Method of Stationary phase

Reference: Hormander vol I

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Method of Stationary Phase

We now describe the method of stationary phase, which gave the estimate

\[ \hat{\chi}(2\pi tN) = O \left( (1 + |tN|)^{-\frac{(n+1)}{2}} \right). \]

This is actually a beautiful and clever application of stationary phase. First we need to describe the basic method.

It consists of two quite distinct ingredients:

- Integration by parts to localize at critical points;

- Comparison to a Gaussian integral to evaluate asymptotically near critical points.
Method of Stationary Phase

We consider a general oscillatory integral

\[ I_k(a, \varphi) = \int_{\mathbb{R}^n} a(x) e^{ikS(x)} \, dx \]

where \( a \in C^\infty_0(\mathbb{R}^n) \). \( S \) is called that phase function and \( a \) is called the amplitude.

The stationary phase points of \( I_k(a, \varphi) \) are the points \( x \) in the supp \( a \) where \( d\varphi(x) = 0 \). Using a partition of unity we can break up the integral into terms where \( S \) has a unique critical point in \( \text{supp} a \). We can choose coordinates so that this point is at the origin.
References


Stationary Phase expansion

Let us write $H$ for the Hessian of $S$ at 0 and $R_3$ for the third order remainder:

$$S(x) = S(0) + \langle Hx, x \rangle + R_3(x).$$

The stationary phase expansion is:

$$I_k(a, \varphi) = \left(\frac{2\pi}{k}\right)^{n/2} e^{i\pi \text{sgn}(H)/4} e^{ikS(0)} \sqrt{|\det H|} Z_{k}^{h\ell}$$

$$Z_{k}^{h\ell} \sim \sum_{j=0}^{\infty} k^{-j} a_j(0),$$

for certain coefficients $a_j(0)$. We will explain how to compute them later on.
Lemma of stationary phase

Lemma 1 If $d\varphi(x) \neq 0$ on $\text{supp}(a)$ then

$$\int_{\mathbb{R}^n} a(x)e^{i\lambda \varphi(x)} dx = O(\lambda^{-K})$$

for all $K > 0$.

The proof is to integrate repeatedly with the operator

$$\frac{1}{\lambda} L = \frac{1}{\lambda} |\nabla \varphi|^{-2} \nabla \varphi \cdot \nabla.$$

It is well defined when $\nabla \varphi \neq 0$ and reproduces the phase. Integration by parts $K$ times proves the Lemma.
Fourier transform of a Gaussian

The simplest case of the stationary phase method, and the basis for the general proof, is the case where the phase function $S(x)$ is purely quadratic, i.e. of the form $S(x) = \langle Ax, x \rangle/2 + i\langle x, \xi \rangle$ for some symmetric $n \times n$ matrix $A$.

In order that the integral be well-defined we need $\langle \Re Ax, x \rangle \geq 0$.

**Theorem 2** In this case,
\[
\int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} e^{-\langle Ax, x \rangle/2} dx
\]

\[
= (2\pi)^{n/2} (\det A)^{-1/2} e^{-\langle A^{-1}\xi, \xi \rangle}.
\]

The square root is defined by
\[
(\det A)^{-1/2} = |\det A|^{-1/2} e^{i\pi/4} \text{sgn} (A),
\]

where $\text{sgn}(A)$ is the signature of $A$ (the number of positive minus the number of negative eigenvalues).
Fourier transform of an imaginary Gaussian

The Gaussian $e^{i\langle Ax, x \rangle}$ is not in $L^2$ so we need to define its Fourier transform.

We recall that a tempered distribution $u \in S'$ is a continuous linear functional on the space $S$ of Schwarz functions.

The Fourier transform is an isomorphism on $S$, and hence extends to $S'$ by duality, i.e. $\hat{u}(\varphi) = u(\hat{\varphi})$. Thus, e.g., $\hat{\delta}_0 \equiv 1$. 
Fourier transform of an imaginary Gaussian

Since \( e^{i\langle Ax, x \rangle} \in S' \), it possesses a Fourier transform. We can calculate it by continuity by replacing \( A \) by \( A + i\epsilon I \) and letting \( \epsilon \to 0 \).

In the case of \( u(x) = e^{-\langle Ax, x \rangle/2} \), we can calculate the Fourier transform by solving a systems of ODE’s. We observe that \( D_j u = -i(AD)_j \hat{u} \). Multiplying by \( A^{-1} \) gives \( i(A^{-1}\xi)_j = D_j \hat{u} \). Thus, \( \hat{u} = C e^{-\langle A^{-1}\xi, \xi \rangle/2} \). When \( A \) is positive definite, \( C = (2\pi)^{n/2}(\det A)^{-1/2} \) and hence the formula holds by continuity.
Gaussian stationary phase

As a warm-up to stationary phase, we prove:

**Theorem 3** Let $A$ be symmetric and non-degenerate and $\Re A \geq 0$. Then for every $k > 0$, and $a(x) \in S$,

\[
\int_{\mathbb{R}^n} a(x) e^{i\lambda \langle Ax, x \rangle / 2} dx = \frac{1}{\sqrt{\det(\lambda A/2\pi i)}} \sum_{j=0}^{k-1} \frac{1}{j!} (2i\lambda)^{-1} \langle A^{-1} D, D \rangle^j a(0) + R_k(\lambda),
\]

\[
R_k(\lambda) = O(\lambda^{-n/2-k} \sum_{|\alpha| \leq 2k} \| D^\alpha a \|_{L^2}).
\]
Proof

We observe that

$$\int_{\mathbb{R}^n} a(x) e^{i\lambda \langle Ax, x \rangle / 2} dx$$

is the pairing of the Schwarz function $a(x)$ with the tempered distribution $e^{i\lambda \langle Ax, x \rangle / 2}$. Plancherel’s theorem $\langle f, g \rangle = \langle F f, F g \rangle$ on $L^2$ extends to the pairing of $S$ and $S'$. Hence, the integral equals

$$\frac{1}{\sqrt{\det(\lambda A/2\pi i)}} \int_{\mathbb{R}^n} \hat{a}(\xi) e^{i\lambda^{-1} \langle A^{-1} \xi, \xi \rangle / 2} dx.$$ 

We now Taylor expand the exponential, using that

$$|e^x - \sum_{j<k} \frac{x^j}{j!}| \leq \frac{|x|^k}{k!}.$$
Proof with poor remainder

It follows that
\[
\frac{1}{\sqrt{\det(\lambda A/2\pi i)}} \int_{\mathbb{R}^n} \hat{a}(\xi) e^{i\lambda^{-1}\langle A^{-1}\xi, \xi \rangle/2} d\xi
\]
\[
= \frac{1}{\sqrt{\det(\lambda A/2\pi i)}} \sum_{j<k} \lambda^{-j} \int_{\mathbb{R}^n} \hat{a}(\xi) \frac{\langle A^{-1}\xi, \xi \rangle^j}{j!} d\xi
\]
\[+ O\left( \frac{1}{\sqrt{\det(\lambda A/2\pi i)}} \lambda^{-j} \int_{\mathbb{R}^n} |\hat{a}(\xi)| \frac{\langle A^{-1}\xi, \xi \rangle^k}{k!} d\xi \right).
\]

The dependence on \( a \) in the remainder is rather strange:
\[
|\hat{a}(\xi)| \frac{\langle A^{-1}\xi, \xi \rangle^k}{k!} = |\mathcal{F}(\frac{\langle A^{-1}D, D \rangle^k}{k!} a)|
\]
so the estimate is in terms of
\[
||\mathcal{F}(\frac{\langle A^{-1}D, D \rangle^k}{k!} a)||_{L^1}.
\]

We will present a better estimate later.
Hörmander proof of stationary phase

Theorem 4 Let $K \subset \mathbb{R}^n$ be compact, let $U$ be an open neighborhood of $K$, and let $k \in \mathbb{N}$. Let $a \in C^\infty_0(K)$, $S \in C^\infty(U)$ with $\mathcal{S} S = 0$. Assume $S'(x_0) = 0$, $\det S''(x_0) \neq 0$, $S' \neq 0$ in $K \setminus \{x_0\}$. Then:

$$\int_{\mathbb{R}^n} a(x) e^{i\lambda S(x)} dx =$$

$$= e^{i\lambda S(x_0)} \sqrt{\det(\lambda S''(x_0)) / 2\pi i} \sum_{j \leq k} \lambda^{-j} L_j a(x_0)$$

$$+ O(\lambda^{-k} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u(x)|).$$

Here, if $g_{x_0}(x) = S(x) - S(x_0) - \langle S''(x_0)(x - x_0), (x - x_0) \rangle / 2$ then

$$L_j a = \sum_{\nu - \mu = j} \sum_{2\nu \geq 2\mu} \frac{i^{-j} 2^{-\nu}}{\mu! \nu!} \langle S''(x_0)^{-1} D, D \rangle^\nu (g_{x_0}^\mu a).$$
Outline of Hörmander proof

Notation:

1. \( M = \sum_{\alpha \leq 2k} \sup |D^\alpha a(x)|. \)

2. \( a_1 = \rho \) times the Taylor expansion of order \( 2k \) at \( x_0 \) of \( a \).

3. \( S(x) = S(x_0) + \langle S'''(x_0)(x-x_0), (x-x_0) \rangle/2 + g_{x_0}(x), g_{x_0} = \) third order Taylor remainder.

4. \( g_{x_0} = G_{x_0} + R_{3k}(x) \) where the remainder vanishes to order \( 3k \)
Outline of Hörmander proof

- Replace $a$ by $a_1$. Integrate by parts to show that $a - a_1$ can be estimated by $M \omega^{-k}$.

- Replace the phase by $S_s(x) = \langle S'''(x_0)(x - x_0), (x-x_0) \rangle / 2 + s g_{x_0}(x)$ and consider $I(s) = \int a_1 e^{i \lambda S_s(x)} \, dx$. Show $I(1) = \sum_{\mu < 2k} I^{\mu}(0) / \mu!$ modulo $M \omega^{-k}$.

- Replace $g_{x_0}$ by $G_{x_0}$ modulo $M \omega^{-k}$.

- This reduces to Gaussian stationary phase where the amplitude involves only $2k$ derivatives of the original amplitude.
Details of step 1

We have: \( S'(x) = S'(x) - S'(x_0) = S'''(x_0)(x - x_0) + O(|x - x_0|^2) \). Hence, for small \( x - x_0 \),

\[
|x - x_0| \leq 2\|S'''(x_0)^{-1}\|\|S'(x)|
\]

\[
\implies \frac{|x-x_0|}{|S'(x)|} \leq C.
\]

When the amplitude \( a - a_1 \) vanishes to order \( 2k \) at \( x_0 \), one can integrate by parts \( k \) times using \( \frac{1}{\lambda}L \) where

\[
L = \frac{1}{\|\nabla S(x)\|^2} \nabla S(x) \nabla.
\]

Let \( O_m \) denote the functions which vanish to order \( k \) at \( x_0 \). Then \( L^t : O_m \to O_{m-2} \). So \( (L^t)^m(a - a_1) \) is integrable for \( m \leq k \) and so

\[
|\int (a - a_1)e^{i\lambda S} dx| \leq CM\lambda^{-k}.
\]
Details of step 2

Introduce $I(s) = \int_{\mathbb{R}^n} a_1(x)e^{iS_s(x)}dx$, where $S_s(x) = \langle S'''(x_0)(x - x_0), (x - x_0) \rangle/2 + s g_{x_0}(x)$. Then

$$I(s) = \sum_{\mu < 2k} I^{(\mu)}(0)/\mu! + O(\sup_{0 < s < 1} |I^{(2k)}(s)|/(2k!)).$$

Here,

$$I^{(2k)}(s) = (i\lambda)^{2k} \int a_1(x)g_{x_0}^{2k}(x)e^{i\lambda S_s(x)}dx.$$

The integrand vanishes to order $6k$ so we can integrate by parts in $L 3k$ times, taking $\lambda^{2k} \rightarrow \lambda^{-k}$. Since $a_1$ involves only $2k$ derivatives of $a$, the remainder is bounded by $M\lambda^{-k}$. 
Details of step 3

The same argument shows that one can replace $g_{x_0}^h$ by $G_{x_0}^h$ since the difference gives an amplitude that vanishes to high order at $x_0$. One now integrates by parts $k + \mu$ times.
The reduction

We are now reduced to

$$\sum_{\mu<2k} I^{(\mu)}(0)/\mu! = \sum_{\mu<2k} \left(\frac{i\lambda}{\mu!}\right)$$

$$\int a_1(x) G_{x_0}^{\mu}(x) e^{i\lambda \langle S''(x_0)(x-x_0),(x-x_0)\rangle}/2 \, dx.$$ 

We use Plancherel to write the $\mu$th term as

$$(\det(\lambda S''(x_0)/2\pi i))^{-1/2}$$

$$\int \mathcal{F}(a_1 G_{x_0}^{\mu})(\xi) e^{i\lambda^{-1} \langle S''(x_0)^{-1}\xi,\xi\rangle}/2 \, d\xi.$$ 

We then Taylor expand to order $\mu + k$. 

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The finite part

The Taylor polynomial of order $\mu + k$ of the $\mu$th term equals

$$(\det(\lambda S''(x_0)/2\pi i))^{-1/2}$$

$$\sum_{\nu \leq \mu+k} (2i\lambda)^{-\nu} \langle S''(x_0)^{-1} D, D \rangle \nu (i\lambda G x_0)^{\mu} a(x_0)/\nu!\mu!.$$  

Then use: Sobolev inequality, Plancherel formula and the estimate

$$|e^w - \sum_{j<k} \frac{w^j}{j!} | \leq \frac{w^k}{k!}$$

to obtain the reminder estimate. A priori it involves $6k = n/2$ derivatives of the phase and amplitude, but due to step one of the reduction, the amplitude now depends only on the $2k$-jet of the original amplitude.
Formula for coefficients

We now give a graphical interpretation of the
coefficients

\[ L_j a = \sum_{\nu - \mu = j} \sum_{2\nu \geq 2\mu} \frac{i^{-j} 2^{-\nu}}{\mu! \nu!} \langle S''(x_0)^{-1} D, D \rangle^\nu (g_{x_0}^\mu a). \]

We associate a labelled graph \((\Gamma, \ell)\) to each term in this sum (and for each \(j\). The graph has two types of vertices: one open one (which may be absent) and closed vertices. Further;

1. \(\mu\) is the number of closed vertices

2. \(\nu\) is the number of edges;

3. Thus, \(-j = \chi(\Gamma')\) where \(\Gamma'\) is \(\Gamma\) minus the open vertex.
The closed vertices correspond to the ‘phase factors’, the open vertex corresponds to the amplitude. It takes 3 derivatives of each phase factor to give a non-zero contribution, since the phase factor $G_3$ vanishes to order 3. Hence, each closed vertex has valency $\geq 3$.

We note that there are only finitely many graphs for each $\chi = -j$ because the valency condition forces $I \geq 3/2V$. Thus, $V \leq 2j, I \leq 3j$.  


By definition, $I_\ell(\Gamma)$ is obtained by the following rule: To each edge with end labels $j, k$ one assigns a factor of $\frac{-1}{ik} h^{jk}$ where $H^{-1} = (h^{jk})$. To each closed vertex one assigns a factor of $ik \frac{\partial^\nu S(0)}{\partial x^{i_1} \cdots \partial x^{i_\nu}}$ where $\nu$ is the valency of the vertex and $i_1, \ldots, i_\nu$ at the index labels of the edge ends incident on the vertex. To the open vertex, one assigns the factor $\frac{\partial^\nu a(0)}{\partial x^{i_1} \cdots \partial x^{i_\nu}}$, where $\nu$ is its valence. Then $I_\ell(\Gamma)$ is the product of all these factors. To the empty graph one assigns the amplitude 1. In summing over $(\Gamma, \ell)$ with a fixed graph $\Gamma$, one sums the product of all the factors as the indices run over $\{1, \ldots, n\}$. 

Feynman amplitudes

By definition, $I_\ell(\Gamma)$ is obtained by the following rule: To each edge with end labels $j, k$ one assigns a factor of $\frac{-1}{ik} h^{jk}$ where $H^{-1} = (h^{jk})$. To each closed vertex one assigns a factor of $ik \frac{\partial^\nu S(0)}{\partial x^{i_1} \cdots \partial x^{i_\nu}}$ where $\nu$ is the valency of the vertex and $i_1, \ldots, i_\nu$ at the index labels of the edge ends incident on the vertex. To the open vertex, one assigns the factor $\frac{\partial^\nu a(0)}{\partial x^{i_1} \cdots \partial x^{i_\nu}}$, where $\nu$ is its valence. Then $I_\ell(\Gamma)$ is the product of all these factors. To the empty graph one assigns the amplitude 1. In summing over $(\Gamma, \ell)$ with a fixed graph $\Gamma$, one sums the product of all the factors as the indices run over $\{1, \ldots, n\}$.
Euler characteristic expansion

As noted above, the terms in the $\lambda^{-j}$ term correspond to graphs with $-j = \chi_{\Gamma'}$, where $\chi_{\Gamma'} = V - I$ equals the Euler characteristic of the graph $\Gamma'$ defined to be $\Gamma$ minus the open vertex. The stationary phase expansion is thus an Euler characteristic expansion

$$L_j a = \sum (\Gamma, \ell) : \chi_{\Gamma'} = -j \frac{I_\ell(\Gamma)}{S(\Gamma)}$$

The function $\ell$ ‘labels’ each end of each edge of $\Gamma$ with an index $i \in \{1, \ldots, n\}$. 
Example: \( j = 1 \)

There are 5 possible graphs with \( \chi = -1 = V - I \). The possibilities are:

1. \( V = 0, I = 1 \): thus, two derivatives on the amplitude \( h^{ij} D_i D_j a \).

2. \( V = 1, I = 2 \). If no open vertex, then two loops at one closed vertex

\[
h^{ij} h^{k\ell} D_i D_j D_k D_\ell S(0).
\]

If one open vertex, then: a loop at the closed vertex plus an edge between the vertices:

\[
h^{ij} h^{k\ell} D_i D_j D_k S(0) D_\ell a(0).
\]
3. $V = 2, I = 3$: Two graphs: one loop at each closed vertex plus one edge between the two:

$$h_{ij} h^{k\ell} h^{mn} D_i D_j D_k S(0) D_\ell D_m D_n S(0).$$

Or three edges from the left closed vertex to the right:

$$h_{ij} h^{k\ell} h^{mn} D_i D_k D_m S(0) D_j D_\ell D_n S(0).$$
Bessel functions

As a first example, let us consider the Bessel integrals

$$I_N(\lambda) = \int_{S^{n-1}} e^{iN\langle \lambda, \omega \rangle} d\omega,$$

where $d\omega$ is the standard Haar measure on $S^{n-1}$. A more elementary formula is

$$J_{n-2}(r) = \int_0^\pi e^{ir \cos \varphi} \sin^{n-1}(\varphi) d\varphi.$$

As is well-known, these integrals have quite different behaviour in even and odd dimensions: in even dimensions, they have the form

$$I_N(\lambda) = \Re \frac{e^{iN|\lambda|}}{|\lambda|^{n-1/2}} P_n(\lambda),$$

where $P_n$ is a polynomial of degree $n$, while in odd dimensions the factor $P_n$ is not a polynomial and the expansion is not exact.