Quantização de Campos Espinoriais

(Peskin 9.5, Ryder 6.7)

No caso da quantização canônica, era necessário (para evitarmos estados de energia negativa) exigir que os campos de férmions satisfizessem relações de anti-comutação:

$$\{\psi(x), \psi(y)\} = 0$$

Isto é simples quando estes campos são operadores, mas agora queremos fazer a troca:

$$\langle \hat{\psi}(x)\,\hat{\psi}(y) \rangle = \int D\psi \; \psi(x)\,\psi(y)\, e^{iS}$$

não são operadores, mas precisamos que anti-comutem

$$\langle \hat{\psi}(x)\,\hat{\psi}(y) \rangle = -\langle \hat{\psi}(y)\,\hat{\psi}(x) \rangle$$

Precisamos de funções definindas em um espaço de números complexos que anti-comutem, o que já havia sido proposto antes por Grassmann. Os números de Grassmann satisfazem a seguinte propriedade:

$$\boxed{\{\theta, \eta\} = \theta\eta + \eta\theta = 0}$$

O que tem diversas consequências:

$$\boxed{\theta^2 = 0}$$

$$\eta_1\eta_2\eta_3 = -\eta_1\eta_3\eta_2 = \eta_3\eta_1\eta_2$$

um par de números de Grassmann se comporta como um número usual

$$f(\theta,\eta) = a_0 + a_1\theta + a_2\eta + a_3\theta\eta + a_4\underbrace{\theta^2}_{0}\eta + a_5\theta\underbrace{\eta^2}_{0}$$

$a_0, a_1, a_2, a_3 \in \mathbb{C}$

Há uma ambiguidade na definição de derivada (temos que decidir se ela age pela direita ou esquerda):

$$\frac{\partial^L}{\partial\theta} f(\theta,\eta) = a_1 + a_3\eta \qquad \frac{\partial^R}{\partial\theta} f(\theta,\eta) = a_1 - a_3\eta$$

$$\frac{\partial^L}{\partial\eta} f(\theta,\eta) = a_2 - a_3\theta \qquad \frac{\partial^R}{\partial\eta} f(\theta,\eta) = a_2 + a_3\theta$$

Definiremos: $\dfrac{\partial}{\partial\eta} = \dfrac{\partial^L}{\partial\eta}$

$$\eta\frac{\partial f}{\partial\eta} = a_2\eta - a_3\eta\theta = \underbrace{a_2\eta + a_3\theta\eta}_{f}$$

$$\eta f = a_0\eta + a_1\eta\theta \Rightarrow \frac{\partial}{\partial\eta}(\eta f) = \underbrace{a_0 + a_1\theta}_{f}$$

$$\left(\eta\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\eta}\eta\right)f = f \longrightarrow \boxed{\left\{\eta_i, \frac{\partial}{\partial\eta_j}\right\} = \delta_{ij}}$$

$$\boxed{\left\{\frac{\partial}{\partial\eta}, \frac{\partial}{\partial\theta}\right\} = 0}$$

Para definir integrais é natural assumir que os infinitesimais de Grassmann também anti-comutam:

$$\{\theta, d\eta\} = 0$$

$$\{d\theta, d\eta\} = 0$$

que resolve outra ambiguidade de sinal (se fizemos primeiro a integral de fora o sinal fica invertido)

$$\int d\theta\, d\eta\, f(\theta,\eta) \equiv \int d\theta\left[\int d\eta\, f(\theta,\eta)\right]$$

$$f(\theta,\eta) = 1 \Rightarrow \left(\int d\theta\right)^2 = \int d\theta \int d\eta = \int d\theta\, d\eta = -\int d\eta\, d\theta = -\left(\int d\theta\right)^2$$

$$\boxed{\int d\theta = 0}$$

$F(\theta) = \int d\theta\, f(\theta)$

$F(\theta)$ tem que ser linear, já que todas expansões aqui só vao até ordem 1

$$\int d\theta\, f(\theta) = \int d\theta\,(a_1 + a_2\theta) = a_2\int d\theta\,\theta \Rightarrow \int d\theta\,\theta = ?$$

Adicionalmente, queremos que:

$$\theta \rightarrow \theta + \eta \Rightarrow \int d\theta \, f(\theta) = \int d\theta \, f(\theta + \eta)$$

$$F(\theta) = F(\theta + \eta)$$

$$A + B\theta = A + B(\theta + \eta)$$

$$\forall \theta, \eta \Rightarrow B = 0$$

para satisfazer essa propriedade $\int d\theta \, \theta$ tem que ser uma constante

$$\boxed{\int d\theta \, \theta = 1}$$

$$\int d\theta \, f(\theta, \eta) = \int d\theta \left(a_0 + a_1 \theta + a_2 \eta + a_3 \theta \eta\right) =$$

$$= a_1 + a_3 \eta = \frac{\partial}{\partial \theta} f(\theta, \eta)$$

Para números de Grassmann complexos:

$$(\theta \eta)^* \equiv \eta^* \theta^* = -\theta^* \eta^* \qquad \begin{cases} \int d\eta = \int d\eta^* = 0 \\ \int d\eta \, \eta = \int d\eta^* \, \eta^* = 1 \end{cases}$$

$$e^{-\eta^* \eta} = 1 - \eta^* \eta$$

$$\int d\eta^* d\eta \, e^{-\eta^* b \eta} = \int d\eta^* d\eta \, (1 - \eta^* b \eta) = \int d\eta^* d\eta \, (1 + \eta \eta^* b) = b$$

$$\boxed{\int d\eta^* d\eta \, e^{-\eta^* b \eta} = b}$$

$$\boxed{\int d\eta^* d\eta \, \eta \eta^* \, e^{-\eta^* b \eta} = \frac{1}{b} \cdot b}$$

compare com (pg 19):

$$\int \frac{dz^*}{(2\pi i)^{1/2}} \frac{dz}{(2\pi i)^{1/2}} e^{-z^* \cdot b z} = \frac{1}{b}$$

$$\int \frac{dz^*}{(2\pi i)^{1/2}} \frac{dz}{(2\pi i)^{1/2}} z^* z \, e^{-z^* \cdot b z} = \frac{1}{b} \cdot \frac{1}{b}$$

Suponha um caso bidimensional:

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \qquad \bar{\eta} = \begin{pmatrix} \eta_1^* & \eta_2^* \end{pmatrix}$$

$$(\bar{\eta}\eta)^2 = (\eta_1^* \eta_1 + \eta_2^* \eta_2)^2 = \eta_1^* \eta_1 \eta_2^* \eta_2 + \eta_2^* \eta_2 \eta_1^* \eta_1 =$$

$$= 2 \eta_1^* \eta_1 \eta_2^* \eta_2$$

$$e^{-\bar{\eta}\eta} = 1 - \eta_1^* \eta_1 + \eta_2^* \eta_2 + \eta_1^* \eta_1 \eta_2^* \eta_2$$

$$d\bar{\eta} d\eta \equiv d\eta_1^* d\eta_1 d\eta_2^* d\eta_2$$

$$\int d\bar{\eta} d\eta \, e^{-\bar{\eta}\eta} = \int d\eta_1^* d\eta_1 d\eta_2^* d\eta_2 \, \eta_1^* \eta_1 \eta_2^* \eta_2 = 1$$

$$\boxed{\int d\bar{\eta} d\eta \, e^{-\bar{\eta}\eta} = 1}$$

Fazendo uma mudança de variáveis,obtemos:

$$\eta = M \alpha \qquad M, N \in \mathbb{C}^{2 \times 2}$$

$$\bar{\eta} = \bar{\alpha} N$$

$$\eta_1 \eta_2 = (M_{11} \alpha_1 + M_{12} \alpha_2)(M_{21} \alpha_1 + M_{22} \alpha_2) =$$

$$= (M_{11} M_{22} + M_{12} M_{21}) \alpha_1 \alpha_2 = \mathrm{DET}[M] \alpha_1 \alpha_2$$

Então, se queremos que: $\int d\eta_1\, d\eta_2\, \eta_1 \eta_2 = \int d\alpha_1\, d\alpha_2\, \alpha_1 \alpha_2$

temos que exigir: $d\eta_1\, d\eta_2 = (\mathrm{DET}[M])^{-1}\, d\alpha_1\, d\alpha_2$

o contrário do que temos normalmente

da mesma forma: $d\eta_1^*\, d\eta_2^* = (\mathrm{DET}[N])^{-1}\, d\alpha_1^*\, d\alpha_2^*$

então:

$$1 = \int d\eta_1^*\, d\eta_1\, d\eta_2^*\, d\eta_2\, e^{-\bar{\eta}\eta} = -\int \frac{d\alpha_1^*\, d\alpha_2^*}{\mathrm{DET}[N]} \frac{d\alpha_1\, d\alpha_2}{\mathrm{DET}[M]} e^{-\bar{\alpha} N M \alpha} =$$

$$= \frac{1}{\mathrm{DET}[\underbrace{NM}_{A}]} \int d\alpha_1^*\, d\alpha_1\, d\alpha_2^*\, d\alpha_2\, e^{-\bar{\alpha} \overbrace{NM}^{A} \alpha}$$

$$\boxed{\int d^n\bar{\alpha}\, d^n\alpha\, e^{-\bar{\alpha}\cdot A\alpha} = \mathrm{DET}[A]}$$

Provamos isso para 2D, mas a mesma coisa poderia ter sido feita para mais dimensões.

compare novamente com a pg 19: $\int \frac{d^n z^*}{(2\pi i)^{n/2}} \frac{d^n z}{(2\pi i)^{n/2}} e^{-\vec{z}^*\cdot A\vec{z}} = (\mathrm{DET}\ A)^{-1}$

Usando derivadas em a, podemos também mostrar que:

$$\boxed{\int d^n\bar{\alpha}\, d^n\alpha\, \alpha_i \bar{\alpha}_j\, e^{-\bar{\alpha}\cdot A\alpha} = (A^{-1})_{ij}\, \mathrm{DET}[A]}$$

Podemos definir uma "função de Grassmann" como:

$$\psi(x) = \sum_i \underbrace{\psi_i}_{\text{coeficientes são números de Grassmann}} \; \underbrace{\phi_i(x)}_{\{\phi_i\}\ \text{base de funções usuais}}$$

E generalizar as integrais funcionais Gaussianas para funções deste tipo:

$$\boxed{\int D\bar{\psi}\, D\psi \; e^{-\int \bar{\psi}\cdot A\,\psi} = \mathrm{Det}[A]}$$

Com isso podemos voltar aos campos de Dirac.

$\phi_i(x) \longrightarrow$ espinores

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \underbrace{+\bar{\eta}\psi + \bar{\psi}\eta}_{\substack{\text{Fontes}\\ \text{(Grassmann)}}}$$

$$Z_0[\bar{\eta},\eta] = \frac{1}{N}\int D\bar{\psi}\, D\psi\, \exp\left\{ i\int d^4x \left[\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + \bar{\eta}\psi + \bar{\psi}\eta \right] \right\}$$

$$N = \int D\bar{\psi}\, D\psi\, \exp\left\{ i\int d^4x \left[\bar{\psi}\underbrace{\left(i\gamma^\mu\partial_\mu - m\right)}_{\hat{A}}\psi \right] \right\}$$

Podemos seguir o mesmo procedimento de completar quadrados usado anteriormente para separar as fontes dos campos. Neste caso obtemos:

$$\psi \longrightarrow \psi' = \psi + \psi_0 \qquad\qquad \bar{\psi} \longrightarrow \bar{\psi}' = \bar{\psi} + \bar{\psi}_0$$

$$\psi_0 = -(A^{-1})\eta \qquad \bar{\psi}_0 = -\bar{\eta}(A^{-1})$$

$$D\psi' = D\psi \qquad D\bar{\psi}' = D\psi$$

$$\bar{\psi}\hat{A}\psi + \bar{\eta}\psi + \bar{\psi}\eta \rightarrow \Big\{(\bar{\psi} - \bar{\eta}A^{-1})\hat{A}(\psi - A^{-1}\eta) + \bar{\eta}\psi +$$

$$- \bar{\eta}A^{-1}\eta + \bar{\psi}\eta - \bar{\eta}A^{-1}\eta\Big\} = \bar{\psi}A\psi - \bar{\eta}A^{-1}\eta$$

$$Z_0[\bar{\eta},\eta] = \frac{1}{N}\int D\bar{\psi}D\psi \exp\left\{i\int d^4x\left[\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \int d^4y(\bar{\eta}A^{-1}\eta)\right]\right\} =$$

$$= \exp\left\{-i\int d^4x\, d^4y\ \bar{\eta}A^{-1}\eta\right\} \cdot \overbrace{\frac{1}{N}\int D\bar{\psi}D\psi \exp\left\{i\int d^4x\left[\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi\right]\right\}}^{N}$$

$$\boxed{Z_0[\bar{\eta},\eta] = \exp\left\{-i\int d^4x\, d^4y\ \bar{\eta}A^{-1}\eta\right\}}$$

(eq. 67.1)

Resta apenas obter A^{-1}:

$$A A^{-1} = (i\gamma^\mu \partial_\mu - m)A^{-1}(x-y) = \delta^4(x-y)$$

$$A^{-1}(x-y) \equiv S(x-y) = (i\gamma_\alpha \partial^\alpha + m)\underbrace{\Delta_F(x-y)}_{\text{o mesmo do campo escalar}}$$

$$(i\gamma^\mu \partial_\mu - m)(i\gamma_\alpha \partial^\alpha + m)\Delta_F(x-y) =$$

$$\gamma^\mu \partial_\mu \gamma_\alpha \partial^\alpha = (2g^\mu{}_\alpha - \gamma_\alpha\gamma^\mu)\partial_\mu\partial^\alpha = 2\Box - \underbrace{\gamma_\alpha\gamma^\mu\partial_\mu\partial^\alpha}$$

$$\Box = \gamma^\mu \partial_\mu \gamma_\alpha \partial^\alpha$$

$$= (-\Box - m^2)\Delta_F(x-y) = \delta^4(x-y)$$

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$$S(x-y) = (i\gamma_\alpha \partial^\alpha + m)\frac{1}{(2\pi)^4}\int \frac{e^{-ik(x-y)}}{k^2-m^2+i\epsilon}\, d^4k =$$

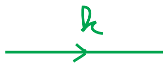
$$= \frac{1}{(2\pi)^4}\int \frac{\overbrace{\gamma_\alpha k^\alpha}^{\not k} + m}{k^2-m^2+i\epsilon}\, e^{-ikx}\, d^4k$$

novamente é preciso atentar para notação, segui aqui o Ryder, no Peskin está definido como:

$$S(k) = \frac{i(\not k + m)}{k^2-m^2+i\epsilon} = \frac{i}{\not k - m + i\epsilon}$$

$$\langle T\{\psi\bar\psi\}\rangle = S(x-y)$$

$$S(k) = \frac{\not k + m}{k^2-m^2+i\epsilon} = \frac{1}{\not k - m + i\epsilon}$$



$$\frac{\not k + m}{k^2-m^2+i\epsilon}$$

O procedimento para encontrar os correlatores é então o mesmo de antes (tomando apenas o cuidado de não comutar os números de Grassmann, lembre-se que definimos a derivada agindo pela esquerda)

$$\langle 0|T\{\psi(x_1)\bar\psi(x_2)\}|0\rangle = \frac{-\delta^2}{\delta\eta(x_1)\,\delta\bar\eta(x_2)} Z_0[\bar\eta,\eta]\Big|_{\eta=\bar\eta=0}$$

$$\frac{1}{i}\frac{\delta}{\delta\bar\eta(x_2)} \exp\Big\{-i\underbrace{\int d^4x\, \bar\eta S \eta}_{I}\Big\} = -\left[\frac{\delta}{\delta\bar\eta(x_2)}\int d^4x\, d^4y\; \bar\eta(x) S(x-y)\eta(y)\right] e^{-iI} =$$

$$= -\left[\int d^4y\; S(x_2-y)\eta(y)\right] e^{-iI}$$

este sinal vem da comutação

$$\frac{1}{i}\frac{\delta}{\delta\bar\eta(x_1)}\frac{1}{i}\frac{\delta}{\delta\eta(x_2)} e^{-iI} = iS(x_2-x_1)\, e^{-iI} \underbrace{+ i\left[\int d^4y\; S(x_2-y)\eta(y)\right]\left(-\frac{\delta}{\delta\bar\eta(x_1)}\right) e^{-iI}}_{0\,/\,\eta=\bar\eta=0}$$

$$\langle 0|T\{\psi(x_1)\bar\psi(x_2)\}|0\rangle = iS(x_2-x_1)$$