Method of Stationary phase

Reference: Hormander vol I

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Method of Stationary Phase

We now describe the method of stationary phase, which gave the estimate

$$(1) \qquad \widehat{\chi}(2\pi tN) = O\left((1+|tN|)^{-\frac{(n+1)}{2}}\right) .$$

This is actually a beautiful and clever application of stationary phase. First we need to describe the basic method.

It consists of two quite distinct ingredients:

- Integration by parts to localize at critical points;
- Comparison to a Gaussian integral to evaluate asymptotically near critical points.

Method of Stationary Phase

We consider a general oscillatory integral

$$I_k(a, \varphi) = \int_{\mathbb{R}^n} a(x) e^{ikS(x)} dx$$

where $a \in C_0^\infty(\mathbb{R}^n)$. S is called that phase function and a is called the amplitude.

The stationary phase points of $I_k(a, \varphi)$ are the points x in the supp a where $d\varphi(x) = 0$. Using a partition of unity we can break up the integral into terms where S has a unique critical point in supp$a$. We can choose coordinates so that this point is at the origin.

References

1. J.J. Duistermaat, Oscillatory integrals, Lagrange immersions and unfolding of singularities. Comm. Pure Appl. Math. 27 (1974), 207–281.

2. Lars Hö rmander, *The analysis of linear partial differential operators. I*. Distribution theory and Fourier analysis. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 256.

Stationary Phase expansion

Let us write H for the Hessian of S at 0 and R_3 for the third order remainder:

$$S(x) = S(0) + \langle Hx, x \rangle + R_3(x).$$

The stationary phase expansion is:

$$I_k(a, \varphi) = (\frac{2\pi}{k})^{n/2} \frac{e^{i\pi sgn(H)/4}}{\sqrt{|detH|}} e^{ikS(0)} Z_k^{h\ell}$$

$$Z_k^{h\ell} \sim \sum_{j=0}^{\infty} k^{-j} a_j(0),$$

for certain coefficients $a_j(0)$. We will explain how to compute them later on.

Lemma of stationary phase

Lemma **1** *If* $d\varphi(x) \neq 0$ *on* $supp(a)$ *then*

$$\int_{\mathbb{R}^n} a(x) e^{i\lambda\varphi(x)} dx = O(\lambda^{-K})$$

for all $K > 0$.

The proof is to integrate repeatedly with the operator

$$\frac{1}{\lambda} L = \frac{1}{\lambda} \ |\nabla\varphi|^{-2} \nabla\varphi \cdot \nabla.$$

It is well defined when $\nabla\varphi \neq 0$ and reproduces the phase. Integration by parts K times proves the Lemma.

Fourier transform of a Gaussian

The simplest case of the stationary phase method, and the basis for the general proof, is the case where the phase function $S(x)$ is purely quadratic, i.e. of the form $S(x) = \langle Ax, x\rangle/2 + i\langle x, \xi\rangle$ for some symmetric $n \times n$ matrix A.

In order that the integral be well-defined we need $\langle \Re A x, x\rangle \geq 0$.

Theorem **2** *In this case,*

$$\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} e^{-\langle Ax,x\rangle/2} dx$$

$$= (2\pi)^{n/2} (\det A)^{-1/2} e^{-\langle A^{-1}\xi,\xi\rangle}.$$

The square root is defined by

$$(\det A)^{-1/2} = |\det A|^{-1/2} e^{\frac{i\pi}{4}\text{sgn }(A)},$$

where $sgn(A)$ is the signature of A (the number of positive minus the number of negative eigenvalues).

Fourier transform of an imaginary Gaussian

The Gaussian $e^{i\langle Ax,x\rangle}$ is not in L^2 so we need to define its Fourier transform.

We recall that a tempered distribution $u \in \mathcal{S}'$ is a continuous linear functional on the space $\mathcal{S}$ of Schwarz functions.

The Fourier transform is an isomorphism on $\mathcal{S}$, and hence extends to $\mathcal{S}'$ by duality, i.e. $\widehat{u}(\varphi) = u(\widehat{\varphi})$. Thus, e.g., $\widehat{\delta_0} \equiv 1$..

Fourier transform of an imaginary Gaussian

Since $e^{i\langle Ax,x\rangle} \in \mathcal{S}'$, it possesses a Fourier transform. We can calculate it by continuity by replacing A by $A + i\epsilon I$ and letting $\epsilon \to 0$.

In the case of $u(x) = e^{-\langle Ax,x\rangle/2}$, we can calculate the Fourier transform by solving a systems of ODE's. We observe that $D_j u = -i(AD)_j \widehat{u}$. Multiplying by A^{-1} gives $i(A^{-1}\xi)_j = D_j \widehat{u}$. Thus, $\widehat{u} = Ce^{-\langle A^{-1}\xi,\xi\rangle/2}$. When A is positive definite, $C = (2\pi)^{n/2}(\det A)^{-1/2}$ and hence the formula holds by continuity.

Gaussian stationary phase

As a warm-up to stationary phase, we prove:

Theorem **3** *Let A by symmetric and non-degenerate and $\Im A \geq 0$. Then for every $k > 0$, and $a(x) \in \mathcal{S}$,*

$$\int_{\mathbb{R}^n} a(x) e^{i\lambda\langle Ax, x\rangle/2} dx = \frac{1}{\sqrt{\det(\lambda A/2\pi i)}}$$

$$\times \sum_{j=0}^{k-1} \frac{1}{j!} (2i\lambda)^{-1} \langle A^{-1}D, D\rangle^j a(0) + R_k(\lambda),$$

$$R_k(\lambda) = O(\lambda^{-n/2-k} \sum_{|\alpha| \leq 2k} ||D^\alpha a||_{L^2}.$$

Proof

We observe that

$$\int_{\mathbb{R}^n} a(x) e^{i\lambda\langle Ax,x\rangle/2} dx$$

is the pairing of the Schwarz function $a(x)$ with the tempered distribution $e^{i\lambda\langle Ax,x\rangle/2}$. Plancherel's theorem $\langle f, g\rangle = \langle \mathcal{F}f, \mathcal{F}g\rangle$ on L^2 extends to the pairing of $\mathcal{S}$ and $\mathcal{S}'$. Hence, the integral equals

$$\frac{1}{\sqrt{\det(\lambda A/2\pi i)}} \int_{\mathbb{R}^n} \widehat{a}(\xi) e^{i\lambda^{-1}\langle A^{-1}\xi,\xi\rangle/2} dx.$$

We now Taylor expand the exponential, using that

$$|e^x - \sum_{j<k} \frac{x^j}{j!}| \leq \frac{|x|^k}{k!}.$$

Proof with poor remainder

It follows that

$$\frac{1}{\sqrt{\det(\lambda A/2\pi i)}}\int_{\mathbb{R}^n}\widehat{a}(\xi)e^{i\lambda^{-1}\langle A^{-1}\xi,\xi\rangle/2}dx$$

$$=\frac{1}{\sqrt{\det(\lambda A/2\pi i)}}\sum_{j<k}\lambda^{-j}\int_{\mathbb{R}^n}\widehat{a}(\xi)\frac{\langle A^{-1}\xi,\xi\rangle^j}{j!}d\xi$$

$$+O(\frac{1}{\sqrt{\det(\lambda A/2\pi i)}}\lambda^{-j}\int_{\mathbb{R}^n}|\widehat{a}(\xi)|\frac{\langle A^{-1}\xi,\xi\rangle^k}{k!}d\xi).$$

The dependence on a in the remainder is rather strange:

$$|\widehat{a}(\xi)|\frac{\langle A^{-1}\xi,\xi\rangle^k}{k!}=|\mathcal{F}(\frac{\langle A^{-1}D,D\rangle^k}{k!}a)|$$

so the estimate is in terms of

$$||\mathcal{F}(\frac{\langle A^{-1}D,D\rangle^k}{k!}a)||_{L^1}.$$

We will present a better estimate later.

Hörmander proof of stationary phase

Theorem **4** *Let $K \subset \mathbb{R}^n$ be compact, let U be an open neighborhood of K, and let $k \in \mathbb{N}$. Let $a \in C_0^\infty)K), S \in C^\infty(U)$ with $\Im S = 0$. Assume $S'(x_0) = 0, \det S''(x_0) \neq 0, S' \neq 0$ in $K \backslash \{x_0\}$. Then:*

$$\int_{\mathbb{R}^n} a(x) e^{i\lambda S(x)} dx =$$

$$= e^{i\lambda S(x_0)} \sqrt{\det(\lambda S''(x_0))/2\pi i} \sum_{j<k} \lambda^{-j} L_j a(x_0)$$

$$+ O(\lambda^{-k} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u(x)|).$$

Here, if $g_{x_0}(x) = S(x) - S(x_0) - \langle S''(x_0)(x - x_0), (x - x_0)\rangle/2$ then

$$L_j a = \sum_{\nu - \mu = j} \sum_{2\nu \geq 2\mu} \frac{i^{-j} 2^{-\nu}}{\mu! \nu!} \langle S''(x_0)^{-1} D, D\rangle^\nu (g_{x_0}^\mu a).$$

Outline of Hörmander proof

Notation:

1. $M = \sum_{\alpha \leq 2k} \sup |D^\alpha a(x)|$.

2. $a_1 =$ a cutoff ρ times the Taylor expansion of order $2k$ at x_0 of a.

3. $S(x) = S(x_0) + \langle S''(x_0)(x - x_0), (x - x_0)\rangle/2 + g_{x_0}(x)$, $g_{x_0} =$ third order Taylor remainder.

4. $g_{x_0} = G_{x_0} + R_{3k}(x)$ where the remainder vanishes to order $3k$

Outline of Hörmander proof

- Replace a by a_1. Integrate by parts to show that $a - a_1$ can be estimated by $M\omega^{-k}$.

- Replace the phase by $S_s(x) = \langle S''(x_0)(x - x_0), (x - x_0)\rangle/2 + s g_{x_0}(x)$ and consider $I(s) = \int a_1 e^{i\lambda S_s(x)} dx$. Show $I(1) = \sum_{\mu<2k} I^{\mu}(0)/\mu!$ modulo $M\omega^{-k}$.

- Replace g_{x_0} by G_{x_0} modulo $M\omega^{-k}$.

- This reduces to Gaussian stationary phase where the amplitude involves only $2k$ derivatives of the original amplitude.

Details of step 1

We have: $S'(x) = S'(x) - S'(x_0) = S''(x_0)(x - x_0) + O(|x - x_0|^2)$. Hence, for small $x - x_0$,

$$|x - x_0| \leq 2||S''(x_0)^{-1}|||S'(x)|$$

$$\implies \frac{|x-x_0|}{|S'(x)|} \leq C.$$

When the amplitude $a - a_1$ vanishes to order $2k$ at x_0, one can integrate by parts k times using $\frac{1}{\lambda}L$ where

$$L = \frac{1}{||\nabla S(x)||^2} \nabla S(x) \nabla.$$

Let O_m denote the functions which vanish to order k at x_0. Then $L^t : O_m \to O_{m-2}$. So $(L^t)^m(a - a_1)$ is integrable for $m \leq k$ and so

$$|\int (a - a_1) e^{i\lambda S} dx| \leq CM\lambda^{-k}.$$

Details of step 2

Introduce $I(s) = \int_{\mathbb{R}^n} a_1(x) e^{iS_s(x)} dx$, where $S_s(x) = \langle S''(x_0)(x - x_0), (x - x_0)\rangle/2 + s g_{x_0}(x)$. Then

$$I(s) = \sum_{\mu<2k} I^{(\mu)}(0)/\mu! + O(\sup_{0<s<1} |I^{(2k)}(s)|/(2k!)).$$

Here,

$$I^{(2k)}(s) = (i\lambda)^{2k} \int a_1(x) g_{x_0}^{2k}(x) e^{i\lambda S_s(x)} dx.$$

The integrand vanishes to order $6k$ so we can integrate by parts in L $3k$ times, taking $\lambda^{2k} \to \lambda^{-k}$. Since a_1 involves only $2k$ derivatives of a, the remainder is bounded by $M\lambda^{-k}$.

Details of step 3

The same argument shows that one can replace $g^{\mu}_{x_0}$ by $G^{\mu}_{x_0}$ since the difference gives an amplitude that vanishes to high order at x_0. One now integrates by parts $k + \mu$ times.

The reduction

We are now reduced to

$$\sum_{\mu<2k} I^{(\mu)}(0)/\mu! = \sum_{\mu<2k} \frac{(i\lambda)^\mu}{\mu!}$$

$$\int a_1(x) G^\mu_{x_0}(x) e^{i\lambda\langle S''(x_0)(x-x_0),(x-x_0)\rangle/2} dx.$$

We use Plancherel to write the μth term as

$$(\det(\lambda S''(x_0)/2\pi i)^{-1/2}$$

$$\int \mathcal{F}(a_1 G^\mu_{x_0})(\xi) e^{i\lambda^{-1}\langle S''(x_0)^{-1}\xi,\xi\rangle/2} d\xi.$$

We then Taylor expand to order $\mu + k$.

The finite part

The Taylor polynomial of order $\mu + k$ of the μth term equals

$(\det(\lambda S''(x_0)/2\pi i)^{-1/2}$

$\sum_{\nu \leq \mu+k}(2i\lambda)^{-\nu}\langle S''(x_0)^{-1}D, D\rangle^{\nu}(i\lambda G_{x_0})^{\mu}a(x_0)/\nu!\mu!.$

Then use: Sobolev inequality, Plancherel formula and the estimate

$$|e^w - \sum_{j<k} \frac{w^j}{j!}| \leq \frac{w^k}{k!}$$

to obtain the reminder estimate. A priori it involves $6k = n/2$ derivatives of the phase and amplitude, but due to step one of the reduction, the amplitude now depends only on the $2k$-jet of the original amplitude.

Formula for coefficients

We now give a graphical interpretation of the coefficients

$$L_j a = \sum_{\nu-\mu=j} \sum_{2\nu \geq 2\mu} \frac{i^{-j} 2^{-\nu}}{\mu!\nu!} \langle S''(x_0)^{-1} D, D \rangle^{\nu} (g^{\mu}_{x_0} a).$$

We associate a labelled graph (Γ, ℓ) to each term in this sum (and for each j. The graph has two types of vertices: one open one (which may be absent) and closed vertices. Further;

1. μ is the number of closed vertices

2. ν is the number of edges;

3. Thus, $-j = \chi(\Gamma')$ where Γ' is Γ minus the open vertex.

Feynman diagrams

The closed vertices correspond to the ‘phase factors’, the open vertex corresponds to the amplitude. It takes 3 derivatives of each phase factor to give a non-zero contribution, since the phase factor G_3 vanishes to order 3. Hence, each closed vertex has valency ≥ 3.

We note that there are only finitely many graphs for each $\chi = -j$ because the valency condition forces $I \geq 3/2V$. Thus, $V \leq 2j, I \leq 3j$.

Feynman amplitudes

By definition, $I_\ell(\Gamma)$ is obtained by the following rule: To each edge with end labels j, k one assigns a factor of $\frac{-1}{ik}h^{jk}$ where $H^{-1} = (h^{jk})$. To each closed vertex one assigns a factor of $ik\frac{\partial^\nu S(0)}{\partial x^{i_1}\cdots\partial x^{i_\nu}}$ where ν is the valency of the vertex and $i_1 \ldots, i_\nu$ at the index labels of the edge ends incident on the vertex. To the open vertex, one assigns the factor $\frac{\partial^\nu a(0)}{\partial x^{i_1}\ldots\partial x^{i_\nu}}$, where ν is its valence. Then $I_\ell(\Gamma)$ is the product of all these factors. To the empty graph one assigns the amplitude 1. In summing over (Γ, ℓ) with a fixed graph Γ, one sums the product of all the factors as the indices run over $\{1, \ldots, n\}$.

Euler characteristic expansion

As noted above, the terms in the λ^{-j} term correspond to graphs with $-j = \chi_{\Gamma'}$, where $\chi_{\Gamma'} = V - I$ equals the Euler characteristic of the graph Γ' defined to be Γ minus the open vertex. The stationary phase expansion is thus an Euler characteristic expansion

$$L_j a = \sum_{(\Gamma,\ell):\chi_{\Gamma'}=-j} \frac{I_\ell(\Gamma)}{S(\Gamma)}$$

The function ℓ 'labels' each end of each edge of Γ with an index $i \in \{1, \ldots, n\}$.

Example: $j = 1$

There are 5 possible graphs with $\chi = -1 = V - I$. The possibilities are:

1. $V = 0, I = 1$: thus, two derivatives on the amplitude $h^{ij} D_i D_j a$.

2. $V = 1, I = 2$. If no open vertex, then two loops at one closed vertex

$$h^{ij} h^{k\ell} D_i D_j D_k D_\ell S(0).$$

If one open vertex, then: a loop at the closed vertex plus an edge between the vertices:

$$h^{ij} h^{k\ell} D_i D_j D_k S(0) D_\ell a(0).$$

3. $V = 2, I = 3$: Two graphs: one loop at each closed vertex plus one edge between the two:

$$h^{ij}h^{k\ell}h^{mn}D_iD_jD_kS(0)D_\ell D_mD_nS(0).$$

Or three edges from the left closed vertex to the right:

$$h^{ij}h^{k\ell}h^{mn}D_iD_kD_mS(0)D_jD_\ell D_nS(0).$$

Bessel functions

As a first example, let us consider the Bessel integrals

(2) $$I_N(\lambda) = \int_{S^{n-1}} e^{iN\langle\lambda,\omega\rangle} d\omega,$$

where $d\omega$ is the standard Haar measure on S^{n-1}. A more elementary formula is

$$J_{\frac{n-2}{2}}(r) = \int_0^\pi e^{ir\cos\varphi} \sin^{n-1}(\varphi) d\varphi.$$

As is well-known, these integrals have quite different behaviour in even and odd dimensions: in even dimensions, they have the form

$$I_N(\lambda) = \Re \frac{e^{iN|\lambda|}}{|\lambda|^{n-1/2}} P_n(\lambda),$$

where P_n is a polynomial of degree n, while in odd dimensions the factor P_n is not a polynomial and the expansion is not exact.