$$= \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\exp\left\{\int dt\left[-\bar{\psi}\,D_F^{-1}\,\psi\right] + i\int dt\left[\bar{\psi}\eta + \bar{\eta}\psi\right]\right\}$$

$$\therefore \boxed{Z[\eta,\bar{\eta}] = Z[0,0]\exp\left\{-\int ds\,d\tau\;\bar{\eta}(s)\,D_F(s,\tau)\,\eta(\tau)\right\}}$$ (eq. 125.1)

$$\boxed{D_F(s,\tau) = \left[-i\left(i\partial_t - \omega\right)\right]^{-1} = i\int \frac{dE}{2\pi}\,\frac{e^{-iE(s-\tau)}}{E-\omega+i\epsilon} = \Theta(s-\tau)\,e^{-i\omega(s-\tau)}}$$ (eq. 125.2)

Note que: (1) Temos apenas um polo, em $E = \omega - i\epsilon$

(2) Isso significa que se fizemos a integral no hemisfério superior ($Im\,E > 0$) ela dá zero, e somos forçados a fazer isso se ($s < \tau$), portanto a integral é zero para ($s < \tau$). No outro hemisfério (obrigatório se $s > \tau$) pegamos o polo e obtemos o resultado não nulo acima.

$$-iE(s-\tau) \sim Im(E)(s-\tau)$$

(3) Basta substituir a última expressão em 125.1 para obter 124.2 (incluindo o limite de integração, que impõe $s > \tau$)

Passando para o espaço Euclideano: $t = -it_E$

$$Z_E[\eta,\bar{\eta}] = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\;\mathrm{Exp}\left\{\int dt_E\;\bar{\psi}_E\left(-\partial_{t_E} - \omega\right)\psi_E + \int dt_E\;\bar{\psi}_E\eta_E + \bar{\eta}_E\psi_E\right\}$$

suprimindo o "E"

$$= Z[0,0]\,\mathrm{Exp}\left\{\int d\tau\,ds\;\bar{\eta}(s)\,D(s,\tau)\,\eta(\tau)\right\}$$

$$\boxed{-S_E = -\bar{\psi}\cdot D^{-1}\psi + \bar{\psi}\cdot\eta + \bar{\eta}\cdot\psi}$$ (eq. 125.3)

$$\boxed{D[s,\tau] = \left(\partial_t + \omega\right)^{-1} = i\int \frac{dE}{2\pi}\,\frac{e^{-iE(s-\tau)}}{E + i\omega}}$$ (eq. 125.4)

E rodando de volta para Minkowski com $E_E = (-i+\epsilon)E$ (para evitar o polo), voltamos ao propagador em 125.2

Agora basta aumentar o número de coordenadas espaciais para obter uma teoria de campo. A lagrangeana (em Mink.) é:

$$\mathcal{L}_F^{(M)} = -\bar{\psi}\left(\not{\partial} + m\right)\psi$$

Passando para o Euclideano:

$$\mathcal{L}_F^{(E)} = \bar{\psi}(\not{\partial} + m)\psi$$

Tomando o cuidado de manter a álgebra de Clifford funcionando: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$(\gamma^0)^2 = -1 \Rightarrow (\gamma^4)^2 = +1$$

$$\boxed{\gamma^4 = i\gamma^0}$$

$$\rightarrow \bar{\psi} = \psi^+ i\gamma^0 = \psi^+\gamma^4$$

E, nesta representação: $\gamma_\mu = \gamma^\mu = (\gamma^\mu)^+$

A função de partição obtida é:

$$Z_F^{(0)}[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{-\int d^4x\, \bar{\psi}(\not{\partial} + m)\psi + \int d^4x (\bar{\eta}\psi + \bar{\psi}\eta)\right\} = \quad \text{(eq. 126.1)}$$

$$= Z(0,0)\, e^{\bar{\eta}(\not{\partial}+m)^{-1}\eta}$$

$$S_F(x,y) = (\not{\partial}_x + m)^{-1} = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{-\not{p} + im} \quad \text{(eq. 126.2)}$$

$$(\not{\partial}_x + m) S_F(x,y) = \delta^4(x-y) \quad \therefore \quad \not{\partial}_x e^{ipx} = i\not{p}\, e^{ipx}$$

Usaremos com frequência a seguinte relação: $\dfrac{1}{-\not{p} + im} = \dfrac{-\not{p} - im}{p^2 + m^2}$

$$\not{p}\not{p} = \gamma^\mu p_\mu \gamma^\nu p_\nu = 2g^{\mu\nu}p_\mu p_\nu - \underbrace{\gamma^\nu\gamma^\mu p_\mu p_\nu}_{\gamma^\mu\gamma^\nu p_\nu p_\mu} \Rightarrow 2p^2 = 2\not{p}\not{p}$$

$$\boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}}$$

Voltando para Minkwoski, obtemos:

$$S_F^{(E)}(x,y) = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{-\not{p} + im} = i\int \frac{d^4p}{(2\pi)^4} \frac{-\not{p} - im}{p^2 + m^2} e^{ip\cdot(x-y)}$$

$$t_E = i t \qquad p_E^0 = (-i+\epsilon)p^0 \qquad p_E^2 + m^2 = p^2 + m^2 - i\epsilon$$

$$\not{p}^E = \gamma^4 p_E^0 + \gamma^i p_i = i\gamma^0(-i+\epsilon)p^0 + \gamma^i p_i = \gamma^0 p^0 + \gamma^i p_i + i\gamma^0\epsilon'$$

$$S_F^{(M)}(x,y) = \int \frac{d^4p}{(2\pi)^4}\, e^{i[p^0(x^0-y^0)+p^i(x^i-y^i)]} \frac{-\gamma^0 p^0 - \gamma^i p_i - im}{p^2+m^2-i\epsilon} =$$

$$\underbrace{=}_{p_i \to -p_i} \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)} \frac{-\not{p} - im}{p^2+m^2-i\epsilon}$$

Teorema de Wick para Campos Fermiônicos

Como um exemplo, consideremos uma teoria com um escalar ϕ (com fonte J) e um férmion ψ, interagindo por meio de um termo $S_I[\bar{\psi},\psi,\phi]$, neste caso poderíamos escrever:

$$\boxed{Z[\bar{\eta},\eta,J] = e^{-S_I\left[-\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta J}\right]} Z_F^{(0)}[\bar{\eta},\eta]\, Z_\phi^{(0)}[J]} \quad \text{(eq. 127.1)}$$

que é obtida segundo exatamente o mesmo procedimento usado na pag 95. A única diferença está no termo $-\frac{\delta}{\delta\eta}$ que tem este sinal pois o termo de fonte tem a forma: $\bar{\eta}\psi + \bar{\psi}\eta$

Logo: $$\psi\, e^{\int d^4x\, \bar{\eta}\psi + \bar{\psi}\eta} = -\frac{\delta}{\delta\eta} e^{\int d^4x\, \bar{\eta}\psi+\bar{\psi}\eta}$$

O lema de Coleman (eq. 100.2) também ganha um sinal pelo mesmo motivo:

$$\boxed{F\left(-\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}}\right) Z[\bar{\eta},\eta] = Z\left[-\frac{\delta}{\delta\psi},\frac{\delta}{\delta\bar{\psi}}\right]\left(F(\bar{\psi},\psi)\, e^{\bar{\psi}\eta+\bar{\eta}\psi}\right)\Bigg|_{\bar{\psi}=\psi=0}} \quad \text{(eq. 127.2)}$$

$$Z[\bar{\eta},\eta,J] = e^{-S_I\left(-\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta J}\right)} e^{\bar{\eta}S_F\eta} Z_\phi^{(0)}[J] =$$

(eq. 127.3)

$$\boxed{\begin{aligned} &= e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}} e^{-S_I\left(\bar{\psi},\psi,\frac{\delta}{\delta J}\right)+\bar{\psi}\cdot\eta+\bar{\eta}\cdot\psi} Z_\phi^{(0)}[J]\Bigg|_{\psi=\bar{\psi}=J=0} = \\ &= e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}} e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\left\{e^{-S_I(\bar{\psi},\psi,\phi)+\bar{\psi}\cdot\eta+\bar{\eta}\cdot\psi+J\cdot\phi}\right\}_{\psi=\bar{\psi}=\phi=0}\end{aligned}}$$

Regras de Feynman para Férmions (Interação de Yukawa)

$$\mathcal{L}_y = g\bar{\Psi}\Psi\phi$$

$$Z[\bar{\eta},\eta,J] = e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}}\, e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\left\{e^{-g\int d^4x\,\bar{\Psi}\Psi\phi+\bar{\Psi}\cdot\eta+\bar{\eta}\cdot\Psi+J\cdot\phi}\right\}_{\Psi=\bar{\Psi}=\phi=0}$$

Comecemos com a função de dois pontos livre (onde os dois pontos são aplicações do campo fermiônico):

$G^{(N)}_{(n,m)}$ N → ordem na expansão perturbativa

m → numero de pontos externos bosônicos

n → numero de pontos externos fermiônicos

$$G^{(0)}_{(2,0)} = \langle 0|\,\psi(x)\bar{\Psi}(y)\,|0\rangle = \frac{\delta}{\delta\bar{\eta}(x)}\left(-\frac{\delta}{\delta\eta(y)}\right)Z_0(\bar{\eta},\eta,J)\Big|_{\eta=\bar{\eta}=J=0}$$

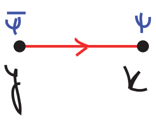
note que, trivialmente, temos $\langle 0|\,\Psi\Psi\,|0\rangle = \langle 0|\,\bar{\Psi}\bar{\Psi}\,|0\rangle = 0$

$$Z_0[\bar{\eta},\eta,J] = e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}}\, \underbrace{e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}}_{\text{Ñ CONTRIBUE}}\left\{e^{\bar{\Psi}\cdot\eta+\bar{\eta}\cdot\Psi+J\cdot\phi}\right\}_{\Psi=\bar{\Psi}=\phi=0} =$$

Só estou interessado na parte que contribui para $G^{(0)}_{(2,0)}$ ou seja com $\eta\cdot\bar{\eta}$

$$= -\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar{\Psi}}\, e^{\bar{\Psi}\eta+\bar{\eta}\cdot\psi} = \bar{\eta}\cdot S_F\cdot\eta\; e^{\bar{\Psi}\eta+\bar{\eta}\cdot\psi}\Big|_{\bar{\Psi}=\psi=0} = \bar{\eta}\cdot S_F\cdot\eta$$

$$G^{(0)}_{(2,0)} = \frac{\delta}{\delta\bar{\eta}(x)}\left(-\frac{\delta}{\delta\eta(y)}\right)\bar{\eta}\cdot S_F\cdot\eta = S_F(x-y)$$



$$= S_F(x-y) = i\int\frac{d^4p}{(2\pi)^4}\,\frac{e^{ip(x-y)}}{-\not{p}+im}$$ (eq. 128.1)

Note que agora o sinal do momento (ou a ordem de x e y) importa!

A regra para o vértice vem trivialmente da função de três pontos:

$$G^{(1)}_{(2,1)} = \frac{\delta}{\delta\bar{\eta}(x)}\left(-\frac{\delta}{\delta\eta(y)}\right)\frac{\delta}{\delta J(y)}\, e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}}\, e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\left\{e^{-g\int d^4w\,\bar{\Psi}\Psi\phi+\bar{\Psi}\cdot\eta+\bar{\eta}\cdot\psi+J\cdot\phi}\right\}_{\substack{\psi=\bar{\Psi}=\phi=0\\ \bar{\eta}=\eta=J=0}} =$$

$$= e^{-\frac{\delta}{\delta\psi} S_F \frac{\delta}{\delta\bar\psi}}\, e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}} \left\{ \phi(z)\,\psi(x)\,\bar\psi(y)\, \underbrace{e^{-g\int d^4w\, \bar\psi\psi\phi}} \right\} =$$

só quero o termo O(g¹)

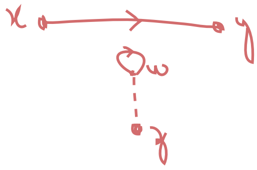
$$= e^{-\frac{\delta}{\delta\psi} S_F \frac{\delta}{\delta\bar\psi}}\, e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}} (-g)\,\phi(z)\,\psi(x)\,\bar\psi(y)\int d^4w\, \bar\psi(w)\psi(w)\phi(w) =$$

$$= -g\int d^4w\, \Delta(z-w)\frac{1}{2}\left(-\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\right)\left(-\frac{\delta}{\delta\psi}\cdot S_F\frac{\delta}{\delta\bar\psi}\right)\psi(x)\bar\psi(y)\,\bar\psi(w)\psi(w) =$$

$$\left(-\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\right)\psi(x)\,\bar\psi(y)\,\bar\psi(w)\,\psi(w) =$$

$$= \int d^4z_1\, d^4z_2 \left(-\frac{\delta}{\delta\psi(z_1)} S_F(z_1-z_2)\right)\left[-\psi(x)\,\delta(z_2-y)\,\bar\psi(w)\,\psi(w) + \psi(x)\,\bar\psi(y)\,\delta(w-z_2)\,\psi(w)\right] =$$

Não quero o diagrama de bolha no vácuo:



$$= \int d^4z_1\, d^4z_2 (-S_F(z_1-z_2))\left[-\psi(x)\,\delta(z_2-y)\,\bar\psi(w)\,\delta(z_1-w) + \delta(x-z_1)\,\bar\psi(y)\,\delta(w-z_2)\,\psi(w)\right] + \text{BOLHAS}$$

$$= +S_F(w-y)\,\psi(x)\,\bar\psi(w) - S_F(x-w)\,\bar\psi(y)\,\psi(w) + \text{BOLHAS}$$

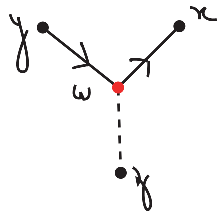
$$= -g\int d^4w\, \Delta(z-w)\frac{1}{2}\int d^4z_3\, d^4z_4 \left(-\frac{\delta}{\delta\psi(z_3)}\cdot S_F(z_3-z_4)\cdot\frac{\delta}{\delta\bar\psi(z_4)}\right)\left[+S_F(w-y)\,\psi(x)\,\bar\psi(w) - S_F(x-w)\,\bar\psi(y)\,\psi(w)\right] =$$

$$= -g\int d^4w\, \Delta(z-w)\frac{1}{2}\int d^4z_3\, d^4z_4 \left(-\frac{\delta}{\delta\psi(z_3)}\cdot S_F(z_3-z_4)\right)\left[-S_F(w-y)\,\psi(x)\,\delta(w-z_4) - S_F(x-w)\,\delta(y-z_4)\,\psi(w)\right] =$$

$$= -g\int d^4w\, \Delta(z-w)\frac{1}{2}\int d^4z_3\, d^4z_4 \left(-S_F(z_3-z_4)\right)\left[-S_F(w-y)\,\delta(x-z_3)\,\delta(w-z_4) - S_F(x-w)\,\delta(y-z_4)\,\delta(w-z_3)\right] =$$

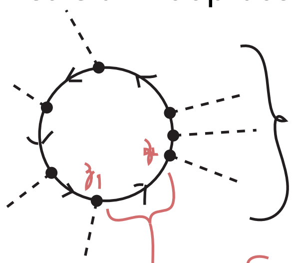
$$= -g\int d^4w\, \Delta(z-w)\frac{1}{2}\left[S_F(x-w)S_F(w-y) + S_F(w-y)S_F(x-w)\right] =$$

$$\boxed{G^{(1)}_{(2,1)} = -g\int d^4w\, \Delta(z-w)\, S_F(x-w)\, S_F(w-y)}$$ (eq. 130.1)



$= -g$ (eq. 130.2)

A importância do ordamento do campo fermiônico cria uma importante diferença entre um loop fermiônico e um loop bosônico, pense no seguinte diagrama:



total de N linhas saindo do loop

$$S_F(z_2-z_1) = -\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\left(\psi\bar\psi\right)$$

Temos vário termos deste tipo:

$$\left(-\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\right)\cdots\left(-\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\right)\left((\bar\Psi\Psi)_1\cdots(\bar\Psi\Psi)_N\right) \qquad (\bar\Psi\Psi)_i = \bar\Psi(x_i)\Psi(x_i)$$

Para obter a combinação cíclica (já que é um loop):

$$S_F(x_N-x_1)S_F(x_1-x_2)\ldots S_F(x_{N-1}-x_N)$$

Temos que trazer o último campo para a primeira posição e então aplicar as derivadas:

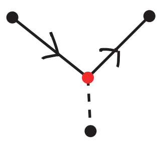
$$\left((\bar\Psi\Psi)_1\cdots(\bar\Psi\Psi)_N\right) = -\Psi(x_N)(\bar\Psi\Psi)_1\cdots(\bar\Psi\Psi)_{N-1}\bar\Psi_N$$

passo por 2N-1 campos

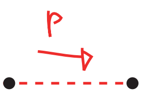
De onde vemos que, além de qualquer sinal que venha dos vértices $(-g)^N$, temos uma regra de Feynman nova, devemos multiplicar por sinal total negativo toda vez que aparecer um loop fermiônico. Você pode checar, por exemplo que o mesmo loop gerado em uma teoria $\lambda\phi^3$ (abaixo) não tem sinal algum além do que vem dos vértices.

Regras de Feymnan para interação de Yukawa:

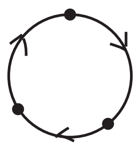
$$= S_F(x-y) = i\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{-\not p + im}$$ (direção é importante)



$= -g$



$$= \frac{1}{p^2+m^2}$$



$\to -1$ (multiplico por -1^L, onde L = # loops)

(eq. 130.3)