Introduction to Superstring Theory

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Preface

These lecture notes are the written version of four lectures delivered at the 3rd ICTP Latin-American String School held in Sao Paulo, Brazil, 1-19 December 2003. They are an introduction to the NSR formulation of superstring theory.

The first lecture introduces the superstring action and analyzes its symmetries, the equations of motion, boundary conditions and mode expansions. Three different quantization methods are briefly discussed in the second lecture where it is indicated how the critical dimension arises and how the spectrum is constructed in the old covariant quantization. The physical states in both the NS and R sectors are analyzed and the GSO projection is discussed. Interactions are the subject of the third lecture where the path integral quantization is sketched and vertex operators are constructed. Bosonization and picture changing are briefly discussed. Finally the last lecture deals with the one-loop partition function and modular invariance. It is shown how the GSO projection arises as the geometrical requirement of modular invariance.

These notes are not self-contained; they heavily rely on many concepts and techniques that were discussed in previous lectures on the bosonic string theory by Luis Alvarez-Gaume. Furthermore I have not attempted to be original. Rather, this is a summary of more extensive and detailed material that can be found in books and reviews. Indispensable general reference are the books by Green, Schwarz and Witten: *Superstring Theory*, Vol. 1 and 2 and by Polchinski: *String Theory*, Vol. 1 and 2. Since I have taken parts from different references the notation might not be consistent throughout the lectures and moreover there could be several mistakes especially in signs and factors. I would appreciate if they were pointed out to me as well as any comments or suggestions you might have.

I have benefitted greatly from discussions with D. Hofman whom I am pleased to acknowledge also for elaborating part of the material contained in these notes.

1 Overview

The bosonic string theory, despite all its beautiful features, has a number of shortcomings. The most obvious of these are the absence of fermions and the presence of tachyons in a spacetime with 26 dimensions.

The tachyon is not an actual physical inconsistency; it indicates at least that the calculations are being performed in an unstable vacuum state. But tachyon exchange contributes infrared divergences in loop diagrams and these divergences make it hard to isolate the ultraviolet behaviour of the effective quantum field theory the bosonic string theory gives rise to and to determine whether it is really satisfactory.

Historically, the solution to the tachyon problem appeared with the solution to the other problem, the absence of fermions. The addition of a new ingredient, supersymmetry on the world-sheet, improves substantially the general picture. In 1977 Gliozzi, Scherk and Olive showed that it was possible to get a model with no tachyons and with equal masses and multiplicities for bosons and fermions. In 1980, Green and Schwarz proved that this model had spacetime supersymmetry. In the completely consistent tachyon free form of the superstring theory it was then possible to show that the one-loop diagrams were completely finite and free of ultraviolet divergences.

It is important to keep in mind the distinction between **world-sheet** and **spacetime supersymmetry**. Whether a particular string theory is spacetime supersymmetric or not will manifest itself, for instance, in the spectrum. Especially, the existence of one or more massless gravitinos will signal spacetime supersymmetry. The formulation of the fermionic string theories which we will present in these lectures is the Ramond, Neveu, Schwarz spinning string. It has manifest world-sheet supersymmetry; spacetime supersymmetry, if present, is not manifest. A string spectrum with spacetime supersymmetry is obtained after a suitable truncation as it was found by Gliozzi, Scherk and Olive. There exists also the so-called Green-Schwarz formalism in which spacetime supersymmetry is manifest at the cost of world-sheet supersymmetry. This formulation will be discussed by Nathan Berkovits at the end of the week.

The particular string theory we are going to describe is based on the introduction of a world-sheet supersymmetry that relates the spacetime coordinates $X^\mu(\sigma,\tau)$ to fermionic partners $\Psi^\mu(\sigma,\tau)$, which are two component world-sheet spinors. We will see that an action principle with N=1 supersymmetry gives rise to a consistent string theory with critical dimension $D=10$. Truncating the spectrum as proposed by GSO gives supersymmetry in $D=10$ spacetime with one or two Majorana-Weyl supercharges (N=1 or N=2) depending on the choice of boundary conditions. This reduction of the critical dimension from $D=26$ in the bosonic string theory to $D=10$ in the superstring is another nice feature of the theory.

It is natural to ask if more supersymmetries can be added on the world-sheet. It turns out that the quantum consistency of the theory with N=2 world-sheet supersymmetries requires $D = 2$ spacetime dimensions and with $N = 4$ supersymmetries leads to the unpleasent requirement of negative spacetime dimensionality.

1.1 The action and its symmetries

In bosonic string theory, the mass shell condition

$$p_\mu p^\mu + m^2 = 0 \tag{1}$$

where $\mu = 0, ..., D-1$, came from the physical state condition

$$L_0|phys>=|phys> \tag{2}$$

(and also $\tilde{L}_0|phys>=|phys>$ in the closed string).

The mass shell condition is the Klein Gordon equation in momentum space. To get spacetime fermions we need the Dirac equation

$$ip_\mu\Gamma^\mu + m = 0, \tag{3}$$

where the gamma matrices satisfy the algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{4}$$

($\eta^{\mu\nu}$ is the spacetime metric which we take with signature $(-++...+)$).

L_0 and $\tilde{L}_0$ are the center-of-mass modes of the world-sheet energy momentum tensor ($T_B, \tilde{T}_B$, where B refers to bosonic). It seems that we need now conserved quantities T_F and $\tilde{T}_F$, whose center of mass modes give the Dirac equation and which play the same role as $T_B, \tilde{T}_B$ in the bosonic string theory. Noting further that the spacetime momenta p^μ are the center-of-mass modes of the world-sheet currents $(\partial X^\mu, \bar{\partial} X^\mu)$, it is natural to guess that the gamma matrices are the center-of-mass modes of an anticommuting world-sheet field Ψ^μ.

With this in mind, let us recall the world-sheet action for the bosonic string, in conformal gauge,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \partial_a X^\mu \partial^a X_\mu \tag{5}$$

This is a free field theory in two dimensions σ^a, $a = 0, 1$. $X^\mu(\sigma^1, \sigma^2)$, $\mu = 0, 1, ..., D-1$ are spacetime coordinates for a string that is propagating in D flat spacetime dimensions.

An obvious generalization of (5) would be the following Euclidean action

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \left(\partial_a X^\mu \partial^a X_\mu - i\alpha'\bar{\Psi}^\mu\gamma^a\partial_a\Psi_\mu\right) \tag{6}$$

where Ψ^μ are a set of Majorana (real) spinors, that in two dimensions have two components. These world-sheet fermions would correspond physically to internal degrees of freedom that are free to propagate along the string. Notice that Ψ^μ are spacetime vectors (whereas the index μ), *i.e.* they transform in the vector representation of the Lorentz group $SO(D-1,1)$.

For a Majorana spinor

$$\bar{\Psi} = \Psi^\dagger \gamma^0 = \Psi^t \gamma^0. \tag{7}$$

The two dimensional Dirac matrices γ^a satisfy the standard anticommutation relations

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\delta^{ab} \quad . \tag{8}$$

A convenient representation is

$$\gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad ; \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad . \tag{9}$$

The action is invariant under the following transformations:

$$\delta X^\mu = \bar{\epsilon} \Psi^\mu \quad ; \quad \delta \Psi^\mu = -\frac{i}{\alpha'} \gamma^a \partial_a X^\mu \epsilon \quad , \tag{10}$$

where the parameter ϵ is again a Majorana spinor. These transformations mix bosonic and fermionic coordinates and are known as supersymmetry transformations.

Most string computations are performed in the complex two dimensional plane. Therefore it is convenient to introduce complex coordinates w and $\bar{w}$ as

$$w = \sigma + i\tau \quad , \quad \bar{w} = \sigma - i\tau \tag{11}$$

Writing the two component spinor as

$$\Psi = \begin{pmatrix} \tilde{\psi} \\ \psi \end{pmatrix} \tag{12}$$

and noticing that

$$\partial = \partial_w = \frac{1}{2}(\partial_{\sigma^1} - i\partial_{\sigma^2}) \quad , \quad \bar{\partial} = \partial_{\bar{w}} = \frac{1}{2}(\partial_{\sigma^1} + i\partial_{\sigma^2}), \tag{13}$$

the action becomes

$$\frac{1}{2\pi\alpha'} \int d^2 w \left[\partial X^\mu \bar{\partial} X_\mu + \frac{\alpha'}{2} \left(\psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi} \partial \tilde{\psi} \right) \right] \quad . \tag{14}$$

For the indices σ^1, σ^2 the metric is the identity and we do not distinguish between upper and lower, while complex indices are raised and lowered with

$$g_{w\bar{w}} = g_{\bar{w}w} = \frac{1}{2} \quad g_{ww} = g_{\bar{w}\bar{w}} = 0 \quad g^{w\bar{w}} = g^{\bar{w}w} = 2 \quad g^{ww} = g^{\bar{w}\bar{w}} = 0 \tag{15}$$

Note also that

$$d^2w = 2d\sigma^1 d\sigma^2 \tag{16}$$

with the factor 2 from the Jacobian. Moreover $d^2w\sqrt{g} = \frac{1}{2}d^2w = d\sigma^1 d\sigma^2$.

Under the Minkowski continuation $\sigma^2 \to i\sigma^2$, a holomorphic field becomes a function only of $\sigma^2 - \sigma^1$ and an antiholomorphic field a function only of $\sigma^2 + \sigma^1$. We thus use as synonyms

$$\begin{aligned} \text{holomorphic} &= \text{left} - \text{moving} && (17) \\ \text{antiholomorphic} &= \text{right} - \text{moving} && (18) \end{aligned}$$

The SUSY transformations can be rewritten as

$$\begin{aligned} &\delta_+ X^\mu = \epsilon_+ \psi^\mu \quad ; \quad \delta_+ \psi^\mu = -\frac{2}{\alpha'} \partial X^\mu \epsilon_+ \quad ; \quad \delta_+ \tilde{\psi}^\mu = 0 \quad ; && (19) \\ &\delta_- X^\mu = \epsilon_- \tilde{\psi}^\mu \quad ; \quad \delta_- \psi^\mu = 0 \quad ; \quad \delta_- \tilde{\psi}^\mu = \frac{2}{\alpha'} \bar{\partial} X^\mu \epsilon_- \quad ; && (20) \end{aligned}$$

where $\delta_\pm$ refer to the two SUSY transformations under the two components of ϵ:

$$\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}. \tag{21}$$

Notice that the classical equations of motion are $\partial\tilde{\psi} = 0$ and $\bar{\partial}\psi = 0$ and their solutions are any holomorphic function $\psi(w)$ and any antiholomorphic function $\tilde{\psi}(\bar{w})$. They have to be supplemented with boundary conditions corresponding to the open and closed string. Moreover, the SUSY $\pm$ transformations are effectively decoupled from each other. The $+$ transformations involve ψ and ∂X whereas the $-$ transformations involve $\tilde{\psi}$ and $\bar{\partial}X$. This important property is at the root of the construction of the heterotic string theories, allowing the possibility of an action with a supersymmetric holomorphic sector (invariant under δ_+ transformations) and a bosonic non-supersymmetric antiholomorphic sector. These $N = 1$ SUSY transformations are sometimes denoted $N = (1, 1)$ in order to better clarify the existence of the two distinct transformations. Theories with $N = (1, 0)$ SUSY, such as the heterotic string, are sometimes denoted $N = 1/2$ theories.

As the bosonic string, the action (6) is invariant under reparametrizations of the world-sheet coordinates σ^a. It is also conformally invariant, so in all it is an $N = 1$

superconformal field theory (SCFT). One can explicitly verify this after computing the energy-momentum tensor T_B, for instance applying Noether's method. For a transformation of the form

$$w \to w + \xi(w, \bar{w}) \quad , \quad \bar{w} \to \bar{w} \tag{22}$$

we get

$$\delta S = \frac{1}{2\pi} \int d^2w \; \bar{\partial}\xi \; T_B(w) \quad . \tag{23}$$

Noting that

$$\delta(\partial X) = -(\partial\xi)(\partial X) \quad ; \quad \delta(\bar{\partial} X) = -(\bar{\partial}\xi)(\partial X) \tag{24}$$

$$\delta\psi = -\frac{1}{2}(\partial\xi)\psi \quad ; \quad \delta\tilde{\psi} = 0 \tag{25}$$

$$\delta(\partial) = -(\partial\xi)\partial \quad ; \quad \delta(\bar{\partial}) = -(\bar{\partial}\xi)\partial \tag{26}$$

and integrating by parts it is straightforward to compute δS. The final result for T_B, the generator of conformal transformations, is

$$T_B(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu \tag{27}$$

and similarly its antiholomorphic $\tilde{T}_B$.

Since T_B is holomorphic, it is automatically conserved: $\bar{\partial} T_B = 0$. Therefore the product of T_B with any holomorphic function, *e.g.* $\xi(w)T_B(w)$ is also conserved, and thus the conformal group is infinite dimensional.

Before proceeding further let us review some facts about conformal field theory. The first thing we should notice is that there are two possible choices of complex coordinates:

1) $w = \sigma^1 + i\sigma^2$.

If we are interested in the closed string, where the coordinate σ^1 is periodic, *i.e.* $\sigma^1 \sim \sigma^1 + 2\pi$ and we take σ^2 as the euclidean time $-\infty < \sigma^2 < \infty$, this defines a cylinder in the w plane where $w \sim w + 2\pi$ (see figure).

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2) $z = e^{-iw} = e^{-i\sigma^1 + \sigma^2}$

This is the map from the cylinder to the plane. The equal time lines in the cylinder are now radial lines. The infinite past $\sigma^2 = -\infty$ is mapped to the origin and similarly the infinite future goes to the point at infinity in the complex plane.

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Fourier expansions in the cylinder (w coordinates) corresponds to Laurent expansions in the plane (z coordinates). Indeed any holomorphic or antiholomorphic operator can be Laurent expanded. Transforming coordinates from the cylinder to the plane, conformal fields transform as

$$\phi(z)_{plane} = z^{-h}\phi(w(z))_{cylinder} \tag{28}$$

Therefore if $\phi(w)_{cylinder}$ has mode expansion

$$\phi(w)_{cylinder} = \sum_{n\in\mathbf{Z}} \phi_n e^{-nw} = \sum_n \phi_n z^{-n} \tag{29}$$

the mode expansion on the complex plane is

$$\phi(z)_{plane} = \sum_{n\in\mathbf{Z}} z^{-n-h}\phi_n \tag{30}$$

with the same coefficients ϕ_n, which can be found inverting this expression as

$$\phi_n = \oint_{C_0} \frac{dz}{2\pi i}\phi(z) z^{n+h-1} \tag{31}$$

where the contour integral must be taken counterclockwise around the origin.

In particular the energy-momentum tensor has the following Laurent expansion

$$T_B(z) = T_{zz}(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \quad , \quad \tilde{T}_B(\bar{z}) = \tilde{T}_{\bar{z}\bar{z}}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}} \tag{32}$$

and $L_m, \tilde{L}_m$ are the Virasoro generators, *i.e.*

$$L_m = \oint_{C_0} \frac{dz}{2\pi i} z^{m+2} T_{zz} \tag{33}$$

and similarly for $\tilde{L}_m$.

With each current we associate a conserved charge

$$T_\xi = \oint \frac{dw}{2\pi i}\xi(w) T_B(w) \tag{34}$$

which generates conformal transformations

$$w \to w' = w + \xi(w) \tag{35}$$

For instance, infinitesimally the transformation (**??**) reads

$$\delta_{\xi\bar{\xi}}\phi(w,\bar{w}) = (h\partial\xi + \tilde{h}\bar{\partial}\tilde{\xi} + \xi\partial + \tilde{\xi}\bar{\partial})\phi(w,\bar{w}) \tag{36}$$

and it is implemented by the commutator of $\phi(w)$ and $T_\xi(w)$ as

$$\delta_\xi\phi(w) = [T_\xi, \phi(w)]\,. \tag{37}$$

Using the prescription of radial ordering:

$$R(\phi_1(z)\phi_2(w)) = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w| \\ \phi_2(w)\phi_1(z) & |z| < |w| \end{cases} \tag{38}$$

equal radius commutators are defined as

$$[\phi_1(z), \phi_2(w)]_{|z|=|w|} = \lim_{\delta\to 0}\left\{[\phi_1(z)\phi_2(w)]_{|z|=|w|+\delta} - [\phi_2(w)\phi_1(z)]_{|z|=|w|-\delta}\right\} \tag{39}$$

Therefore

$$\begin{aligned} \delta_\xi\phi(w) &= \oint_{C_0;|z|>|w|}\frac{dz}{2\pi i}\xi(z)T(z)\phi(w) - \oint_{C_0;|z|<|w|}\frac{dz}{2\pi i}\xi(z)T(z)\phi(w) = \\ &= \oint_{C_w}\frac{dz}{2\pi i}\xi(z)T(z)\phi(w) \end{aligned} \tag{40}$$

(see the figure).

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Thus from (36) and using Cauchy-Riemann equation (one of the most famous mathematical formulae :)) we find that any conformal field must have the following OPE with T_B

$$T_B(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + finite\ terms \tag{41}$$

These OPEs must always be thought of as inserted into correlation functions.

Using this relation one can easily find out that the conformal dimension of $\partial X(\bar{\partial} X)$ is $h=1(\tilde{h}=1)$ and

$$h(\psi)=\tilde{h}(\tilde{\psi})=\frac{1}{2} \quad , \quad \tilde{h}(\psi)=h(\tilde{\psi})=0 \quad . \tag{42}$$

We have learned that radial quantization allows to identify *commutators* with *OPE*s. We can now examine the conformal transformation properties of T_B itself.

Exercise 1: Using

$$[\delta_{\xi_1},\delta_{\xi_2}]=\delta_{(\xi_1\partial\xi_2-\xi_2\partial\xi_1)} \tag{43}$$

find the OPE

$$T_B(z)T_B(w)=\frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{(z-w)}+finite\ terms \tag{44}$$

This is equivalent to the Virasoro algebra with central charge c. Indeed using (32)and (32) one can work out the Virasoro algebra.

Exercise 2: From the OPE (44) obtain

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}m(m^2-1)\delta_{m,-n} \tag{45}$$

Let's go back to the superstring. The action (6) posseses a new symmetry whose conserved currents are the *world sheet supercurrents*

$$T_F(z)=i\sqrt{\frac{2}{\alpha'}}\psi^\mu(z)\partial X_\mu(z) \quad , \quad \tilde{T}_F(\bar{z})=i\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^\mu(\bar{z})\bar{\partial} X_\mu(\bar{z}) \tag{46}$$

These are respectively holomorphic and antiholomorphic, since they are just products of (anti)holomorphic fields, and therefore they are automatically conserved. Normal order is implicit througout these lectures.

Exercise 3: Compute the generator of superconformal transformations $T_F(z)$ from the SUSY transformations (20) with $\epsilon_\pm=\epsilon_\pm(z,\bar{z})$.

This gives the desired result: the zero modes ψ_0^μ and $\tilde{\psi}_0^\mu$ will satisfy the gamma matrix algebra and the centers of mass of T_F and $\tilde{T}_F$ will have the form of Dirac operators. We will see that the resulting string theory has spacetime fermions as well as bosons and that the tachyon is gone after performing the GSO projection.

The conformal and superconformal transformations close to form the *superconformal algebra*. In terms of currents, this means that the OPEs of T_F and T_B close, that is, only T_B and T_F appear in the singular terms:

$$T_B(z)T_B(0)\sim\frac{3D}{4z^4}+\frac{2}{z^2}T_B(0)+\frac{1}{z}\partial T_B(0), \tag{47}$$

$$T_B(z)T_F(0)\sim\frac{3}{2z^2}T_F(0)+\frac{1}{z}\partial T_F(0), \tag{48}$$

$$T_F(z)T_F(0)\sim\frac{D}{z^3}+\frac{2}{z}T_B(0), \tag{49}$$

and similarly for the antiholomorphic currents. The $T_B T_F$ OPE implies that T_F is a tensor of weight $(\frac{3}{2}, 0)$. Each scalar contributes 1 to the central charge and each fermion $\frac{1}{2}$, for a total

$$c = (1 + \frac{1}{2})D = \frac{3}{2}D. \tag{50}$$

This can be checked by computing the propagators. In the covariant gauge the dynamics of the coordinates $X^\mu(z)$ and $\psi^\mu(z)$ are given by a free two dimensional Klein-Gordon equation and a free Dirac equation supplemented by certain constraints. The quantization of these coordinates is just that of a free two dimensional field theory. Thus we obtain the standard propagators.

$$< X^\mu(z,\bar{z}) X^\nu(0,0) > \;=\; -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z|^2 \tag{51}$$

$$< \psi(z)\psi(w) >= \frac{1}{z-w} \quad , \quad < \tilde{\psi}(\bar{z})\tilde{\psi}(\bar{w}) >= \frac{1}{\bar{z}-\bar{w}} \tag{52}$$

$$< \psi(z)\tilde{\psi}(\bar{w}) > \;=\; 0 \tag{53}$$

Exercise 4: Verify the OPE (47)-(49).

This enlarged algebra with T_F and $\tilde{T}_F$ as well as T_B and $\tilde{T}_B$ will play the same role as the conformal algebra in the bosonic string. That is we will impose it on the states as constraint algebra – it must annihilate physical states. Because of the Minkowski signature of spacetime the timelike ψ^0 and $\tilde{\psi}^0$, like X^0, have opposite sign commutators and lead to negative norm states. The fermionic constraints T_F and $\tilde{T}_F$ will remove these states from the spectrum.

Similarly as in the bosonic string theory the equation of motion for the metric is

$$T_B = 0 \quad , \quad \tilde{T}_B = 0. \tag{54}$$

We cannot obtain these equations from the action (5) because it is written in the conformal gauge (*i.e.* $h_{ab} = \delta_{ab}$). But in view of the algebra (49) we can hardly expect to set $T_B, \tilde{T}_B$ to zero without setting $T_F, \tilde{T}_F$ to zero as well. Indeed this can be systematically derived by gauge fixing a two dimensional Lagrangian with local supersymmetry as well as general covariance. The equations $T_F = 0$ and $\tilde{T}_F = 0$ then arise as the equations of motion of the world-sheet gravitino. We will discuss this in the third lecture.

1.2 Boundary conditions and Mode Expansions

Let us now consider the equations of motion arising from the action (14). The spacetime coordinate X^μ satisfies the same free wave equation as in the bosonic string theory,

$$\bar{\partial}\partial X = 0. \tag{55}$$

The possible boundary conditions correspond to open or closed strings and the resulting mode expansions are completely unchanged from before and therefore do not need to be repeated here.

We have already found the equations of motion for the fermions in complex coordinates.

$$\bar{\partial}\psi = 0 \to \psi = \psi(z) \quad , \quad \partial\tilde{\psi} = 0 \to \tilde{\psi} = \tilde{\psi}(\bar{z}). \tag{56}$$

As anticipated this simply implies that ψ ($\tilde{\psi}$) is a (anti) holomorphic field.

In order to obtain the possible boundary conditions it is convenient to consider the Lorentzian theory, *i.e.* in terms of $\sigma^{\pm} = \sigma^2 \pm \sigma^1$

$$S_F \sim \int d^2\sigma \left(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+\right) \tag{57}$$

where the index μ has been suppressed, ψ_- and ψ_+ denote right- and left- moving modes and $\partial_\pm = \frac{1}{2}(\partial^2_\sigma \pm \partial^1_\sigma)$.

$$\begin{aligned} \delta S_F &= \frac{1}{4\pi}\int d^2\sigma \left[\delta\psi_-\partial_+\psi_- + \delta\psi_+\partial_-\psi_+ + \psi_-\partial_+\delta\psi_- + \psi_+\partial_-\delta\psi_+\right] & (58)\\ &= \frac{1}{4\pi}\int d^2\sigma \left[2(\delta\psi_-\partial_+\psi_- + \delta\psi_+\partial_-\psi_+) + \partial_+(\psi_-\delta\psi_-) + \partial_-(\psi_+\delta\psi_+)\right]. & (59)\end{aligned}$$

The vanishing of the first two terms gives the equations of motion for ψ_+ and ψ_-. In light-cone coordinates these are $\partial_\pm\psi^\mu_\mp = 0$.

Vanishing of the surface terms requires that $\psi_-\delta\psi_- - \psi_+\delta\psi_+ = 0$ at each end of the open string. So the condition that the surface terms in the equations of motion vanish allows the possibilities

$$\psi^\mu_-(0,\sigma^2) = \pm\psi^\mu_+(0,\sigma^2) \quad , \quad \psi^\mu_-(\pi,\sigma^2) = \pm\psi_+(\pi,\sigma^2) \tag{60}$$

at each end. The overall relative sign between ψ_+ and ψ_- is a matter of convention, so without loss of generality we set

$$\psi^\mu_-(0,\sigma^2) = \psi^\mu_+(0,\sigma^2) \quad . \tag{61}$$

The relative sign at the other end now becomes meaningful and there are two cases to be considered:

$$\text{Ramond (R)}: \quad \psi^\mu_-(\pi,\sigma^2) = \psi^\mu_+(\pi,\sigma^2) \quad , \tag{62}$$

$$\text{Neveu} - \text{Schwarz (NS)}: \quad \psi^\mu_-(\pi,\tau) = -\tilde{\psi}^\mu(\pi,\sigma^2) \quad . \tag{63}$$

For closed strings vanishing of the surface terms is achieved by either periodic or antiperiodic fermions, since the fields enter quadratically in the constraint. Periodicity thus allows two possible choices for ψ^μ:

$$\text{(NS)}: \quad \psi^\mu(\sigma^2,\sigma^1+2\pi) = -\psi^\mu(\sigma^2,\sigma^1) \quad , \quad \tilde{\psi}^\mu(\sigma^2,\sigma^1+2\pi) = -\tilde{\psi}^\mu(\sigma^2,\sigma^1), \tag{64}$$

$$\text{(R)}: \quad \psi^\mu(\sigma^2,\sigma^1+2\pi) = +\psi^\mu(\sigma^2,\sigma^1) \quad , \quad \tilde{\psi}^\mu(\sigma^2,\sigma^1+2\pi) = +\tilde{\psi}^\mu(\sigma^2,\sigma^1) \tag{65}$$

These conditions can be written as

$$\psi^\mu(w+2\pi) = e^{2\pi i\nu}\psi^\mu(w) \quad , \quad \tilde{\psi}^\mu(\bar{w}+2\pi) = e^{2\pi i\tilde{\nu}}\tilde{\psi}^\mu(\bar{w}) \tag{66}$$

where $\nu(\tilde{\nu})$ equals 0 for R and $1/2$ for NS. The conditions for the two spinor components ψ and $\tilde{\psi}$ can be chosen independently, leading to a total of four possibilities, each of which will lead to a different Hilbert space – essentially there are four different kinds of closed superstring. We will denote these by NS–NS, NS–R, R–NS, R–R.

To study the spectrum in a given sector consider the Laurent expansions:

$$\psi^\mu(z) = \sum_{r\in\mathbf{Z}+\nu} \frac{\psi^\mu_r}{z^{r+1/2}} \quad , \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbf{Z}+\tilde{\nu}} \frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+1/2}} \quad . \tag{67}$$

As mentioned this is just the same as an ordinary Fourier expansion in the cylinder (w frame at time $\sigma^2 = 0$). Here $\nu, \tilde{\nu} = 0, \frac{1}{2}$ correspond to R or NS sector. (Notice the $1/2$ corresponding to the conformal weight).

Let us recall the corresponding bosonic expansions

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha^\mu_m}{z^{m+1}} \quad , \quad \bar{\partial} X^\mu(\bar{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}^\mu_m}{\bar{z}^{m+1}}, \tag{68}$$

where $\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = (\alpha'/2)^{1/2} p^\mu$ in the closed string and $\alpha^\mu_0 = (2\alpha')^{1/2} p^\mu$ in the open string.

For T_F and T_B the Laurent expansions are

$$T_B(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \quad , \quad \tilde{T}_B(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}} \quad , \tag{69}$$

$$T_F(z) = \sum_{r\in\mathbf{Z}+\nu} \frac{G_r}{z^{r+3/2}} \quad , \quad \tilde{T}_F(\bar{z}) = \sum_{r\in\mathbf{Z}+\tilde{\nu}} \frac{\tilde{G}_r}{\bar{z}^{r+3/2}} \quad . \tag{70}$$

The usual CFT contour calculation gives the mode algebra

$$\{\psi^\mu_r, \psi^\nu_s\} = \{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}, \tag{71}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m,-n}. \tag{72}$$

and similarly for T_B, T_F

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \quad , \tag{73}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r,-s} \quad , \tag{74}$$

$$[L_m, G_r] = \frac{m-2r}{2} G_{m+r} \quad . \tag{75}$$

Exercise 5: Obtain this algebra from the OPEs (47)-(49).

This is known as the Ramond algebra for r, s integer and the Neveu-Schwarz algebra for r, s half-integer. The antiholomorphic fields give a second copy of these algebras. The superconformal generators in either sector can be written in terms of the bosonic and fermionic oscillators as

$$L_m = \frac{1}{2}\sum_{n\in\mathbf{Z}} :\alpha^\mu_{m-n}\alpha_{\mu n}: \ + \ \frac{1}{4}\sum_{r\in\mathbf{Z}+\nu}(2r-m) :\psi^\mu_{m-r}\psi_{\mu r}: + a\delta_{m,0} \quad , \tag{76}$$

$$G_r \ = \ \sum_{n\in\mathbf{Z}} \alpha^\mu_n \psi_{\mu r-n} \quad . \tag{77}$$

Here : : denotes creation-annihilation normal ordering, placing all lowering operators to the right of all raising operators, with a minus sign whenever anticommuting operators are switched. The normal ordering constant a is due to the normal ordering ambiguity in L_0. There are several methods to determine it. The simplest computation of normal ordering constant uses the super Virasoro algebra and it requires defining the vacuum state as:

$$\alpha^\mu_n|0>=\tilde{\alpha}^\mu_n|0>=0 \quad n\geq 0 \tag{78}$$

$$\psi^\mu_r|0>=\tilde{\psi}^\mu_r|0>=0 \quad r>0 \tag{79}$$

Let's apply (73)

$$[L_1, L_{-1}] = 2L_0 \tag{80}$$

on this vacuum state:

$$2L_0|0>=(L_1L_{-1}-L_{-1}L_1)|0> \quad . \tag{81}$$

Using the mode expansion (76) for L_1 and L_{-1}, every term of each contains either a lowering operator or p^μ and so annihilates $|0>$ except for a couple of terms in the R sector. One thus obtains

$$\text{R}: \quad a=\frac{D}{16} \quad , \quad \text{NS}: \quad a=0 \tag{82}$$

Exercise 6: Work out these normal ordering constants.

2 Spectrum of the superstring

The quantization of the theory can be performed using the same techniques as for the bosonic string. There are basically three methods to obtain the spectrum: Old Covariant Quantization, light-cone gauge and path integral formalism. They all lead to the same spectrum. Consistent quantization implies the existence of a critical dimension ($D = 10$). The super-Virasoro constraints on physical states are implemented and analyzed in essencially the same way as the Virasoro constraints were in the bosonic string (recall that we have introduced the action in the gauge fixed form, so it is not possible to see these constraints arising from (6)). One new feature is the appearence of two sectors, bosonic and fermionic, that need to be studied separately. Eventually, the spectrum should be truncated by the GSO conditions and the two sectors become related by space-time supersymmetry.

In the bosonic string the classical equation of motion from variation of the two dimensional metric g_{ab} was $T_{ab} = 0$. After gauge fixing, this does not hold as an operator equation: we have a missing equation of motion because we do not vary g_{ab} in the gauge-fixed theory. A physical amplitude should not depend on the gauge choice, therefore one imposes

$$< phys|T^{ab}|phys' >= 0 \tag{83}$$

for arbitrary physical states. When one varies the gauge in a modern covariant quantization the change in the Fadeev-Popov determinant must be taken into account, so the energy momentum tensor in the matrix element is the sum of the matter and ghost contributions:

$$T_{ab} = T^m_{ab} + T^g_{ab}. \tag{84}$$

The **OCQ** consists in promoting the bosonic and fermionic oscillators of the classical theory to operators, and the Poisson brakets to commutators or anticommutators. The physical states are created applying the creation operators on the ground state. However in this way one generates negative norm states acting with the time components α^0_n, ψ^0_r for $n < 0, r \leq 0$ (recall that $[\alpha^\mu_n, \alpha^\nu_n] = \eta^{\mu\nu}\delta_{m,-n}$ and $\{\psi^\mu_r, \psi^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}$). These negative norm states nevertheless decouple imposing the condition (83) in a rather ad hoc way: one ignores the ghosts and tries to restrict the matter Hilbert space so that the missing equation of motion $T^m_{ab} = 0$ holds for matrix elements. In terms of Laurent coefficients this is $L^m_n = 0$. One might try to require physical states to satisfy $L^m_n|phys >= 0$ for all n, but this is too strong; acting on this equation with L^m_m and forming the commutator, one encounters an inconsistency due to the central charge of the Virasoro algebra. However it is sufficient that only the Virasoro lowering and zero operator annihilate physical states,

$$(L^m_n + A\delta_{n,0})|phys >= 0 \quad , \quad n \geq 0. \tag{85}$$

(Recall that at $n = 0$ one has to include the possibility of an ordering constant A).

In the superstring theory one has to consider in addition the constraints arising from the supercurrent, namely

$$G_r|phys>= 0, r \geq 0 \tag{86}$$

Since these conditions are in one to one correspondence with timelike oscillators their number is just sufficient for having a chance of leaving a positive definite Fock space.

Using $L_n^{m\dagger} = L_{-n}^m$, as follows from the Hermiticity of the energy-momentum tensor, one can see that a state of the form

$$|\chi>= \sum_{n=1}^{\infty} L_{-n}^m|\chi_n> \tag{87}$$

is orthogonal to all physical states for any $|\chi_n>$. Such a state is called spurious. A state that is both spurious and physical is called null. If $|\zeta>$ is physical and $|\chi>$ is null, then $|\zeta>+|\chi>$ is also physical and its inner product with any physical state is the same as that of $|\zeta>$. Therefore these two states are physically indistinguishable, and we identify

$$|\zeta>= |\zeta>+|\chi> \tag{88}$$

The real physical Hilbert space is then the set of equivalence classes

$$\mathcal{H}_{OCQ} = \frac{\mathcal{H}_{phys}}{\mathcal{H}_{null}}. \tag{89}$$

Light-cone quantization: In the bosonic theory we saw that even in the covariant gauge $g_{ab} = \delta_{ab}$ there are residual gauge invariances that allow further choices, such as light-cone gauge. The NSR model as we have been discussing it is really a gauge-fixed version of a model with local world-sheet supersymmetry.

In the bosonic string theory one defines spacetime light-cone coordinates, *e.g.*

$$X^{\pm}(\sigma,\tau) = \frac{X^0 \pm X^{D-1}}{\sqrt{2}}, \tag{90}$$

where we have singled out a spacial coordinate X^{D-1} and thus Lorentz invariance has been broken. We will have to check at the end that it is restored.

The residual reparametrization invariance that preserves the covariant gauge choice is just sufficient to be able to gauge away the + nonzero mode oscillators α_n^+ so that

$$X^+ = x^+ + p^+\tau \tag{91}$$

The reasoning applies so long as X^+ satisfies the two-dimensional wave equation, so it applies equally well in the present context for the superstring. However there is now also the freedom of applying local supersymmetry transformations that preserve the gauge choices. They turn out to be just sufficient to gauge away ψ^+ completely so that we make the gauge choice

$$\psi^+ = 0. \tag{92}$$

As a consistency check we note that under a global supersymmetry transformation

$$\delta X^+ = \bar{\epsilon}\psi^+ = 0, \tag{93}$$

since $\psi^+ = 0$ so that the X^+ gauge choice is not altered in the process.

The subsidiary constraints implied by the vanishing of T_F and T_B can be solved for the light-cone components ψ^- and $\partial_+ X^-$ in terms of the transverse oscillators α_n^i and ψ_r^i.

The light-cone gauge is not consistent with the quantum structure of the theory in general. In fact, we can deduce restrictions on the spacetime dimension D and the parameter A, just as in the bosonic string theory, by requiring that the Lorentz algebra be satisfied in the light-cone gauge. It turns out that the Lorentz generators

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + E^{\mu\nu} + \Sigma^{\mu\nu} \tag{94}$$

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu) \tag{95}$$

$$\Sigma^{\mu\nu} = -\frac{i}{2}\sum_{r=0}^{\infty} [\psi_{-r}^\mu, \psi_r^\nu] \tag{96}$$

satisfy the usual Lorentz algebra only if

$$m\left(\frac{D-2}{8}\right) + \frac{1}{m}\left(2a - \frac{D-2}{8}\right) = 0 \tag{97}$$

for all m, in the NS sector; thus $D = 10$ and $a = 1/2$. Similarly one finds $D = 10$ and $a = 0$ in the R sector.

Therefore in the light-cone gauge all string excitations are generated by the transverse oscillators α_n^i, ψ_r^i, $i = 1, ..., D-2$ and thus there are no negative norm states. Moreover, the states come in representations of the transverse rotation group $SO(8)$ (or spin(8) in the case of the fermions). However one can prove that the massive states fill out complete multipletes of $SO(9)$. There is a no-ghost theorem stating that the spectrum of the theory in the light-cone gauge coincides with the covariant spectrum after applying the super Virasoro constraints.

We will discuss the path integral quantization in the next lecture and now construct the spectrum in the OCQ.

2.1 NS and R spectra

We are now ready to derive the spectrum of the superstring. We consider the spectrum generated by a single set of NS or R modes, corresponding to the open string or to one side of the closed string.

Let us recall the (anti)commutation relations for the Laurent coefficients

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m,-n} \tag{98}$$

$$\{\psi_r^\mu, \psi_s^\nu\} = \{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\} = \eta^{\mu\nu}\delta_{r,-s}. \tag{99}$$

Remember that r, s take semi-integer or integer values in the NS and R sectors respectively.

The oscillator ground state in both sectors is defined as

$$\alpha_n^\mu|0, p >= \tilde{\alpha}_n^\mu|0, p >= 0 \quad n > 0 \tag{100}$$

$$\psi_r^\mu|0, p >= \tilde{\psi}_r^\mu|0, p >= 0 \quad r > 0 \tag{101}$$

The properties of the ground state are very different in both sectors. The NS sector is easier because it is very similar to the bosonic string. There is no $r = 0$ mode, so we define the ground state to be annihilated by all $r > 0$ modes. The modes with $r < 0$ then act as raising operators. Since these are anticommuting, each mode can only be excited once. Moreover it is important to note that the NS ground state has spin 0. This may be easily verified applying the Lorentz generators on the ground state. Recall that

$$\Sigma^{\mu\nu} = -\frac{i}{2}\sum_{r\in\mathbf{Z}+\nu}\left[\psi_r^\mu, \psi_{-r}^\nu\right]. \tag{102}$$

satisfy the $SO(D-1,1)$ algebra. Furthermore all the oscillators are spacetime vectors (both α_n^μ and ψ_r^μ for all n and r), and consequently all excited states in this sector will be spacetime bosons.

In the R sector the situation is very different. We have the ψ_0^μ zero modes. Since

$$\{\psi_r^\mu, \psi_0^\nu\} = 0 \tag{103}$$

in particular for $r > 0$, the ψ_0^μ take ground states into ground states. ($\psi_r^\mu\psi_0^\nu|0 >= \{\psi_r^\mu, \psi_0^\nu\}|0 >= 0$). They satisfy the Dirac gamma matrix algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \quad , \tag{104}$$

(recall the anticommutation relations $\{\psi_r^\mu, \psi_s^\nu\} = \eta^{\mu\nu}\delta_{r,-s}$). We can thus identify

$$\Gamma^\mu \approx 2^{1/2}\psi_0^\mu \quad . \tag{105}$$

The ground states thus form a representation of the gamma matrix algebra which in $D = 10$ has dimension 32. Therefore as we will see the R ground state is a spacetime spinor of $SO(9,1)$.

This representation can be explicitly constructed grouping the fermion zero modes as

$$\psi^{0\pm} = \frac{1}{\sqrt{2}}\left(\pm\psi^0 + \psi^1\right) \tag{106}$$

$$\psi^{a\pm} = \frac{1}{\sqrt{2}}\left(\psi^{2a} \pm i\psi^{2a+1}\right) \quad , \quad a = 1, ..., k \tag{107}$$

where we ommited for simplicity the subscript 0 in each operator and $k = \frac{D}{2} - 1$. They satisfy

$$\{\psi^{a+}, \psi^{b-}\} = \delta^{ab} \quad , \tag{108}$$

$$\{\psi^{a+}, \psi^{b+}\} = \{\psi^{a-}, \psi^{b-}\} = 0 \quad . \tag{109}$$

In particular, $(\psi^{a+})^2 = (\psi^{a-})^2 = 0$. It follows that by acting repeatedly with the ψ^{a-} we can reach a state $|\zeta >$ annihilated by all the ψ^{a-},

$$\psi^{a-}|\zeta >= 0 \quad \text{for all } a \quad . \tag{110}$$

Starting from $|\zeta >$ one obtains a representation of dimension 2^{k+1} by acting in all possible ways with the ψ^{a+}, at most once each. This is the dimension of a Dirac or Majorana spinor in D dimensions. We will label these with $\mathbf{s} \equiv (s_0, s_1, ..., s_k)$, where each of the s_a is $\pm\frac{1}{2}$:

$$|\mathbf{s} >\equiv \left(\psi^{k+}\right)^{s_k+1/2} ... \left(\psi^{0+}\right)^{s_0+1/2} |\zeta > \quad . \tag{111}$$

In particular, the original $|\zeta >$ corresponds to all $s_a = -\frac{1}{2}$. There are therefore 32 different ground states in $D = 10$ $(k = 4)$ that we will denote

$$|\mathbf{s} >= |s_0, s_1, s_2, s_3, s_4 > \quad \textit{with } s_a = \pm\frac{1}{2} \tag{112}$$

It is important to note that these are spin 1/2 states since it is a dimension 32 multiplet in a 10 dimensional spacetime. More formally one can apply the Lorentz generators $\Sigma^{\mu\nu}$ on this state and verify this. The generators $\Sigma^{2a,2a+1}$ commute and can be simultaneously diagonalized. In terms of the raising and lowering operators,

$$S_a \equiv i^{\delta_{a,0}}\Sigma^{2a,2a+1} = \Gamma^{a+}\Gamma^{a-} - \frac{1}{2} \tag{113}$$

so $|\mathbf{s}>$ is a simultaneous eigenstate of the S_a with eigenvalues s_a. The half-integer values show that this is a spinor representation. The spinors form the 2^{k+1}-dimensional Dirac representation of the Lorentz algebra $SO(2k+1,1)$.

The Dirac representation is reducible as a representation of the Lorentz algebra. Because $\Sigma^{\mu\nu}$ is quadratic in the Γ matrices, the $|\mathbf{s}>$ with even and odd numbers of $+\frac{1}{2}$ do not mix. To see this define

$$\Gamma = i^{-k}\Gamma^0\Gamma^1...\Gamma^{D-1} \quad , \tag{114}$$

which has the properties

$$(\Gamma)^2 = 1 \quad , \quad \{\Gamma, \Gamma^\mu\} = 0 \quad , \quad [\Gamma, \Sigma^{\mu\nu}] = 0 \quad . \tag{115}$$

The eigenvalues of Γ are ± 1. The conventional notation for Γ in $D=4$ is γ_5, but this is inconvenient in general D. Noting that

$$\Gamma = 2^{k+1} S_0 S_1 ... S_k \quad , \tag{116}$$

we see that $\Gamma_{ss'}$ is diagonal, taking the value $+1$ when the s_a include an even number of $-\frac{1}{2}$ and -1 for an odd number of $-\frac{1}{2}$. The 2^k states with Γ eigenvalue (*chirality*) $+1$ form a Weyl representation of the Lorentz algebra, and the 2^k states with eigenvalue -1 form a second, inequivalent, Weyl representation. For $D=4$, the Dirac representation is the familiar four dimensional one, which separates into 2 two-dimensional Weyl representations,

$$\mathbf{4}_{\mathrm{Dirac}} = \mathbf{2} + \mathbf{2}' \quad . \tag{117}$$

Here we have used a common notation, labeling a representation by its dimension (in boldface). In $D=10$ the representations are

$$\mathbf{32}_{\mathrm{Dirac}} = \mathbf{16} + \mathbf{16}' \tag{118}$$

Now we can go back to the Ramond vacuum states. As anticipated they form a representation of the gamma matrix algebra. We can thus take a basis of eigenstates of the Lorentz generators S_a,

$$|s_0, s_1, ..., s_4>_R \equiv |\mathbf{s}>_R \quad , \quad s_a = \pm\frac{1}{2} \quad . \tag{119}$$

The half integral values show that these are indeed spacetime spinors. In the R sector of the open string not only the ground state but all states have half-integer spacetime spins, because the raising operators α^μ_{-n}, ψ^μ_{-r} are vectors and change the S_a by integers.

As mentioned the Dirac representation $\mathbf{32}$ is reducible to two Weyl representations $\mathbf{16} + \mathbf{16}'$, distinguished by their eigenvalue under Γ. This has a natural extension to the full string spectrum. The distinguishing property of Γ is that it anticommutes with all Γ^μ. Since the Dirac matrices are now the center-of-mass modes of ψ^μ, we need an operator that anticommutes with the full ψ^μ. We will call this operator

$$exp(\pi i F) = (-1)^F \quad , \tag{120}$$

where F, the *world-sheet fermion number*, is defined only mod 2. The operator $(-1)^F$ is defined to anticommute with the fermion field $(-1)^F\psi(z) = -\psi(z)(-1)^F$ and to satisfy $((-1)^F)^2 = 1$. So $(-1)^F$ will have eigenvalue ± 1 acting on states with even or odd numbers of fermion creation operators. Since ψ^μ changes F by one it anticommutes with the exponential.

One can explicitly define

$$(-1)^F = \Gamma(-1)^{\sum_{r>0}\psi^\mu_{-r}\psi_{\mu r}} \tag{121}$$

in the R sector. The exponent has the form of a number operator. Effectively it counts the number of operators ψ^μ_r acting on the ground state, thus it counts the ψ^μ's, linear combinations of ψ^μ_r and ψ_0. Thus this operator is a product of operators: one anticommutes with the zero mode ψ^μ_0 whereas the other one anticommutes with the rest of the modes. The overall result is that $(-1)^F$ anticommutes with ψ^μ as we wished.

The definition can be extended to the NS sector of the theory where the natural definition for $(-1)^F$ is

$$(-1)^F = -(-1)^{\sum_{r>0}\psi^\mu_{-r}\psi_{\mu r}} \tag{122}$$

and the (-1) is due to the ghosts (to be discussed in the next lecture).

When we include the ghost part of the states we will see that it contributes to the total F, so that on the total matter plus ghost ground state one has

$$exp(\pi i F)|0>_{NS} = -|0>_{NS} \quad , \quad exp(\pi i F)|\mathbf{s}>_R = -|\mathbf{s}'>_R \Gamma_{\mathbf{s}'\mathbf{s}} \quad . \tag{123}$$

The ghost ground state contributes a factor -1 in the NS sector and $-i$ in the R sector.

A similar analysis holds for the right moving sector. Tensoring the two sectors, taking into account the level matching condition $L_0 = \tilde{L}_0$, one obtains the full spectrum. Since both the left and right moving R vacua are fermionic, both the NS-NS and R-R sectors give spacetime bosons while the NS-R and R-NS sectors give spacetime fermions.

So now we can construct the physical states of the theory. We have to implement the constraints. They are

$$L_n|phys>=0 \quad , \quad n>0 \qquad , \qquad G_r|phys>=0 \quad , \quad r \geq 0, \tag{124}$$

$$(L_0 + a)|phys> \;=\; 0 \tag{125}$$

where a is the normal ordering constant to be determined. There is of course a second set of conditions for the left movers and we also have to demand that

$$(L_0 - \tilde{L}_0)|phys >= 0 \tag{126}$$

which expresses the fact that no point on a closed string is distinct (there is no preferred point to be taken as the origin of the two dimensional coordinates).

The L_0 constraint gives the mass-shell condition

$$\alpha' m^2 = N + a \tag{127}$$

where

$$N = \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{\mu n} + \frac{1}{2}\sum_{r=1-\nu}^{\infty} r\psi^{\mu}_{-r}\psi_{\mu r} \tag{128}$$

Notice that the index r is semi-integer in the NS sector and integer in the R sector. The fermionic zero modes ψ_0^{μ} do not change the mass of a given state, thus the states $|0>$ and $\psi_0^{\mu}|0>$ are degenerate in mass.

Let us first discuss the open string or equivalently the right-moving part of the closed string spectrum. They are in fact identical, up to a mass rescaling by a factor of 2.

The mass shell condition

$$(L_0 + a)|phys >= 0 \tag{129}$$

in ten flat dimensions and considering the normal ordering constants that were obtained in the light-cone formalism (namely $a^R = 0, a^{NS} = -1/2$) becomes

$$\left(\alpha' p^2 + N - \frac{1}{2}\right)|0, p >= 0 \quad \text{(NS)} \tag{130}$$

$$\left(\alpha' p^2 + N\right)|0, p >= 0 \quad \text{(R)} \tag{131}$$

Let us start analysing the NS sector.

i) NS-sector:

The NS spectrum is simple. The lowest state is $|0; p >_{NS}$, labeled by the matter state and the momentum p. The only non-trivial condition arises from L_0:

$$L_0|0; p >_{NS} = \left(\alpha' p^2 - \frac{1}{2}\right)|0; p >_{NS} = 0 \Rightarrow m^2 = -p^2 = -\frac{1}{2\alpha'}. \tag{132}$$

This is a tachyon. We can compute its eigenvalue under the generalized chirality operator $(-1)^F$ and obtain -1 (actually at this point we can take this as a definition). So we will say that this state belongs to the sector NS$-$.

The first excited state is obtained by acting on the vacuum with the creation operator $\psi^{\mu}_{-1/2}$,

$$|\zeta; p >_{NS} = \zeta_{\mu}\psi^{\mu}_{-1/2}|0; p >_{NS}, \tag{133}$$

where ζ^μ is a polarization vector.

We now get two non-trivial conditions:

$$(L_0 - \frac{1}{2})|\zeta; p>_{NS} \;=\; \left(\alpha' p^2 + \frac{1}{2} - \frac{1}{2}\right)|\zeta; p>_{NS} \;= 0 \Rightarrow m^2 = -p^2 = 0; \quad (134)$$

$$G_{1/2}|\zeta; p>_{NS} \;=\; \left(\frac{\alpha'}{2}\right)^{1/2} p^\mu \zeta_\mu(p)|\zeta; p>_{NS} = 0 \Rightarrow \zeta_\mu p^\mu = 0. \quad (135)$$

At this level there is an extra condition leading to a spurious state, namely

$$G_{-1/2}|0, p>_{NS} = (2\alpha')^{1/2} p \cdot \psi_{-1/2}|0, p>_{NS} \quad (136)$$

This is a null state and thus we can identify

$$\zeta_\mu = \zeta_\mu + p_\mu \quad (137)$$

Then the first excited state in the NS sector is a vector of $SO(8)$. Its eigenvalue under $(-1)^F$ is $+1$, therefore we say that it belongs to the sector NS+.

Further applying α^μ_{-n} and/or ψ^μ_{-r} creation operators we can get the whole tower of string states with increasing mass. At the next excitation level we have the states $\alpha^\mu_{-1}|0>$ and $\psi^\mu_{-1/2}\psi^\nu_{-1/2}$ with $\alpha' m^2 = 1/2$, comprising $8 + 28$ bosonic states after identifying the spurious states. It can be shown that these and all other massive states are tensors of $SO(9)$, the little group for massive states in ten dimensions.

Let us discuss the R spectrum. The R ground state is, as discussed above, a spacetime massless fermion with 32 components and it actually describes a Majorana spinor. We will denote it as

$$|u; p>_R = |\mathbf{s}; p>_R u_\mathbf{s} \quad (138)$$

where $u_\mathbf{s}$ is the polarization, and the sum on $\mathbf{s}$ is implicit.

The physical state conditions in this case are

$$L_0|u; p>_R = \alpha' p^2|u; p>_R = 0 \Rightarrow m^2 = -p^2 = 0 \quad (139)$$

$$G_0|u; p>_R = \sqrt{\alpha'}|\mathbf{s}'; p>_R p_\mu \Gamma^\mu_{\mathbf{s}'\mathbf{s}} u_\mathbf{s} = 0 \Rightarrow p_\mu \Gamma^\mu_{\mathbf{s}'\mathbf{s}} u_\mathbf{s} = 0 \quad (140)$$

This implies that the ground state in this sector is massless, and as we said it is degenerate due to ψ^μ_0. The condition obtained from G_0 is extremely important: it is the Dirac equation for a massless spin 1/2 fermion. This actually confirms that the R ground state is a spacetime fermion as argued before. This is the equation we were looking for when we introduced anticommuting fields and supersymmetrized the theory.

This state posseses the full representation of the Clifford algebra and thus contains both positive and negative chirality states. The sectors containing these states are called R+ and R− respectively.

We can take a frame with $p_0 = p_1$. Thus the massless Dirac operator becomes

$$p_0\Gamma^0 + p_1\Gamma^1 = -p_1\Gamma^0(\Gamma^0\Gamma^1 - 1) = -2p_1\Gamma^0(\Sigma^{0,1} - \frac{1}{2}) \tag{141}$$

and the physical state condition is thus

$$\left(\Sigma^{01} - \frac{1}{2}\right) |\mathbf{s}, p >_R u_{\mathbf{s}} = 0 \tag{142}$$

Therefore only the states with $s_0 = \frac{1}{2}$ survive. The decomposition of $SO(9,1)$ under the subgroups $SO(1,1) \times SO(8)$ gives

$$\mathbf{16} \to \left(+\frac{1}{2}, 8\right) + \left(-\frac{1}{2}, 8'\right) \tag{143}$$

$$\mathbf{16'} \to \left(+\frac{1}{2}, 8'\right) + \left(-\frac{1}{2}, 8\right) \tag{144}$$

$$\tag{145}$$

In $D = 10$ massless particle states are classified by their behavior under SO(8) rotations that leave the momentum invariant. In the NS sector the massless physical states are the eight transverse polarizations forming the vector representation $\mathbf{8}_v$ of SO(8). In the R sector the Dirac equation leaves an $\mathbf{8}$ with $(-1)^F = 1$ and an $\mathbf{8'}$ with $(-1)^F = -1$.

As in the NS sector by further applying α^μ_{-n} and/or ψ^μ_{-r} creation operators we can get an infinite tower of massive states, which are all fermionic as mentioned

To obtain the spectrum of the closed string we just have to consider products of the states discussed above. The mass-shell condition can be summarized as

$$\frac{\alpha'}{4} m^2 = N - \nu = \tilde{N} - \tilde{\nu} \tag{146}$$

Notice that the condition relating the holomorphic and antiholomorphic parts forbids the product between NS$-$ and NS+,R$\pm$. Therefore the only the possibilities are (NS$-$,NS$-$) and any other combination not containing NS$-$. The resulting states transform as the product of the states we have described. For instance the massless state in (NS+,NS+) is a second rank tensor, equivalent to a scalar, an antisymmetric tensor and a symmetric traceless tensor. Similarly one obtains the rest of the states.

So we now have spacetime fermions. But we still have a tachyon in the NS-NS sector with mass $m^2 = -2/\alpha'$ for the closed string while $m^2 = -1/2\alpha'$ for the open string.

Even though all these sectors arise in the theory one can show that only some combinations can appear in a consistent theory. These come from three requirements:

i) The vertex operators (to be discussed later) must be mutually local in pairs;

ii) All operators must form a closed OPE;

iii) Modular invariance of the partition function.

The last condition implies that there must be at least one left-moving R sector and at least one right-moving R sector (see the fourth lecture).

Here we will explain how to obtain the appropriate combinations using heuristic arguments.

There are three important points to notice in the theory we have just described. First, there is a tachyon and this is a bad feature clearly. Second, the states in the NS sector are strange: they are bosonic states created by anticommuting fields ψ^μ. Even though this is not strictly incorrect, it seems unnatural. Notice that, whereas one field ψ^μ is anticommuting, any pair of fields ψ^μ is commuting. Finally there is supersymmetry. It is clear that it is manifest on the world-sheet, but it is not at all clear that the theory is spacetime supersymmetric. Counting the number of degrees of freedom of the massless states it is easy to see that there are 16 fermionic and 8 bosonic degrees of freedom. It looks as if there is one more R sector than there should be at the massless level.

The three problems above should be corrected. The solution is to project over just some of the sectors. This is known as the GSO projection (Gliozzi, Scherk and Olive). One way to solve all these problems together is to keep only those sectors with $(-1)^F = 1$ both for right and left movers. In this way we have the sectors: (NS+,NS+), (R+,NS+), (NS+,R+), R+,R+) . Notice that the tachyon does not belong to any of these sectors. Moreover, projecting over $(-1)^F = 1$ implies in the NS sector keeping only states with an even number of ψ^μ operators acting on the vacuum. Regarding supersymmetry one can check the massless spectrum as a test. It is

(NS+,NS+): one scalar (1), one antisymmetric tensor of rank 2 (28) and one traceless symmetric tensor of rank 2 (35).

(NS+,R+): one spin 1/2 fermion (8) and one spin 3/2 fermion (56).

(R+,NS+): one spin 1/2 fermion (8) and one spin 3/2 fermion (56).

(R+,R+): one scalar (1), one antisymmetric tensor of rank 2 (28) and one antisymmetric selfdual tensor of rank 4 (35).

Exercise: Obtain the tensors in (R+,R+) from the spinor products decomposition.

The counting of degrees of freedom agrees with a supersymmetric theory. Actually we will see that the symmetric rank 2 tensor in NS+NS+ can be identified with the graviton and also we can identify the gravitino in NS-R.

The theory above is known as Type IIB. Notice that the holomorphic and antiholomorphic parts have the same chirality, thus it is called a chiral theory. One can construct another theory, in this case non-chiral, which also solves the problems above. This is defined so that $(-1)^F = 1$ for the holomorphic part and $(-1)^F = (-1)^{1-2\tilde{\nu}}$ for the antiholomorphic part. This makes sense since both chiralities are equivalent in R whereas this is not so in NS where there is only one tachyon. In this case the sectors allowed are (NS+,NS+), (R+,NS+), (NS+,R−), (R+,R−). The spectrum of the theory is the same as above for the first three sectors. However there is a relative minus sign in the (R,R) sector which changes the spectrum.

(R+,R−): one vector (8) and one antisymmetric tensor of rank 3 (56).

Exercise: Obtain these tensors from the spinor product decomposition.

This theory is known as Type IIA and it is non chiral since it posseses fermions with both chiralities.

These are the only two consistent closed superstring theories solving the problems mentioned above with supersymmetry in both holomorphic and antiholomorphic sectors. The Heterotic theory is also spacetime supersymmetric but it is the product of a supersymmetric sector and a bosonic one.

Now suppose that there is no R-NS sector. By *iii*) there must be at least one R-R sector. In fact the combination of (NS+,NS+) with any single R-R sector solves $i), ii)$ and $iii)$, but these turn out not to be modular invariant. Proceeding further one readily finds the only other solutions:

$$0A: \quad (NS+, NS+), (NS-, NS-), (R+, R-), (R-, R-), \tag{147}$$

$$0B: \quad (NS+, NS+), (NS-, NS-), (R+, R+), (R-, R-). \tag{148}$$

These are modular invariant, but they both have a tachyon and there are no spacetime fermions.

In conclusion we have two potentially interesting theories, the *type IIA* and *type IIB* superstring theories. One finds the massless spectra:

$$IIA: \quad [0] + [1] + [2] + [3] + (2) + \mathbf{8} + \mathbf{8'} + \mathbf{56} + \mathbf{56'} \quad , \tag{149}$$

$$IIB: \quad [0]^2 + [2]^2 + [4]_+ + (2) + \mathbf{8'}^2 + \mathbf{56}^2. \tag{150}$$

where $[n]$ and (n) denote an antisymmetric and a symmetric rank n tensors respectively.

The IIB theory is defined by keeping all sectors with

$$exp(\pi i F) = exp(\pi i \tilde{F}) = +1, \tag{151}$$

and the IIA theory by keeping all sectors with

$$exp(\pi i F) = +1 \quad , \quad exp(\pi i \tilde{F}) = (-1)^{1-2\tilde{\nu}} \tag{152}$$

The construction that we have discussed is the Ramond-Neveu-Schwarz form of the superstring (RNS). It is rather ad hoc. In particular one might expect that the spacetime supersymmetry should be manifest from the start. There is certainly no truth in this, but the existing supersymmetric formulation (the Green-Schwarz superstring) seems to be even more unwieldy.

Note that the world-sheet and spacetime supersymmetries are distinct, and that the connection between them is indirect. The world-sheet supersymmetry parameter $\epsilon(z)$ is a spacetime scalar and world-sheet spinor, while the spacetime supersymmetry parameter is a spacetime spinor and world-sheet scalar.

Unoriented and open superstrings

The IIB superstring, with the same chiralities on both sides, has a world-sheet parity symmetry Ω. We can gauge this symmetry to obtain an unoriented closed string theory. In the NS-NS sector this eliminates the [2], leaving $[0]+(2)$, just as in the unoriented bosonic theory. The fermionic NS-R and R-NS sectors of the IIB theory have the same spectra, so the Ω-projection picks out the linear combination (NS-R)+(R-NS), with massless states $\mathbf{8}'+\mathbf{56}$. In particular one gravitino survives the projection. Finally the existence of the gravitino means that there must be equal numbers of massless bosons and fermions, so a consistent definition of the world-sheet parity operator must select the [2] from the R-R sector to give 64 of each. Thus, projecting onto $\Omega=+1$ picks out the antisymmetric [2] from NS-NS. The result is the *type I closed unoriented theory* with spectrum

$$[0]+[2]+(2)+\mathbf{8}'+\mathbf{56}=\mathbf{1}+\mathbf{28}+\mathbf{35}+\mathbf{8}'+\mathbf{56}. \tag{153}$$

However this theory is inconsistent. It has tadpole divergences at one loop. But these can cancel including the open sector. So let us discuss the open superstring.

Closure of the OPE in open + open $\to$ closed scattering implies that any open string that couples consistently to type I or type II closed superstrings must have a GSO projection in the open string sector. The two possibilities and their massless spectra are

$$I: NS+, R+ = \mathbf{8}_v+\mathbf{8}, \tag{154}$$

$$\tilde{I}: NS+, R- = \mathbf{8}_v+\mathbf{8}'. \tag{155}$$

Adding Chan-Paton factors, the gauge group will be $U(n)$ in the oriented case and $SO(n)$ or $Sp(n)$ in the unoriented case. The $\mathbf{8}$ or $\mathbf{8}'$ are gauginos.

We can anticipate that not all these theories will be consistent. The open string multiplets, with 16 states, are representations of D=10, N=1 supersymmetry, but not of N=2 supersymmetry. Thus the open superstring cannot couple to the oriented closed superstring theories, which have two gravitinos. It can only couple to the unoriented closed string theory and so the open string theory must also be unoriented for consistent interactions. The final result is the unoriented *type I open plus closed superstring theory*, with massless content

$$[0]+[2]+(2)+\mathbf{8}'+\mathbf{56}+(\mathbf{8}_v+\mathbf{8})_{SO(n)\ or\ Sp(k)}. \tag{156}$$

There is a further inconsistency in all but the SO(32) theory. For all other groups as well as the purely closed unoriented theory there is a one-loop divergence and superconformal anomaly.

Thus we have found precisely three tachyon-free and nonanomalous string theories: type IIA, type IIB and type I SO(32).

3 Interactions

Up to now we have discussed the free theory. The natural way to introduce interactions in string theory is the Feynman path integral. In path integral quantization amplitudes are given by summing over all possible histories interpolating between the initial and final states. Each history is weighted by

$$e^{iS_{cl}/h} \tag{157}$$

Thus one defines the amplitude in string theory by summing over all world-sheets connecting inicial and final curves.

The only interactions that are allowed are those that are already implicit in the sum over world-sheets: one string decaying into two or two strings merging into one.

This is the basic closed string interaction. In closed string theory all particle interactions are obtained as various states of excitation of the string and all interactions (gauge, gravitational, Yukawa, for example) arise from the single process of the figure.

There is no distinguished point where the interaction occurs. The interaction arises only from the global topology of the world-sheet, while the local properties of the world-sheet are the same as they were in the free case. It is this smearing of the interaction that cuts off the short distance divergencies of gravity.

There are 4 ways to define the sum over world-sheets which correspond to the 4 theories: 1) Closed oriented; 2) Closed unoriented; 3) Closed + open oriented; 4)

Closed + open unoriented. All theories with open strings include closed strings.

The idea to sum over all world-sheets bounded by initial and final curves seems like a natural one. However it is difficult to define this consistently with the local world-sheet symmetries, and the resulting amplitudes are rather complicated. There is one special case when the amplitudes simplify: this is the limit when the sources are taken to infinity. This corresponds to a scattering amplitude, an S-matrix element with the incoming and outgoing strings specified. So most string calculations confine to this S-matrix and are on shell.

So we will consider processes like

where the sources are pulled to infinity. The cylinder has a conformally equivalent description in terms of the coordinate z

$$z = e^{-iw} \quad e^{-2\pi t} \leq |z| \leq 1. \tag{158}$$

In this picture the long cylinder is mapped into the unit disk, where the external string state is a tiny circle in the center. The process looks like:

In the limit $t \to \infty$ the tiny circles shrink to points and the world-sheet reduces to a sphere with a point-like insertion for each external state. The same idea holds for the open string

Each string source becomes a local disturbance on the world-sheet. To a given incoming or outgoing string with D momentum p^μ and internal state j, there corresponds a local vertex operator $V_j(p)$ determined by the limiting process.

Sum over topologies (genus)

In order to include interactions we have to introduce the *vertex operators*. These operators are in one to one correspondence with the physical states of the theory. They are very important since they are necessary to compute the scattering amplitudes among different states in the path integral formalism. Let us recall this construction in the case of the bosonic string. A general scattering amplitude is given by

$$\mathcal{A}_{j_1,\ldots,j_n}(p_1,\ldots,p_n) = \sum_{compact\ \ topologies} \int \frac{\mathcal{D}X\mathcal{D}g}{V_{diff\times Weyl}} e^{-S-\lambda\chi} \prod_{i=1}^{n} \int d^2 z_i (g(z_i))^{1/2} V_{j_i}(p_i, z_i) \tag{159}$$

where S refers to the action without gauge fixing; j_i denote the quantum numbers of the vertex operators; the sum over topologies indicates that one not only has to consider local variations of the world-sheet but also those involving a change in its global properties. Each topology is weighted by a factor χ, the Euler number of the manifold and λ is a multiplicative constant (actually it is defined by the background of the theory). This expression for the scattering amplitude is thus a perturbative expansion in χ or the genus of the compact surface. As we have already mentioned one has to divide by the volume of the symmetry group (similarly to what one does in electromagnetism in the path integral formalism). This is precisely the origin of the ghosts in the theory. In the bosonic string theory these are the anticommuting bc ghosts of reparametrizations; here there will be commuting ghosts of supersymmetry. Finally, the vertex operators $V_{j_i}(p_i, z_i)$ create physical states: they can contain matter fields as well as ghosts. Thus the vacuum amplitudes in string theory are the expectation values of a product of vertex operators.

3.1 Path Integral Quantization

In the path integral quantization of the theory, one fixes the conformal gauge by introducing auxiliary fields called *ghosts*. Let us consider this issue in the case of the bosonic string. The Polyakov path integral runs over all Euclidean two dimensional metrics $g_{ab}(\sigma,\tau)$ and over all embeddings $X^\mu(\sigma,\tau)$ of the world-sheet in Minkowski spacetime:

$$\int \mathcal{D}X\mathcal{D}g exp(-S) \tag{160}$$

The path integral (160) is not quite right. It contains an enormous overcounting, because configurations (X,g) and (X',g') that are related to one another by the local diff $\times$ Weyl symmetry represent the same physical configuration. We need to divide by the volume of this local symmetry group,

$$\int \frac{\mathcal{D}X\mathcal{D}g}{V_{\text{diff}\times\text{Weyl}}} exp(-S) \tag{161}$$

We carry this out by gauge-fixing, integrating over a slice that cuts through each gauge equivalence class once and obtaining the correct measure on the slice by the Fadeev-Popov method.

Note that the metric, being symmetric, has three independent components, and that there are three gauge functions, the two coordinates and the local scale of the metric. Thus there is just enough gauge freedom to eliminate the integration over the metric, fixing it at some specific functional form which we will call the *fiducial* metric, $\hat{g}_{ab}(\sigma)$. Any metric can be brought to the flat form at least locally, in a given neighborhood on the world-sheet.

Let us discuss the gauge fixing procedure directly in the superstring case. We have already mentioned that the action (6) is the gauge fixed form of a more general action. The underlying original action is a locally supersymmetric action, *i.e.* a two dimensional supergravity action. Here we just introduce it without derivation:

$$\begin{aligned} S_{sugra} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\ e\Big[&g^{ab}\partial_a X^\mu \partial_b X_\mu - i\alpha'\bar{\Psi}^\mu\gamma^a D_a\Psi_\mu + \\ &4\bar{\chi}_a\gamma^b\gamma^a\Psi^\mu\partial_b X_\mu + \frac{1}{2}\bar{\Psi}^\mu\Psi_\mu\bar{\chi}_a\gamma^b\gamma^a\chi_b\Big]. \end{aligned} \tag{162}$$

χ_a is a two dimensional gravitino, e is the determinant of the *zweibein* e_a^α (α is a flat index and a is a curved one), g_{ab} is the world-sheet metric and D_a is the usual covariant derivative for fermions in curved space. Notice that there are no kinetic terms for the graviton or the gravitino in the action above. This is a special feature of two dimensions where these fields have no dynamics; indeed the Einstein-Hilbert term $\int d^2\sigma e R$ is a topological invariant quantity (the Euler number of the world-sheet surface) whereas the would-be gravitino kinetic term $\bar{\chi}\gamma^{abc}D_b\chi_c$ trivially vanishes because $\gamma^{abc} = 0$ in two dimensions. Similarly, the fermion covariant derivative can be replaced by a normal derivative because the term proportional to the spin connection vanishes.

The action above is invariant under the following local supersymmetry transformations:

$$\begin{aligned} \delta X^\mu = \bar{\epsilon}\Psi^\mu \quad , \quad \delta\Psi^\mu &= -\frac{i}{\alpha'}\gamma_a\epsilon(\partial_a X^\mu - \bar{\Psi}^\mu\chi_a); \\ \delta e_a^\alpha &= -\frac{i}{\alpha'}\bar{\epsilon}\gamma\chi_a \quad , \quad \delta\chi_a = D_a\epsilon, \end{aligned} \tag{163}$$

conformal transformations,

$$\delta X^\mu = 0, \quad \delta\Psi^\mu = -\frac{1}{2}\Lambda\Psi^\mu, \quad \delta e_a^\alpha = \Lambda e_a^\alpha, \quad \delta\chi_a = \frac{1}{2}\Lambda\chi_a, \tag{164}$$

and local fermionic transformations:

$$\delta e_a^\alpha = \delta X^\mu = \delta\Psi^\mu = 0, \quad \delta\chi_a = i\gamma_a\eta, \tag{165}$$

where ϵ and η are arbitrary spinors and Λ an arbitrary function on the world-sheet. All these symmetries allow to fix $e_a^\alpha = \delta_a^\alpha$ and $\chi_a = 0$. In this way it is easy to show that the action reduces to (6). However we can now understand the constraints: they arise from the equations of motion for e_a^α and χ_a evaluated in the above gauge:

$$\begin{aligned} e_a^\alpha \frac{\delta S}{\delta e_\alpha^b} &\sim (T_B)_{ab} = 0 \\ \frac{\delta S}{\delta\chi_a} &\sim \gamma^b\gamma^a\partial_b X^\mu\Psi_\mu = 0, \end{aligned} \tag{166}$$

which in complex coordinates are $T_B = \tilde{T}_B = T_F = \tilde{T}_F = 0$.

Gauge fixing in the path integral approach gives rise to jacobians that can be rewritten as Fadeev-Popov ghosts. In this case we get the anticommuting bc ghosts associated to the conformal gauge fixing and commuting $\beta\gamma$ ghosts associated to the superconformal symmetries. The full path integral can be roughly written as

$$\begin{aligned} \int \mathcal{D}X\mathcal{D}\Psi\mathcal{D}g\mathcal{D}\chi\delta(g_{zz})\delta(g_{\bar{z}\bar{z}})\det\left(\frac{\delta g_{zz}}{\delta\xi_z}\right)\det\left(\frac{\delta g_{\bar{z}\bar{z}}}{\delta\xi_{\bar{z}}}\right) \times \\ \delta(\epsilon_z)\delta(\epsilon_{\bar{z}})\det\left(\frac{\delta\chi_{\bar{z}}}{\delta\epsilon_{\bar{z}}}\right)\det\left(\frac{\delta\chi_z}{\delta\epsilon_z}\right) e^{-S(X,\Psi,g=\delta,\chi=0)} \end{aligned} \tag{167}$$

where $\xi_z, \xi_{\bar{z}}$ are the holomorphic and antiholomorphic parameters of the change of coordinates and $\epsilon_z, \epsilon_{\bar{z}}$ the two spinorial components of the local supersymmetry transformations in complex coordinates.

The determinants above can be expressed as

$$\det\left(\frac{\delta g_{\bar{z}\bar{z}}}{\delta \xi_{\bar{z}}}\right) \sim \int \mathcal{D}c\mathcal{D}b e^{-\frac{1}{2\pi}\int d^2 z b\bar{\partial}c},$$
$$\det\left(\frac{\delta \chi_{\bar{z}}}{\delta \epsilon_{\bar{z}}}\right) \sim \int \mathcal{D}\gamma\mathcal{D}\beta e^{-\frac{1}{2\pi}\int d^2 z \beta\bar{\partial}\gamma}. \tag{168}$$

The anticommuting ghost action is therefore

$$S^{bc}_{\rm ghost} = \int d^2 z\, [b_{zz}\partial_{\bar{z}}c^z + c.c.] \quad , \tag{169}$$

and the equations of motion

$$\partial_{\bar{z}}c^z = \partial_{\bar{z}}b_{zz} = 0 \tag{170}$$

imply that b and c are analytic fields. Hence the ghost system is also a conformal field theory. The fields b, c are effectively free fermions of the wrong spin; the propagator on the plane is easily found to be

$$< b_{zz}c^w >= \frac{1}{z-w} \quad . \tag{171}$$

From (169) one finds the traceless energy-momentum tensor of the ghosts

$$T_{gh}(z) = c\partial b + 2(\partial c)b \quad . \tag{172}$$

Using the two point function (171) one may calculate the operator product

$$T_{gh}(z)T_{gh}(w) \sim \frac{-13}{(z-w)^4} + \frac{2}{(z-w)^2}T_{gh}(z) + \frac{1}{z-w}\partial T_{gh}(z) + nonsingular \tag{173}$$

demonstrating that $c = -26$ for this system. When the number of spacetime dimensions $D = 26$, the total anomaly $c_X + c_{gh}$ vanishes and allows the consistent application of the Virasoro gauge conditions to decouple the unphysical states in the bosonic string.

Similarly in the superstring theory, the Fadeev-Popov (super)determinant compensates for fixing the intrinsic supermetric on the world surface of the fermionic string. This superdeterminant is the jacobian for the change of variables used to factor out the super-reparametrization group. The determinant of the differential operators may be represented by a path integral over a conjugate pair of dimension $-1, \frac{3}{2}$ ghost fields. The full reparametrization ghost action is

$$S_{BC} = \frac{1}{2\pi}\int d^2 z(b\bar{\partial}c + \beta\bar{\partial}\gamma + c.c.) \quad , \tag{174}$$

and

$$T_B = -(\partial b)c - 2b\partial c - \frac{1}{2}(\partial\beta)\gamma - \frac{3}{2}\beta\partial\gamma \quad , \tag{175}$$
$$T_F = (\partial\beta)c + \frac{3}{2}\beta\partial c - 2b\gamma \quad . \tag{176}$$

The ghost central charge is then $-26 + 11 = -15$, and the condition that the total central charge vanish gives the critical dimension

$$0 = \frac{3}{2}D - 15 \longrightarrow D = 10 \quad . \tag{177}$$

Exercise: Compute the ghost central charge.

The ghosts β and γ must have the same periodicity as the generator T_F with which they are associated,

$$T_F(w + 2\pi) = exp(2\pi i\nu)T_F(w) \tag{178}$$
$$\tilde{T}_F(\bar{w} + 2\pi) = exp(-2\pi i\tilde{\nu})\tilde{T}_F(\bar{w}) \tag{179}$$

This is necessary to make the BRST current periodic, so that it can be integrated to give the BRST charge. The mode expansions of these conformal fields are thus

$$\beta(z) = \sum_{r\in\mathbf{Z}+\nu} \frac{\beta_r}{z^{r+3/2}} \quad , \quad \gamma(z) = \sum_{r\in\mathbf{Z}+\nu} \frac{\gamma_r}{z^{r-1/2}} \quad , \tag{180}$$
$$b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+2}} \quad , \quad c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m-1}} \quad , \tag{181}$$

and similarly for the antiholomorphic fields. The (anti) commutators are

$$[\gamma_r, \beta_s] = \delta_{r,-s} \quad , \quad \{b_m, c_n\} = \delta_{n,-m} \quad . \tag{182}$$

Define the ground states $|0>_{NS,R}$ by

$$\beta_r|0>_{NS} = 0 \quad , \quad r \geq \frac{1}{2} \quad , \quad \gamma_r|0>_{NS} = 0 \quad , \quad r \geq \frac{1}{2} \quad , \tag{183}$$
$$\beta_r|0>_R = 0 \quad , \quad r \geq 0 \quad , \quad \gamma_r|0>_R = 0 \quad , \quad r \geq 1 \quad , \tag{184}$$
$$b_m|0>_{NS,R} = 0 \quad , \quad m \geq 0 \quad , \quad c_m|0>_{NS,R} = 0 \quad , \quad m \geq 1 \quad . \tag{185}$$

We have grouped β_0 with the lowering operators and γ_0 with the raising ones, in parallel with the bosonic case. The spectrum is built as usual by acting on the ground states with the raising operators. The generators are

$$L_m^g = \sum_{n\in\mathbf{Z}}(m+n) : b_{m-n}c_n : + \sum_{r\in\mathbf{Z}+\nu} \frac{1}{2}(m+2r) : \beta_{m-r}\gamma_r : + a^g\delta_{m,0} \quad , \tag{186}$$
$$G_r^g = -\sum_{n\in\mathbf{Z}} \left[\frac{1}{2}(2r+n)\beta_{r-n}c_n + 2b_n\gamma_{r-n}\right] \quad . \tag{187}$$

The normal ordering constant is determined by the usual methods to be

$$\mathrm{R}: \quad a^g = -\frac{5}{8} \quad , \quad \mathrm{NS}: \quad a^g = -\frac{1}{2} \quad . \tag{188}$$

Exercise: Obtain these normal ordering constants.

3.2 Vertex operators in bosonic string theory

We can see from (159) that the vertex operators must be conformal weight one operators (both for the holomorphic and antiholomorphic part). Recall that the conformal dimension $(h, \tilde{h})$ indicates the transformation properties of the fields under $z \to z' = f(z)$. It is clear that the scattering amplitude must be invariant under conformal transformations (since this is a symmetry of the action). Consequently the integral of the vertex operators on the world-sheet must have conformal weight 0 (it must be a scalar under these transformations). Since d^2z transforms with weight $(-1,-1)$, V must be a weight $(1,1)$ field.

We can now find the vertex operators creating physical states. They must satisfy in addition the physical state conditions imposed on the states by the Virasoro constraints. One can easily verify that any weight one operator satisfies them automatically. In general, if the operator behaves as a tensor it implies certain transversality conditions on the polarization tensor. Therefore, consistency of the operator formalism and the path integral asures that it is sufficient to find all possible operators of weight (1,1) and they should coincide with the states in the spectrum.

Since we want to have states with definite momentum, a first guess could be

$$V_0 \propto e^{ip\cdot X(z,\bar{z})} \tag{189}$$

This is in fact a tensor with weight

$$h = \tilde{h} = \frac{\alpha' p^2}{4} \tag{190}$$

and thus this is a good vertex operator if $p^2 = \frac{4}{\alpha'} = -m^2$. Consequently it corresponds to the tachyon vertex operator in the bosonic theory. The subindex indicates the number of excitations according to the number operator N. Normal ordering is implicit in these expresions.

Excitations can be obtained inserting the conformal field ∂X^μ. For instance

$$V_1 \propto \zeta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ip\cdot X} \tag{191}$$

This operator corresponds to the dilaton, graviton and antisymmetric tensor, *i.e.* the massless states. This can be verified computing the conformal dimension $h = \tilde{h} = 1 + \frac{\alpha' p^2}{4}$ and thus $p^2 = 0$. Notice further that the spacetime indices correspond to a rank two tensor. The tensor $\zeta_{\mu\nu}$ denotes the polarization of the state. Moreover this operator is weight one if $p^\mu \zeta_{\mu\nu} = 0$ and $p^\nu \zeta_{\mu\nu} = 0$. In this way, adding derivatives one can build the vertex operators for arbitrary states in the theory.

As we mentioned, in order to factorize the volume of the symmetry group one has to fix the gauge on the world-sheet and introduce ghost fields. However there are still residual symmetries (as in electromagnetism once one fixes the divergence). For example, on the sphere (which is the topology associated with scattering amplitudes at tree level in the closed string theory) these can be computed and they lead to 6 transformations defined by 3 complex parameters. This group of transformations is known as the Möbius group and the parameters are known as the *conformal Killing vectors*. These transformations are not evident, but it is easy to visualize an SO(3) subgroup, the rotation group. It is clear that if one rotates the sphere with the vertex operator insertions the scattering amplitude cannot change. Therefore in order to factorize these residual symmetries in the sphere one fixes three vertex operators in three given points. Notice that now there are 3 vertex operators that are not integrated. We required the vertex operators to have weight one to cancel the -1 from the integrating measure d^2z (so that everything is invariant). This is a problem because now there are three insertions which are not invariant. To make them invariant we should insert other fields with conformal dimension -1. Here is where the ghosts come in. The field c has weight -1 and we could thus consider operators such as:

$$V'_N = c\bar{c}V_N \tag{192}$$

Therefore the amplitude on the sphere is built with 3 primed vertex operators V' and the rest unprimed V, integrating only over these last ones. This result can be obtained more formally and one can see that the c insertions are actually necessary to cancel the volume of the symmetry group.

The same reasoning can be applied to an arbitrary topology and the results obtained are similar. In more general cases one has to insert also b fields (which have weight $+2$ and are associated to the moduli space of the surface).

There is an equivalent way to understand all this. The ghost action

$$S_g = \frac{1}{2\pi}\int d^2z(b\bar{\partial}c + \beta\bar{\partial}\gamma) \tag{193}$$

has a symmetry that we have not considered yet. It is invariant under a U(1) group with the following conserved current:

$$j(z) = -:b(z)c(z): = \sum_n z^{-n-1}j_n \tag{194}$$

where

$$j_n = \sum_m :c_{n-m}b_m: \tag{195}$$

The fields b and c are charged under this current. The charge of a general operator ϕ is

$$j(z)\phi(w) \sim \frac{N_g\phi(w)}{z-w} \tag{196}$$

(and it is clearly $N_g = j_0\phi$. The fields b and c have $N_g = -1$ and 1 respectively.

Even though the current j is classically conserved, it has a quantum anomaly. This can be interpreted as a background charge of the theory. We can compute this background charge analyzing the OPE

$$T(z)j(w) \sim \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{z-w} \tag{197}$$

The background charge depends on the nature of the b, c fields. In particular, it is a function of their conformal weights and it depends on their commuting or anticommuting properties. In this case it is $Q = -3$. The OPE above can be rewritten as the following anomalous commutation relations:

$$[L_n, j_m] = \frac{1}{2}Qn(n+1)\delta_{n,-m} - mj_{n+m} \tag{198}$$

It is clear from the definition that $j_n^\dagger = -j_{-n}$. From this and (198) we can compute $j_0^\dagger$:

$$j_0^\dagger = -\left[L_{-1}, j_1\right]^\dagger = -\left[L_1, j_{-1}\right] = -j_0 - Q \tag{199}$$

Now it is interesting to observe the following. Take a charge q operator O_q satisfying $[j_0, O_q] = qO_q$. Consider the vacuum expectation value of O_q multiplied by the charge q:

$$q < 0|O_q|0 > = < 0|\left[j_0, O_q\right]|0 > = -Q < 0|O_q|0 > \tag{200}$$

Therefore O_q must have charge q equal to $-Q$ if the expectation value is not zero. Thus, the j anomaly is canceled and as a result the vertex operators inserted in the path integral must cancel the background charges so that the amplitudes are invariant under this U(1) symmetry. In particular, for the bosonic string this implies that the vertex operators must add 3 units of charge in all. This is achieved inserting 3 c fields of charge $+1$ each.

Up to now we have summarized the vertex operators in the bosonic string theory. Our interest though is in the superstring. So we now extend the above considerations to this case.

3.3 Vertex operators in the superstring and bosonization

The vertex operators are constructed in a similar way. The additional considerations we have to take into account are: the presence of fermionic fields and the different sectors in the theory.

Since the theory posseses fermions these give possible excitations of the spectrum and must therefore be taken into account in the vertex operators. It will be very useful, especially due to the R sector, to define the *bosonized* vertex operators. Indeed it may be shown that it is possible to reproduce all the OPEs of a theory of two free real fermions in terms of a free bosonic theory as follows.

Consider the CFT of two Majorana-Weyl fermions $\psi^{1,2}(z)$, and form the complex combinations

$$\psi = 2^{-1/2}(\psi^1 + i\psi^2) \quad ,\bar{\psi} = 2^{-1/2}(\psi^1 - i\psi^2) \quad . \tag{201}$$

These have the properties

$$\psi(z)\bar{\psi}(0) \sim \frac{1}{z} \quad ,\psi(z)\psi(0) = O(z) \quad ,\bar{\psi}(z)\bar{\psi}(0) = O(z) \quad . \tag{202}$$

Now let $H(z)$ be the holomorphic part of a scalar field,

$$H(z)H(0) \sim -\ln\, z \quad . \tag{203}$$

Consider the basic operators $e^{\pm iH(z)}$. These have the OPE

$$e^{iH(z)}e^{-iH(0)} \sim \frac{1}{z} \quad , \tag{204}$$

$$e^{iH(z)}e^{iH(0)} = O(z) \quad , \tag{205}$$

$$e^{-iH(z)}e^{-iH(0)} = O(z) \quad . \tag{206}$$

Eqs. (206) and (202) are identical in form, and so the expectation values of $\psi(z)$ on the sphere are identical to those of $e^{iH(z)}$. We will write

$$\psi(z) \approx e^{iH(z)} \quad , \quad \bar{\psi}(z) \approx e^{-iH(z)} \tag{207}$$

to indicate this. Of course all of this extends to the antiholomorphic case,

$$\tilde{\psi}(\bar{z}) \approx e^{i\tilde{H}(\bar{z})} \quad , \quad \bar{\tilde{\psi}}(\bar{z}) \approx e^{-i\tilde{H}(\bar{z})} \quad . \tag{208}$$

In order for these theories to be the same as CFTs, the energy-momentum tensors must be equivalent. The easiest way to show this is via the operator products

$$e^{iH(z)}e^{-iH(-z)} = \frac{1}{2z} + i\partial H(0) + 2zT_B^H(0) + O(z^2) \quad , \tag{209}$$

$$\psi(z)\bar{\psi}(-z) = \frac{1}{2z} + \psi\bar{\psi}(0) + 2zT_B^{\psi}(0) + O(z^2) \quad . \tag{210}$$

This rather surprising equivalence is known as *bosonization.* Equivalence between field theories with very different actions and fields occurs frequently in two dimensions, especially in CFTs because holomorphicity puts strong constraints on the theory.

For the superstring in ten dimensions we need five bosons, H^a for $a = 0, ..., 4$. Then [1]

$$2^{-1/2}\left(\pm\psi^0 + \psi^1\right) \approx e^{\pm iH^0} \tag{211}$$

$$2^{-1/2}\left(\psi^{2a} \pm i\psi^{2a+1}\right) \approx e^{\pm iH^a} \quad , \quad a = 1, ..., 4 \tag{212}$$

The general bc CFT, renaming $\psi \to b$ and $\bar{\psi} \to c$ is obtained by modifying the energy-momentum tensor of the $j = \frac{1}{2}$ theory (see lectures by L. Alvarez Gaume). Since the inner product for the reparameterization ghosts makes b and c Hermitean, the bosonic field H must be anti Hermitean in this application. The bosonization of the ghosts is usually written in terms of a Hermitean field with the opposite OPE,

$$H \to i\rho \quad ; \quad c \approx e^{\rho} \quad , \quad b \approx e^{-\rho} \quad . \tag{213}$$

In a similar way we can bosonize the $\beta\gamma$ *system.* Of course β and γ are already bosonic, but bosonization here refers to a rewriting of the theory in a way that is similar to, but a bit more intricate than, the bosonization of the anticommuting bc theory.

Start with the current $\beta\gamma$. Similarly as for the bc system there is a U(1) symmetry with current:

$$j^{\beta\gamma}(z) = - : \beta(z)\gamma(z) := \sum_n z^{-n-1} j_n^{\beta\gamma} \tag{214}$$

The operator product

$$\beta\gamma(z)\beta\gamma(0) \sim -\frac{1}{z^2} \tag{215}$$

is the same as that of $\partial\phi$, where $\phi(z)\phi(0) \sim -\ln z$ is a holomorphic scalar. Holomorphicity then implies that this equivalence extends to all correlation functions,

$$\beta\gamma(z) \approx \partial\phi(z) \quad . \tag{216}$$

The OPE of the current with β and γ then suggests

$$\beta(z) \approx e^{-\phi(z)} \quad , \quad \gamma(z) \approx e^{\phi(z)} \quad . \tag{217}$$

For the bc system we would be finished: this approach leads to the same bosonization as before. For the $\beta\gamma$ system, however, the sign of the current-current OPE and therefore of the $\phi\phi$ OPE is changed. The would-be bosonization above gives the wrong OPEs: it would imply

$$\beta(z)\beta(0) = O(z^{-1}), \quad \beta(z)\gamma(0) = O(z), \quad \gamma(z)\gamma(0) = O(z^{-1}), \tag{218}$$

[1] The precise operator definition has a subtlety when there are several species of fermion. The H^a for different a are independent and so the exponentials commute rather than anticommute. A cocycle is needed (more later).

whereas the correct OPE is

$$\beta(z)\beta(0) = O(z^0), \quad \beta(z)\gamma(0) = O(z^{-1}), \quad \gamma(z)\gamma(0) = O(z^0). \tag{219}$$

To repair this, additional factors are added,

$$\beta(z) \approx e^{-\phi(z)}\partial\xi(z), \quad \gamma \approx e^{\phi(z)}\eta(z) \tag{220}$$

In order not to spoil the OPE with the current $\beta\gamma \approx \partial\phi(z)$, the new fields $\eta(z)$ and $\xi(z)$ must be nonsingular with respect to ϕ, which means that the $\eta\xi$ theory is a new CFT, decoupled from the ϕ CFT. Further, the equivalence above will hold – all OPEs will be correct – if η and ξ satisfy

$$\eta(z)\xi(0) \sim \frac{1}{z}, \quad \eta(z)\eta(0) = O(z), \quad \partial\xi(z)\partial\xi(0) = O(z) \quad . \tag{221}$$

This identifies the $\eta\xi$ theory as a holomorphic CFT of the bc type: the OPE of like fields has a zero due to the anticommutativity, and the conformal dimensions are $h(\eta) = 1$ and $h(\xi) = 0$.

The analysis is now equivalent to that of the ghost current in the bosonic string. The differences are the weights of the fields: $h(\beta) = \frac{3}{2}$ and $h(\gamma) = -\frac{1}{2}$; and the background charge which in this case is $Q^{\beta\gamma} = 2$. The U(1) charges of the $\beta\gamma$ fields are as those of the bc system: -1 for β and 1 for γ. Therefore any non vanishing vacuum expectation value must have a total charge -2.

Exercise: Compute the background charge for the $\beta\gamma$ system.

Now we have to check that the identification (220) correctly gives the charge of the bosonic fields. This can be done computing the OPE of $j^{\beta\gamma}$ with $e^{q\phi}$. It turns out that $e^{q\phi}$ has ghost charge q as expected according to the identifications. Finally the conformal weights also agree.

This is all we need to define vertex operators in the NS sector. The physical state conditions

$$(L_n + a\delta_{n,0}|phys >= 0, \quad n \geq 0 \quad , \quad G_r|phys >= 0, \quad r \geq 0 \tag{222}$$

are more restrictive than in the bosonic case. Therefore it is not enough to require conformal dimension 1 for the vertex operators. This would take into account the first condition above. In order to assure the second constraint we have to ask that the OPE of T_F with the vertex operators does not posses poles higher that 1. This is enough since inverting the mode expansion we have

$$G_r = \oint \frac{dz}{2\pi i} T_F(z) z^{r+1/2} \tag{223}$$

Thus we nust have

$$G_r V(w) = \oint_w \frac{dz}{2\pi i} z^{r+1/2} T_F(z) V(w) = 0 \tag{224}$$

For $r \geq 1/2$ the expression above vanishes if the OPE has no higher poles than 1.

We can now construct a few vertex operators in the NS sector. A general vertex operator can be written as (we omit bosonic excitations):

$$V \propto e^{i\lambda_a H^a} e^{q\phi} e^{ip_\mu X^\mu} \tag{225}$$

This has conformal dimension:

$$h(V) = \frac{1}{2}\lambda^2 - \frac{1}{2}q^2 - q + \frac{\alpha' p^2}{4}. \tag{226}$$

Thus the operator corresponding to the tachyon must have $p^2 = -m^2 = \frac{2}{\alpha'}$ and one does not expect fermionic excitations, therefore we can fix $\lambda^2 = 0$. Therefore the weight one condition fixes $q = -1$. The tachyon vertex operator is then

$$V_T^{-1} \propto e^{-\phi} e^{ip_\mu X^\mu} \quad \text{with} \quad p^2 = \frac{2}{\alpha'} \tag{227}$$

The superindex -1 denotes the $\beta\gamma$ ghost charge of the operator. It is easy to see that this operator verifies the correct OPE with T_F to satisfy (224). Therefore it satisfies all the necessary conditions to be a good vertex operator and it effectively creates the tachyon.

Similarly we can obtain the operator for the massless state in the NS sector. In this case we expect $p^2 = 0$ and one fermionic excitation $\lambda^2 = 1$. Therefore the conformal weight 1 condition requires again $q = -1$. The vertex operator is then

$$V_{massless}^{-1} \propto \zeta_\mu \psi^\mu e^{-\phi} e^{ip_\mu X^\mu} \quad \text{with} \quad p^2 = 0. \tag{228}$$

The constraint (224) requires that $\zeta_\mu p^\mu = 0$. This condition is equivalent to the requirement $G_{\frac{1}{2}} V_{massless}^{-1}|0>= 0$.

Picture changing

In this way one can construct all the vertex operators associated to the NS sector. However notice that in this way we can only construct non vanishing amplitudes of two vertex operators, so that the total charge is -2. So we have to find other operators with different ghost charge. This can be achieved with a different choice of the free parameters in the general vertex operator (225). The components of the vector λ must be always integer since they represent fermions ψ. Therefore we can choose for the tachyon $q = 0$ and $\lambda^2 = 1$ so that the new vertex operator is

$$V_T^0 \propto -ip_\mu \psi^\mu e^{ip\cdot X} \quad \text{with} \quad p^2 = \frac{2}{\alpha'}. \tag{229}$$

We contracted the field ψ^μ with p_μ since there should not be free indices and, since this is a tachyon, we cannot insert a polarization tensor, thus p was the unique option. We say that the operator V_T^0 is in the 0 *picture* whereas those computed before are in the -1 *picture*. It is easy to verify that they all have conformal dimension 1, and only differ in their ghost charge.

We can proceed similarly with the other states. For example the vertex operator for the massless states is

$$V^0_{massless} \propto -\zeta_\mu(\partial X^\mu + \psi^\mu(p_\rho \psi^\rho))e^{ip\cdot X} \quad \text{with} \quad p^2 = 0 \tag{230}$$

Notice that we used bosonic excitations. In this case there were two operators compatible with the mass and conformal weight conditions. We used a lineal combination of them. There is a unique way to choose these combinations. This can be obtained from a superfield analysis or using the BRST charge. We do not have time to present either of them here. Nevertheless we introduce a *picture changing operator* which produces the correct results when it is applied to the -1 *picture* repeatedly:

$$P_{+1}(z) = e^{\phi}\psi^\mu \partial X_\mu(z) \tag{231}$$

$$V_{q+1} = lim_{w\to z} P_{+1}(w)V_q(z). \tag{232}$$

Exercise: Verify that the vertices above can be obtained using the picture changing operator.

We now have to construct the vertex operators in the R sector.

Bosonization extends readily to the R sector. In this case there are the two degenerate ground states. The essential point to consider is that the fermionic fields ψ^μ have a *branch cut* in this sector since the Laurent expansions involve semi integer powers and the fields may thus be multivalued. Therefore the vertex operator must contain a *non local* field introducing this feature.

The field with these features is introduced by analogy with the bosonization of ψ^μ in the NS sector. Since we need square roots introducing the branch cut we try the following operator:

$$|s> \approx e^{isH} \quad , \quad s = \pm\frac{1}{2} \quad . \tag{233}$$

The vertex operator Θ_s for a R state $|\mathbf{s}>$ is

$$\Theta_s \approx exp\left[i\sum_a s_a H^a\right] \quad . \tag{234}$$

This operator, which produces a branch cut in ψ^μ, is sometimes called a *spin field.* For closed string states, this is combined with the appropriate antiholomorphic vertex operator, built from $\tilde{H}^a$.

This can be formally derived using Lie algebra lattices. It is then possible to see that the indeces chosen correspond to vector (ψ^μ) and spinor ($\Theta_{\mathbf{s}}$) representations of SO(10). These properties should be expected if we recall that the R sector constains the spacetime fermions of the theory.

The operator Θ_s has conformal dimension $\frac{5}{8}$. Therefore in order to build vertex operators we have to add other fields to get total dimension 1. As usual we have to include an exponential $e^{ip\cdot X}$. For the massless state this has conformal weight zero. Therefore the additional insertion cannot come from matter fields. Proceeding as in the NS sector we propose the following vertex operator:

$$V^{-1/2}_{massless} = u^{\mathbf{s}}(p) e^{-\frac{\phi}{2}} \Theta_{\mathbf{s}}(z) e^{ip\cdot X} \tag{235}$$

We have inserted the square root of a ghost field. This is again a spinor representation, this time for the ghost fields. The charge of this operator is $-\frac{1}{2}$ as indicated by the superindex. The physical state conditions are satisfied for $p^2 = 0$ and $u^{\mathbf{s}}$ has to verify the Dirac equation. This is a direct consequence of the OPEs of the spin fields.

Similarly as in the NS sector one can build vertex operators with different ghost charge acting with P_{+1}. The vertex operator with ghost charge $+\frac{1}{2}$ is important to compute scattering amplitudes. It turns out to be

$$V^{\frac{1}{2}}_{massless} = \frac{1}{\sqrt{2}} u_{\mathbf{s}}(p) e^{-\frac{\phi}{2}} \{\partial X^\mu + \frac{i}{4}(p_\rho \psi^\rho)\psi^\mu\} \Gamma^{\mathbf{ss}'}_\mu S_{\mathbf{s}'}(z) e^{ip\cdot X} \tag{236}$$

Adding the contribution -1 of the bc ghosts, the weight of $e^{-\phi/2}$ and of $e^{-\phi}$ agree with the values $a^g = -\frac{5}{8}$ in R and $a^g = -\frac{1}{2}$ in NS. The R ground state vertex operators are then

$$\mathcal{V}_s = e^{-\phi/2}\Theta_s, \tag{237}$$

with the spin field Θ_s having been defined above.

We need to extend the definition of the world-sheet fermion number F to be odd for β and γ. The ultimate reason is that it anticommutes with the supercurrent T_F and we will need it to commute with the BRST operator, which contains terms such as γT_F. The natural definition for F is then that it be the charge associated with the current $\beta\gamma$, which is l for $e^{l\phi}$. Again it is conserved by the OPE. This accounts for the ghost contributions in $exp(\pi i F)|0>_{NS} = -|0>_{NS}$, $exp(\pi i F)|s>_R = |s'>_R \Gamma_{s's}$. Note that this definition is based on spin rather than statistics, since the ghosts have the wrong spin-statistics relation; it would therefore be more appropriate to call F the *world-sheet spinor number.*

Using the techniques we have introduced it is possible in principle to compute arbitrary scattering amplitudes.

4 One loop partition function and modular invariance

At one-loop there are 4 Riemann surfaces with Euler number zero. The torus is the only closed oriented surface with Euler number zero. If we include unoriented surfaces we have the Klein bottle for the closed string (two crosscaps). In the open string we have surfaces with boundaries: the cylinder and Möbius strip.

We now compute the simplest one-loop amplitude in the closed oriented string theory: the partition function or vacuum amplitude. The possibility to assign to the world-sheet fermions periodic or antiperiodic boundary conditions leads to the concept of *spin structures*. The GSO projection is then shown to be the geometric constraint of modular invariance.

In Field Theory one usually does not pay much attention to the one-loop vacuum amplitude. This is a function of the masses of the finite number of fields of a given model, fully determined by the free spectrum that, aside from its relation to the cosmological constant, does not embody important structural information. On the other hand, strings describe infinitely many modes, and their vacuum amplitudes satisfy a number of geometric constraints, that in a wide class of models essentially determine the full perturbative spectrum. There is a great deal of physics in the amplitude with no physical operators $< 1 >_{T^2(\tau)} \equiv Z(\tau)$

To describe a torus we need to identify two periods in w. Alternatively we can cut the torus along the two cycles and map it to the plane. We thus describe it as the complex plane with metric

$$ds^2 = dw d\bar{w} \tag{238}$$

and identifications

$$w \approx w + 2\pi \approx w + 2\pi\tau \tag{239}$$

There are two moduli, the real and imaginary parts of $\tau = \tau_1 + i\tau_2$ and two CKV, the translation. See figure

The conformal gauge $g_{ab} = \delta_{ab}$ can be fixed locally but *globally* there is a missmatch between the space of metrics and the world-sheet gauge group. The point particle is a good example. Consider the path integral

$$\int \mathcal{D}e\mathcal{D}X exp\left[-\frac{1}{2}\int d\tau\left(e^{-1}\dot{X}^\mu\dot{X}_\mu + em^2\right)\right] \tag{240}$$

Take a path forming a closed loop in spacetime, so the topology is a circle. The parameter τ can be taken to run from 0 to 1 with the endpoints identified. That is $X^\mu(\tau)$ and $e(\tau)$ are periodic on $0 \leq \tau \leq 1$. The tetrad $e(\tau)$ has one component and there is one local symmetry, the choice of parameter, so as for the string there is just enough local symmetry to fix the tetrad completely. The tetrad transforms as $e'd\tau' = ed\tau$. The gauge choice $e' = 1$ thus gives a differential equation for $\tau'(\tau)$,

$$\frac{\partial \tau'}{\partial \tau} = e(\tau). \tag{241}$$

Integrating this with the boundary condition $\tau'(0) = 0$ determines

$$\tau'(\tau) = \int_0^\tau d\tau'' e(\tau''). \tag{242}$$

The complication is that in general $\tau'(1) \neq 1$ so the periodicity is not preserved. In fact

$$\tau'(1) = \int_0^1 d\tau e(\tau) = l \tag{243}$$

is the invariant length of the circle. So we cannot simultaneouly set $e' = 1$ and keep the coordinate region fixed. We can hold the coordinate region fixed and set e' to the constant value $e' = l$ or set $e' = 1$ and let the coordinate region vary:

$$e' = l, \quad 0 \leq \tau \leq 1, \tag{244}$$

$$e' = 1, \quad 0 \leq \tau \leq l \tag{245}$$

In either case after fixing the gauge invariance we are left with an ordinary integral over l. In other words, not all tetrads on the circle are diff-equivalent. There is a one parameter family of inequivalent tetrads, parametrized by l.

Both descriptions have analogs in the string. Take the torus with coordinate region

$$0 \leq \sigma^1 \leq 1 \quad , \quad 0 \leq \sigma^2 \leq 1, \tag{246}$$

with $X^\mu(\sigma^1, \sigma^2)$ and $g_{ab}(\sigma^1, \sigma^2)$ periodic in both directions. Equivalently we can think of this as the σ plane with the identification of points,

$$(\sigma^1, \sigma^2) \approx (\sigma^1, \sigma^2) + (m, n) \tag{247}$$

for integer m and n.

To what extent is the field space diff $\times$ Weyl redundant? The theorem is that it is *not* possible to bring a general metric to unit form by a diff $\times$ Weyl transformation that leaves invariant the periodicity (246), but it is possible to bring it to the form

$$ds^2 = |d\sigma^1 + \tau d\sigma^2|^2 \tag{248}$$

where τ is a complex constant. For $\tau = i$ this would be the unit metric δ_{ab}.

Alternatively one can take the flat metric. By coordinate and Weyl transformations we can keep the metric flat but it is not waranteed that this will leave the periodicity unchanged. Rather we may now have

$$\tilde{\sigma}^a \sim \tilde{\sigma}^a + (mu^a + nv^a) \tag{249}$$

with general translation vectors u^a and v^a. By rotating and rescaling the coordinate system accompanied by a shift in the Weyl factor we can always set $u = (1, 0)$. This leaves two parameters, the components of v. Thus defining $w = \tilde{\sigma}^1 + i\tilde{\sigma}^2$ the metric is $ds^2 = dwd\bar{w}$ and the periodicity is $w \sim w + (m + n\tau)$ where $\tau = v^1 + iv^2$. The torus now is the parallelogram in the w plane with periodic boundary conditions.

Alternatively define σ^a by $w = \sigma^1 + \tau\sigma^2$. The original periodicity

$$(\sigma^1, \sigma^2) \sim (\sigma^1, \sigma^2) + (m, n) \tag{250}$$

is preserved but the metric now takes the more general form (248).

The integration over metrics reduces to two ordinary integrals, over the real and imaginary parts of τ. The metric (248) is invariant under complex conjugation of τ and degenerate for τ real, so we can restrict attention to Im $\tau > 0$. As in the case of the circle we can put these parameters either in the metric or the periodicity. The parameter τ is known as a *Teichmuller parameter* or a *modulus*.

There is some additional redundancy which does not have an analog in the point-particle case. The value $\tau + 1$ generates the same set of identifications $w \approx w + 2\pi(m + n\tau)$ as τ, replacing $(m, n) \to (m - n, n)$. So does $-1/\tau$, defining $w' = \tau w$ and replacing $(m, n) \to (n, -m)$. Repeated application of these two transformations

$$T: \quad \tau' = \tau + 1, \quad S: \quad \tau' = -1/\tau, \tag{251}$$

generates

$$\tau' = \frac{a\tau + b}{c\tau + d} \tag{252}$$

for all integer a, b, c, d such that $ad-bc = 1$. This group of conformal transformations is known as the conformal group or $SL(2,Z)/Z_2 = PSL(2,Z)$, because τ' is not changed if all the signs of a, b, c, d are reversed.

Using the modular transformations it can be shown that every τ is equivalent to exactly one point in the region $\mathcal{F}_0$:

$$-\frac{1}{2} \leq Re\tau \leq \frac{1}{2} \quad , \quad |\tau| \geq 1 \tag{253}$$

This is called a fundamental region and it is one representation of the moduli space of (diff $\times$ Weyl)-inequivalent metrics.

In order to compute the partition function it is convenient to start from the simplest case of a scalar mode of mass M in a field theory in D dimensions, for which

$$S = \int d^D x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2\right). \tag{254}$$

After a Euclidean rotation, the path integral defines the vacuum energy Z as

$$e^{-Z} = \int \mathcal{D}\phi e^{-S_E} \sim \det{}^{-\frac{1}{2}}\left(-\Delta + M^2\right), \tag{255}$$

whose M dependence may be extracted using the identity

$$\log(\det(A)) = -\int_\epsilon^\infty \frac{dt}{t}\mathrm{tr}\left(e^{-tA}\right), \tag{256}$$

where ϵ is an ultraviolet cutoff and t is a Schwinger parameter. In our case, the complete set of momentum eigenstates diagonalizes the kinetic operator, and

$$Z = -\frac{V}{2}\int_\epsilon^\infty \frac{dt}{t}e^{-tM^2}\int \frac{d^D p}{(2\pi)^D}e^{-tp^2}, \tag{257}$$

where V denotes the volume of spacetime. Performing the Gaussian momentum integral then yields

$$Z = -\frac{V}{2(4\pi)^{D/2}}\int_\epsilon^\infty \frac{dt}{t^{D/2+1}}e^{-tM^2}, \tag{258}$$

while similar steps for a Dirac fermion of mass M in D dimensions would result in

$$Z = \frac{V 2^{D/2}}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}} e^{-tM^2}, \tag{259}$$

with an opposite sign, on account of the Grassmann nature of the fermionic path integral. These results can be easily extended to generic Bose and Fermi fields, since Z is only sensitive to their physical modes, and is proportional to their number. Therefore in the general case they are neatly summarized in the expression

$$Z = -\frac{V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}} \mathrm{Str}\left(e^{-tM^2}\right), \tag{260}$$

where Str counts the signed multiplicities of Bose and Fermi states. When applying this equation to the bosonic string in the critical dimension $D = 26$, whose spectrum is encoded in

$$M^2 = \frac{2}{\alpha'}(L_0 + \tilde{L}_0 - 2), \tag{261}$$

subject to the constraint $L_0 = \tilde{L}_0$ (260) the gives

$$Z = -\frac{V}{2(4\pi)^{13}} \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \int_\epsilon^\infty \frac{dt}{t^{14}} \mathrm{tr}\left(e^{-\frac{2}{\alpha'}(L_0+\tilde{L}_0-2)t} e^{2\pi i(L_0-\tilde{L}_0)s}\right), \tag{262}$$

where we have introduced a δ-function constraint. Defining the complex Schwinger parameter

$$\tau = \tau_1 + i\tau_2 = s + i\frac{t}{\alpha'\pi}, \tag{263}$$

and letting $q = e^{2\pi i\tau}, \bar{q} = e^{-2\pi i\bar{\tau}}$ it can be rewritten as

$$Z = -\frac{V}{2(4\pi^2\alpha')^{13}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_\epsilon^\infty \frac{d\tau_2}{\tau_2^{14}} \mathrm{tr}\left(q^{L_0-1}\bar{q}^{\tilde{L}_0-1}\right). \tag{264}$$

Actually, at one loop a closed string sweeps a torus, whose Teichmuller parameter is naturally identified with the complex Schwinger parameter τ but not all values of τ correspond to distinct torii as we have just seen. One has to restrict the integration domain to a fundamental region of the modular group, for instance the region $\mathcal{F}_0 = \{-\frac{1}{2} < \tau_1 \leq \frac{1}{2}, |\tau| \geq 1\}$ and this introduces an effective ultraviolet cutoff. After a final rescaling one is led to the torus amplitude, that defines the partition function for the closed bosonic string as

$$\begin{aligned} \mathsf{Z}_T &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{12}} \mathrm{tr}\left(q^{L_0-1}\bar{q}^{\tilde{L}_0-1}\right) && (265) \\ &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_T(\tau,\bar{\tau}). && (266) \end{aligned}$$

This expression is modular covariant, *i.e.*

$$Z_T(\tau+1,\bar\tau+1) = Z_T(\tau,\bar\tau) \tag{267}$$

$$Z_T(-\frac{1}{\tau},-\frac{1}{\bar\tau}) = |\tau|^2 Z_T(\tau,\bar\tau) \tag{268}$$

so that its transformations compensate those of the measure.

Exercise 1: Verify the modular transformation properties of (266).

It is instructive to recall the explicit expresion for the vacuum amplitude in the bosonic string. Recall that L_0 and $\tilde L_0$ are effectively number operators for two infinite sets of harmonic oscillators. In particular, for each transverse spacetime dimension

$$L_0 = \sum_n :\alpha_{-n}\alpha_n: \tag{269}$$

while for each n

$$\mathrm{tr} q^{\alpha_{-n}\alpha_n} = 1 + q^n + q^{2n} + ... = \frac{1}{1-q^n}, \tag{270}$$

and putting all these contributions together for the full spectrum gives

$$\mathsf{Z}_T = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\frac{1}{\tau_2^{12}}\frac{1}{|\eta(\tau)|^{48}}, \tag{271}$$

where we have defined the Dedekind η function

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n). \tag{272}$$

In the case at hand all string states are oscillator excitations of the tachyonic vacuum, while the factor τ_2^{-12} can be recovered from the integral over the continuum of transverse momentum modes, as

$$(\alpha')^{12}\int d^{24}p e^{-\pi\alpha'\tau_2 p^2} \tag{273}$$

Deriving the partition function in this way it is evident that $Z(\tau,\bar\tau)$ contains the information about the level density of string states, *i.e.* the number of states of each mass level. Expanding $Z(\tau,\bar\tau)$ in powers of q one gets a power series of the form $\sum d_{mn}q^n\bar q^m$ where d_{nm} is simply the number of states with $m^2 = n$ and $\tilde m = m$. The first few terms of the expansion are

$$Z(\tau,\bar\tau) \sim |\eta(\tau)|^{-48} = \frac{1}{|q|^2} + 576 + ... \tag{274}$$

The first term corresponds to the negative (mass)^2 tachyon and the constant term to the massless string states (graviton, dilaton and antisymmetric tensor). Due to the tachyon pole one finds that the one-loop cosmological constant fo the closed bosonic string is infinite.

We are now ready to perform the explicit computation of $Z_T(\tau,\bar\tau)$ for the superstring. The trace includes a sum over the different sectors of the superstring Hilbert space. In each sector it breaks up into a product of independent sums over the transverse X,ψ and $\tilde\psi$ oscillators and the transverse Hamiltonian similarly breaks up into a sum.

Let us parametrize the torus by two coordinates $\sigma^1,\sigma^2 \in [0,1]$ and define complex coordinates $z=\sigma^1+\tau\sigma^2$ and $\bar z=\sigma^1+\bar\tau\sigma^2$ in terms of which the metric is $ds^2=|dz|^2$. Recall that modular transformations are those changes of τ which lead to identical complex structures. They are generated by $S:\tau\to -1/\tau$ and $T:\tau\to\tau+1$. Under a general modular transformation

$$\tau \to \frac{a\tau+b}{c\tau+d} \tag{275}$$

the metric changes to

$$ds^2 \to \frac{1}{|c\tau+d|^2}|d\sigma^{1\prime}+\tau d\sigma^{2\prime}|^2 \tag{276}$$

where $(\sigma^{1\prime},\sigma^{2\prime})=(d\sigma^1+b\sigma^2, c\sigma^1+a\sigma^2)$. We have the possible boundary conditions for fermions, leading to four spin structures:

$$\psi(\sigma^1+1,\sigma^2) = \pm\psi(\sigma^1,\sigma^2) \tag{277}$$

$$\psi(\sigma^1,\sigma^2+1) = \pm\psi(\sigma^1,\sigma^2) \tag{278}$$

Periodic boundary conditions in σ^1 correspond to the Ramond sector and antiperiodic boundary conditions to the NS sector. The boundary conditions may be separately periodic or antiperiodic in the σ^1 and σ^2 directions. We shall denote the spin structure with periodic boundary condition in σ^2 and antiperiodic boundary condition in σ^1 by

Under an S transformation with modular matrix

$$S=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : (\sigma^1,\sigma^2)\to(\sigma^2,-\sigma^1) \tag{279}$$

that is basically σ^1 and σ^2 are exchanged. This means that the fermions transform as

$$\psi(\sigma^1,\sigma^2)\to\psi'(\sigma^1,\sigma^2)\propto\psi(\sigma^2,-\sigma^1) \tag{280}$$

from which we easily derive the following action of S on the boundary conditions or spin structures

$$S: \quad (++) \to (++) \quad \begin{array}{ll} (--) & \to (--) \\ (+-) & \to (-+) \\ (-+) & \to (+-) \end{array} \tag{281}$$

In the same way we find that under $\tau \to \tau + 1$ the torus transforms as

which leads to the following action of T on the spin structures:

$$T: \quad (++) \to (++) \quad \begin{array}{ll} (--) & \to (-+) \\ (+-) & \to (+-) \\ (-+) & \to (--) \end{array} \tag{282}$$

We can see this as follows. Starting from , shifting the upper edge of the box one unit to the right means that the new time direction, from the lower left to upper right, sees both the formerly anti-periodic boundary conditions, to give an overall periodic boundary condition. In the absence of any projection, loop amplitudes contain the factor

$$\mathrm{Tr} e^{2\pi i \tau H} \tag{283}$$

for propagation through imaginary time τ. It is essential now to remember that in the path integral formulation of quantum statistical mechanics, the partition function of the fermions is computed using *antiperiodic* boundary conditions in the σ^2 direction. The trace above is thus naturally represented by a path integral with antiperiodic $(-)$ boundary conditions in the σ^2 direction. If on the other hand we wish to calculate the quantity

$$\mathrm{Tr}(-1)^F e^{2\pi i \tau H} \tag{284}$$

with $(-1)^F$ being the operator used in the GSO projection that counts the number of world-sheet fermions modulo 2, then we must use the $+$ boundary conditions in the σ^2 direction.

Therefore in the absence of the GSO projection, the contribution of the NS sector to a loop amplitude corresponds to $(--)$ boundary conditions, while the contribution of the R sector corresponds to $(+-)$ boundary conditions. The combination of partition functions $(--)$ and $(+-)$ is not modular invariant. To get a modular invariant theory, they must be supplemented by $(-+)$. But $(-+)$ is a partition function for NS states

($-$ boundary conditions in the σ^1 direction) with an insertion of $(-1)^F$ (+ boundary conditions in the σ^2 direction).

The periodicity conditions in the time direction are rather unusual in the context of field theory, but may be expressed as conditions on correlation functions on the torus. Consider the generic correlation function of fermions

$$< \psi(\sigma^1, \sigma^2) X > \tag{285}$$

where X stands for the product of an *odd* number of fermion fields at various positions, so that the correlator is nonzero (the correlator of an odd number of fermions is zero since they are Grassmann numbers). We take this fermion from its position σ^2 to σ^2+1 via some continuous path. Within the operator formalism, this means that the ψ will go through all the possible instants of time (modulo the periodicity) and will have to be passed over all the other fermions in X in succession, because of the time-ordering. Since a minus sign is generated each time, there will be an overall factor of -1 generated by this translation, and therefore the usual correspondence between the path integral and the Hamiltonian approach leads naturally to the antiperiodic condition when the theory is defined on a torus. To implement the periodic condition we need to modify the usual correspondence by inserting in all the correlators an operator that anticommutes with $\psi(\sigma^1, \sigma^2)$. Such an operator is $(-1)^F$, where F is the fermion number. To make sure that this feature is built into the partition function, we simply insert $(-1)^F$ in the definition of the partition function, within the trace, in the time-periodic case. This prescription implies the following expressions for the holomorphic partition functions:

$$\begin{aligned}
Z_{(--)}(\tau) &= \xi_{(--)}\mathrm{Tr}_{NS}\left[q^H\right], && (286)\\
Z_{(-+)}(\tau) &= \xi_{(-+)}\mathrm{Tr}_{NS}\left[exp(\pi i F)q^H\right], && (287)\\
Z_{(+-)}(\tau) &= \xi_{(+-)}\mathrm{Tr}_{R}\left[q^H\right], && (288)\\
Z_{(++)}(\tau) &= \xi_{(++)}\mathrm{Tr}_{R}\left[exp(\pi i F)q^H\right]. && (289)
\end{aligned}$$

where the ξ are phases to be determined by modular invariance.

The Hamiltonians are

$$H_R = \sum_{r=1}^{\infty} r\psi^i_{-r}\psi^i_r + \frac{1}{3} \tag{290}$$

$$H_{NS} = \sum_{r=1/2}^{\infty} r\psi^i_{-r}\psi^i_r - \frac{1}{6} \tag{291}$$

The normal ordering constants follow by subtracting the bosonic contribution $-\frac{D-2}{24}$ from the total normal ordering constant in each sector, namely 0 (R) and $-\frac{1}{2}$ (NS).

It is now easy to evaluate the different contributions to the partition function. For instance for $Z^{\psi}_{(--)}(\tau)$ we get

$$
\begin{aligned}
Z^{\psi}_{(--)}(\tau) &= \xi_{(--)}\mathrm{Tr}q^{H_{NS}} && (292)\\
&= \xi_{(--)}q^{-1/6}\mathrm{Tr}q^{\sum_{r=1/2}^{\infty} r\psi^i_{-r}\psi^i_r} && (293)\\
&= \xi_{(--)}q^{-1/6}\prod_r \left(\sum_{N_r} q^{rN_r}\right)^8 && (294)\\
&= \xi_{(--)}q^{-1/6}\left(\prod_{r=1}^{\infty}(1+q^{r-1/2})\right)^8 . && (295)
\end{aligned}
$$

The calculation is completely analogous to the bosonic case only that the occupation numbers are now restricted by the Pauli principle to $N_r = 0$ and 1.

For the fermionic oscillators we have

$$
\mathrm{tr}\left(q^{\sum_r r\psi_{-r}\psi_r}\right) = \prod_r \mathrm{tr}\left(q^{r\psi_{-r}\psi_r}\right) = \prod_r (1+q^r)^8, \tag{296}
$$

since the Pauli exclusion principle allows at most one fermion in each of these states. This expression actually applies to both the NS and R sectors, provided r is turned into an integer in the second case.

For a given fermionic mode, there are only two states and the traces are trivially calculated:

$$
\begin{aligned}
\mathrm{Tr}q^{r\psi_{-r}\psi_r} &= 1+q^r && (297)\\
\mathrm{Tr}q^{r\psi_{-r}\psi_r}(-1)^F &= 1-q^r . && (298)
\end{aligned}
$$

We can now write

$$
\begin{aligned}
q^{-1/6}\prod_{r=1}^{\infty}\left(1+q^{r-1/2}\right)^8 &= \left\{q^{-1/24}\prod_{r=1}^{\infty}(1-q^r)^{-1}\right\}^4 \left\{\prod_{r=1}^{\infty}(1-q^r)^4\left(1+q^{r-1/2}\right)^8\right\}\\
&= \frac{\Theta_3^4(0|\tau)}{\eta^4(\tau)} && (299)
\end{aligned}
$$

where $\eta(\tau)$ is the Dedekind η function and Θ_3 is one of the four Jacobi theta functions.

In general we can define the theta functions

$$
\begin{aligned}
\Theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](0|\tau) &= \eta(\tau)e^{2\pi i\alpha\beta}q^{\frac{\alpha^2}{2}-\frac{1}{24}}\prod_{r=1}^{\infty}\left(1+q^{r+\alpha-1/2}e^{2\pi i\beta}\right)\left(1+q^{r-\alpha-1/2}e^{-2\pi i\beta}\right) && (300)\\
&= \sum_{r=-\infty}^{+\infty} exp\left[i\pi(r+\alpha)^2\tau + 2\pi i(r+\alpha)\beta\right]. && (301)
\end{aligned}
$$

Through the one-loop partition function the Θ-functions for arbitrary α and β are in correspondence to the generalized fermion boundary conditions as

$$\psi(\sigma^1+1,\sigma^2) = -e^{-2\pi i\alpha}\psi(\sigma^1,\sigma^2), \tag{302}$$
$$\psi(\sigma^1,\sigma^2+1) = -e^{-2\pi i\beta}\psi(\sigma^1,\sigma^2). \tag{303}$$

The different spin structures then correspond to

$$(++) \qquad \alpha=\beta=1/2 \qquad \Theta\begin{bmatrix}1/2\\1/2\end{bmatrix}=\Theta_1 \tag{304}$$
$$(+-) \qquad \alpha=1/2,\beta=0 \qquad \Theta\begin{bmatrix}1/2\\0\end{bmatrix}=\Theta_2 \tag{305}$$
$$(--) \qquad \alpha=\beta=0 \qquad \Theta\begin{bmatrix}0\\0\end{bmatrix}=\Theta_3 \tag{306}$$
$$(-+) \qquad \alpha=0,\beta=1/2 \qquad \Theta\begin{bmatrix}0\\1/2\end{bmatrix}=\Theta_4 \tag{307}$$

The Jacobi theta functions and their generalizations to higher genus Riemann surfaces (the Riemann theta functions) play an important role in string theory and conformal field theory. They satisfy many amazing identities. At one loop, one of the most important ones is

$$\Theta_2^4(0|\tau)-\Theta_3^4(0|\tau)+\Theta_4^4(0|\tau)=0 \tag{308}$$

It is easy to see that $\Theta_1(0|\tau)=0$. In the same way that we have derived the partition function for the $(--)$ spin structure we easily show that

$$Z^{\psi}_{(--)}(\tau)=\xi_{(--)}\frac{\Theta_3^4(0|\tau)}{\eta^4(\tau)}, \tag{309}$$
$$Z^{\psi}_{(-+)}(\tau)=\xi_{(-+)}\frac{\Theta_4^4(0|\tau)}{\eta^4(\tau)}, \tag{310}$$
$$Z^{\psi}_{(+-)}(\tau)=\xi_{(+-)}\frac{\Theta_2^4(0|\tau)}{\eta^4(\tau)}, \tag{311}$$
$$Z^{\psi}_{(++)}(\tau)=\xi_{(++)}\frac{\Theta_1^4(0|\tau)}{\eta^4(\tau)}=0. \tag{312}$$

The modular properties of the partition function established above should be reflected by the transformation properties of the Θ- functions. And indeed they do.

Let us now determine the phases ξ. We will first require that the spin structure sum is modular invariant separately both in the left and right moving sectors. Since only the relative phases are relevant we will arbitrarily set $\xi_{(--)}=+1$, *i.e.*

$$Z^{\psi}_{(--)}(\tau)=\frac{\theta_3^4(0|\tau)}{\eta^4(\tau)}. \tag{313}$$

Using the transformation rules of the theta and eta functions we easily find

$$Z^{\psi}_{(--)}(\tau+1) = Z^{\psi}_{(-+)}(\tau) = \frac{\theta_4^4(0|\tau)}{\eta^4(\tau)} e^{-i\pi/3}. \tag{314}$$

The contribution from the eight transverse bosonic degrees of freedom is $\sim \frac{1}{\eta^8(\tau)}$ which contributes an extra factor of $e^{-2\pi i/3}$ so that we get for the phase $\xi_{(-+)} = -1$. Similarly we show that $\xi_{(+-)} = -1$. Clearly, $\xi_{(++)}$ cannot be determined from modular invariance; we will show later that it has to be ± 1. With these phases the contribution of the right-moving world-sheet fermions to the superstring partition function is

$$\begin{aligned} Z(\tau) &= \mathrm{Tr} e^{2\pi i \tau H_{NS}} \frac{1}{2}\left(1 + (-1)^{F+1}\right) - \mathrm{Tr} e^{2\pi i \tau H_R} \frac{1}{2}\left(1 - \xi_{(++)}(-1)^F\right) & (315)\\ &= \frac{1}{2}\frac{1}{\eta(\tau)^4}\left\{\Theta_3^4(0|\tau) - \Theta_4^4(0|\tau) - \Theta_2^4(0|\tau) + \xi_{(++)}\Theta_1^4(0|\tau)\right\} & (316)\end{aligned}$$

with a similar expression for the left-movers. The relative sign between the two sectors reflects the fact that states in the NS sector are bosons whereas states in the R sector are fermions. The $\frac{1}{2}(1-(-1)^F)$ in the NS sector is just the GSO projection. In the R sector it is $(-1)^F = \pm 1$ according to $\xi_{(++)} = \pm 1$. Due to the Riemann identity and the vanishing of Θ_1, the partition function vanishes. This reflects a supersymmetric spectrum: the contributions from spacetime bosons and fermions cancel.

There is another modular invariant combination of boundary conditions: it consists of summing over the same boundary conditions for the left and right movers. It follows that the left and right moving sectors are not separately modular invariant due to the non-trivial connection between their boundary conditions. This leads to the following partition function:

$$\begin{aligned} Z(\tau) = \mathrm{Tr} \quad & e^{2\pi i \tau H_{NS} - 2\pi i \bar{\tau} \bar{H}_{NS}} \frac{1}{2}(1 + (-1)^{F+\bar{F}}) - \\ & \mathrm{Tr} e^{2\pi i \tau H_R - 2\pi i \bar{\tau} \bar{H}_R} \frac{1}{2}(1 - \xi_{(++)}(-1)^{F+\bar{F}}) \end{aligned} \tag{317}$$

If we include the contribution from the bosons we get

$$\chi(\tau,\bar{\tau}) = \frac{1}{2}\frac{1}{(Im\tau)^4}\frac{|\Theta_2(0|\tau)|^8 + |\Theta_3(0|\tau)|^8 + |\Theta_4(0|\tau)|^8}{|\eta(\tau)|^{24}}. \tag{318}$$

Modular invariance of this expression can be easily checked. This theory has only spacetime bosons and contains a tachyon. The GSO projection in the NS sector is $(-1)^{F+\bar{F}} = 1$ which does allow the tachyon.

Let us now give the argument why the phase $\xi_{(++)} = \pm 1$. Clearly, for the partition function to have an interpretation as a sum over states we can only allow $\xi_{(++)} = 0$ or

± 1. If we look at the partition function at two loops it will be expressible in terms of the appropriate Riemann theta functions. In the limit in which the genus two surface degenerates to two torii, the genus two theta functions become simply products of Jacobi theta functions. Especially $\Theta_1(\tau_1)\Theta_1(\tau_2)$, where $\tau_{1,2}$ are Teichmüller parameters of the two resulting tori, is the degeneration limit of an even theta function at genus two which has to be part of the partition function since the even theta functions transform irreducibly under global diffeomorphisms of the genus two surface; that means that $\xi_{(++)}$ is excluded.