Integral Asymptotics 3: Stationary Phase

1. We consider the behavior (for $\lambda \gg 1$) of

$$I(\lambda) = \int_a^b f(t) e^{i\lambda g(t)}\, dt \tag{1}$$

where f and g are smooth enough to admit Taylor approximations near some appropriate point in $[a,b]$, and g is real-valued.

2. Suppose that $g'(c) = 0$ at some point $c \in (a,b)$, and that $g'(t) \neq 0$ everywhere else in the closed interval. Assume moreover that $g''(c) \neq 0$ and $f(c) \neq 0$. Let μ be the sign of $g''(c)$. Thus

$$\mu g''(c) = |g''(c)|.$$

We rewrite $I(\lambda)$ as

$$I(\lambda) = e^{i\lambda g(c)} \int_a^b f(t) e^{i\lambda[g(t)-g(c)]}\, dt.$$

By the Coates-Euler formula, $\exp\left(i\lambda\left[g(t) - g(c)\right]\right)$ is highly oscillatory for $t \neq c$ and $\lambda \gg 1$. The oscillation gives rise to cancellation which in turn causes the integral to decay rapidly except in a small neighborhood of c. Thus,

$$\begin{aligned}
I(\lambda) &\approx e^{i\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{i\lambda[g(t)-g(c)]}\, dt \\
&\approx f(c) e^{i\lambda g(c)} \int_{c-\varepsilon}^{c+\varepsilon} e^{\frac{i\lambda}{2} g''(c)(t-c)^2}\, dt \\
&\approx f(c) e^{i\lambda g(c)} \int_{-\infty}^{\infty} e^{\frac{i\lambda}{2} g''(c)(t-c)^2}\, dt \\
&= f(c) e^{i\lambda g(c)} \int_{-\infty}^{\infty} e^{\frac{i\lambda}{2} g''(c)s^2}\, ds \\
&= f(c) e^{i\lambda g(c)} \sqrt{\frac{2\pi i}{\lambda g''(c)}} \\
&= f(c) e^{i\lambda g(c)} \sqrt{\frac{2\pi}{\lambda |g''(c)|}}\, (i\mu)^{\frac{1}{2}} \\
&= f(c) e^{i\lambda g(c)} \sqrt{\frac{2\pi}{\lambda |g''(c)|}}\, e^{\frac{\pi i \mu}{4}},
\end{aligned}$$

for $\lambda \gg 1$. So to leading order

$$I(\lambda) \sim f(c) e^{i\lambda g(c)} \sqrt{\frac{2\pi}{\lambda |g''(c)|}}\, e^{\frac{\pi i \mu}{4}}, \quad \text{as } \lambda \to \infty. \tag{2}$$

Since the main contribution to the integral comes from a region of a point c at which the *phase* $g(t)$ is *stationary*, (2) is called the stationary phase approximation.

3. If $g(t)$ is stationary at an endpoint (say $t = a$) then by the usual modification we obtain the stationary phase approximation

$$I(\lambda) \sim f(a) e^{i\lambda g(a)} \sqrt{\frac{\pi}{2\lambda |g''(a)|}}\, e^{\frac{\pi i \mu}{4}}, \quad \text{as } \lambda \to \infty, \tag{3}$$

where μ is the sign of $g''(a)$.

4. **Example**: For a fixed integer n, the Bessel function of the first type has the integral representation

$$\begin{aligned} J_n(\lambda) &= \int_0^1 \cos\left(n\pi t - \lambda \sin \pi t\right) dt \\ &= \Re\left\{ \int_0^1 e^{n\pi i t}\, e^{-i\lambda \sin \pi t}\, dt \right\}. \end{aligned}$$

In the interval $[0, 1]$ the phase $g(t) = -\sin \pi t$ is stationary only at the interior point

$$c = \frac{1}{2},$$

with

$$g(c) = -1, \quad g''(c) = \pi^2, \quad \text{and} \quad \mu = 1.$$

Set

$$f(t) = e^{n\pi i t},$$

so that

$$f(c) = e^{\frac{n\pi i}{2}}.$$

Hence, to leading order,

$$\begin{aligned} J_n(\lambda) &\sim \Re\left\{ e^{\frac{n\pi i}{2}}\, e^{-i\lambda} \sqrt{\frac{2}{\pi\lambda}}\, e^{\frac{i\pi}{4}} \right\}, \\ &= \sqrt{\frac{2}{\pi\lambda}}\, \Re\left\{ e^{-i\left(\lambda - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \right\} \\ &= \sqrt{\frac{2}{\pi\lambda}} \cos\left(\lambda - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad \text{as } \lambda \to \infty. \end{aligned}$$