{}_{\mu}) \\
&= -\, \dot{T}^{\rho}{}_{\mu a} + h_{c}{}^{\rho} \dot{A}^{c}{}_{a\mu} - h_{c}{}^{\lambda} \dot{A}^{c}{}_{a\lambda} \delta^{\rho}{}_{\mu}.
\end{aligned}
\tag{A.19}
$$

Considering all results above, after some algebraic manipulations, the functional derivative (A.14) is found to be

$$
\frac{\partial \dot{\mathcal{L}}}{\partial h^{a}{}_{\rho}} = \frac{h}{k}\, h_{c}{}^{\sigma} \dot{T}^{c}{}_{\nu a}\, \dot{S}_{\sigma}{}^{\rho\nu} + h_{a}{}^{\rho} \dot{\mathcal{L}} + \frac{h}{k} \dot{A}^{c}{}_{a\nu} \dot{S}_{c}{}^{\nu\rho}. \tag{A.20}
$$

The energy-momentum current (A.6) is, consequently, given by

$$\stackrel{\bullet}{\jmath}_a{}^{\rho} = \frac{1}{k}\, h_c{}^{\sigma}\, \stackrel{\bullet}{T}{}^{c}{}_{\nu a}\, \stackrel{\bullet}{S}_{\sigma}{}^{\nu\rho} - \frac{h_a{}^{\rho}}{h}\, \stackrel{\bullet}{\mathcal{L}} - \frac{1}{k} A^{c}{}_{a\nu}\, \stackrel{\bullet}{S}_{c}{}^{\nu\rho}. \tag{A.21}$$

Appendix B

Dirac Equation

B.1 Relativistic Fields

Let us start by recalling that a field is — by definition — a relativistic field if it belongs to some representation of the Poincaré group [179], the semi–direct product of the Lorentz group by the group of translations on Minkowski spacetime: $\mathcal{P} = \mathcal{L} \oslash \mathcal{T}$. The Poincaré group $\mathcal{P}$ is the group of isometries (or motions) on Minkowski, which means that its transformations preserve the Lorentz metric η_{ab}.

The Lorentz generators S^{ab} and the translation generators P^a constitute a base for the Lie algebra of $\mathcal{P}$, and obey the basic commutation rules

$$\begin{aligned} \left[P^a, P^b\right] &= 0 && \text{(B.1)} \\ \left[P^c, S^{ab}\right] &= i\left(P^a\,\eta^{bc} - P^b\,\eta^{ac}\right) && \text{(B.2)} \\ \left[S^{ab}, S^{de}\right] &= i\left(S^{ad}\,\eta^{be} + S^{be}\,\eta^{ad} - S^{bd}\,\eta^{ae} - S^{ae}\,\eta^{bd}\right). && \text{(B.3)} \end{aligned}$$

Under a Lorentz transformation, in particular, a field will transform by the action of a representation $U(\Lambda)$, given as

$$\psi'(x') = U(\Lambda)\,\psi(x) = \exp\left[-\frac{i}{2}\,\omega_{ab}\,S^{ab}\right]\psi(x), \qquad \text{(B.4)}$$

with ω_{ab} the transformation (rotation, boost) parameters.

A carrier space is any space on which the transformations take place. The set of operators *representing* the group Lie algebra on a given carrier space is a *representation*. The Poincaré group $\mathcal{P}$ is a group of rank two. This means that there are at most two operators which commute with all its generators and are, consequently, invariant under the transformations they generate. Of course, a function of invariants is itself invariant, so that it is possible to choose the invariant operators which have the most direct physical meaning. A fundamental result is that the eigenvalues of the two invariant operators classify all the representations: members (vectors, functions) of the space carrying a representation are transformed into each other, so that those eigenvalues are kept the same.

In Field Theory, representations are characterized, or classified, by the eigenvalues of two chosen invariant operators:

$$\eta_{ab}\, P^a P^b = m^2 c^2,$$

with m the rest mass of the particle with four–momentum P^a, and

$$\eta_{ab}\, W^a W^b = -\, m^2 c^2\, s(s+1),$$

where s is the spin and W^a is the Pauli–Lubanski operator

$$W_d = -\tfrac{1}{2}\, \epsilon_{abcd}\, S^{ab} P^c. \tag{B.5}$$

Every relativistic field must have fixed values of $P_a P^a$ and $W_a W^a$, that is, well–defined values of mass and spin. Particles appear in field theory as the quanta of the fundamental fields. Such fields are, before quantization, wave-functions $\psi(\vec{x}, t)$ representing the states of a system in Quantum Mechanics. They are fields in the sense that $\psi(\vec{x}, t)$ stands for a continuous infinity of possible values, one at each point of spacetime.

Comment B.1 Actually, it is the covering $SL(2, C)$ of $\mathcal{L} = SO(3,1)$ which is at work, but we shall not go into these details. Let us only say that the covering is necessary to include half-integer spins. This generalizes the case of the group of usual rotations in ordinary euclidean 3-dimensional space, which is isomorphic to $SO(3)$, the group of 3×3 orthogonal matrices with determinant $= +1$. This group has rank one: it has only one invariant, which is chosen to be

$$J^2 = J_x^2 + J_y^2 + J_z^2,$$

whose eigenvalues are $j(j+1)$, with j an integer. Each value of this invariant characterizes a representation. Its covering, the group $SU(2)$ of unitary 2×2 matrices with determinant $= +1$, has all the representations of $SO(3)$ *plus* many others, characterized by $J^2 = j(j+1)$, with j a half-integer. An example of representation of $SU(2)$, which is not a representation of $SO(3)$, is the $SU(2)$ fundamental (that is, lowest-dimensional, $j = \frac{1}{2}$) representation, generated by $\frac{1}{2}\sigma_a$, with σ_a the 2×2 Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.6}$$

B.2 Dirac Fields

In the non–relativistic case, time evolution is ruled by the prototype of wave equation, the Schrödinger equation. It is obtained from Classical Mechanics through the so-called quantization rules, by which classical quantities become operators acting on $\psi(\vec{x}, t)$. Depending on the "representation", some quantities become differential operators and other are given by a simple product. As it is, $\psi(\vec{x}, t)$ corresponds to the configuration–space representation, in which $\vec{x}$ is the operator acting on $\psi(\vec{x}, t)$ according to

$$\psi(\vec{x}, t) \to \vec{x}\ \psi(\vec{x}, t).$$

The Hamiltonian and the 3-momenta, on the other hand, are given respectively by

$$H \to i\ \hbar\ \frac{\partial}{\partial t} \tag{B.7}$$

and

$$\vec{p} \to \frac{\hbar}{i}\ \vec{\nabla}. \tag{B.8}$$

In the case of a free particle, for which $H = \vec{p}^2/2m$, these rules lead to the free Schrödinger equation

$$i\hbar\ \frac{\partial\psi(\vec{x}, t)}{\partial t} = -\ \frac{\hbar^2}{2m}\ \vec{\nabla}^2\ \psi(\vec{x}, t)\ . \tag{B.9}$$

Now, the Schrödinger equation is the non–relativistic limit of the Klein–Gordon equation, which we write in the form

$$-\ \hbar^2\ \frac{\partial^2\psi(\vec{x}, t)}{\partial t^2} = -\ \hbar^2 c^2\ \vec{\nabla}^2\psi(\vec{x}, t) + m^2c^4\psi(\vec{x}, t). \tag{B.10}$$

Every relativistic field must obey this equation, because it corresponds to the expression

$$H^2 = \vec{p}^2c^2 + m^2c^4,$$

which is compulsory because it says simply that the field is an eigenstate of the Poincaré group invariant operator

$$P_aP^a = H^2/c^2 - \vec{p}^2.$$

with eigenvalue m^2c^2. A field corresponding to a particle of mass m must satisfy this condition. Of course, once we use H^2, we shall be introducing negative energy solutions for a free system: there is no reason to exclude

$$H = -\ \sqrt{\mathbf{p}^2c^2 + m^2c^4}.$$

Following Dirac, we can look for another way to "extract the square root" of the operator H^2. In other words, we look for a linear, first-order equation both in t and $\mathbf{x}$. We write [180]

$$\sqrt{\mathbf{p}^2c^2 + m^2c^4} = c\,\vec{\alpha}\cdot\vec{p} \;+\; \beta mc^2, \tag{B.11}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β are constants to be found. Taking the square, we arrive at the conditions

$$\begin{aligned}
&{\alpha_1}^2 = {\alpha_2}^2 = {\alpha_3}^2 = \beta^2 = 1 \\
&\alpha_k\,\beta + \beta\,\alpha_k = 0, \quad \text{for } k = 1,2,3 \\
&\alpha_i\alpha_j + \alpha_j\alpha_i = 0, \quad \text{for } i,j = 1,2,3 \quad \text{with} \quad i \neq j.
\end{aligned} \tag{B.12}$$

These conditions cannot be met if α_k and β are real or complex numbers, but can be satisfied if they are matrices. In that case, as the equation corresponding to (B.11) is the matrix equation

$$H\psi(\vec{x},t) = i\hbar\,\frac{\partial\psi(\vec{x},t)}{\partial t} = \frac{\hbar}{i}\,c\,\vec{\alpha}\cdot\vec{\nabla}\psi(\vec{x},t) + m\,c^2\,\beta\,\psi(\vec{x},t), \tag{B.13}$$

the wavefunction will be necessarily a column-vector, on which the matrices act. Notice that, once conditions (B.12) are satisfied, ψ will also obey the mandatory Klein-Gordon equation. We must thus look at (B.13) as an equation involving four matrices (complex, $n \times n$ for the time being) and the n-vector ψ. As H should be hermitian, so should α_k and β be:

$$\alpha_k^\dagger = \alpha_k \qquad \beta^\dagger = \beta.$$

It turns out that the minimum value of n necessary to have four matrices that are hermitian, independent and distinct from the identity, is $n = 4$. Thus, α_k and β will be 4×4 matrices. The four components ψ correspond to particles and antiparticles with spins components $+1/2$ and $-1/2$ (whence the name "bispinor representation").

B.3 Covariant Form of the Dirac Equation

Equation (B.13) is the Hamiltonian form of the Dirac equation, in which time and space play distinct roles. To go into the so-called covariant form, we first define new matrices, the celebrated Dirac's gamma matrices, as

$$\gamma^0 = \beta \quad \text{and} \quad \gamma^i = \beta\alpha^i. \tag{B.14}$$

In terms of the gamma matrices, conditions (B.12) acquire the compact form,

$$\gamma^a \gamma^b + \gamma^b \gamma^a = \{\gamma^a, \gamma^b\} = 2\,\eta^{ab}\, I, \tag{B.15}$$

where I stands for the unit 4×4 matrix. In the Pauli–Dirac representation, the γ's have the forms

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{B.16}$$

in which σ_i are the Pauli matrices (B.6). Any other set of matrices γ^a obtained from those by a similarity transformation is equally acceptable [180, 181]. We shall here only use the above representation.

The commutator of gamma matrices

$$\sigma_{ab} = \tfrac{i}{2}\,(\gamma_a \gamma_b - \gamma_b \gamma_a) = \tfrac{i}{2}\,[\gamma_a, \gamma_b] \tag{B.17}$$

has an important role. In effect, these σ_{ab} are such that

$$\left[\tfrac{1}{2}\,\sigma_{ab}, \tfrac{1}{2}\,\sigma_{cd}\right] = i\left(\eta_{bc}\,\tfrac{1}{2}\,\sigma_{ad} - \eta_{ac}\,\tfrac{1}{2}\,\sigma_{bd} + \eta_{ad}\,\tfrac{1}{2}\,\sigma_{bc} - \eta_{bd}\,\tfrac{1}{2}\,\sigma_{ac}\right). \tag{B.18}$$

Comparison with (B.3) shows that

$$S_{ab} = \tfrac{1}{2}\,\sigma_{ab}$$

is a Lorentz generator. Just as each matrix S_{ab} of Eq.(1.25) is a generator of the vector ($j = 1$) representation of the Lie algebra of the Lorentz group, each matrix $\sigma_{ab}/2$ is a generator of the spinor ($j = 1/2$) representation of the Lie algebra of the Lorentz group. In fact, it is found that the covariance of the Dirac equation under the transformation

$$\psi'(x') = U(\Lambda)\psi(x) = \exp\left[-\tfrac{i}{4}\,\omega^{ab}\,\sigma_{ab}\right]\psi(x) \tag{B.19}$$

requires that the Dirac field $\psi(x)$ belong to this bispinor representation generated by $\sigma_{ab}/2$. It is clear, however, that the particular form of the matrices σ_{ab} depend on the "representation" we are using for the matrices γ. In the representation (B.16) the σ_{ab} are particularly simple:

$$\sigma_{ij} = \begin{pmatrix} \epsilon_{ijk}\sigma_k & 0 \\ 0 & \epsilon_{ijk}\sigma_k \end{pmatrix} \quad \sigma^{0i} = \begin{pmatrix} 0 & i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}. \tag{B.20}$$

Currents of Dirac fields have the general form

$$\bar{\psi}(x)\,\hat{\mathcal{O}}\psi(x),$$

where

$$\bar{\psi}(x) = \psi^{\dagger}(x)\,\gamma_0 \tag{B.21}$$

is the adjoint wavefunction, and $\hat{\mathcal{O}}$ is an operator. Such expressions, called bilinear forms, are the only combinations that can appear in lagrangians. Only currents with a well-defined behavior under a Lorentz transformation are acceptable. From (B.19) it follows that

$$\bar{\psi}'(x')\,\hat{\mathcal{O}}\,\psi'(x') = \bar{\psi}(x)\,U^{-1}\,\hat{\mathcal{O}}U\psi(x)$$

This means that the behavior of a current is fixed by the behavior of the operator $\hat{\mathcal{O}}$ itself. It is possible to choose as a basis for the 4×4 matrices a set of matrices, *each one with a well–known behavior.* Such a set is formed by the following 16 matrices:

$$\begin{aligned}
&\Gamma^S = I \\
&\Gamma^V{}_a = \gamma_a \\
&\Gamma^T{}_{ab} = \sigma_{ab} \\
&\Gamma^P = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 = \gamma_5 \\
&\Gamma^A{}_a = \gamma^5\,\gamma_a,
\end{aligned}$$

where we have introduced the notation

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{B.22}$$

The superscript letter denote the corresponding behavior: scalar, vector, tensor, pseudoscalar and axial. As matrices Γ form a basis, any operator $\hat{\mathcal{O}}$ will have the form

$$\hat{\mathcal{O}} = \sum_{n=1}^{16} c_n\,\Gamma_n,$$

and only expressions of type

$$\bar{\psi}(x)\hat{\mathcal{O}}\psi(x) = \sum_{n=1}^{16} c_n\,\bar{\psi}(x)\,\Gamma_n\,\psi(x)$$

can appear in lagrangians.

Mltiplying the Dirac equation (B.13) by β/c on the left, after some algebraic manipulation we find

$$i\hbar\,\gamma^a\partial_a\psi(x) - mc\psi(x) = 0. \tag{B.23}$$

This is the covariant form of the Dirac equation. It follows from the lagrangian

$$\mathcal{L} = \tfrac{i}{2}\,\hbar c\left[\bar{\psi}\,\gamma^a\;\partial_a\psi - (\partial_a\bar{\psi})\,\gamma^a\,\psi\right] - mc^2\;\bar{\psi}\,\psi. \tag{B.24}$$

The probability current, on the other hand, is of the form

$$j^\mu(x) = \bar{\psi}(x)\,c\gamma^\mu\psi(x). \tag{B.25}$$

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