Introduction to quantum field theory I

lectures by
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1 Lecture 1: Review of classical field theory: Lagrangeans, Lorentz group and its representations, Noether theorem

In these lectures, I will assume that classical field theory and quatum mechanics are known, and I will only review of few notions immediately useful in the first two lectures. In this lecture, I will start by describing what is QFT, and then I will review a few things about classical field theory.

What is and why Quantum Field Theory?

Quantum mechanics deals with the quantization of particles, and is a nonrelativistic theory: time is treated as special, and for the energy we use nonrelativistic formulas.

On the other hand, we want to do *quantum field theory*, which is an application of quantum mechanics to *fields* instead of particles, and it has the property of being *relativistic* as well.

Quantum field theory is often called (when derived from first principles) *second quantization*, the idea being that:

-the *first* quantization is when we have a single particle and we quantize its behaviour (its motion) in terms of a wavefunction describing probabilities.

-the *second* quantization is when we quantize the wavefunction itself (instead of a function now we have an operator), the quantum object now being the number of particles the wavefunction describes, which is an arbitrary (a variable) quantum number. Therefore now the field is a description of an arbitrary number of particles (and *antiparticles*), and this number can *change*, i.e. it is not a constant.

People have tried to build a *relativistic quantum mechanics*, but it was quickly observed that if we do that we cannot describe a single particle.

-First, the relativistic relation $E = mc^2$, together with the existence (experimentally confirmed) of *antiparticles* which annihilate with particles giving only energy (photons), means that if we have an energy $E > m_p c^2 + m_{\bar{p}} c^2$ we can create a particle-antiparticle pair, and therefore the number of particles cannot be a constant in a relativistic theory.

-Second, even if $E < m_p c^2 + m_{\bar{p}} c^2$, the particle-antiparticle pair can still be created for a short time. Indeed, Heisenberg's uncertainty principle in the (E, t) sector (as opposed to the usual (x, p) sector) means that $\Delta E \cdot \Delta t \sim \hbar$, meaning that for a short time $\Delta t \sim \hbar/\Delta E$ I can have an uncertainty in the energy ΔE, for instance such that $E + \Delta E > m_p c^2 + m_{\bar{p}} c^2$. That means that I can create a pair of *virtual particles*, i.e. particles that are forbidden by energy and momentum conservation to exist as asymptotic particles, but can exist as quantum fluctuations for a short time.

-Thirdly, causality is violated by a single particle propagating via usual quantum mechanics formulas, even with the relativistic formula for the energy, $E = \sqrt{\vec{p}^2 + m^2}$.

The amplitude for propagation from $\vec{x}_0$ to $\vec{x}$ in a time t is in quantum mechanics

$$U(t) = <\vec{x}|e^{-iHt}|\vec{x}_0> \tag{1.1}$$

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and replacing E, the eigenvalue of H, by $\sqrt{\vec{p}^2 + m^2}$, we obtain

$$U(t) = < \vec{x}|e^{-it\sqrt{\vec{p}^2+m^2}}|\vec{x}_0 > = \frac{1}{(2\pi)^3}\int d^3\vec{p}\, e^{-it\sqrt{\vec{p}^2+m^2}}e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)} \tag{1.2}$$

But

$$\int d^3\vec{p}\, e^{i\vec{p}\cdot\vec{x}} = \int p^2 dp \int 2\pi\sin\theta d\theta e^{ipx\cos\theta} = \int p^2 dp\left[\frac{2\pi}{ipx}(e^{ipx}-e^{-ipx})\right] = \int p^2 dp\left[\frac{4\pi}{px}\sin(px)\right] \tag{1.3}$$

therefore

$$U(t) = \frac{1}{2\pi^2|\vec{x}-\vec{x}_0|}\int_0^\infty p\, dp\, \sin(p|\vec{x}-\vec{x}_0|)e^{-it\sqrt{p^2+m^2}} \tag{1.4}$$

For $x^2 \gg t^2$, we use a saddle point approximation, which is the idea that the integral $I = \int dx e^{f(x)}$ can be approximated by the gaussian around the saddle point, i.e. $I \simeq e^{f(x_0)}\int d\delta x e^{f''(x_0)\delta x^2} \simeq e^{f(x_0)}\sqrt{\pi/f''(x_0)}$, where x_0 is the saddle point, where $f'(x_0) = 0$. Generally, if we are interested in leading behaviour in some large parameter, the function $e^{f(x_0)}$ dominates over $\sqrt{\pi/f''(x_0)}$ and we can just approximate $I \sim e^{f(x_0)}$.

In our case, we obtain

$$\frac{d}{dp}(ipx - it\sqrt{p^2+m^2}) = 0 \Rightarrow x = \frac{tp}{\sqrt{p^2+m^2}} \Rightarrow p = p_0 = \frac{imx}{\sqrt{x^2-t^2}} \tag{1.5}$$

Since we are at $x^2 \gg t^2$, we obtain

$$U(t) \propto e^{ip_0 x - it\sqrt{p_0^2+m^2}} \sim e^{-\sqrt{x^2-t^2}} \neq 0 \tag{1.6}$$

So we see that even much outside the lightcone, at $x^2 \gg t^2$, we have nonzero probability for propagation, meaning a breakdown of causality.

However, we will see that this problem is fixed in quantum field theory, which will be causal.

In quantum field theory, the fields describe many particles. One example of this that is easy to understand is the electromagnetic field, $(\vec{E}, \vec{B}) \to F_{\mu\nu}$, which describes many photons. Indeed, we know from the correspondence principle of quantum mechanics that a classical state is equivalent to a state with many photons, and also that the number of photons is not a constant in any sense: we can define a (quantum) average number of photons that is related to the classical intensity of a electromagnetic beam, but the number of photons is not a classically measurable quantity.

We will describe processes involving many particles by *Feynman diagrams*, which we will be an important part of this course. In quantum mechanics, a particle propagates forever, so its "Feynman diagram" is always a single line, as in Fig.1

In quantum field theory however, we will derive the mathematical form of Feynman diagrams, but the simple physical interpretation for which Feynman introduced them is that we can have processes where for instance a particle splits into two (or more) (see Fig.1 a)), two (or more) particles merge into one (see Fig. 1 b)), or two (or more) particles of one type
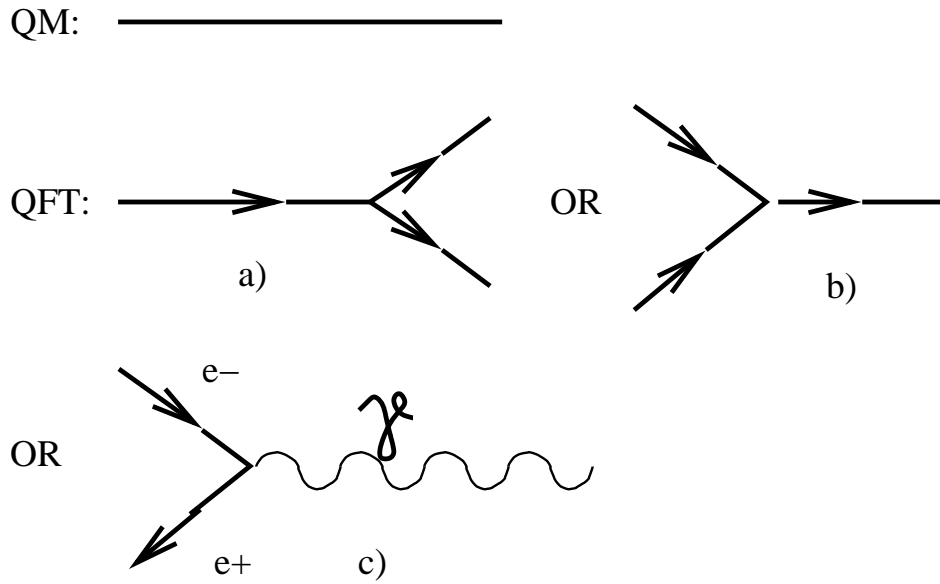
Figure 1: Quantum Mechanics: particle goes on forever. Quantum Field Theory: particles can split (a), join (b), and particles of different types can appear and disappear, like in the QED process (c).

disappear and another type is created, like for instance in the annihilation of a e^+ (positron) with an e^- (electron) into a photon (γ) as in Fig. 1 c), etc.

Moreover, we can have (as we mentioned) virtual processes, like a photon γ creating a e^+e^- pair, which lives for a short time Δt and then annihilates into a γ, creating a e^+e^- virtual loop inside the propagating γ, as in Fig.2. Of course, $E, \vec{p}$ conservation means that $(E, \vec{p})$ is the same for the γ before and after the loop.

Now we review a few notions of classical field theory. To understand that, we begin with **classical mechanics**.

We have a Lagrangean $L(q_i, \dot{q}_i)$, and the corresponding action

$$S = \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t)) \tag{1.7}$$

By varying the action with fixed boundary values, $\delta S = 0$, we obtain the Lagrange equations (equations of motion)

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{1.8}$$

We can also do a Legendre transformation from $L(q_i, \dot{q}_i)$ to $H(q_i, p_i)$ in the usual way,

$$H(p, q) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i) \tag{1.9}$$

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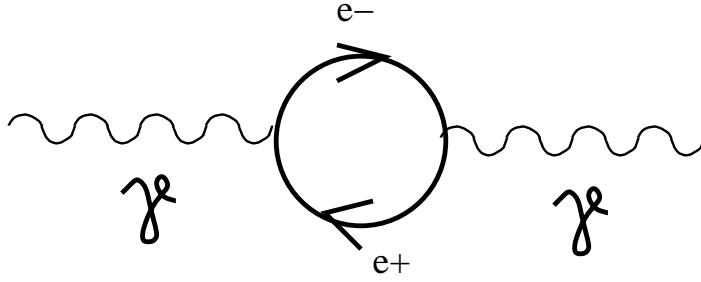
Figure 2: Virtual particles can appear for a short time in a loop. Here a photon creates a virtual electron-positron pair, that then annihilates back into the photon.

where

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{1.10}$$

is a momentum conjugate to the coordinate q_i.

Differentiating the Legendre transformation formula we get the first order Hamilton equations (instead of the second order Lagrange equations)

$$\frac{\partial H}{\partial p_i} = \dot{q}_i$$
$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i \tag{1.11}$$

The generalization to **field theory** is that instead of a set $\{q_i(t)\}_i$, which is a collection of given particles, we now have fields $\phi(\vec{x}, t)$, where $\vec{x}$ is a generalization of i, and not a coordinate of a particle!!

We will be interested in *local* field theories, which means all objects are integrals over $\vec{x}$ of functions defined at a point, in particular the Lagrangean is

$$L(t) = \int d^3\vec{x} \mathcal{L}(\vec{x}, t) \tag{1.12}$$

Here $\mathcal{L}$ is called the *Lagrange density*, but by an abuse of notation, one usually refers to it also as the Lagrangean.

We are also interested in *relativistic field theories*, which means that $\mathcal{L}(\vec{x}, t)$ is a relativistically invariant function of fields and their derivatives,

$$\mathcal{L}(\vec{x}, t) = \mathcal{L}(\phi(\vec{x}, t), \partial_\mu \phi(\vec{x}, t)) \tag{1.13}$$

Considering also several ϕ_a, we have an action

$$S = \int L dt = \int d^4 x \mathcal{L}(\phi_a, \partial_\mu \phi_a) \tag{1.14}$$

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where $d^4x = dt d^3\vec{x}$ is the relativistically invariant volume element for spacetime.

The Lagrange equations are obtained in the same way, as

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right] = 0 \tag{1.15}$$

Note that you could think of $L(q_i)$ as a discretization over $\vec{x}$ of $\int d^3\vec{x} \mathcal{L}(\phi_a)$, but it is not particularly useful.

In the Lagrangean we have relativistic fields, i.e. fields that have a well defined transformation property under Lorentz transformations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.16}$$

namely

$$\phi'_i(x') = R_i{}^j \phi_i(x) \tag{1.17}$$

where i is some index for the fields, related to its Lorentz properties. We will come back to this later, but for now let us just observe that for a scalar field there is no i and $R \equiv 1$, i.e. $\phi'(x') = \phi(x)$.

In these notes I will use the convention for the spacetime metric with "mostly plus" on the diagonal, i.e. the Minkowski metric is

$$\eta_{\mu\nu} = diag(-1, +1, +1, +1) \tag{1.18}$$

Note that this is the convention that is the most natural in order to make heavy use of Euclidean field theory via Wick rotation, as we will do (by just redefining the time t by a factor of i), and so is very useful if we work with the functional formalism, where Euclidean field theory is essential.

On the other hand, for various reasons, people connected with phenomenology and making heavy use of the operator formalism often use the "mostly minus" metric ($\eta_{\mu\nu} = diag(+1, -1, -1, -1)$), for instance Peskin and Schroeder does so, so one has to be very careful when translating results from one convention to the other.

With this metric, the Lagrangean for a scalar field is generically

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - V(\phi) \\ &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\vec{\nabla}\phi|^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \end{aligned} \tag{1.19}$$

so is of the general type $\dot{q}^2/2 - \tilde{V}(q)$, as it should (where $1/2|\vec{\nabla}\phi|^2 + m^2\phi^2/2$ are also part of $\tilde{V}(q)$).

To go to the Hamiltonian formalism, we first must define the momentum conjugate to the field $\phi(\vec{x})$ (remembering that $\vec{x}$ is a label like i),

$$p(\vec{x}) = \frac{\partial L}{\partial \dot{\phi}(\vec{x})} = \frac{\partial}{\partial \dot{\phi}(\vec{x})} \int d^3\vec{y} \mathcal{L}(\phi(\vec{y}), \partial_\mu \phi(\vec{y})) = \pi(\vec{x}) d^3\vec{x} \tag{1.20}$$

where

$$\pi(\vec{x}) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(\vec{x})} \qquad (1.21)$$

is a conjugate momentum density, but by an abuse of notation again will be called just conjugate momentum.

Then the Hamiltonian is

$$\begin{aligned} H &= \sum_{\vec{x}} p(\vec{x})\dot{\phi}(\vec{x}) - L \\ &\to \int d^3\vec{x}[\pi(\vec{x})\dot{\phi}(\vec{x}) - \mathcal{L}] \equiv \int d^3\vec{x}\mathcal{H} \end{aligned} \qquad (1.22)$$

where $\mathcal{H}$ is a Hamiltonian density.

Noether theorem

The statement of the Noether theorem is that for every symmetry of the Lagrangean L, there is a corresponding conserved charge.

The best known examples are the time translation $t \to t + a$ invariance, corresponding to conserved energy E, and space translation $\vec{x} \to \vec{x} + \vec{a}$, corresponding to conserved momentum $\vec{p}$, together spacetime translation $x^\mu \to x^\mu + a^\mu$, corresponding to conserved 4-momentum P^μ. The *currents* corresponding to these charges form the energy-momentum tensor $T_{\mu\nu}$.

Consider the symmetry $\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi$ that takes the Lagrangean density

$$\mathcal{L} \to \mathcal{L} + \alpha\partial_\mu J^\mu \qquad (1.23)$$

such that the action $S = \int d^4x\mathcal{L}$ is invariant, if the fields vanish on the boundary, usually considered at $t = \pm\infty$, since the boundary term

$$\int d^4x\partial_\mu J^\mu = \oint_{bd} dS_\mu J^\mu = \int d^3\vec{x}J^0|_{t=-\infty}^{t=+\infty} \qquad (1.24)$$

is then zero. In this case, there exists a conserved current j^μ, i.e.

$$\partial_\mu j^\mu(x) = 0 \qquad (1.25)$$

where

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu \qquad (1.26)$$

For linear symmetries (linear in ϕ), we can define

$$(\alpha\Delta\phi)^i \equiv \alpha^a(T^a)^i{}_j\phi^j \qquad (1.27)$$

such that, if $J^\mu = 0$, we have the Noether current

$$j^{\mu,a} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(T^a)^i{}_j\phi^j \qquad (1.28)$$

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Applying to translations, $x^\mu \to x^\mu + a^\mu$, we have for an infinitesimal parameter a^μ

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi \tag{1.29}$$

which are the first terms in the Taylor expansion. The corresponding conserved current is then

$$T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L}\delta^\mu_\nu \tag{1.30}$$

where we have added a term $J^\mu_{(\nu)} = \mathcal{L}\delta^\mu_\nu$ to get the conventional definition of the *energy-momentum tensor* or *stress-energy tensor*. The conserved charges are integrals of the energy-momentum tensor, i.e. P^μ. Note that the above translation can be considered as giving also the term $J^\mu_{(\nu)}$ from the general formalism, since we can check that for $\alpha^\nu = a^\nu$, the Lagrangean changes by $\partial_\mu J^\mu_{(\nu)}$.

Lorentz representations

The Lorentz group is $SO(1,3)$, i.e. an orthogonal group that generalizes $SO(3)$, the group of rotations in the (euclidean) 3 spatial dimensions.

Its basic objects in the fundamental representation, defined as the representation that acts onto coordinates x^μ (or rather dx^μ), are called $\Lambda^\mu{}_\nu$, and thus

$$dx'^\mu = \Lambda^\mu{}_\nu dx^\nu \tag{1.31}$$

If η is the matrix $\eta_{\mu\nu}$, the Minkowski metric $diag(-1,+1,+1,+1)$, the orthogonal group $SO(1,3)$ is the group of elements Λ that satisfy

$$\Lambda \eta \Lambda^T = \eta \tag{1.32}$$

Note that the usual rotation group $SO(3)$ is an orthogonal group satisfying

$$\Lambda \Lambda^T = \mathbf{1} \Rightarrow \Lambda^{-1} = \Lambda^T \tag{1.33}$$

but we should write is actually as

$$\Lambda \mathbf{1} \Lambda^T = \mathbf{1} \tag{1.34}$$

which admits a generalization to $SO(p,q)$ groups as

$$\Lambda g \Lambda^T = g \tag{1.35}$$

where $g = diag(-1, ..., -1, +1, ... +1)$ with p minuses and q pluses. In the above, Λ satisfies the group property, namely if Λ_1, Λ_2 belong in the group, then

$$\Lambda_1 \cdot \Lambda_2 \equiv \Lambda \tag{1.36}$$

is also in the group.

General representations are a generalization of (1.31), namely instead of acting on x, the group acts on a vector space ϕ^a by

$$\phi'^a(\Lambda x) = R(\Lambda)^a{}_b \phi^b(x) \tag{1.37}$$

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such that it respects the group property, i.e.

$$R(\Lambda_1)R(\Lambda_2) = R(\Lambda_1 \cdot \Lambda_2) \tag{1.38}$$

Group elements are represented for infinitesimally small parameters β^a as exponentials of the *Lie algebra generators* in the R representation, $t_a^{(R)}$, i.e.

$$R(\beta) = e^{i\beta^a t_a^{(R)}} \tag{1.39}$$

The statement that $t_a^{(R)}$ form a Lie algebra is the statement that we have a relation

$$[t_a^{(R)}, t_b^{(R)}] = if_{ab}{}^c t_c^{(R)} \tag{1.40}$$

where $f_{ab}{}^c$ are called the *structure constants*. Note that the factor of i is conventional, with this definition we can have hermitian generators, for which $\text{tr}(t_a t_b) = \delta_{ab}$; if we redefine t_a by an i we can remove it from there, but then $\text{tr}(t_a t_b)$ can be put only to $-\delta_{ab}$ (antihermitian generators).

The representations of the Lorentz group are:

-bosonic: scalars ϕ, for which $\phi'(x') = \phi(x)$; vectors like the electromagnetic field $A_\mu = (\phi, \vec{A})$ that transform as ∂_μ (covariant) or dx^μ (contravariant), and representations which have products of indices, like for instance the electromagnetic field strength $F_{\mu\nu}$ which transforms as

$$F'_{\mu\nu}(\Lambda x) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}(x) \tag{1.41}$$

where $\Lambda_\mu{}^\nu = \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma$. and for fields with more indices, $B_{\mu_1\ldots\mu_k}^{\nu_1\ldots\nu_j}$ it transforms as the appropriate products of Λ.

-fermionic: spinors, which will be treated in more detail later on in the course. For now, let us just say that fundamental spinor representations ψ are acted upon by gamma matrices γ^μ.

The Lie algebra of the Lorentz group $SO(1,3)$ is

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\eta_{\mu\rho}J_{\nu\sigma} + i\eta_{\mu\sigma}J_{\nu\rho} - i\eta_{\nu\sigma}J_{\mu\rho} + i\eta_{\nu\rho}J_{\mu\sigma} \tag{1.42}$$

Note that if we denote $a \equiv (\mu\nu), b \equiv (\rho\sigma)$ and $c \equiv (\lambda\pi)$ we then have

$$f_{ab}{}^c = -\eta_{\mu\rho}\delta^\lambda_{[\nu}\delta^\pi_{\sigma]} + \eta_{\mu\sigma}\delta^\lambda_{[\nu}\delta^\pi_{\rho]} - \eta_{\nu\sigma}\delta^\lambda_{[\mu}\delta^\pi_{\rho]} + \eta_{\nu\rho}\delta^\lambda_{[\mu}\delta^\pi_{\sigma]} \tag{1.43}$$

so (1.42) is indeed of the Lie algebra type.

The Lie algebra $SO(1,3)$ is (modulo some global subtleties) the same as a products of two $SU(2)$'s, i.e. $SU(2) \times SU(2)$, which can be seen by first defining

$$J_{0i} \equiv K_i; \quad J_{ij} \equiv \epsilon_{ijk}J_k \tag{1.44}$$

where $i, j, k = 1, 2, 3$, and then redefining

$$M_i \equiv \frac{J_i + iK_i}{2}; \quad N_i \equiv \frac{J_i - iK_i}{2} \tag{1.45}$$

after which we obtain

$$
\begin{aligned}
[M_i, M_j] &= i\epsilon_{ijk}M_k \\
[N_i, N_j] &= i\epsilon_{ijk}N_k \\
[M_i, N_j] &= 0
\end{aligned}
\tag{1.46}
$$

which we leave as an exercise to prove.

Important concepts to remember

- Quantum field theory is a relativistic quantum mechanics, which necessarily describes an arbitrary number of particles.

- Particle-antiparticle pairs can be created and disappear, both as real (energetically allowed) and virtual (energetically disallowed, only possible due to Heisenberg's uncertainty principle).

- If we use the usual quantum mechanics rules, even with $E = \sqrt{p^2 + m^2}$, we have causality breakdown: the amplitude for propagation is nonzero even much outside the lightcone.

- Feynman diagrams represent the interaction processes of creation and annihilation of particles.

- When generalizing classical mechanics to field theory, the label i is generalized to $\vec{x}$ in $\phi(\vec{x}, t)$, and we have a Lagrangean density $\mathcal{L}(\vec{x}, t)$, conjugate momentum density $\pi(\vec{x}, t)$ and Hamiltonian density $\mathcal{H}(\vec{x}, t)$.

- For relativistic and local theories, $\mathcal{L}$ is a relativistically invariant function defined at a point x^μ.

- The Noether theorem associates a conserved current ($\partial_\mu j^\mu = 0$) with a symmetry of the Lagrangean L, in particular the energy-momentum tensor T^μ_ν with translations $x^\mu \to x^\mu + a^\mu$.

- Lorentz representations act on the fields ϕ^a, and are the exponentials of Lie algebra generators.

- The Lie algebra of $SO(1,3)$ splits into two $SU(2)$'s.

Further reading: See chapters 2.1 and 2.2 in [2] and 1 in [1].

Exercises, Lecture 1

1) Prove that for the Lorentz Lie algebra,

$$[J_{\mu\nu}, J_{\rho\sigma}] = -\left(-i\eta_{\mu\rho}J_{\nu\sigma} + i\eta_{\mu\sigma}J_{\nu\rho} - i\eta_{\nu\sigma}J_{\mu\rho} + i\eta_{\nu\rho}J_{\mu\sigma}\right), \tag{1.47}$$

defining

$$J_{0i} \equiv K_i; \quad J_{ij} \equiv \epsilon_{ijk}J_k$$
$$M_i \equiv \frac{J_i + iK_i}{2}; \quad N_i \equiv \frac{J_i - iK_i}{2} \tag{1.48}$$

we obtain that the M_i and N_i satisfy

$$[M_i, M_j] = i\epsilon_{ijk}M_k$$
$$[N_i, N_j] = i\epsilon_{ijk}N_k$$
$$[M_i, N_j] = 0 \tag{1.49}$$

2) Consider the action in Minkowski space

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}(\slashed{D} + m)\psi - (D_\mu\phi)^*D^\mu\phi\right) \tag{1.50}$$

where $D_\mu = \partial_\mu - ieA_\mu$, $\slashed{D} = D_\mu\gamma^\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\bar{\psi} = \psi^\dagger i\gamma_0$, ψ is a spinor field and ϕ is a scalar field, and γ_μ are the gamma matrices, satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Consider the electromagnetic $U(1)$ transformation

$$\psi'(x) = e^{ie\lambda(x)}\psi(x); \quad \phi'(x) = e^{ie\lambda(x)}\phi(x); \quad A'_\mu(x) = A_\mu(x) + \partial_\mu\lambda(x) \tag{1.51}$$

Calculate the Noether current.

2 Lecture 2: Quantum mechanics: harmonic oscillator and QM in terms of path integrals

"The career of a young theoretical physicist
consists of treating the harmonic oscillator
in ever-increasing levels of abstraction"
Sidney Coleman

In this lecture, I will review some facts about the harmonic oscillator in quantum mechanics, and then present how to do quantum mechanics in terms of path integrals, something that should be taught in a standard quantum mechanics course, though it does not always happen.

As the quote above shows, understanding really well the harmonic oscillator is crucial: we understand everything if we understand really well this simple example, such that we can generalize it to more complicated systems. Similarly, most of the issues of quantum field theory in path integral formalism can be described by using the simple example of the quantum mechanical path integral.

Harmonic oscillator

The harmonic oscillator is the simplest possible nontrivial quantum system, with a quadratic potential, i.e. with the Lagrangean

$$L = \frac{\dot{q}^2}{2} - \omega^2 \frac{q^2}{2} \tag{2.1}$$

giving the Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) \tag{2.2}$$

Using the definition

$$
\begin{aligned}
a &= \frac{1}{\sqrt{2\omega}}(\omega q + ip) \\
a^\dagger &= \frac{1}{\sqrt{2\omega}}(\omega q - ip)
\end{aligned}
\tag{2.3}
$$

inverted as

$$
\begin{aligned}
p &= -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \\
q &= \frac{1}{\sqrt{2\omega}}(a + a^\dagger)
\end{aligned}
\tag{2.4}
$$

we can write the Hamiltonian as

$$H = \frac{\omega}{2}(aa^\dagger + a^\dagger a) \tag{2.5}$$

where, even though we are now at the classical level, we have been careful to keep the order of $a, a^\dagger$ as it is. Of course, classically we could then write

$$H = \omega a^\dagger a \tag{2.6}$$

In **classical mechanics**, one can define the Poisson bracket of two functions $f(p, q)$ and $g(p, q)$ as

$$\{f, g\}_{P.B.} \equiv \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{2.7}$$

With this definition, we can immediately check that

$$\{p_i, q_j\}_{P.B.} = -\delta_{ij} \tag{2.8}$$

The Hamilton equations of motion then become

$$
\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i} = \{q_i, H\} \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i} = \{p_i, H\}
\end{aligned}
\tag{2.9}
$$

Then, **canonical quantization** is simply the procedure of replacing the c-number variables (q, p) with the operators $(\hat{q}, \hat{p})$, and replacing the Poisson brackets $\{, \}_{P.B.}$ with $1/(i\hbar)[,]$ (commutator).

In this way, in theoretical physicist's units, with $\hbar = 1$, we have

$$[\hat{p}, \hat{q}] = -i \tag{2.10}$$

Replacing in the definition of $a, a^\dagger$, we find also

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{2.11}$$

One thing that is not obvious from the above is the picture we are in. We know that we can describe quantum mechanics in the Schrödinger picture, with operators independent of time, or in the Heisenberg picture, where operators depend on time. There are other pictures, in particular the interaction picture that will be relevant for us later, but are not important at this time.

In the Heisenberg picture, we can translate the classical Hamilton equations in terms of Poisson brackets into equations for the time evolution of the Heisenberg picture operators, obtaining

$$
\begin{aligned}
i\hbar \frac{d\hat{q}_i}{dt} &= [\hat{q}_i, H] \\
i\hbar \frac{d\hat{p}_i}{dt} &= [\hat{p}_i, H]
\end{aligned}
\tag{2.12}
$$

For the quantum Hamiltonian of the harmonic oscillator, we write from (2.5),

$$\hat{H}_{qu} = \frac{\hbar\omega}{2} \left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} \right) = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{2.13}$$

where we have reintroduced $\hbar$ just so we remember that $\hbar\omega$ is an energy. The operators $\hat{a}$ and $\hat{a}^\dagger$ are called destruction (annihilation) or lowering, and creation or raising operators, since the eigenstates of the harmonic oscillator hamiltonian are defined by a occupation number n, such that

$$\hat{a}^\dagger|n> \propto |n+1>; \quad \hat{a}|n> \propto |n-1> \tag{2.14}$$

such that

$$\hat{a}^\dagger\hat{a}|n> \equiv \hat{N}|n> = n|n> \tag{2.15}$$

That means that in the vacuum, for occupation number $n=0$, we still have an energy

$$E_0 = \frac{\hbar\omega}{2} \tag{2.16}$$

called vacuum energy or zero point energy. On the other hand, the remainder is called *normal ordered Hamiltonian* $: \hat{H} :$,

$$: \hat{H} := \hbar\omega\hat{a}^\dagger\hat{a} \tag{2.17}$$

where we define the *normal order* such that $\hat{a}^\dagger$ is to the left of $\hat{a}$, i.e.

$$: \hat{a}^\dagger\hat{a} := \hat{a}^\dagger\hat{a}; \quad : \hat{a}\hat{a}^\dagger := \hat{a}^\dagger\hat{a} \tag{2.18}$$

Feynman path integral in quantum mechanics in phase space

We now turn to the discussion of the Feynman path integral.

Given a position q at time t, an important quantity is the probability to find the particle at point q' and time t',

$$F(q',t';q,t) = {}_H<q',t'|q,t>_H \tag{2.19}$$

where $|q,t>_H$ is the state, eigenstate of $\hat{q}(t)$ at time t, in the Heisenberg picture.

Let us remember a bit about pictures in Quantum Mechanics. There are more pictures, but for now we will be interested in the two basic ones, the Schrödinger picture and the Heisenberg picture. In the Heisenberg picture, operators depend on time, in particular we have $\hat{q}_H(t)$, and the state $|q,t>_H$ is independent of time, and t is just a label, which means that it is an eigenstate of $\hat{q}_H(t)$ at time t, i.e.

$$\hat{q}_H(\tau = t)|q,t>_H = q|q,t>_H \tag{2.20}$$

and it is *not* an eigenstate for $\tau \neq t$. The operator in the Heisenberg picture $\hat{q}_H(t)$ is related to the one in the Schrödinger picture $\hat{q}_S$ by

$$\hat{q}_H(t) = e^{i\hat{H}t}\hat{q}_S e^{-i\hat{H}t} \tag{2.21}$$

and the Schrödinger picture state is related as

$$|q> = e^{-i\hat{H}t}|q,t>_H \tag{2.22}$$

and is an eigestate of $\hat{q}_S$, i.e.

$$\hat{q}_S|q> = q|q> \tag{2.23}$$

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In terms of the Schrödinger picture we then have

$$F(q', t'; q, t) = < q' | e^{-i\hat{H}(t'-t)} | q >$$ (2.24)

From now on we will drop the index H and S for states, since it is obvious, if we write $|q, t >$ we are in the Heisenberg picture, if we write $|q >$ we are in the Schrödinger picture.

Let us now derive the path integral representation.

Divide the time interval between t and t' into a large number $n + 1$ of equal intervals, and denote

$$\epsilon \equiv \frac{t' - t}{n + 1}; \qquad t_0 = t, t_1 = t + \epsilon, ..., t_{n+1} = t'$$ (2.25)

But at any fixed t_i, the set $\{|q_i, t_i >_H | q_i \in \mathbb{R}\}$ is a complete set, meaning that we have the completeness relation

$$\int dq_i |q_i, t_i >< q_i, t_i| = \mathbf{1}$$ (2.26)

We then introduce n factors of $\mathbf{1}$, one for each t_i, $i = 1, ..., n$ in $F(q', t'; q, t)$ in (2.19), obtaining

$$F(q, t'; q, t) = \int dq_1...dq_n < q', t' | q_n, t_n >< q_n, t_n | q_{n-1}, t_{n-1} > ... | q_1, t_1 >< q_1, t_1 | q, t >$$ (2.27)

where $q_i \equiv q(t_i)$ give us a *regularized path* between q and q'.

But note that this is not a classical path, since at any t_i, q can be anything ($q_i \in \mathbb{R}$), independent of q_{i-1}, and independent of how small ϵ is, whereas classically we have a continuous path, meaning as ϵ gets smaller, $q_i - q_{i-1}$ can only be smaller and smaller. But integrating over all $q_i = q(t_i)$ means we integrate over these quantum paths, where q_i is arbitrary (independent on q_{i-1}), as in Fig.3. Therefore we denote

$$\mathcal{D}q(t) \equiv \prod_{i=1}^{n} dq(t_i)$$ (2.28)

and this is an "integral over all paths", or "path integral".

On the other hand, considering that

$$|q> = \int \frac{dp}{2\pi} |p >< p|q >$$
$$|p> = \int dq |q >< q|p >$$ (2.29)

(note the factor of 2π, which is necessary, since $< q|p > = e^{ipq}$ and $\int dq e^{iq(p-p')} = 2\pi\delta(p-p')$), we have

$$_H\langle q(t_i), t_i | q(t_{i-1}), t_{i-1}\rangle_H = \langle q(t_i) | e^{-i\epsilon\hat{H}} | q(t_{i-1})\rangle = \int \frac{dp(t_i)}{2\pi} \langle q(t_i) | p(t_i)\rangle \langle p(t_i) | e^{-i\epsilon\hat{H}} | q(t_{i-1})\rangle$$ (2.30)
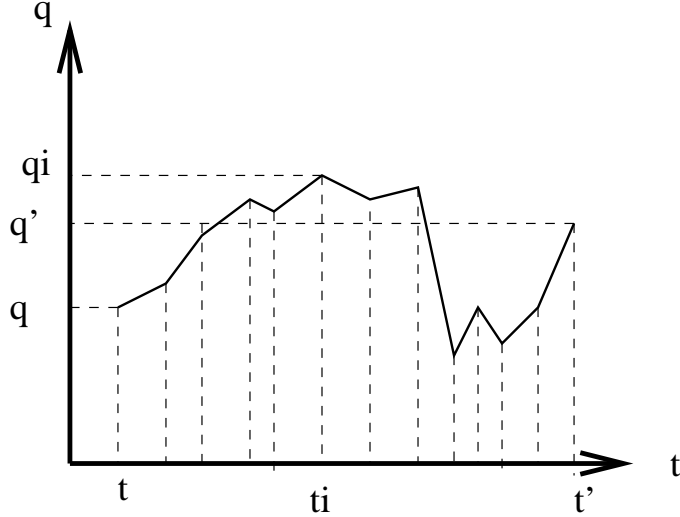
Figure 3: In the quantum mechanical path integral, we integrate over discretized paths. The paths are not necessarily smooth, as classical paths are: we divide the path into a large number of discrete points, and then integrate over the positions of these points.

Now we need a technical requirement on the quantum Hamiltonian: it has to be ordered such that all the $\hat{p}$'s are to the left of the $\hat{q}$'s.

Then, to the first order in ϵ, we can write

$$< p(t_i)|e^{-i\epsilon\hat{H}(\hat{p},\hat{q})}|q(t_{i-1}) >= e^{-i\epsilon H(p(t_i),q(t_{i-1}))} < p(t_i)|q(t_{i-1}) >= e^{-i\epsilon H(p(t_i),q(t_{i-1}))}e^{-ip(t_i)q(t_{i-1})}$$

(2.31)

since $\hat{p}$ will act on the left on $< p(t_i)|$ and $\hat{q}$ will act on the right on $|q(t_{i-1}) >$. Of course, to higher order in ϵ, we have $\hat{H}(\hat{p},\hat{q})\hat{H}(\hat{p},\hat{q})$ which will have terms like $\hat{p}\hat{q}\hat{p}\hat{q}$ which are more complicated. But since we have $\epsilon \to 0$, we only need the first order in ϵ.

Then we get

$$
\begin{aligned}
F(q',t';q,t) &= \int \prod_{i=1}^{n} \frac{dp(t_i)}{2\pi} \prod_{j=1}^{n} dq(t_j)\langle q(t_{n+1}|p(t_{n+1})\rangle\langle p(t_{n+1})|e^{-i\epsilon\hat{H}}|q(t_n)\rangle... \\
&\quad ...\langle q(t_1)|p(t_1)\rangle\langle p(t_1)|e^{-i\epsilon\hat{H}}|q(t_0)\rangle \\
&= \mathcal{D}p(t)\mathcal{D}q(t)\exp\left\{i\left[p(t_{n+1})(q(t_{n+1}) - q(t_n)) + ... + p(t_1)(q(t_1) - q(t_0))\right.\right. \\
&\quad \left.\left. -\epsilon(H(p(t_{n+1}),q(t_n)) + ... + H(p(t_1),q(t_0)))\right]\right\} \\
&= \mathcal{D}p(t)\mathcal{D}q(t)\exp\left\{i\int_{t_0}^{t_{n+1}} dt[p(t)\dot{q}(t) - H(p(t),q(t))]\right\}
\end{aligned}
$$

(2.32)

where we have used that $q(t_{i+1}) - q(t_i) \to dt\dot{q}(t_i)$. The above expression is called the *path integral in phase space.*

We note that this was derived rigorously (for a physicist, of course...). But we would like a path integral in configuration space. For that however, we need one more technical

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requirement: we need the Hamiltonian to be quadratic in momenta, i.e.

$$H(p, q) = \frac{p^2}{2} + V(q) \tag{2.33}$$

If this is not true, we have to start in phase space and see what we get in configuration space. But if we have only quadratic terms in momenta, we can use gaussian integration to derive the path integral in configuration space. Therefore we will make a math interlude to define some gaussian integration formulas that will be useful here and later.

Gaussian integration

The basic gaussian integral is

$$I = \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \tag{2.34}$$

Squaring this integral formula, we obtain also

$$I^2 = \int dx dy e^{-(x^2 + y^2)} = \int_0^{2\pi} d\phi \int_0^{\infty} r dr e^{-r^2} = \pi \tag{2.35}$$

We can generalize it as

$$\int d^n x e^{-\frac{1}{2} x_i A_{ij} x_j} = (2\pi)^{n/2} (\det A)^{-1/2} \tag{2.36}$$

which can be proven for instance by diagonalizing the matrix A, and since $\det A = \prod_i \alpha_i$, with α_i the eigenvalues of A, we get the above.

Finally, consider the object

$$S = \frac{1}{2} x^T A x + b^T x \tag{2.37}$$

(which will later on in the course be used as an action). Considering it as an action, the classical solution will be $\partial S / \partial x_i = 0$,

$$x_c = -A^{-1} b \tag{2.38}$$

and then

$$S(x_c) = -\frac{1}{2} b^T A^{-1} b \tag{2.39}$$

which means that we can write

$$S = \frac{1}{2} (x - x_c)^T A (x - x_c) - \frac{1}{2} b^T A^{-1} b \tag{2.40}$$

and thus we find

$$\int d^n x e^{-S(x)} = (2\pi)^{n/2} (\det A)^{-1/2} e^{-S(x_c)} = (2\pi)^{n/2} (\det A)^{-1/2} e^{+\frac{1}{2} b^T A^{-1} b} \tag{2.41}$$

Path integral in configuration space

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We are now ready to go to configuration space. The gaussian integration we need to do is then the one over $\mathcal{D}p(t)$, which is

$$\int \mathcal{D}p(\tau)e^{i\int_t^{t'} d\tau[p(\tau)\dot{q}(\tau)-\frac{1}{2}p^2(\tau)]} \tag{2.42}$$

which discretized is

$$\prod_i \frac{dp(t_i)}{2\pi} \exp\left[i\Delta\tau(p(t_i)\dot{q}(t_i) - \frac{1}{2}p^2(\tau_i))\right] \tag{2.43}$$

and therefore $x_i = p(t_i)$, $A_{ij} = i\Delta\tau\delta_{ij}$, $b = -i\Delta\tau\dot{q}(t_i)$, giving

$$\int \mathcal{D}p(\tau)e^{i\int_t^{t'} d\tau[p(\tau)\dot{q}(\tau)-\frac{1}{2}p^2(\tau)]} = \mathcal{N}e^{i\int_t^{t'} d\tau \frac{\dot{q}(\tau)^2}{2}} \tag{2.44}$$

where $\mathcal{N}$ contains constant factors of $2, \pi, i, \Delta\tau$, which we will see are irrelevant.

$$\begin{aligned}
F(q't; q, t) &= \mathcal{N}\int \mathcal{D}q \exp\left\{i\int_t^{t'} d\tau\left[\frac{\dot{q}^2(\tau)}{2} - V(q)\right]\right\} \\
&= \mathcal{N}\int \mathcal{D}q \exp\left\{i\int_t^{t'} d\tau L(q(\tau), \dot{q}(\tau))\right\} \\
&= \mathcal{N}\int \mathcal{D}q e^{iS[q]}
\end{aligned} \tag{2.45}$$

This is the path integral in configuration space that we were seeking. But we have to remember that this is valid only if the Hamiltonian is quadratic in momenta.

Correlations functions

We have found how to write the probability of transition between (q, t) and (q', t'), and that is good. But there are other observables that we can construct which are of interest, for instance the correlations functions.

The simplest one is the one-point function

$$< q', t'|\hat{q}(\bar{t})|q, t > \tag{2.46}$$

where we can make it such that $\bar{t}$ coincides with a t_i in the discretization of the time interval. The calculation now proceeds as before, but in the step (2.27) we introduce the **1**'s such that we have (besides the usual products) also the expectation value

$$< q_{i+1}, t_{i+1}|\hat{q}(\bar{t})|q_i, t_i > = q(\bar{t}) < q_{i+1}, t_{i+1}|q_i, t_i > \tag{2.47}$$

since $\bar{t} = t_i \Rightarrow q(\bar{t}) = q_i$. Then the calculation proceeds as before, since the only new thing is the appearance of $q(\bar{t})$, leading to

$$< q', t'|\hat{q}(\bar{t})|q, t > = \int \mathcal{D}q e^{iS[q]} q(\bar{t}) \tag{2.48}$$

Consider next the two-point functions,

$$< q', t'|\hat{q}(\bar{t}_2)\hat{q}(\bar{t}_1)|q, t > \tag{2.49}$$

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If we have $\bar{t}_1 < \bar{t}_2$, we can proceed as before. Indeed, remember that in (2.27) the $\mathbf{1}$'s introduced in time order, such that we can do the rest of the calculation and get the path integral. Therefore, if $\bar{t}_1 < \bar{t}_2$, we can choose $\bar{t}_1 = t_i, \bar{t}_2 = t_j$, such that $j > i$, and then we have

$$
\begin{aligned}
&... < q_{j+1}, t_{j+1}|\hat{q}(\bar{t}_2)|q_j, t_j > ... < q_{i+1}, t_{i+1}|\hat{q}(\bar{t}_1)|q_i, t_i > ... \\
= \ &...q(\bar{t}_2) < q_{j+1}, t_{j+1}|q_j, t_j > ...q(\bar{t}_1) < q_{i+1}, t_{i+1}|q_i, t_i > ...
\end{aligned}
\tag{2.50}
$$

besides the usual products, leading to

$$
\int \mathcal{D}q e^{iS[q]} q(\bar{t}_2) q(\bar{t}_1)
\tag{2.51}
$$

Then reversely, the path integral leads to the two-point function where the $q(t)$ are ordered according to time (time ordering), i.e.

$$
\int \mathcal{D}q e^{iS[q]} q(\bar{t}_1) q(\bar{t}_2) = < q', t'|T\{\hat{q}(\bar{t}_1)\hat{q}(\bar{t}_2)\}|q, t >
\tag{2.52}
$$

where time ordering is defined as

$$
\begin{aligned}
T\{\hat{q}(\bar{t}_1)\hat{q}(\bar{t}_2)\} &= \hat{q}(\bar{t}_1)\hat{q}(\bar{t}_2) \ \text{ if } \ \bar{t}_1 > \bar{t}_2 \\
&= \hat{q}(\bar{t}_2)\hat{q}(\bar{t}_1) \ \text{ if } \ \bar{t}_2 > \bar{t}_1
\end{aligned}
\tag{2.53}
$$

which has an obvious generalization to

$$
T\{\hat{q}(\bar{t}_1)...\hat{q}(\bar{t}_N)\} = \hat{q}(t_1)...\hat{q}(\bar{t}_N) \ \text{ if } \ \bar{t}_1 > \bar{t}_2 > ... > \bar{t}_N
\tag{2.54}
$$

and otherwise they are ordered in the order of their times.

Then we similarly find that the *n-point function* or *correlation function* is

$$
G_n(\bar{t}_1, ..., \bar{t}_n) \equiv < q', t'|T\{\hat{q}(\bar{t}_1)....q(\bar{t}_n)\}|q, t > = \int \mathcal{D}q e^{iS[q]} q(\bar{t}_1)...q(\bar{t}_n)
\tag{2.55}
$$

In math, for a set $\{a_n\}_n$, we can define a *generating function $F(z)$*,

$$
F(z) \equiv \sum_n \frac{1}{n!} a_n z^n
\tag{2.56}
$$

such that we can find a_n from its derivatives,

$$
a_n = \frac{d^n}{dz^n} F(z)|_{z=0}
\tag{2.57}
$$

Similarly now, we can define a generating functional

$$
Z[J] = \sum_{N \geq 0} \int dt_1 ... \int dt_N \frac{i^N}{N!} G_N(t_1, ..., t_N) J(t_1)...J(t_N)
\tag{2.58}
$$

As we see, the difference is that now we have $G_N(t_1, ..., t_N)$ instead of a_N, so we needed to integrate over dt, and instead of z, introduce $J(t)$, and i was conventional. Using (2.55), the integrals factorize, and we obtain just a product of the same integral,

$$Z[J] = \int \mathcal{D}q e^{iS[q]} \sum_{N \geq 0} \frac{1}{N!} \left[\int dt i q(t) J(t) \right]^N \tag{2.59}$$

so finally

$$Z[J] = \int \mathcal{D}q e^{iS[q,J]} = \int \mathcal{D}q e^{iS[q] + i \int dt J(t) q(t)} \tag{2.60}$$

We then find that this object indeed generates the correlation functions by

$$\frac{\delta^N}{i\delta J(t_1)...i J(t_N)} Z[J]|_{J=0} = \int \mathcal{D}q e^{iS[q]} q(t_1)...q(t_N) = G_N(t_1, ..., t_N) \tag{2.61}$$

Important concepts to remember

- For the harmonic oscillator, the Hamiltonian in terms of $a, a^\dagger$ is $H = \omega/2(aa^\dagger + a^\dagger a)$.

- Canonical quantization replaces classical functions with quantum operators, and Poisson brackets with commutators, $\{,\}_{P.B.} \to 1/(i\hbar)[,]$.

- At the quantum level, the harmonic oscillator Hamiltonian is the sum of a normal ordered part and a zero point energy part.

- The transition probability from (q, t) to (q', t') is a path integral in phase space, $F(q', t'; q, t) = \int \mathcal{D}q \mathcal{D}p e^{i \int (p\dot{q} - H)}$.

- If the Hamiltonian is quadratic in momenta, we can go to the path integral in configuration space and find $F(q', t'; q, t) = \int \mathcal{D}q e^{iS}$.

- The n-point functions or correlation functions, with insertions of $q(t_i)$ in the path integral, gives the expectation value of time-ordered $\hat{q}(t)$'s.

- The n-point functions can be found from the derivatives of the generating function $Z[J]$.

Further reading: See chapters 1.3 in [4] and 2 in [3].

Exercises, Lecture 2

1) Let

$$L(q, \dot{q}) = \frac{\dot{q}^2}{2} - \frac{\lambda}{4!}q^4 \qquad (2.62)$$

Write down the *Hamiltonian* equations of motion and the path integral *in phase space* for this model.

2) Let

$$\ln Z[J] = \int dt \frac{J^2(t)}{2} f(t) + \lambda \int dt \frac{J^3(t)}{3!} + \tilde{\lambda} \int dt \frac{J^4(t)}{4!} \qquad (2.63)$$

Calculate the 3-point function and the 4-point function.

3 Lecture 3: Canonical quantization of scalar fields

As we saw, in quantum mechanics, for a particle with Hamiltonian $H(p,q)$, we replace the Poisson bracket

$$\{f,g\}_{P.B.} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{3.1}$$

with $\{p_i, q_j\}_{P.B.} = -\delta_{ij}$, with the commutator, $\{,\}_{P.B.} \to \frac{1}{i\hbar}[,]$, and all functions of (p,q) become quantum operators, in particular $[\hat{p}_i, \hat{q}_j] = -i\hbar$ and for the harmonic oscillator we have

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{3.2}$$

Then, in order to generalize to field theory, we must first define the Poisson brackets. As we already have a definition in the case of a set of particles, we will discretize *space*, in order to use that definition. Therefore we consider the coordinates and conjugate momenta

$$\begin{aligned} q_i(t) &= \sqrt{\Delta V} \phi_i(t) \\ p_i(t) &= \sqrt{\Delta V} \pi_i(t) \end{aligned} \tag{3.3}$$

where $\phi_i(t) \equiv \phi(\vec{x}_i, t)$ and similarly for $\pi_i(t)$. We should also define how to go from the derivatives in the Poisson brackets to functional derivatives. The recipe is

$$\begin{aligned} \frac{1}{\Delta V} \frac{\partial f_i(t)}{\partial \phi_j(t)} &\to \frac{\delta f(\phi(\vec{x}_i, t), \pi(\vec{x}_i, t))}{\delta \phi(\vec{x}_j, t)} \\ \Delta V &\to d^3 x \end{aligned} \tag{3.4}$$

where *functional derivatives* are defined such that, for instance

$$H(t) = \int d^3 x \frac{\phi^2(\vec{x}, t)}{2} \Rightarrow \frac{\delta H(t)}{\delta \phi(\vec{x}, t)} = \phi(\vec{x}, t) \tag{3.5}$$

i.e., by dropping the integral sign and then taking normal derivatives.

Replacing these definitions in the Poisson brackets (3.1), we get

$$\{f,g\}_{P.B.} = \int d^3 x \left[\frac{\delta f}{\delta \phi(\vec{x}, t)} \frac{\delta g}{\delta \pi(\vec{x}, t)} - \frac{\delta f}{\delta \pi(\vec{x}, t)} \frac{\delta g}{\delta \phi(\vec{x}, t)} \right] \tag{3.6}$$

and then we immediately find

$$\begin{aligned} \{\phi(\vec{x}, t), \pi(\vec{x}', t)\}_{P.B.} &= \delta^3(\vec{x} - \vec{x}') \\ \{\phi(\vec{x}, t), \phi(\vec{x}', t)\}_{P.B.} &= \{\pi(\vec{x}, t), \pi(\vec{x}', t)\} = 0 \end{aligned} \tag{3.7}$$

where we note that these are *equal time commutation relations*, in the same way that we had before really $\{q_i(t), p_j(t)\}_{P.B.} = \delta_{ij}$.

We can now easily do *canonical quantization* of this scalar field. We just replace classical fields $\phi(\vec{x}, t)$ with quantum Heisenberg operators $\phi_H(\vec{x}, t)$ (we will drop the H, understanding that if there is time dependence we are in the Heisenberg picture and if we don't have

time dependence we are in the Schrödinger picture), and $\{,\}_{P.B.} \to 1/(i\hbar)[,]$, obtaining the fundamental equal time commutation relations

$$
\begin{aligned}
[\phi(\vec{x},t), \pi(\vec{x}',t)] &= i\hbar\delta^3(\vec{x} - \vec{x}') \\
[\phi(\vec{x},t), \phi(\vec{x}',t)] &= [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0
\end{aligned}
\tag{3.8}
$$

We further define the Fourier transforms

$$
\begin{aligned}
\phi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{x}\cdot\vec{p}} \phi(\vec{p},t) \\
&\Rightarrow \phi(\vec{p},t) = \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \phi(\vec{x},t) \\
\pi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{x}\cdot\vec{p}} \pi(\vec{p},t) \\
&\Rightarrow \pi(\vec{p},t) = \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \pi(\vec{x},t)
\end{aligned}
\tag{3.9}
$$

We further define, using the same formulas as we did for the harmonic oscillator, but now for the momentum modes,

$$
\begin{aligned}
a(\vec{k},t) &= \sqrt{\frac{\omega_k}{2}}\phi(\vec{k},t) + \frac{i}{\sqrt{2\omega_k}}\pi(\vec{k},t) \\
a^\dagger(\vec{k},t) &= \sqrt{\frac{\omega_k}{2}}\phi^\dagger(\vec{k},t) - \frac{i}{\sqrt{2\omega_k}}\pi^\dagger(\vec{k},t)
\end{aligned}
\tag{3.10}
$$

where we will find later that we have $\omega_k = \sqrt{\vec{k}^2 + m^2}$, but for the moment we will only need that $\omega_k = \omega(|\vec{k}|)$. Then replacing these definitions in ϕ and π, we obtain

$$
\begin{aligned}
\phi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a(\vec{p},t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p},t)e^{-i\vec{p}\cdot\vec{x}}) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} e^{i\vec{p}\cdot\vec{x}} (a(\vec{p},t) + a^\dagger(-\vec{p},t)) \\
\pi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3} \left(-i\sqrt{\frac{\omega_p}{2}}\right) (a(\vec{p},t)e^{i\vec{p}\cdot\vec{x}} - a^\dagger(\vec{p},t)e^{-i\vec{p}\cdot\vec{x}}) \\
&= \int \frac{d^3p}{(2\pi)^3} \left(-i\sqrt{\frac{\omega_p}{2}}\right) (a(\vec{p},t) - a^\dagger(-\vec{p},t))e^{i\vec{p}\cdot\vec{x}}
\end{aligned}
\tag{3.11}
$$

In terms of $a(\vec{p},t)$ and $a^\dagger(\vec{p},t)$ we obtain the commutators

$$
\begin{aligned}
[a(\vec{p},t), a^\dagger(\vec{p}',t)] &= (2\pi)^3\delta^3(\vec{p} - \vec{p}') \\
[a(\vec{p},t), a(\vec{p}',t)] &= [a^\dagger(\vec{p},t), a^\dagger(\vec{p}',t)] = 0
\end{aligned}
\tag{3.12}
$$

We can check, for instance, that the $[\phi, \pi]$ commutator is correct with these commutators:

$$
[\phi(\vec{x},t), \pi(\vec{x}',t)] = \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3} \left(-\frac{i}{2}\sqrt{\frac{\omega_{p'}}{\omega_p}}\right) ([a^\dagger(-\vec{p},t), a(\vec{p}',t)] - [a(\vec{p},t), a^\dagger(-\vec{p}',t)])e^{i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{x}')}
$$

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$$= i\delta^3(\vec{x} - \vec{x}'). \tag{3.13}$$

We note now that the calculation above was independent on the form of ω_p $(= \sqrt{p^2 + m^2})$, we only used that $\omega_p = \omega(|\vec{p}|)$, but otherwise we just used the definitions from the harmonic oscillator. We also have not written any explicit time dependence, it was left implicit through $a(\vec{p}, t), a^\dagger(\vec{p}, t)$. Yet, we obtained the same formulas as for the harmonic oscillator.

We should now understand the dynamics, which will give us the formula for ω_p.

We therefore go back, and start systematically. We work with a *free scalar field*, i.e. one with $V = 0$, with Lagrangean

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{3.14}$$

and action $S = \int d^4x\mathcal{L}$. Partially integrating $-1/2 \int \partial_\mu\phi\partial^\mu\phi = +1/2 \int \phi\partial_\mu\partial^\mu\phi$, we obtain the *Klein-Gordon (KG)* equation of motion

$$(\partial_\mu\partial^\mu - m^2)\phi = 0 \Rightarrow (-\partial_t^2 + \partial_{\vec{x}}^2 - m^2)\phi = 0 \tag{3.15}$$

Going to momentum $(\vec{p})$ space via a Fourier transform, we obtain

$$\left[\frac{\partial^2}{\partial t^2} + (\vec{p}^2 + m^2)\right]\phi(\vec{p}, t) = 0 \tag{3.16}$$

which is the equation of motion for a harmonic oscillator with $\omega = \omega_p = \sqrt{p^2 + m^2}$.

This means that the hamiltonian is

$$H = \frac{1}{2}(p^2 + \omega_p^2\phi^2) \tag{3.17}$$

and we can use the transformation

$$\phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger); \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \tag{3.18}$$

and we have $[a, a^\dagger] = 1$. Therefore now we can justify the transformations that we did a posteriori.

Let's also calculate the Hamiltonian. As explained in the first lecture, using the discretization of space, we write

$$\begin{aligned} H &= \sum_{\vec{x}} p(\vec{x}, t)\dot{\phi}(\vec{x}, t) - L \\ &= \int d^3x[\pi(\vec{x}, t)\dot{\phi}(\vec{x}, t) - \mathcal{L}] \equiv d^3x\mathcal{H} \end{aligned} \tag{3.19}$$

and from the Lagrangean (3.14) we obtain

$$\pi(\vec{x}, t) = \dot{\phi}(\vec{x}, t) \Rightarrow$$

$$\mathcal{H} \;=\; \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{3.20}$$

Substituing ϕ, π inside this, we obtain

$$
\begin{aligned}
H &= \int d^3x \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3} e^{i(\vec{p}+\vec{p}')\vec{x}} \left\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4}(a(\vec{p},t) - a^\dagger(-\vec{p},t))(a(\vec{p}',t) - a^\dagger(-\vec{p}',t)) \right.\\
&\quad \left. +\frac{-\vec{p}\cdot\vec{p}' + m^2}{4\sqrt{\omega_p\omega_{p'}}}(a(\vec{p},t) + a^\dagger(-\vec{p},t))(a(\vec{p}',t) + a^\dagger(-\vec{p}',t)) \right\}\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{\omega_p}{2}(a^\dagger(\vec{p},t)a(\vec{p},t) + a(\vec{p},t)a^\dagger(\vec{p},t)) \tag{3.21}
\end{aligned}
$$

where in the last line we have first done the integral over $\vec{x}$, obtaining $\delta^3(\vec{p}+\vec{p}')$ and then we integrate over $\vec{p}'$, obtaining $\vec{p}' = -\vec{p}$. We have finally reduced the Hamiltonian to an infinite (continuum, even) sum over harmonic oscillators.

We have dealt with the first point observed earlier, about the dynamics of the theory. We now address the second, of the explicit time dependence.

We have Heisenberg operators, for which the time evolution is

$$i\frac{d}{dt}a(\vec{p},t) = [a(\vec{p},t), H] \tag{3.22}$$

Calculating the commutator from the above Hamiltonian (using the fact that $[a, aa^\dagger] = [a, a^\dagger a] = a$), we obtain

$$i\frac{d}{dt}a(\vec{p},t) = \omega_p a(\vec{p},t) \tag{3.23}$$

More generally, the time evolution of the Heisenberg operators in field theories is given by

$$\mathcal{O}(x) = \mathcal{O}_H(\vec{x},t) = e^{iHt}\mathcal{O}(\vec{x})e^{-iHt} \tag{3.24}$$

which is equivalent to

$$i\frac{\partial}{\partial t}\mathcal{O}_H = [\mathcal{O}, H] \tag{3.25}$$

via

$$i\frac{d}{dt}(e^{iAt}Be^{-iAt}) = [B, A] \tag{3.26}$$

The solution of (3.23) is

$$
\begin{aligned}
a(\vec{p},t) &= a_{\vec{p}}e^{-i\omega_p t}\\
a^\dagger(\vec{p},t) &= a_{\vec{p}}^\dagger e^{+i\omega_p t} \tag{3.27}
\end{aligned}
$$

Replacing in $\phi(\vec{x},t)$ and $\pi(\vec{x},t)$, we obtain

$$
\begin{aligned}
\phi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\vec{p}}e^{ip\cdot x} + a_{\vec{p}}^\dagger e^{-ip\cdot x})|_{p^0=E_p}\\
\pi(\vec{x},t) &= \frac{\partial}{\partial t}\phi(\vec{x},t) \tag{3.28}
\end{aligned}
$$

so we have formed the Lorentz invariants $e^{\pm ip \cdot x}$, though we haven't written an explicitly Lorentz invariant formula. We will do that next lecture. Here we have denoted $E_p = \sqrt{\vec{p}^2 + m^2}$ remembering that it is the relativistic energy of a particle of momentum $\vec{p}$ and mass m.

Finally, of course, if we want the Schrödinger picture operators, we have to remember the relation between the Heisenberg and Schrödinger pictures,

$$\phi_H(\vec{x}, t) = e^{iHt} \phi(\vec{x}) e^{-iHt} \tag{3.29}$$

Discretization

Continuous systems are hard to understand, so it would be better if we could find a rigorous way to discretize the system. Luckily, there is such a method, namely we consider a space of finite volume V, i.e. we "put the system in a box". Obviously, this doesn't discretize space, but it does discretize *momenta*, since in a direction z of length L_z, allowed momenta will be only $k_n = 2\pi n / L_z$.

Then, the discretization is defined, as always, by

$$\int d^3 k \to \frac{1}{V} \sum_{\vec{k}}$$
$$\delta^3(\vec{k} - \vec{k}') \to V \delta_{\vec{k}\vec{k}'} \tag{3.30}$$

to which we add the redefinition

$$a_{\vec{k}} \to \sqrt{V(2\pi)^3} \alpha_{\vec{k}} \tag{3.31}$$

which allows us to keep the usual orthonormality condition in the discrete limit, $[\alpha_{\vec{k}}, \alpha_{\vec{k}'}^\dagger] = \delta_{\vec{k}\vec{k}'}$.

Using these relations, and replacing the time dependence, which cancels out of the Hamiltonian, we get the Hamiltonian of the free scalar field in a box of volume V,

$$H = \sum_{\vec{k}} \frac{(\hbar)\omega_k}{2} (\alpha_{\vec{k}}^\dagger \alpha_{\vec{k}} + \alpha_{\vec{k}} \alpha_{\vec{k}}^\dagger) = \sum_{\vec{k}} h_{\vec{k}} \tag{3.32}$$

where $h_{\vec{k}}$ is the Hamiltonian of a single harmonic oscillator,

$$h_{\vec{k}} = \omega_{\vec{k}} \left(N_{\vec{k}} + \frac{1}{2} \right), \tag{3.33}$$

$N_{\vec{k}} = \alpha_{\vec{k}}^\dagger \alpha_{\vec{k}}$ is the number operator for mode $\vec{k}$, with eigenstates $|n_{\vec{k}} >$,

$$N_{\vec{k}} |n_{\vec{k}} > = n_{\vec{k}} |n_{\vec{k}} > \tag{3.34}$$

and the orthonormal eigenstates are

$$|n> = \frac{1}{\sqrt{n!}} (\alpha^\dagger)^n |0>; \quad <n|m> = \delta_{mn} \tag{3.35}$$

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Here $\alpha_{\vec{k}}^{\dagger}$ = raising/creation operator and $\alpha_{\vec{k}}$=lowering/annihilation(desctruction) operator, named so since they create and annihilate a particle, respectively, i.e.

$$
\begin{aligned}
\alpha_{\vec{k}}|n_{\vec{k}}> &= \sqrt{n_{\vec{k}}}|n_{\vec{k}}-1> \\
\alpha_{\vec{k}}^{\dagger}|n_{\vec{k}}> &= \sqrt{n_{\vec{k}}+1}|n_{\vec{k}}+1> \\
h_{\vec{k}}|n> &= \omega_{\vec{k}}\left(n+\frac{1}{2}\right)|n>
\end{aligned}
\tag{3.36}
$$

Therefore, as we know from quantum mechanics, there is a ground state $|0>$, $\Leftrightarrow n_{\vec{k}} \in \mathbf{N}_+$, in which case $n_{\vec{k}}$ is called occupation number, or number of particles in the state $\vec{k}$.

Fock space

The Hilbert space of states in terms of eigenstates of the number operator is called *Fock space*, or *Fock space representation.* The Fock space representation for the states of a single harmonic oscillator is $\mathcal{H}_{\vec{k}} = \{|n_{\vec{k}}>\}$.

Since the total Hamiltonian is the sum of the Hamiltonians for each mode, the total Hilbert space is the direct product of the Hilbert spaces of the Hamiltonians for each mode, $\mathcal{H} = \otimes_{\vec{k}}\mathcal{H}_{\vec{k}}$. Its states are then

$$
|\{n_{\vec{k}}\}> = \prod_{\vec{k}}|n_{\vec{k}}> = \left(\prod_{\vec{k}}\frac{1}{\sqrt{n_{\vec{k}}!}}(\alpha_{\vec{k}}^{\dagger})^{n_{\vec{k}}}\right)|0>
\tag{3.37}
$$

Note that we have defined a unique vacuum for all the Hamiltonians, $|0>$, such that $a_{\vec{k}}|0> = 0, \forall\vec{k}$, instead of denoting it as $\prod_{\vec{k}}|0>_{\vec{k}}$.

Normal ordering

For a single harmonic oscillator mode, the ground state energy, or zero point energy, is $\hbar\omega_{\vec{k}}/2$, which we may think it could have some physical significance. But for a free scalar field, even one in a box ("discretized"), the total ground state energy is $\sum_{\vec{k}}\hbar\omega_{\vec{k}}/2 = \infty$, and since an observable of infinite value doesn't make sense, we have to consider that it is *unobservable*, and put it to zero.

In this simple model, that's no problem, but consider the case where this free scalar field is coupled to gravity. In a gravitational theory, energy is equivalent to mass, and gravitates, i.e. it can be measured by its gravitational effects. So how can we drop a constant piece from the energy? Are we allowed to do that? In fact, this is part of one of the biggest problems of modern theoretical physics, the *cosmological constant problem*, and the answer to this question is far from obvious. At this level however, we will not bother with this question anymore, and drop the infinite constant.

But it is also worth mentioning, that while infinities are of course unobservable, the finite difference between two infinite quantities might be observable, and in fact one such case was already measured. If we consider the difference in the zero point energies between fields in two different boxes, one of volume V_1 and another of volume V_2, that *is* measurable, and leads to the so-called Casimir effect, which we will discuss at the end of the course.

We are then led to define the *normal ordered Hamiltonian*,

$$
: H := H - \frac{1}{2}\sum_{\vec{k}}\hbar\omega_{\vec{k}} = \sum_{\vec{k}}\hbar\omega_{\vec{k}}N_{\vec{k}}
\tag{3.38}
$$

by dropping the infinite constant. The *normal order* is to have always $a^\dagger$ before a, i.e. $: a^\dagger a := a^\dagger a, : a a^\dagger := a^\dagger a$. Since a's commute among themselves, as do $a^\dagger$'s, and operators from different modes, in case these appear, we don't need to bother with their order. For instance then, $: a a^\dagger a^\dagger a a a a^\dagger := a^\dagger a^\dagger a^\dagger a a a$.

We then consider that only normal ordered operators have physical expectation values, i.e. for instance

$$< 0| : \mathcal{O} : |0 > \tag{3.39}$$

is measurable.

One more observation to make is that in the expansion of ϕ we have

$$(a_{\vec{p}} e^{ip \cdot x} + a_{\vec{p}}^\dagger e^{-ip \cdot x})_{p^0 = E_p} \tag{3.40}$$

and here $E_p = +\sqrt{\vec{p}^2 + m^2}$, but we note that the second has *positive frequency* (energy), $a_{\vec{p}}^\dagger e^{-iE_p t}$, whereas the first has *negative frequency* (energy), $a e^{+iE_p t}$, i.e. we create $E > 0$ and destroy $E < 0$, which means that in this context we have only positive energy excitations. But we will see in the next lecture that in the case of the complex scalar field, we create both $E > 0$ and $E < 0$ and similarly destroy, leading to the concept of *anti-particles*. At this time however, we don't have that.

Bose-Einstein statistics

Since $[a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger] = 0$, for a general state defined by a wavefunction $\psi(\vec{k}_1, \vec{k}_2)$,

$$
\begin{aligned}
|\psi > & = \sum_{\vec{k}_1, \vec{k}_2} \psi(\vec{k}_1, \vec{k}_2) \alpha_{\vec{k}_1}^\dagger \alpha_{\vec{k}_2}^\dagger |0 > \\
& = \sum_{\vec{k}_1, \vec{k}_2} \psi(\vec{k}_2, \vec{k}_1) \alpha_{\vec{k}_1}^\dagger \alpha_{\vec{k}_2}^\dagger |0 >
\end{aligned} \tag{3.41}
$$

where in the second line we have commuted the two $\alpha^\dagger$'s and then renamed $\vec{k}_1 \leftrightarrow \vec{k}_2$.

We then obtain *Bose-Einstein statistics*,

$$\psi(\vec{k}_1, \vec{k}_2) = \psi(\vec{k}_2, \vec{k}_1) \tag{3.42}$$

i.e. for undistiguishable particles (permuting them we obtain the same state).

As an aside, note that the Hamiltonian of the free (bosonic) oscillator is $1/2(a a^\dagger + a^\dagger a)$ (and of the free fermionic oscillator is $1/2(b^\dagger b - b b^\dagger)$), so in order to have a well defined system we must have $[a, a^\dagger] = 1, \{b, b^\dagger\} = 1$. In turn, $[a, a^\dagger] = 1$ leads to Bose-Einstein statistics, as above.

Important concepts to remember

- The commutation relations for scalar fields are defined at equal time. The Poisson brackets are defined in terms of integrals of functional derivatives.

- The canonical commutation relations for the free scalar field imply that we can use the same redefinitions as for the harmonic oscillator, for the momentum modes, to obtain the $[a, a^\dagger] = 1$ relations.

- The Klein-Gordon equation for the free scalar field implies the Hamiltonian of the free harmonic oscillator for each of the momentum modes.

- Putting the system in a box, we find a sum over discrete momenta of harmonic oscillators, each with a Fock space.

- The Fock space for the free scalar field is the direct product of the Fock space for each mode.

- We must use normal ordered operators, for physical observables, in particular for the Hamiltonian, in order to avoid unphysical infinities.

- The scalar field is quantized in terms of the bose-einstein statistics.

Further reading: See chapters 2.3 in [2] and 2.1 and 2.3 in [1].

Exercises, Lecture 3

1) Consider the classical Hamiltonian

$$H = \int d^3x \left\{ \frac{\pi^2(\vec{x}, t)}{2} + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{\lambda}{3!}\phi^3(\vec{x}, t) + \frac{\tilde{\lambda}}{4!}\phi^4(\vec{x}, t) \right\} \tag{3.43}$$

Using the Poisson brackets, write the Hamiltonian equations of motion. Quantize canonically the *free* system and compute the equations of motion for the Heisenberg operators.

2) Write down the Hamiltonian above in terms of $a(\vec{p}, t)$ and $a^\dagger(\vec{p}, t)$ *at the quantum level*, and then write down the *normal ordered* Hamiltonian.

4 Lecture 4: Propagators for free scalar fields

Relativistic invariance

We still need to understand the relativistic invariance of the quantization of the free scalar field. The first issue is the normalization of the states. We saw that in the discrete version of the quantized scalar field, we had in each mode states

$$|n_{\vec{k}}> = \frac{1}{\sqrt{n_k!}}(\alpha_{\vec{k}}^\dagger)^{n_{\vec{k}}}|0> \tag{4.1}$$

normalized as $<m|n> = \delta_{mn}$. In discretizing, we had $a_{\vec{k}} \to \sqrt{V}\alpha_{\vec{k}}$ and $\delta^3(\vec{k} - \vec{k}') \to V\delta_{\vec{k}\vec{k}'}$.

However, we want to have a *relativistic normalization*.

$$<\vec{p}|\vec{q}> = 2E_{\vec{p}}(2\pi)^3\delta^3(\vec{p} - \vec{q}) \tag{4.2}$$

or in general, for occupation numbers in all momentum modes,

$$<\{\vec{k}_i\}|\{\vec{q}_j\}> = \sum_{\pi(j)}\prod_i 2\omega_{\vec{k}_i}(2\pi)^3\delta^3(\vec{k}_i - \vec{q}_{\pi(j)}) , \tag{4.3}$$

where $\{\pi(j)\}$ are permutations of the $\{j\}$ indices. We see that we are missing a factor of $2\omega_k V$ in each mode in order to get $2\omega_k\delta^3(\vec{k} - \vec{k}')$ instead of $\delta_{\vec{k}\vec{k}'}$. We therefore take the normalized states

$$\prod_{\vec{k}}\frac{1}{\sqrt{n_{\vec{k}}!}}(\sqrt{2\omega_{\vec{k}}}\sqrt{V(2\pi)^3}\alpha_{\vec{k}})^{n_{\vec{k}}}|0> \to \prod_{\vec{k}}\frac{1}{\sqrt{n_{\vec{k}}!}}[a_{\vec{k}}^\dagger\sqrt{2\omega_{\vec{k}}}]^{n_{\vec{k}}}|0> \equiv |\{\vec{k}_i\}> \tag{4.4}$$

We now prove that we have a *relativistically invariant* formula.

First, we look at the normalization. It is obviously invariant under rotations, so we need only look at boosts,

$$p_3' = \gamma(p_3 + \beta E); \quad E' = \gamma(E + \beta p_3) \tag{4.5}$$

Since

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0) \tag{4.6}$$

and a boost acts only on p_3, but not on p_1, p_2, we have

$$\delta^3(\vec{p} - \vec{q}) = \delta^3(\vec{p}' - \vec{q}')\frac{dp_3'}{dp_3} = \delta(\vec{p}' - \vec{q}')\gamma\left(1 + \beta\frac{dE_3}{dp_3}\right) = \delta^3(\vec{p}' - \vec{q}')\frac{\gamma}{E}(E + \beta p_3) = \delta^3(\vec{p}' - \vec{q}')\frac{E'}{E} \tag{4.7}$$

That means that $E\delta^3(\vec{p} - \vec{q})$ is relativistically invariant, as we wanted.

Also the expansion of the scalar field,

$$\phi(\vec{x}, t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\vec{p}}e^{ip\cdot x} + a_{\vec{p}}^\dagger e^{-ip\cdot x})|_{p^0=E_p} \tag{4.8}$$

contains the relativistic invariants $e^{\pm i p \cdot x}$ and $\sqrt{2E_p} a_{\vec{p}}$ (since they create a relativistically invariant normalization, these operators are relativistically invariant), but we also have a relativistically invariant measure

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} = \int \frac{d^4 p}{(2\pi)^4} (2\pi) \delta(p^2 + m^2)|_{p^0 > 0} \tag{4.9}$$

(since $\delta(p^2 + m^2) = \delta(-(p^0)^2 + E_p^2)$ and then we use (4.6)), allowing to write

$$\phi(x) \equiv \phi(\vec{x}, t) = \int \frac{d^4 p}{(2\pi)^4} (2\pi) \delta(p^2 + m^2)|_{p^0 > 0} (\sqrt{2E_p} a_{\vec{p}} e^{i p \cdot x} + \sqrt{2E_p} a_{\vec{p}}^\dagger e^{-i p \cdot x})|_{p^0 = E_p} \tag{4.10}$$

Complex scalar field

We now turn to the quantization of the complex scalar field, in order to understand better the physics of propagation in quantum field theory.

The Lagrangean is

$$\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi^* - m^2 |\phi|^2 - U(|\phi|^2) \tag{4.11}$$

This Lagrangean has a $U(1)$ global symmetry $\phi \to \phi e^{i\alpha}$, or in other words ϕ is *charged under the $U(1)$ symmetry.*

Note the absence of the factor $1/2$ in the kinetic term for ϕ with respect to the real scalar field. The reason is that we treat ϕ and ϕ^* as independent fields. Then the equation of motion of ϕ is $(\partial_\mu \partial^\mu - m^2)\phi^* = \partial U / \partial \phi$. We could write the Lagrangean as a sum of two real scalars, but then with a factor of $1/2$, $-\partial_\mu \phi_1 \partial^\mu \phi_1 / 2 - \partial_\mu \phi_2 \partial^\mu \phi_2 / 2$, since then we get $(\partial_\mu \partial^\mu - m^2)\phi_1 = \partial U / \partial \phi_1$.

Exactly paralleling the discussion of the real scalar field, we obtain an expansion in terms of a and $a^\dagger$ operators, just that now we have complex fields, with twice as many degrees of freedom, so we have $a_\pm$ and $a_\pm^\dagger$, with half of the degrees of freedom in ϕ and half in $\phi^\dagger$,

$$\begin{aligned}
\phi(\vec{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_+(\vec{p}, t) e^{i\vec{p}\cdot\vec{x}} + a_-^\dagger(\vec{p}, t) e^{-i\vec{p}\cdot\vec{x}}) \\
\phi^\dagger(\vec{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_+^\dagger(\vec{p}, t) e^{-i\vec{p}\cdot\vec{x}} + a_-(\vec{p}, t) e^{i\vec{p}\cdot\vec{x}}) \\
\pi(\vec{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \left(-i\sqrt{\frac{\omega_p}{2}}\right) (a_-(\vec{p}, t) e^{i\vec{p}\cdot\vec{x}} - a_+^\dagger(\vec{p}, t) e^{-i\vec{p}\cdot\vec{x}}) \\
\pi^\dagger(\vec{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \left(i\sqrt{\frac{\omega_p}{2}}\right) (a_-^\dagger(\vec{p}, t) e^{-i\vec{p}\cdot\vec{x}} - a_+(\vec{p}, t) e^{i\vec{p}\cdot\vec{x}})
\end{aligned} \tag{4.12}$$

As before, this ansatz is based on the harmonic oscillator, whereas the form of ω_p comes out of the KG equation. Substituting this ansatz inside the canonical quantization commutators, we find

$$[a_\pm(\vec{p}, t), a_\pm^\dagger(\vec{p}', t)] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \tag{4.13}$$

and the rest zero. Again, we note the equal time for the commutators. Also, the time dependence is the same as before.

We can calculate the $U(1)$ charge operator (left as an exercise), obtaining

$$Q = \int \frac{d^3k}{(2\pi)^3}[a^\dagger_{+\vec{k}}a_{+\vec{k}} - a^\dagger_{-\vec{k}}a_{-\vec{k}}] \tag{4.14}$$

Thus as expected from the notation used, a_+ has charge $+$ and a_- has charge $-$, and therefore we have

$$Q = \int \frac{d^3k}{(2\pi)^3}[N_{+\vec{k}} - N_{-\vec{k}}] \tag{4.15}$$

(the number of $+$ charges minus the number of $-$ charges).

We then see that ϕ creates $-$ charge and annihilates $+$ charge, and $\phi^\dagger$ creates $+$ charge and annihilates $-$ charge.

Since in this simple example there are no other charges, we see that $+$ and $-$ particles are *particle/antiparticle pairs*, i.e. pairs which are equal in everything, except they have opposite charges. As promised last lecture, we have now introduced the concept of antiparticle, and it is related to the existence of positive and negative frequency modes.

Therefore in a real field, the particle is its own antiparticle.

We consider the object

$$< 0|\phi^\dagger(x)\phi(y)|0 > \tag{4.16}$$

corresponding to propagation from $y = (t_y, \vec{y})$ to $x = (t_x, \vec{x})$, the same way as in quantum mechanics $< q', t'|q, t >$ corresponds to propagation from (q, t) to (q', t'). This object corresponds to a measurement of the field ϕ at y, then of $\phi^\dagger$ at x.

For simplicity, we will analyze the real scalar field, and we will use the complex scalar only for interpretation. Substituting the expansion of $\phi(x)$, since $a|0 >=< 0|a^\dagger = 0$, and we have $< 0|(a + a^\dagger)(a + a^\dagger)|0 >$, only

$$< 0|a_{\vec{p}}a^\dagger_{\vec{q}}|0 > e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} \tag{4.17}$$

survives in the sum, and, as $< 0|aa^\dagger|0 >=< 0|a^\dagger a|0 > +[a, a^\dagger] < 0|0 >$, we get $(2\pi)^3\delta(\vec{p} - \vec{q})$ from the expectation value. Then finally we obtain for the scalar propagation from y to x,

$$D(x - y) \equiv< 0|\phi(x)\phi(y)|0 >= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip\cdot(x-y)} \tag{4.18}$$

We now analyze what happens for varying $x - y$. By Lorentz transformations, we have only two cases to analyze.

a) For timelike separation, we can put $t_x - t_y = t$ and $\vec{x} - \vec{y} = 0$. In this case, using $d^3p = d\Omega p^2 dp$ and $dE/dp = p/\sqrt{p^2 + m^2}$,

$$\begin{aligned}
D(x - y) &= 4\pi \int_0^\infty \frac{p^2 dp}{(2\pi)^3}\frac{1}{2\sqrt{p^2 + m^2}}e^{-i\sqrt{p^2 + m^2}t} \\
&= \frac{1}{4\pi^2} \int_m^\infty dE\sqrt{E^2 - m^2}e^{-iEt} \overset{t\to\infty}{\propto} e^{-imt}
\end{aligned} \tag{4.19}$$

which is oscillatory, i.e. it doesn't vanish. But that's OK, since in this case, we remain in the same point as time passes, so the probability should be large.

b) For spacelike separation, $t_x = t_y$ and $\vec{x} - \vec{y} = \vec{r}$, we obtain

$$
\begin{aligned}
D(\vec{x} - \vec{y}) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}} = 2\pi \int_0^\infty \frac{p^2 dp}{2E_p (2\pi)^3} \int_{-1}^1 d(\cos\theta) e^{ipr\cos\theta} \\
&= 2\pi \int_0^\infty \frac{p^2 dp}{2E_p (2\pi)^3} \left[\frac{e^{ipr} - e^{-ipr}}{ipr} \right] = \frac{-i}{(2\pi)^2 2r} \int_{-\infty}^{+\infty} p\, dp \frac{e^{ipr}}{\sqrt{p^2 + m^2}} \quad (4.20)
\end{aligned}
$$

where in the last line we have redefined in the second term $p \to -p$, and then added up $\int_0^\infty$ to $\int_{-\infty}^0$.

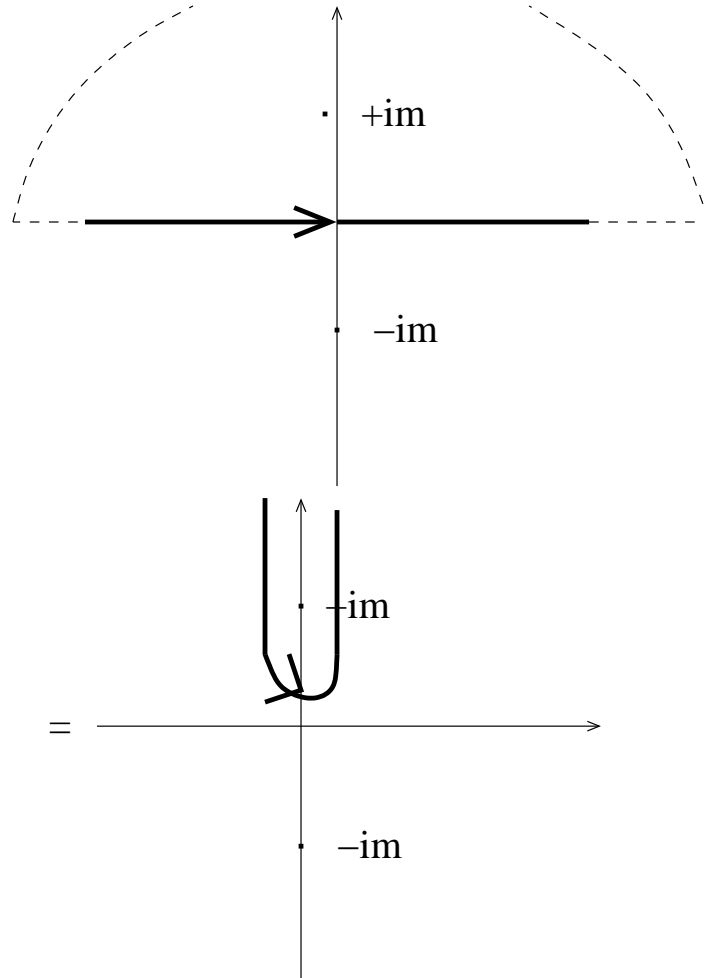Figure 4: By closing the contour in the upper-half plane, we pick up the residue of the pole in the upper-half plane, at $+im$.

In the last form, we have e^{ipr} multiplying a function with poles at $p = \pm im$, so we know, by a theorem from complex analysis, that we can consider the integral in the complex p plane, and add for free the integral on an infinite semicircle in the upper half plane, since

then $e^{ipr} \propto e^{-Im(p)r} \to 0$. Thus closing the contour, we can use the residue theorem and say that our integral equals the residue in the upper half plane, i.e. at $+im$, see Fig.4. Looking at the residue for $r \to \infty$, its leading behaviour is

$$D(\vec{x} - \vec{y}) \propto e^{i(im)r} = e^{-mr} \tag{4.21}$$

But for spacelike separation, at $r \to \infty$, we are much outside the lightcone (in no time, we move in space), and yet, propagation gives a small but nonzero amplitude, which is not OK.

But the relevant question is, will measurements be affected? We will see later why, but the only relevant issue is whether the commutator $[\phi(x), \phi(y)]$ is nonzero for spacelike separation. We thus compute

$$
\begin{aligned}
[\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_q}} [(a_{\vec{p}} e^{ip \cdot x} + a_{\vec{p}}^\dagger e^{-ip \cdot x}), (a_{\vec{q}} e^{iq \cdot y} + a_{\vec{q}}^\dagger e^{-iq \cdot y})] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{ip \cdot (x-y)} - e^{ip \cdot (y-x)}) \\
&= D(x-y) - D(y-x) \tag{4.22}
\end{aligned}
$$

But if $(x-y)^2 > 0$ (spacelike), $(x-y) = (0, \vec{x} - \vec{y})$ and we can make a Lorentz transformation (a rotation, really) $(\vec{x} - \vec{y}) \to -(\vec{x} - \vec{y})$, leading to $(x-y) \to -(x-y)$. But since $D(x-y)$ is Lorentz invariant, it follows that for spacelike separation we have $D(x-y) = D(y-x)$, and therefore

$$[\phi(x), \phi(y)] = 0 \tag{4.23}$$

and we have causality. Note that this is due to the existence of negative frequency states ($e^{ip \cdot x}$) in the scalar field expansion. On the other hand, we should also check that for timelike separation we have a nonzero result. Indeed, for $(x-y)^2 < 0$, we can set $(x-y) = (t_x - t_y, 0)$ and so $-(x-y) = (-(t_x - t_y), 0)$ corresponds to time reversal, so is not a Lorentz transformation, therefore we have $D(-(x-y)) \neq D(x-y)$, and so $[\phi(x), \phi(y)] \neq 0$.

Klein-Gordon propagator

We are finally ready to describe the propagator. Consider the c-number

$$
\begin{aligned}
[\phi(x), \phi(y)] &= < 0|[\phi(x), \phi(y)]|0 > = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{ip \cdot (x-y)} - e^{-ip \cdot (x-y)}) \\
&= \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2E_p} e^{ip \cdot (x-y)}|_{p^0 = E_p} + \frac{1}{-2E_p} e^{ip \cdot (x-y)}|_{p^0 = -E_p} \right] \tag{4.24}
\end{aligned}
$$

For $x^0 > y^0$, we can write it as

$$\int \frac{d^3p}{(2\pi)^3} \int_C \frac{dp^0}{2\pi i} \frac{1}{p^2 + m^2} e^{ip \cdot (x-y)} \tag{4.25}$$

where the contour C is on the real line, except it avoids slightly above the two poles at $p^0 = \pm E_p$, and then, in order to select both poles, we need to close the contour below, with an infinite semicircle in the lower half plane, as in Fig.5. Closing the contour below works, since for $x^0 - y^0 > 0$, we have $e^{-ip^0(x^0 - y^0)} = e^{Im(p^0)(x^0 - y^0)} \to 0$. Note that this way
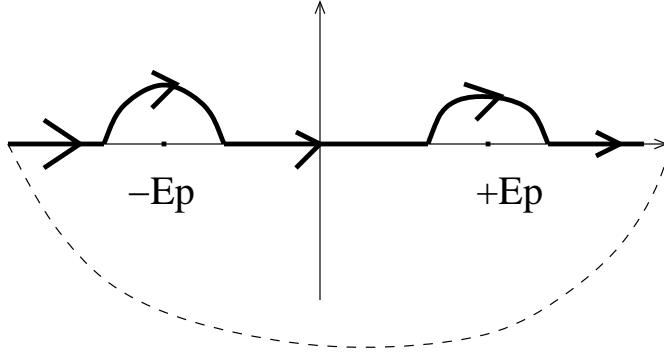
Figure 5: For the retarded propagator, the contour is such that closing it in the lower-half plane picks up both poles at $\pm E_p$.

we get a contour closed clockwise, hence its result is minus the residue (plus the residue is for a contour closed anti-clockwise), giving the extra minus sign for the contour integral to reproduce the right result.

On the other hand, for this contour, if we have $x^0 < y^0$ instead, the same reason above says we need to close the contour above (with an infinite semicircle in the upper half plane). In this case, there are no poles inside the contour, therefore the integral is zero.

We can then finally write for the *retarded propagator* (vanishes for $x^0 < y^0$)

$$
\begin{aligned}
D_R(x-y) &\equiv \theta(x^0 - y^0) < 0|[\phi(x), \phi(y)]|0> = \int \frac{d^3p}{(2\pi)^3} \frac{dp^0}{2\pi i} \frac{1}{p^2 + m^2} e^{ip\cdot(x-y)} \\
&= \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + m^2} e^{ip\cdot(x-y)}
\end{aligned}
\tag{4.26}
$$

This object is a Green's functions from the KG operator. This is easier to see in momentum space. Indeed, making a Fourier transform,

$$
D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} D_R(p)
\tag{4.27}
$$

we obtain

$$
D_R(p) = \frac{-i}{p^2 + m^2} \Rightarrow (p^2 + m^2) D_R(p) = -i
\tag{4.28}
$$

which means that in x space (the Fourier transform of 1 is $\delta(x)$, and the Fourier transform of p^2 is $-\partial^2$),

$$
(\partial^2 - m^2) D_R(x-y) = i\delta^4(x-y) \leftrightarrow -i(\partial^2 - m^2) D_R(x-y) = \delta^4(x-y)
\tag{4.29}
$$

The Feynman propagator

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Consider now a different "$i\epsilon$ prescription" for the contour of integration C. Consider a contour that avoids slightly below the $-E_p$ pole and avoids slightly above the $+E_p$ pole, as in Fig.6. This is equivalent with the Feynman prescription,

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2+m^2-i\epsilon} e^{ip\cdot(x-y)} \tag{4.30}$$

Since $p^2 + m^2 - i\epsilon = -(p^0)^2 + E_p^2 - i\epsilon = -(p^0 + E_p - i\epsilon/2)(p^0 - E_p + i\epsilon/2)$, we have poles at $p^0 = \pm(E_p - i\epsilon/2)$, so in this form we have a contour along the real line, but the poles are modified instead ($+E_p$ is moved below, and $-E_p$ is moved above the contour).
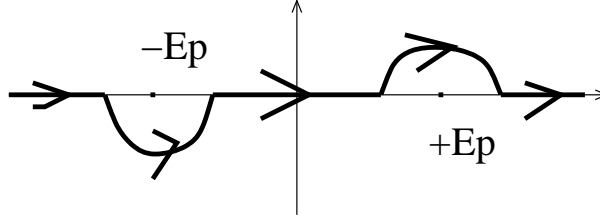

Figure 6: The contour for the Feynman propagator avoids $-E_p$ from below and $+E_p$ from above.

For this contour, as for D_R (for the same reasons), for $x^0 > y^0$ we close the contour below (with an infinite semicircle in the lower half plane). The result for the integration is then the residue inside the closed contour, i.e. the residue at $+E_p$. But we saw that the clockwise residue at $+E_p$ is $D(x-y)$ (and the clockwise residue at $-E_p$ is $-D(y-x)$).

For $x^0 < y^0$, we need to close the contour above (with an infinite semicircle in the upper half plane), and then we get the anticlockwise residue at $-E_p$, therefore $+D(y-x)$. The final result is then

$$D_F(x-y) = \theta(x^0-y^0) <0|\phi(x)\phi(y)|0> + \theta(y^0-x^0) <0|\phi(y)\phi(x)|0> \equiv <0|T(\phi(x)\phi(y))|0> \tag{4.31}$$

This is then the Feynman propagator, which again is a Green's function for the KG operator, with the *time ordering operator* in the two-point function, the same that we defined in the n-point functions in quantum mechanics, e.g. $<q',t'|T[\hat{q}(t_1)\hat{q}(t_2)]|q,t>$.

As suggested by the quantum mechanical case, the Feynman propagator will appear in the Feynman rules, and has a physical interpretation as propagation of the particle excitations of the quantum field.

Important concepts to remember

- The expansion of the free scalar field in quantum fields is relativistically invariant, as is the relativistic normalization.

- The complex scalar field is quantized in terms of $a_\pm$ and $a_\pm^\dagger$, which correspond to $U(1)$ charge ± 1. ϕ creates minus particles and destroys plus particles, and $\phi^\dagger$ creates plus particles and destroys minus particles.

- The plus and minus particles are particle/antiparticle pairs, since they only differ by their charge.

- The object $D(x-y) = <0|\phi(x)\phi(y)|0>$ is nonzero much outside the lightcone, however $[\phi(x), \phi(y)]$ is zero outside the lightcone, and since only this object leads to measurable quantities, QFT is causal.

- $D_R(x-y)$ is a retarded propagator and corresponds to a contour of integration that avoids the poles $\pm E_p$ from slightly above, and is a Green's function for the KG operator.

- $D_F(x-y) = <0|T[\phi(x)\phi(y)]|0>$, the Feynman propagator, corresponds to the $-i\epsilon$ prescription, i.e. avoids $-E_p$ from below and $+E_p$ from above, is also a Green's function for the KG operator, and it will appear in Feynman diagrams. It has the physical interpretation of propagation of particle excitations of the quantum field.

Further reading: See chapters 2.4 in [2] and 2.4 in [1].

Exercises, Lecture 4

1) For

$$\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi^* - m^2 |\phi|^2 - U(|\phi|^2) \tag{4.32}$$

calculate the Noether current for the $U(1)$ symmetry $\phi \to \phi e^{i\alpha}$ in terms of $\phi(\vec{x}, t)$ and show that it then reduces to the expression in the text,

$$Q = \int \frac{d^3 k}{(2\pi)^3} [a^\dagger_{+\vec{k}} a_{+\vec{k}} - a^\dagger_{-\vec{k}} a_{-\vec{k}}] \tag{4.33}$$

2) Calculate the *advanced* propagator, $D_A(x - y)$, by using the integration contour that avoids the $\pm E_p$ poles from below, instead of the contours for D_R, D_F, as in Fig.7.
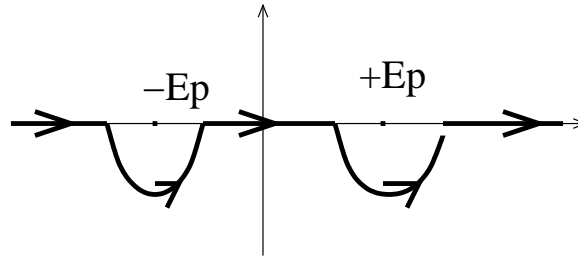

Figure 7: The contour for the advanced propagator avoids both poles at $\pm E_p$ from below.

5 Lecture 5: Interaction picture and Wick theorem for $\lambda\phi^4$ in operator formalism

Quantum mechanics pictures

One can transform states and opertors (similar to the canonical transformations in classical mechanics) with unitary operators W ($W^\dagger = W^{-1}$) by

$$|\psi_W >= W|\psi >$$
$$A_W = WAW^{-1} \tag{5.1}$$

Then the Hamiltonian in the new formulation is

$$\mathcal{H}' = \mathcal{H}_W + i\hbar\frac{\partial W}{\partial t}W^\dagger \tag{5.2}$$

where $\mathcal{H}_W$ stands for the original Hamiltonian, transformed as an operator by W. The statement is that $\mathcal{H}'$ generates the new time evolution, so in particular we obtain for the time evolution of the transformed operators

$$i\hbar\frac{\partial A_W}{\partial t} = i\hbar\left(\frac{\partial A}{\partial t}\right)_W + [i\hbar\frac{\partial W}{\partial t}W^\dagger, A_W] \tag{5.3}$$

and that the new evolution operator is

$$U'(t,t_0) = \hat{W}(t)U(t,t_0)W^\dagger(t_0) \neq \hat{U}_W \tag{5.4}$$

(note that on the left we have time t and on the right t_0, so we don't obtain U_W). Here the various pictures are assumed to coincide at time $t = 0$.

With this general formalism, let's review the usual pictures.

Schrödinger picture (usual)

In the Schrödinger picture, $\psi_S(t)$ depends on time, and usual operators don't $\frac{\partial \hat{A}_S}{\partial t} = 0$. We could have at most a c-number (non-operator) time dependence, like $A_s(t)\cdot\mathbf{1}$, with $A_s(t)$ a c-number function, maybe.

Heisenberg picture

In the Heisenberg picture, wavefunctions (states) are independent on time, $\partial\psi_H/\partial t = 0$, which is the same as saying that $\mathcal{H}' = 0$, the Hamiltonian generating the evolution of states vanishes. That means that we also have

$$\mathcal{H}_H = \mathcal{H}_S = -i\hbar\frac{\partial W}{\partial t}W^\dagger \tag{5.5}$$

giving the time evolution of operators,

$$i\hbar\frac{\partial}{\partial t}\hat{A}_H(t) = [\hat{A}_H(t), H] \tag{5.6}$$

If at t_0, the Schrödinger picture equals the Heisenberg picture, then

$$W(t) = U_S^{-1}(t,t_0) = U_S(t_0,t) \tag{5.7}$$

where $U_S(t, t_0)$ is the evolution operator (giving the time evolution of states) in the Schrödinger picture.

Dirac (interaction) picture

The need for the general theory of QM pictures was so that we could define the picture used in perturbation theory, the Dirac, or interaction, picture. We need it when we have

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \tag{5.8}$$

where $\hat{H}_0$ is a free quadratic piece, and $\hat{H}_1$ is an interacting piece. Then, we would like to write formulas like the ones already written, for the quantization of fields. Therefore we need the time evolution of operators to be only in terms of $\hat{H}_0$, leading to

$$i\hbar \frac{\partial}{\partial t}|\psi_I(t)> = \hat{H}_{1I}|\psi_I(t)>$$
$$i\hbar \frac{\partial}{\partial t}\hat{A}_I(t) = [\hat{A}_I(t), \hat{H}_0] \tag{5.9}$$

where note that $H_{0,I} = H_{0,S}$. Note that here we have denoted the interaction piece by H_1 instead of H_i, as we will later on, in order not to confuse the subscript with I, meaning in the interaction picture. Once the distinction will become clear, we will go back to the usual notation H_i.

Why is this picture, where operators (for instance the quantum fields) evolve with the free part ($\hat{H}_0$) only, preferred? Because of the usual physical set-up. The interaction region, where there is actual interaction between fields (or particles), like for instance in a scattering experiment at CERN, is finite both in space and in time. That means that at $t = \pm\infty$, we can treat the states as free. And therefore, it would be useful if we could take advantage of the fact that asymptotically the states are the free states considered in the previous lectures.

We must distinguish then between the true vacuum of the interacting theory, a state that we will call $|\Omega>$, and the vacuum of the free theory, which we will call $|0>$ as before, that satisfies $a_{\vec{p}}|0> = 0, \forall \vec{p}$. For the full theory we should use $|\Omega>$, but to use perturbation theory we will relate to $|0>$.

The basic objects to study are correlators, like the two-point function $< \Omega|T[\phi(x)\phi(y)]|\Omega >$, where $\phi(x)$ are Heisenberg operators. The interaction picture field is related to the field at the reference time t_0, where we define that the interaction picture field is the same as the Heisenberg and the Schrödinger picture field, by

$$\begin{aligned}
\phi_I(t, \vec{x}) &= e^{iH_0(t-t_0)}\phi(t_0, \vec{x})e^{-iH_0(t-t_0)} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}e^{ip\cdot x} + a_{\vec{p}}^\dagger e^{-ip\cdot x})|_{x^0=t-t_0, p^0=E_{\vec{p}}}
\end{aligned} \tag{5.10}$$

since, as explained in a previous lecture, $i\hbar\partial A/\partial t = [A, H_0]$ is equivalent to

$$A(t) = e^{iH_0(t-t_0)}A(t_0)e^{-iH_0(t-t_0)}, \tag{5.11}$$

so the first line relates $\phi_I(t)$ with $\phi_I(t_0)$, and then by the definition of the reference point t_0 as the one point of equality of pictures, we can replace with the Schrödinger picture operator,

with no time dependence and the usual definition of space dependence in terms of a and $a^\dagger$ (as in the canonical quantization lecture). Then the action of H_0 is replaced with the usual (free) p^0 in the exponent of $e^{ip\cdot x}$ (again as in the canonical quantization lecture).

$\lambda\phi^4$ *theory*

We will specialize to $\lambda\phi^4$ theory, with the free KG piece, plus an interaction,

$$H = H_0 + H_1 = H_0 + \int d^3x \frac{\lambda}{4!}\phi^4 \tag{5.12}$$

for simplicity.

Let us consider the interaction picture evolution operator (note that here t_0 is the reference time when the pictures are all equal; compare with (5.4))

$$U_I(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{5.13}$$

where $e^{-iH(t-t_0)}$ is the Schrödinger picture evolution operator $U_S(t, t_0)$, i.e.

$$\begin{aligned} |\psi_S(t)> &= U_S(t, t_0)|\psi_S(t_0)> \\ |\psi_I(t)> &= U_I(t, t_0)|\psi_I(t_0)> \end{aligned} \tag{5.14}$$

Since the Heisenberg operator has $\phi(t, \vec{x}) = e^{iH(t-t_0)}\phi(t_0, \vec{x})e^{-iH(t-t_0)}$, we have

$$\begin{aligned} \phi_I(t, \vec{x}) &= e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\phi(t, \vec{x})e^{iH(t-t_0)}e^{-iH_0(t-t_0)} \Rightarrow \\ \phi(t, \vec{x}) &= U_I^\dagger(t, t_0)\phi_I(t, \vec{x})U_I(t, t_0) \end{aligned} \tag{5.15}$$

The goal is to find an expression for $U_I(t, t_0)$, but to do that, we wil first prove a differential equation:

$$i\frac{\partial}{\partial t}U_I(t, t_0) = H_{i,I}U_I(t, t_0) \tag{5.16}$$

Let's prove this. When we take the derivative on U_I, we choose to put the resulting H and H_0 in the same place, in the middle, as:

$$i\frac{\partial}{\partial t}U_I(t, t_0) = e^{iH_0(t-t_0)}(H - H_0)e^{-iH(t-t_0)} \tag{5.17}$$

But $H - H_0 = H_{i,S}$ (in the Schrödinger picture), and by the relation between Schrödinger and interaction picture operators,

$$H_{i,I} = e^{iH_0(t-t_0)}H_{i,S}e^{-iH_0(t-t_0)} \tag{5.18}$$

and therefore

$$i\frac{\partial}{\partial t}U_I(t, t_0) = H_{i,I}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} = H_{i,I}U_I(t, t_0) \tag{5.19}$$

q.e.d. We then also note that

$$H_{i,I} = e^{iH_0(t-t_0)}H_{i,S}e^{-iH_0(t-t_0)} = \int d^3x \frac{\lambda}{4!}\phi_I^4 \tag{5.20}$$

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We now try to solve this differential equation for something like $U \sim e^{-iH_I t}$ by analogy with the Schrödinger picture. However, now we need to be more careful. We can write for the first few terms in the solution

$$U_I(t, t_0) = 1 + (-i) \int_{t_0}^t dt_1 H_{i,I}(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_{i,I}(t_1) H_{i,I}(t_2) + ... \qquad (5.21)$$

and naming these the zeroth term, the first term and the second term, we can easily check that we have

$$i\partial_t(\text{first term}) = H_{i,I}(t) \times \text{zeroth term}$$
$$i\partial_t(\text{second term}) = H_{i,I}(t) \times \text{first term} \qquad (5.22)$$

and this is like what we have when we write for c-numbers for instance $\partial_x e^{ax} = ae^{ax}$, and expanding $e^{ax} = \sum_n (ax)^n/n!$, we have a similar relation.
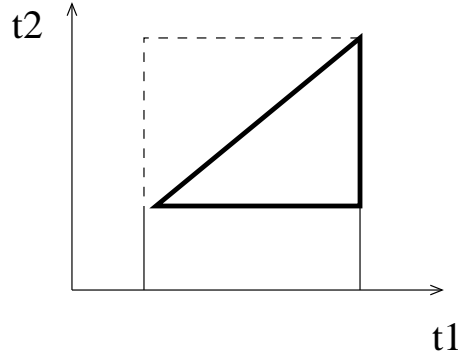

Figure 8: The integration in (t_1, t_2) is over a triangle, but can be written as half the integration over the rectangle.

But the integration region is a problem. In the (t_1, t_2) plane, the integration domain is the lower right-angle triangle in between t_0 and t, i.e. half the rectangle bounded by t_0 and t. So we can replace the triangular domain by half the rectangular domain, see Fig.8. But we have to be careful since $[H_I(t), H_I(t')] \neq 0$, and since in the integral expression above, we actually have time ordering, since $t_1 > t_2$, to obtain equality suffices to put time ordering in front, i.e.

$$U_I(t, t_0) = 1 + (-i) \int_{t_0}^t dt_1 H_{i,I}(t_1) + \frac{(-i)^2}{2!} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 T\{H_{i,I}(t_1) H_{i,I}(t_2)\} + ... \qquad (5.23)$$

and the higher order terms can be written in the same way, leading finally to

$$U_I(t, t_0) = T\left\{ \exp\left[-i \int_{t_0}^t dt' H_{i,I}(t') \right] \right\} \qquad (5.24)$$

Now that we have an expression for U_I, we still want an expression for the vacuum of the full theory, $|\Omega>$. We want to express it in terms of the free vacuum, $|0>$. Consider

$< \Omega|H|\Omega >= E_0$ (the energy of the full vacuum), and we also have states $|n >$ of higher energy, $E_n > E_0$. On the other hand for the free vacuum, $H_0|0 >= 0$. The completeness relation for the full theory is then

$$\mathbf{1} = |\Omega >< \Omega| + \sum_{n \neq 0} |n >< n| \tag{5.25}$$

and introducing it in between e^{-iHT} and $|0 >$ in their product, we obtain

$$e^{-iHT}|0 >= e^{-iE_0 T}|\Omega >< \Omega|0 > + \sum_{n \neq 0} e^{-iE_n T}|n >< n|0 > \tag{5.26}$$

Then consider $T \to \infty(1 - i\epsilon)$, where we take a slightly complex time in order to dampen away the higher $|n >$ modes, since then $e^{-iE_n T} \to e^{-iE_n \infty} \times e^{-E_n \infty \epsilon}$ and $e^{-E_n \infty \epsilon} \ll e^{-E_0 \infty \epsilon}$, so we can drop the terms with $n \neq 0$, with $e^{-iE_n T}$. Thus we obtain

$$
\begin{aligned}
|\Omega > &= \lim_{T \to \infty(1-i\epsilon)} \frac{e^{-iH(T+t_0)}|0 >}{e^{-iE_0(T+t_0)} < \Omega|0 >} \\
&= \lim_{T \to \infty(1-i\epsilon)} \frac{e^{-iH(T+t_0)} e^{iH_0(T+t_0)}|0 >}{e^{-iE_0(T+t_0)} < \Omega|0 >} \\
&= \lim_{T \to \infty(1-i\epsilon)} \frac{U_I(t_0, -T)|0 >}{e^{-iE_0(T+t_0)} < \Omega|0 >}
\end{aligned} \tag{5.27}
$$

where in the first line we isolated $T \to T + t_0$, in the second line we have introduced for free a term, due to $H_0|0 >= 0$, and in the third line we have formed the evolution operator.

Similarly, we obtain for $< \Omega|$,

$$< \Omega| = \lim_{T \to \infty(1-i\epsilon)} \frac{< 0|U_I(T, t_0)}{e^{-iE_0(T-t_0)} < 0|\Omega >} \tag{5.28}$$

We now have all the ingredients to calculate the two-point function.

Consider in the case $x^0 > y^0 > t_0$ the quantity

$$
\begin{aligned}
< \Omega|\phi(x)\phi(y)|\Omega > &= \lim_{T \to \infty(1-i\epsilon)} \frac{< 0|U_I(T, t_0)}{e^{-iE_0(T-t_0)} < 0|\Omega >} U_I^\dagger(x^0, t_0)\phi_I(x)U_I(x^0, t_0) \times \\
&\quad \times U_I^\dagger(y^0, t_0)\phi(y)U_I(y^0, t_0) \frac{U_I(t_0, -T)|0 >}{e^{-iE_0(T+t_0)} < \Omega|0 >} \\
&= \lim_{T \to \infty(1-i\epsilon)} \frac{< 0|U_I(T, x^0)\phi_I(x)U_I(x^0, y^0)\phi_I(y)U_I(y^0, -T)|0 >}{e^{-iE_0(2T)}| < 0|\Omega > |^2} \tag{5.29}
\end{aligned}
$$

where we have used $U^\dagger(t, t') = U^{-1}(t, t') = U(t', t)$ and $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$.

On the other hand, multiplying the expression we have for $|\Omega >$ and $< \Omega|$, we get

$$1 =< \Omega|\Omega >= \frac{< 0|U_I(T, t_0)U_I(t_0, -T)|0 >}{| < 0|\Omega > |^2 e^{-iE_0(2T)}} \tag{5.30}$$

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We divide then (5.29) by it, obtaining

$$< \Omega | \phi(x)\phi(y) | \Omega > = \lim_{T \to \infty(1-i\epsilon)} \frac{< 0 | U_I(T, x^0)\phi_I(x)U_I(x^0, y^0)\phi_I(y)U_I(y^0, -T) | 0 >}{< 0 | U_I(T, -T) | 0 >} \quad (5.31)$$

We now observe that, since we used $x^0 > y^0 > t_0$ in our calculation, both the left hand side and the right hand side are time ordered, so we can generalize this relation with T symbols on both lhs and rhs. But then, since $T(AB) = T(BA)$ (the time ordering operators orders temporally, independent of their initial order inside it), we can permute how we want the operators inside $T()$, in particular by extracting the field operators to the left and then multiplying the resulting evolution operators, $U_I(T, x^0)U_I(x^0, y^0)U_I(y^0, -T) = U_I(T, -T)$. We then use the exponential expression for the evolution operator, to finally get our desired result

$$< \Omega | T\{\phi(x)\phi(y)\} | \Omega > = \lim_{T \to \infty(1-i\epsilon)} \frac{< 0 | T\left\{ \phi_I(x)\phi_I(y) \exp\left[-i \int_{-T}^{T} dt H_I(t) \right] \right\} | 0 >}{< 0 | T\left\{ \exp\left[-i \int_{-T}^{T} dt H_I(t) \right] \right\} | 0 >} \quad (5.32)$$

This is called **Feynman's theorem** and we can generalize it to a product of any number of ϕ's, and in fact any insertions of Heisenberg operators $\mathcal{O}_H$, so we can write

$$< \Omega | T\{\mathcal{O}_H(x_1)...\mathcal{O}_H(x_n)\} | \Omega > = \lim_{T \to \infty(1-i\epsilon)} \frac{< 0 | T\left\{ \mathcal{O}_I(x_1)...\mathcal{O}_I(x_n) \exp\left[-i \int_{-T}^{T} dt H_I(t) \right] \right\} | 0 >}{< 0 | T\left\{ \exp\left[-i \int_{-T}^{T} dt H_I(t) \right] \right\} | 0 >}$$
$$(5.33)$$

Wick's theorem

From Feynman's theorem, we see we need to calculate $< 0 | T\{\phi_I(x_1)...\phi_I(x_n)\} | 0 >$ (from explicit insertions, or the expansion of the exponential of $H_I(t)$'s). To do that, we split

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x) \quad (5.34)$$

where

$$\phi_I^+ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\vec{p}} e^{ipx}$$
$$\phi_I^- = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\vec{p}}^\dagger e^{-ipx} \quad (5.35)$$

Note that then $\phi_I^+ | 0 > = 0 = < 0 | \phi_I^-$. We defined normal order, $:()$:, which we will denote here (since in the Wick's theorem case it is usually denoted like this) by $N()$, which means ϕ_I^- to the left of ϕ_I^+.

Consider then, in the $x^0 > y^0$ case,

$$\begin{aligned} T(\phi_I(x)\phi_I(y)) &= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + [\phi_I^+(x), \phi_I^-(y)] \\ &= N(\phi_I(x)\phi_I(y)) + [\phi_I^+(y), \phi_I^-(y)] \end{aligned} \quad (5.36)$$

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In the $y^0 > x^0$ case, we need to change $x \leftrightarrow y$.

We now define the *contraction*

$$
\begin{aligned}
\overline{\phi_I(x)\phi_I}(y) &= [\phi_I^+(x), \phi_I^-(y)], \;\; \text{for} \;\; x^0 > y^0 \\
&= [\phi_I^+(y), \phi_I^-(x)] \;\; \text{for} \;\; y^0 > x^0
\end{aligned}
\tag{5.37}
$$

Then we can write

$$
T[\phi_I(x)\phi_I(y)] = N[\phi_I(x)\phi_I(y) + \overline{\phi_I(x)\phi_I}(y)]
\tag{5.38}
$$

But since $[\phi^+, \phi^-]$ is a c-number, we have

$$
[\phi^+, \phi^-] = < 0|[\phi^+, \phi^-]|0 > = < 0|\phi^+\phi^-|0 > = D(x-y)
\tag{5.39}
$$

where we also used $< 0|\phi^- = \phi^+|0 > = 0$. Then we get

$$
\overline{\phi(x)\phi}(y) = D_F(x-y)
\tag{5.40}
$$

Generalizing (5.38), we get *Wick's theorem*,

$$
T\{\phi_I(x_1)...\phi_I(x_n)\} = N\{\phi_I(x_1)...\phi_I(x_n) + \text{all possible contractions}\}
\tag{5.41}
$$

where all possible contractions means not only full contractions, but also partial contractions. Note that for instance if we contract 1 with 3, we obtain

$$
N(\overline{\phi_1\phi_2\phi_3}\phi_4) = D_F(x_1-x_3)N(\phi_2\phi_4)
\tag{5.42}
$$

Why is the Wick theorem useful? Since for any nontrivial operator, $< 0|N(anything)|0 > = 0$, as $< 0|\phi^- = \phi^+|0 > = 0$, so only the *fully contracted* terms (c-numbers) remain in the $< 0|N()|0 >$, giving a simple result. Consider for instance the simplest nontrivial term,

$$
\begin{aligned}
T(\phi_1\phi_2\phi_3\phi_4) = &\; N(\phi_1\phi_2\phi_3\phi_4 + \overline{\phi_1\phi_2}\phi_3\phi_4 + \overline{\phi_1\phi_2\phi_3}\phi_4 + \overline{\phi_1\phi_2\phi_3\phi_4} \\
&+ \phi_1\overline{\phi_2\phi_3}\phi_4 + \phi_1\overline{\phi_2\phi_3\phi_4} + \phi_1\phi_2\overline{\phi_3\phi_4} \\
&+ \overline{\phi_1\phi_2}\;\overline{\phi_3\phi_4} + \overline{\phi_1\phi_2\phi_3\phi_4} + \overline{\phi_1\phi_2\phi_3\phi_4})
\end{aligned}
\tag{5.43}
$$

Which then gives under the vacuum expectation value,

$$
\begin{aligned}
< 0|T(\phi_1\phi_2\phi_3\phi_4)|0 > = &\; D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_4)D_F(x_2-x_3) \\
&+ D_F(x_1-x_3)D_F(x_2-x_4)
\end{aligned}
\tag{5.44}
$$

For the proof of the Wick's theorem, we use induction. We have proved the initial step, for $n = 2$, so it remains to prove the step for n, assuming the step for $n - 1$. This is left as an exercise.

Important concepts to remember

- Unitary operators transform states and operators in quantum mechanics, changing the picture. The new Hamiltonian is different than the transformed Hamiltonian.

- In QFT, we use the Dirac or interaction picture, where operators evolve in time with H_0, and states evolve with $H_{i,I}$.

- Perturbation theory is based on the existence of asymptotically free states (at $t = \pm\infty$ and/or spatial infinity) where we can use canonical quantization of the free theory, and use the vacuum $|0>$ to perturb around, while the full vacuum is $|\Omega>$.

- In the interacting theory, the interaction picture field $\phi_I(t, \vec{x})$ has the same expansion as the Heisenberg field in the free KG case studied before.

- The time evolution operator of the interaction picture is written as $U_I(t, t_0) = T \exp[-i \int dt H_{i,I}(t)]$

- Feynman's theorem expresses correlators (n-point functions) of the full theory as the ratio of VEV *in the free vacuum* of T of interaction picture operators, with an insertion of $U_I(T, -T)$, divided by the same thing without the insertion of operators.

- the contraction of two fields gives the Feynman propagator.

- Wick's theorem relates the time order of operators with the normal order of a sum of terms: the operators, plus all their possible contractions.

- In $< 0|T(...)|0 >$ that appears in Feynman's theorem as the thing to calculate, only the full contractions give a nonzero result.

Further reading: See chapters 4.2,4.3 in [2] and 2.5 in [1].

Exercises, Lecture 5

1) For an interaction Hamiltonian

$$H_I = \int d^3x \frac{\lambda}{4!} \phi_I^4, \tag{5.45}$$

write down the *explicit* form of the Feynman theorem for the operators $\mathcal{O}_I(x)\mathcal{O}_I(y)\mathcal{O}_I(z)$, where $\mathcal{O}_I(x) = \phi^2(x)$, in order λ^2 for the numerator, and then the Wick theorem for the numerator (not all the terms, just a few *representative* ones).

2) Complete the induction step of Wick's theorem and then write all the *nonzero* terms of the Wick theorem for

$$< 0|T\{\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6\}|0 > \tag{5.46}$$

6 Lecture 6: Feynman rules for $\lambda \phi^4$ from the operator formalism

We saw that the Feynman theorem relates the correlators to a ratio of vacuum expectation values (VEVs) in the free theory, of time orderings of the interaction picture fields, i.e. things like

$$< 0|T\{\phi(x_1)...\phi(x_n)|0 > \tag{6.1}$$

which can then be evaluated using Wick's theorem, as the sum of all possible contractions (products of Feynman propagators).

This includes insertions of

$$\left[-i \int dt H_{i,I}(t)\right]^n \tag{6.2}$$

inside the time ordering.

We can make a diagrammatic representation for

$$\begin{aligned}< 0|T(\phi_1\phi_2\phi_3\phi_4)|0 > &= D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3) \\ &\quad + D_F(x_1 - x_3)D_F(x_2 - x_4)\end{aligned} \tag{6.3}$$

as the sum of 3 terms: one where the pairs of points $(12), (34)$ are connected by lines, the others for $(13), (24)$ and $(14), (23)$, as in Fig.9. We call these the *Feynman diagrams*, and we will see later how to make this in general.
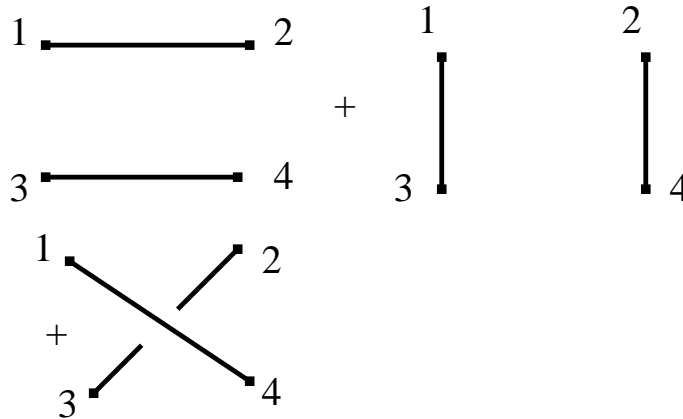
Figure 9: Feynman diagrams for free processes for the 4-point function. We can join the external points in 3 different ways.

These quantities that we are calculating are not yet physical, we will later relate it to some physical scattering amplitudes, but we can still use a physical interpretation of particles propagating, being created and annihilated.

We will later define more precisely the so-called S-matrices that relate to physical scattering amplitudes, and define their relation to the correlators we compute here, but for the moment we will just note that we have something like $S_{fi} = < f|S|i >$, where $S = U_I(+\infty, -\infty)$.

So in some sense this corresponds to propagating the state $|i>$ from $-\infty$ to $+\infty$, and then computing the probability to end up in the state $|f>$. For now, we will just study the abstract story, and leave the physical interpretation for later.

Let us consider a term with a $H_{i,I}$ in it, for instance the first order term (order λ term) in $H_{i,I}$ for the 2-point functions,

$$< 0|T\left\{\phi(x)\phi(y)\left[-i\int dtH_{i,I}\right]\right\}|0> \tag{6.4}$$

where $\int dtH_{i,I}(t) = \int d^4z\lambda/4!\phi_I^4$. We then write this term as

$$< 0|T\left\{\phi(x)\phi(y)\left(-i\frac{\lambda}{4!}\right)\int d^4z\phi(z)\phi(z)\phi(z)\phi(z)\right\}|0> \tag{6.5}$$

In this, there are 2 independent Wick contractions we can consider:

-contracting x with y and z's among themselves: $\overparen{\phi(x)\phi(y)}$ and $\phi(z)\phi(z)\phi(z)\phi(z)$

-contracting x with z and y with z: $\phi(x)\phi(z)$, $\phi(y)\phi(z)$, $\phi(z)\phi(z)$.

The first contraction can be done in 3 ways, because we can choose in 3 ways which is the contraction partner for the first $\phi(z)$ (and the last contraction is then fixed).

The second contraction can be done in $4 \times 3 = 12$ ways, since we have 4 choices for the $\phi(z)$ to contract with $\phi(x)$, then 3 remaining choices for the $\phi(z)$ to contract with $\phi(y)$. In total we have $3 + 12 = 15$ terms, corresponding to having the 6 ϕ's, meaning 5 ways to make the first contraction between them, 3 ways to make the second, and one way to make the last.

The result is then

$$< 0|T\left\{\phi(x)\phi(y)\left[-i\int dtH_{i,I}\right]\right\}|0> = 3\left(-i\frac{\lambda}{4!}\right)D_F(x-y)\int d^4zD_F(z-z)D_F(z-z)$$
$$+12\left(-i\frac{\lambda}{4!}\right)\int d^4zD_F(x-z)D_F(y-z)D_F(z-z) \tag{6.6}$$

The corresponding Feynman diagrams for these terms are: a line between x and y, and a figure eight with the middle point being z, and for the second term, a line between x and y with a z in the middle, and an extra loop starting and ending again at z, as in Fig.10.

Consider now a more complicated contraction: In the $\mathcal{O}(\lambda^3)$ term

$$< 0|T\left\{\phi(x)\phi(y)\frac{1}{3!}\left(-i\frac{\lambda}{4!}\right)^3\int d^4z\phi(z)\phi(z)\phi(z)\phi(z)\int d^4w\phi(w)\phi(w)\phi(w)\phi(w)\int d^4u\times\right.$$
$$\left.\times\phi(u)\phi(u)\phi(u)\phi(u)\right\}|0> \tag{6.7}$$

contract $\phi(x)$ with a $\phi(z)$, $\phi(y)$ with a $\phi(w)$, a $\phi(z)$ with a $\phi(w)$, two $\phi(w)$'s with two $\phi(u)$'s, and the remaing $\phi(z)$'s between them, the remaining $\phi(u)$'s between them. This gives the result

$$\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4zd^4wd^4uD_F(x-z)D_F(z-z)D_F(z-w)D_F(w-u)D_F(w-u)D_F(u-u)D_F(w-y) \tag{6.8}$$
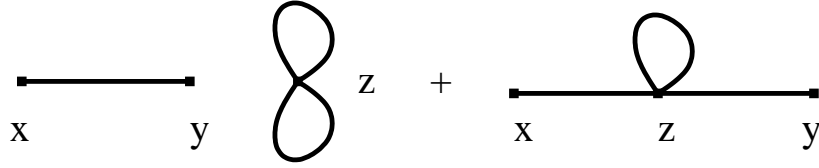
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Figure 10: The two Feynman diagrams at order λ in the expansion (first order in the interaction vertex).

The corresponding Feynman diagram then has the points x and y with a line in betwen them, on which we have the points z and w, with an extra loop starting and ending at z, two lines forming a loop between w and an extra point u, and an extra loop starting and ending at u, as in Fig.11.
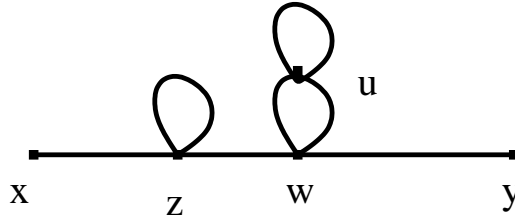
Figure 11: A Feynman diagram at order λ^3, forming 3 loops.

Let us count the number of identical contractions for this diagram: there are 3! ways of choosing which to call z, w, u (permutations of 3 objects). Then in $\int d^4z\phi\phi\phi$, we can choose the contractions with $\phi(x)$ and with $\phi(w)$ in 4×3 ways, in $\int d^4w\phi\phi\phi\phi$ we can choose the contractions with z, y and u, u in $4 \times 3 \times 2 \times 1$ ways, and then choose the contractions in $\int d^4u\phi\phi\phi\phi$ with $\phi(w)\phi(w)$ in 4×3 ways. But then, we overcounted in choosing the two $w - u$ contractions, by counting a 2 in both w and u, so we must divide by 2, for a total of

$$3! \times (4 \times 3) \times (4 \times 3 \times 2) \times (4 \times 3) \times \frac{1}{2} = \frac{3! \times (4!)^3}{8} \tag{6.9}$$

We note that this almost cancels the $1/(3! \times (4!)^3)$ in front of the Feynman diagram result. We therefore define the *symmetry factor*, in general

$$\frac{\text{denominator}}{\text{nr. of contractions}} = \frac{p!(4!)^p}{\# \ of \ contractions} \tag{6.10}$$

which in fact equals the number of symmetries of the Feynman diagram. Here the symmetry factor is

$$2 \times 2 \times 2 = 8 \tag{6.11}$$

which comes because we can interchange the two ends of $D_F(z-z)$ in the Feynman diagram, obtaining the same diagram, then we can interchange the two ends of $D_F(u-u)$ in the same way, and we can also interchange the two $D_F(w-u)$ obtaining the same diagram.

Let us see a few more examples. Consider the one loop diagram with a line between x and y, with a point z in the middle, with an extra loop starting and ending at the same z, see Fig.12a. We can interchange the two ends of $D_F(z-z)$ obtaining the same diagram, thus the symmetry factor is $S=2$. Consider then the figure eight diagram (vacuum bubble) with a central point z, see Fig.12b. It has a symmetry factor of $S=2\times 2\times 2=8$, since we can interchange the ends of one of the $D_F(z-z)$, also of the other $D_F(z-z)$, and we can also interchange the two $D_F(z-z)$ between them. Consider then the "setting sun" diagram, see Fig.12c, with $D_F(x-z)$, theree $D_F(z-w)$'s, and then $D_F(w-y)$. We can permute in 3! ways the three $D_F(z-w)$'s, obtaining the same diagram, thus $S=3!=6$. Finally, consider the diagram with $D_F(x-z)$, $D_F(z-y)$, $D_F(z-u)$, $D_F(z-w)$ and three $D_F(u-w)$'s, see Fig.12d. It has a symmetry factor of $S=3!\times 2=12$, since there are 3! ways of interchanging the three $D_F(u-w)$'s, and we can rotate the subfigure touching at point z, thus interchanging $D_F(z-u)$ with $D_F(z-w)$ and u with w.
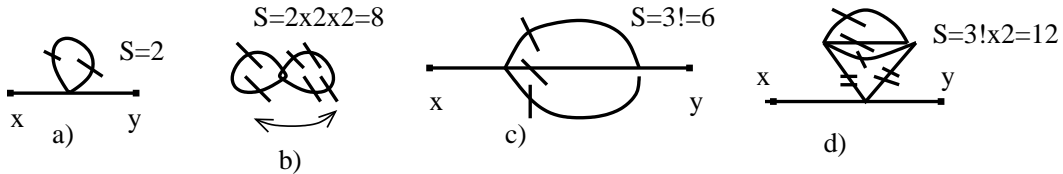
Figure 12: Examples of symmetry factors. For diagram a, $S=2$, for diagram b, $S=8$, for diagram c, $S=6$ and for diagram d, $S=12$.

x-space Feynman rules for $\lambda\phi^4$

We are now ready to state the x-space Feynman rules for the numerator of the Feynman theorem,

$$< 0|T\left\{\phi_I(x_1)...\phi_I(x_n)\exp\left[-i\int H_{i,I}(t)\right]\right\}|0>,\tag{6.12}$$

see Fig.13.
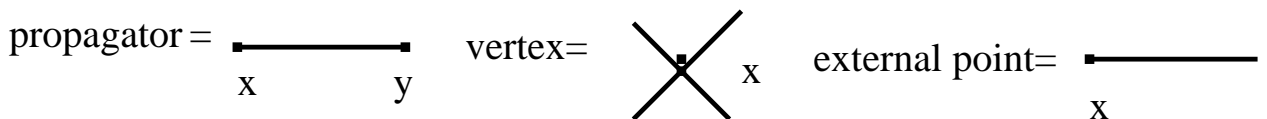
Figure 13: Pictorial representations for Feynman rules in x space.

-For the *propagator* we draw a line between x and y, corresponding to $D_F(x-y)$.

-For the *vertex* we draw a point z with 4 lines coming out of it, and it corresponds to $(-i\lambda) \int d^4z$.

-For the *external point* (line) at x, a we draw a line ending at x, corresponding to a factor of 1 (nothing new).

-After drawing the Feynman diagram according to the above, we divide the result by the symmetry factor.

-Then summing over all possible diagrams of all orders in λ, we get the full result for the numerator above.

Let us now turn to *p-space*.

The Feynman propagator is written as

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + m^2 - i\epsilon} e^{ip\cdot(x-y)} \tag{6.13}$$

so that the p-space propagator is

$$D_F(p) = \frac{-i}{p^2 + m^2 - i\epsilon} \tag{6.14}$$
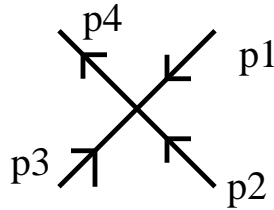
Figure 14: Examples of momenta at a vertex, for definition of convention.

We must choose a direction for the propagator, and we choose arbitrarily to be from x to y (note that $D_F(x-y) = D_F(y-x)$, so the order doesn't matter). Then for instance the line going into a point y corresponds to a factor of e^{-ipy} (see the expression above). Consider that at the 4-point vertex z, for instance p_4 goes out, and p_1, p_2, p_3 go in, as in Fig.14. Then we have for the factors depending on z

$$\int d^4z \, e^{-ip_1z} e^{-ip_2z} e^{-ip_3z} e^{+ip_4z} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4) \tag{6.15}$$

i.e., *momentum conservation.*

p-space Feynman rules

Thus the Feynman rules in p-space are (see Fig.15)

-For the *propagator* we write a line with an arrow in the direction of the momentum, corresponding to $D_F(p)$.

-For the *vertex*, we write 4 lines going into a point, corresponding to a factor of $-i\lambda$.

-For the *external point* (line), we write an arrow going into the point x, with momentum p, corresponding to a factor of $e^{-ip\cdot x}$.

-We then impose momentum conservation at each vertex.
-We integrate over the internal momenta, $\int d^4p/(2\pi)^4$.
-We divide by the symmetry factor.

propagator= $\longrightarrow$ $\qquad$ =vertex $\qquad$ external point= $\longleftarrow$

p $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ x $\quad$ p
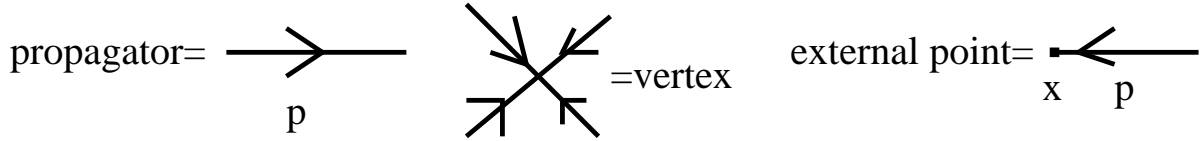
Figure 15: Pictorial representation of Feynman rules in p space.

We should note something: why do we have the *Feynman* propagator, i.e. the Feynman contour? The Feynman contour avoids the $+E_p$ pole from above and the $-E_p$ pole from below, and is equivalent with the $-i\epsilon$ prescription, which takes $p^0 \propto 1 + i\epsilon$, see Fig.16.
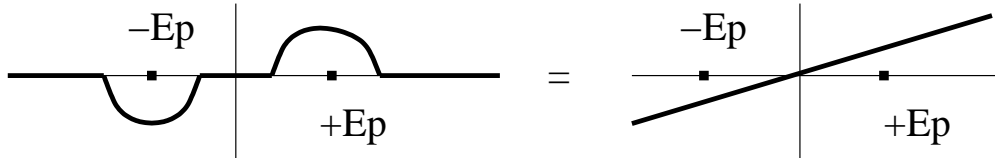
Figure 16: The Feynman contour can be understood in two different ways. On the rhs, the contour is a straight line.

It is needed since without it, due to $T \to \infty(1 - i\epsilon)$, in $\int_{-T}^{T} dz^0 d^3z e^{i(p_1+p_2+p_3-p_4)\cdot z}$, we would have a factor of $e^{\pm(p_1^0+p_2^0+p_3^0-p_4^0)\epsilon}$ which blows up on one of the sides (positive or negative p^0). But with it, we have in the exponent $(1 + i\epsilon)(1 - i\epsilon) = 1 + \mathcal{O}(\epsilon^2)$, so we have no factor to blow up in order ϵ.

Cancelling of the vacuum bubbles in numerator vs. denominator

Vacuum bubbles give infinities, for instance the p-space double figure eight vacuum diagram, with momenta p_1 and p_2 between points z and w, and also a loop of momentum p_3 going out and then back inside z and another loop of momentum p_4 going out and then back inside w, as in Fig.17. Then momentum conservation at z and w, together with the integration over p_1 gives

$$\int \frac{d^4p_1}{(2\pi)^4}(2\pi)^4\delta^4(p_1 + p_2) \times (2\pi)^4\delta^4(p_1 + p_2) = (2\pi)^4\delta^4(0) = \infty \qquad (6.16)$$

which can also be understood as $\int d^4w(\text{const}) = 2T \cdot V$ (time $T \to \infty$ and volume $V \to \infty$).

But these unphysical infinities cancel out between the numerator and denominator of the Feynman theorem: they factorize and exponentiate in the same way in the two. Let us see that.
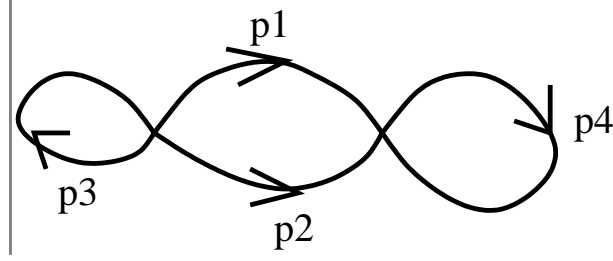
Figure 17: Vacuum bubble diagram, giving infinities.

In the numerator, for the two point function with $\phi(x)$ and $\phi(y)$, we have the connected pieces, like the setting sun diagram, factorized, multiplied by a sum of all the possible vacuum bubbles (i.e., with no external points x and y): terms like the figure eight bubble, that we will call V_1, a product of two figure eights, $(V_1)^2$, the same times a setting sun bubble (4 propagators between two internal points) called V_2, etc., see Fig.18.

Figure 18: The infinite vacuum bubbles factorize in the calculation of n-point functions (here, 2-point function).

Then in general we have (the $1/n_i!$ is for permutations of identical diagrams)

$$\lim_{T\to\infty(1-i\epsilon)} <0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0>$$

$$=\quad \text{connected piece}\times\sum_{\text{all }\{n_i\}\ sets}\prod_i\frac{1}{n_i!}(V_i)^{n_i}$$

$$=\quad \text{connected pieces}\times\left(\sum_{n_1}\frac{1}{n_1!}(V_1)^{n_1}\right)\left(\sum_{n_2}\frac{1}{n_2!}(V_2)^{n_2}\right)\times...$$

$$=\quad \text{connected pieces}\times\prod_i\left(\sum_{n_i}\frac{1}{n_i!}(V_i)^{n_i}\right)$$

$$=\quad \text{connected pieces}\times e^{\sum_i V_i}, \tag{6.17}$$

as in Fig.19.

But the same thing happens for the denominator, obtaining

$$\lim_{T\to\infty(1-i\epsilon)} <0|T\left\{\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0>= e^{\sum_i V_i} \tag{6.18}$$

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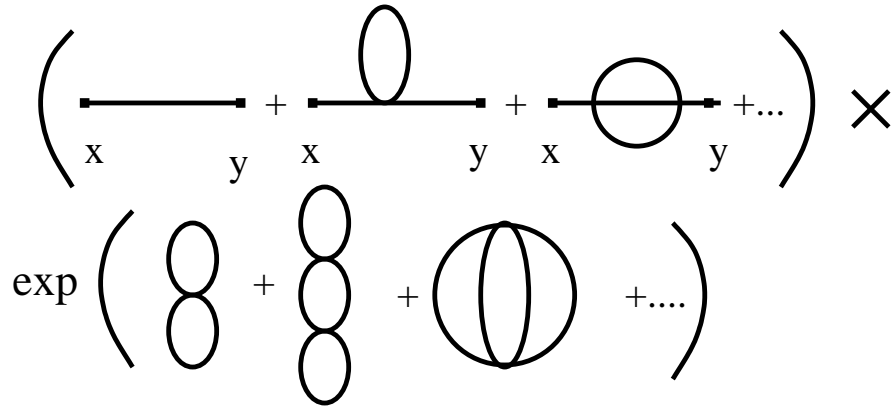
Figure 19: The final result for the factorization of the vacuum bubble diagrams in the 2-point function, with an exponential that cancels against the same one in the denominator.

and cancels with the numerator, obtaining that

$$< \Omega|T\{\phi(x)\phi(y)\}|\Omega > \tag{6.19}$$

is the sum of only the *connected diagrams.*

Thus in the same way, in general,

$$< \Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega > = \text{sum of all connected diagrams} \tag{6.20}$$

This is the last thing to add to the Feynman rules: for the correlators, take only connected digrams.

Important concepts to remember

- Feynman diagrams are pictorial representations of the contractions of the perturbative terms in the Feynman theorem. They have the interpretation of particle propagation, creation and annihilation, but at this point, it is a bit abstract, since we have not yet related to the physical S-matrices for particle scattering.

- The symmetry factor is given by the number of symmetric diagrams (by a symmetry, we get the same diagram).

- The Feynman rules in x-space have $D_F(x - y)$ for the propagator, $-i\lambda \int d^4z$ for the vertex and 1 for the external point, and dividing by the symmetry factor at the end.

- The Feynman rules in p-space have $D_F(p)$ for the propagator, $-i\lambda$ for the vertex, $e^{-ip\cdot x}$ for the external line going in x, momentum conservation at each vertex, integration over internal momenta, and divide by the symmetry factor.

- Vacuum bubbles factorize, exponentiate and cancel between the numerator and denominator of the Feynman theorem, leading to the fact that the n-point functions have only connected diagrams.

Further reading: See chapter 4.4 in [2].

Exercises, Lecture 6

1) Apply the x-space Feynman rules to write down the expression for the Feynman diagram in Fig.20.
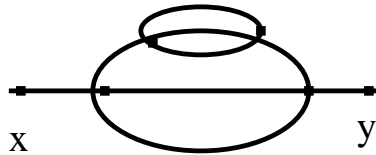
Figure 20: x-space Feynman diagram.

2) Idem, for the p-space diagram in Fig.21.
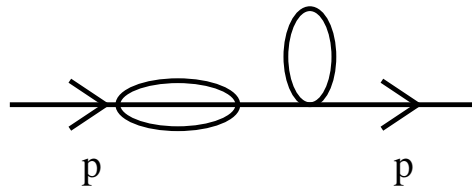
Figure 21: p space Feynman diagram.

7 Lecture 7: The driven (forced) harmonic oscillator

We have seen that for a quantum mechanical system, we can write the transition amplitudes as path integrals, via

$$
\begin{aligned}
F(q', t'; q, t) &\equiv {}_H\!< q', t'|q, t >_H =< q'|e^{-i\hat{H}(t'-t)}|q > \\
&= \int \mathcal{D}p(t)\mathcal{D}q(t) \exp\left\{ i \int_{t_0}^{t_{n+1}} dt[p(t)\dot{q}(t) - H(q(t), q(t))] \right\}
\end{aligned} \tag{7.1}
$$

and, if the Hamiltonian is quadratic in momenta, $H(p, q) = p^2/2 + V(q)$, then

$$
F(q', t'; q, t) = \mathcal{N} \int \mathcal{D}q e^{iS[q]} \tag{7.2}
$$

where $\mathcal{N}$ is a constant. The important objects to calculate in general (in quantum mechanics $F(q't; q't)$ would be sufficient, but not in quantum field theory) are the correlators or n-point functions,

$$
\begin{aligned}
G_N(\bar{t}_1, ..., \bar{t}_N) &= < q', t'|T\{\hat{q}(\bar{t}_1)...\hat{q}(\bar{t}_N)\}|q, t > \\
&= \int \mathcal{D}q(t) e^{iS[q]} q(\bar{t}_1)...q(\bar{t}_N)
\end{aligned} \tag{7.3}
$$

We can compute all the correlators from their generating functional,

$$
Z[J] = \int \mathcal{D}q e^{iS[q;J]} \equiv \int \mathcal{D}q e^{iS[q]+i\int dt J(t)q(t)} \tag{7.4}
$$

by

$$
G_N(t_1, ..., t_N) = \frac{\delta}{i\delta J(t_1)}...\frac{\delta}{i\delta J(t_N)}Z[J]|_{J=0} \tag{7.5}
$$

In the above, the object $J(t)$ was just a mathematical artifice, useful only to obtain the correlators through derivatives of $Z[J]$. But actually, considering that we have the action $S[q; J] = S[q] + \int dt J(t)q(t)$, we see that a nonzero $J(t)$ acts as a source term for the classical $q(t)$, i.e. an external driving force for the harmonic oscillator, since its equation of motion is now

$$
0 = \frac{\delta S[q; J]}{\delta q(t)} = \frac{\delta S[q]}{\delta q(t)} + J(t) \tag{7.6}
$$

So in the presence of nonzero $J(t)$ we have a driven harmonic oscillator.

We can consider then as a field theory primer the (free) harmonic oscillator driven by an external force, with action

$$
S[q; J] = \int dt \left[\frac{1}{2}(\dot{q}^2 - \omega^2 q^2) + J(t)q(t)\right] \tag{7.7}
$$

which is quadratic in q, therefore the path integral is gaussian, of the type we already performed in Lecture 2 (when we went from the phase space path integral to the configuration space path integral). That means that we will be able to compute the path integral exactly.

But there is an important issue of boundary conditions for $q(t)$. We will first make a naive treatment, then come back to do it better.

Sloppy treatment

If we can partially integrate $\dot{q}^2/2$ in the action without boundary terms (not quite correct, see later), then we have

$$S[q;J] = \int dt \left\{ -\frac{1}{2} q(t) \left[\frac{d^2}{dt^2} + \omega^2 \right] q(t) + J(t)q(t) \right\} \tag{7.8}$$

so then the path integral is of the form

$$Z[J] = \mathcal{N} \int \mathcal{D}q \, e^{-\frac{1}{2} q \Delta^{-1} q + i J \cdot q} \tag{7.9}$$

where we have defined $iS = -1/2 q\Delta^{-1} q + ...$, so that

$$\begin{aligned}
\Delta^{-1} q(t) &\equiv i \left[\frac{d^2}{dt^2} + \omega^2 \right] q(t) \\
J \cdot q &\equiv \int dt \, J(t)q(t)
\end{aligned} \tag{7.10}$$

Here Δ is the *propagator*. Then, remembering the general gaussian integration formula,

$$\begin{aligned}
S &= \frac{1}{2} x^T A x + b^T x \Rightarrow \\
\int d^n x \, e^{-S(x)} &= (2\pi)^{n/2} (\det A)^{-1/2} e^{\frac{1}{2} b^T A^{-1} b}
\end{aligned} \tag{7.11}$$

where now $b = -iJ(t)$ and $A = \Delta^{-1}$, we obtain

$$Z[J] = \mathcal{N}' e^{-\frac{1}{2} J \cdot \Delta \cdot J} \tag{7.12}$$

where $\mathcal{N}'$ contains besides $\mathcal{N}$ and factors of 2π, also $(\det \Delta)^{1/2}$, which is certainly nontrivial, however it is J-independent, so it is put as part of the overall constant. Also,

$$J \cdot \Delta \cdot J \equiv \int dt \int dt' J(t) \Delta(t,t') J(t') \tag{7.13}$$

Here we find

$$\Delta(t,t') = i \int \frac{dp}{2\pi} \frac{e^{-ip(t-t')}}{p^2 - \omega^2} \tag{7.14}$$

since

$$\begin{aligned}
\Delta^{-1} \Delta(t,t') &= i \left[\frac{d^2}{dt^2} + \omega^2 \right] i \int \frac{dp}{2\pi} \frac{e^{-ip(t-t')}}{p^2 - \omega^2} = -\int \frac{dp}{2\pi} \frac{-p^2 + \omega^2}{p^2 - \omega^2} e^{-ip(t-t')} \\
&= \int \frac{dp}{2\pi} e^{-ip(t-t')} = \delta(t-t')
\end{aligned} \tag{7.15}$$

But we note that there is a singularity $p^2 = \omega^2$. In fact, we have seen before that we avoided these singularities using a certain integration contour in complex p^0 space, giving the various propagators. Let's think better what this means. The question is, in the presence of this singularity, is the operator Δ^{-1} invertible, so that we can write down the above formula for Δ? An operator depends also on the space of functions on which it is defined, i.e. in the case of quantum mechanics, on the Hilbert space of the theory.

If we ask whether Δ^{-1} is invertible on the space of *all* the functions, then the answer is obviously no. Indeed, there are zero modes, i.e. eigenfunctions with eigenvalue zero for Δ^{-1}, namely ones satisfying

$$\left[\frac{d^2}{dt^2} + \omega^2 \right] q_0(t) = 0 \tag{7.16}$$

and since $\Delta^{-1} q_0 = 0$, obviously on q_0 we cannot invert Δ^{-1}. Moreover, these zero modes are not even pathological, so that we can say we neglect them, but rather these are the classical solutions for the free oscillator!

So in order to find an invertible operator Δ^{-1} we must exclude these zero modes from the Hilbert space *on which Δ^{-1} acts* (they are classical solutions, so obviously they exist in the theory), by imposing some boundary conditions that exclude them.

We will argue that the correct result is

$$\Delta_F(t, t') = i \int \frac{dp}{2\pi} \frac{e^{-ip(t-t')}}{p^2 - \omega^2 + i\epsilon} \tag{7.17}$$

which is the Feynman propagator (in $0 + 1$ dimensions, i.e. with only time, but no space). Indeed, check that, with $p \cdot p = -(p^0)^2$, and p^0 called simply p in quantum mechanics, we get this formula from the previously defined Feynman propagator.

Note that if Δ_F^{-1} has $\{q_i(t)\}$ eigenfunctions with $\lambda_i \neq 0$ eigenvalues, such that the eigenfunctions are orthonormal, i.e.

$$q_i^* \cdot q_j \equiv \int dt q_i(t)^* q_j(t) = \delta_{ij} \tag{7.18}$$

then we can write

$$\Delta_F^{-1}(t, t') = \sum_i \lambda_i q_i(t) q_i(t)^* \tag{7.19}$$

inverted to

$$\Delta_F(t, t') = \sum_i \frac{1}{\lambda_i} q_i(t) q_i(t')^* \tag{7.20}$$

More generally, for an operator A with eigenstates $|q>$ and eigenvalues a_q, i.e. $A|q> = a_q|q>$, with $<q|q'> = \delta_{qq'}$, then

$$A = \sum_q a_q |q> < q| \tag{7.21}$$

Our operator is roughly of this form, but not quite. We have something like $q_i(t) \sim e^{-ipt}$, $\sum_i \sim \int dp/(2\pi)$ and $\lambda_i \sim (p^2 - \omega^2 + i\epsilon)$, but we must be more precise. Note that

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if at $t \to +\infty$, *all* $q_i(t) \to$ same function, then $\Delta_F(t \to +\infty)$ gives the same, and analogously for $t \to -\infty$.

We can in fact do the integral in Δ_F as in previous lectures, since we have poles at $p = \pm(\omega - i\epsilon)$, corresponding to the Feynman contour for the p integration. For $t - t' > 0$, we can close the contour below, since we have exponential decay in the lower half plane, and pick up the pole at $+\omega$, whereas for $t - t' < 0$ we close the contour above and pick up the pole at $-\omega$, all in all giving

$$\Delta_F(t, t') = \frac{1}{2\omega}e^{-i\omega|t-t'|} \tag{7.22}$$

We see that for $t \to \infty$, $\Delta_F \sim e^{-i\omega t}$, whereas for $t \to -\infty$, $\Delta_F \sim e^{+i\omega t}$, therefore the boundary conditions for the $q(t)$'s are

$$\begin{aligned} q(t) &\sim e^{-i\omega t}, \quad t \to \infty \\ q(t) &\sim e^{+i\omega t}, \quad t \to -\infty \end{aligned} \tag{7.23}$$

Note that $[d^2/dt^2 + \omega^2]e^{\pm i\omega t} = 0$, though of course $[d^2/dt^2 + \omega^2]q(t) \neq 0$ (since it is nonzero at finite time).

That means that we must define the space of functions as functions satisfying (7.23), and we do the path integral on this space.

So, a *better definition of the path integral* is in terms of

$$q(t) = q_{cl}(t; J) + \tilde{q}(t) \tag{7.24}$$

where the classical solution $q_{cl}(t; J)$ satisfies the correct boundary conditions, and the quantum fluctuation $\tilde{q}(t)$ satisfies zero boundary conditions, so that it doesn't modify the ones of $q_{cl}(t; J)$. But we know that if we have

$$\begin{aligned} S(x; J) &= \frac{1}{2}Ax^2 + J \cdot x \Rightarrow \\ S(x; J) &= S(x_0; J) + \frac{1}{2}A(x - x_0)^2 = S(x_0; J) + S(x - x_0; 0) \end{aligned} \tag{7.25}$$

where $x_0 = x_0(J)$ is the extremum. Therefore in our case we have

$$S[q; J] = S[q_{cl}; J] + S[\tilde{q}; 0] \tag{7.26}$$

and the path integral is

$$Z[q; J] = \int \mathcal{D}q\, e^{iS[q;J]} = e^{iS[q_{cl};J]} \int \mathcal{D}\tilde{q}\, e^{iS[\tilde{q};0]} \tag{7.27}$$

and the path integral over $\tilde{q}$ is now J-independent, i.e. part of $\mathcal{N}$. The classical equation of motion

$$\Delta^{-1}q_{cl}(t; J) = iJ(t) \tag{7.28}$$

is solved by

$$q_{cl}(t; J) = q_{cl}(t; 0) + i(\Delta \cdot J)(t) \tag{7.29}$$

where $q_{cl}(t; 0)$ is a zero mode, satisfying

$$\Delta^{-1} q_{cl}(t; 0) = 0 \tag{7.30}$$

Note the appearance of the zero mode $q_{cl}(t; 0)$ that doesn't satisfy the correct boundary conditions. But the quantum solutions $q(t)$ on which we invert Δ_F^{-1} have correct boundary conditions instead, and do not include these zero modes. This is so since $q_{cl}(t; J)$ has the boundary conditions of the $i(\Delta \cdot J) = i \int dt' \Delta(t, t') J(t')$ term, i.e. of $\Delta(t, t')$ (the only nontrivial t dependence), as opposed to $q_{cl}(t; 0)$ which has trivial boundary conditions.

Then we have

$$\frac{\delta_{\text{full}} S[q_{cl}; J]}{\delta_{\text{full}} J(t)} = \int dt' \left[\frac{\delta S}{\delta q(t')} \Big|_{q=q_{cl}} \frac{\delta q_{cl}(t; J)}{\delta J(t)} + \frac{\delta S}{\delta J(t)} \right] \tag{7.31}$$

and the first term is zero by the classical equations of motion, and the second is equal to $q_{cl}(t; J)$, giving

$$\frac{\delta_{\text{full}} S[q_{cl}; J]}{\delta_{\text{full}} J(t)} = q_{cl}(t; J) = q_{cl}(t; 0) + i(\Delta \cdot J)(t) \Rightarrow$$

$$S[q_{cl}(J); J] = S[q_{cl}(0); 0] + q_{cl}(0) \cdot J + \frac{i}{2} J \cdot \Delta \cdot J \tag{7.32}$$

where in the second line we have integrated over J the first line. Then we obtain for the path integral

$$Z[J] = \mathcal{N}'' e^{-\frac{1}{2} J \cdot \Delta \cdot J + i q_{cl}(0) \cdot J} \tag{7.33}$$

where the second term is new, and it depends on the fact that we have nontrivial boundary conditions. Indeed, this was a zero mode that we needed to introduce in q_{cl}, as it has the correct boundary conditions, as opposed to the quantum fluctuations which have zero boundary conditions.

All of the above however was more suggestive than rigorous, as we were trying to define the boundary conditions for our functions, but we didn't really need the precise boundary conditions.

Correct treatment: harmonic phase space

The correct treatment gives the path integral in terms of a modified phase space path integral, called harmonic phase space. Let us derive it.

The classical hamiltonian of the free harmonic oscillator is

$$H(p, q) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \tag{7.34}$$

Making the definitions

$$q(t) = \frac{1}{\sqrt{2\omega}} [a(t) + a^\dagger(t)]$$

$$p(t) = -i\sqrt{\frac{\omega}{2}} [a(t) - a^\dagger(t)], \tag{7.35}$$

inverted as

$$
\begin{aligned}
a(t) &= \frac{1}{\sqrt{2\omega}}[\omega q - ip] = a(0)e^{-i\omega t} \\
a^\dagger(t) &= \frac{1}{\sqrt{2\omega}}[\omega q + ip] = a^\dagger(0)e^{+i\omega t},
\end{aligned}
\tag{7.36}
$$

the Hamiltonian is

$$
H(a, a^\dagger) = \omega a^\dagger a
\tag{7.37}
$$

In quantum mechanics, we quantize by $[\hat{a}, \hat{a}^\dagger] = 1$, and we write the Fock space representation, in terms of states

$$
|n> = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0>
\tag{7.38}
$$

which are eigenstates of the Hamiltonian.

But we can also define *coherent states*

$$
|\alpha> = e^{\alpha \hat{a}^\dagger}|0> = \sum_{n\geq 0} \frac{\alpha^n}{n!}(\hat{a}^\dagger)^n|0>
\tag{7.39}
$$

which are a linear combination of all the Fock space states. These states are eigenstates of $\hat{a}$, since

$$
\hat{a}|\alpha> = [\hat{a}, e^{\alpha \hat{a}^\dagger}]|0> = \alpha e^{\alpha \hat{a}^\dagger}|0> = \alpha|\alpha>
\tag{7.40}
$$

where we have used

$$
\begin{aligned}
[\hat{a}, e^{\alpha \hat{a}^\dagger}] &= \sum_{n\geq 0} \frac{\alpha^n}{n!}([\hat{a}, \hat{a}^\dagger](\hat{a}^\dagger)^{n-1} + \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger](\hat{a}^\dagger)^{n-2} + ...(\hat{a}^\dagger)^{n-1}[\hat{a}, \hat{a}^\dagger]) \\
&= \alpha \sum_{n\geq 0} \frac{\alpha^{n-1}}{(n-1)!}(\hat{a}^\dagger)^{n-1} = \alpha e^{\alpha a^\dagger}
\end{aligned}
\tag{7.41}
$$

Thus these coherent states are eigenstates of a, with eigenvalue α. Similarly, defining the bra state

$$
<\alpha^*| \equiv <0|e^{\alpha^* \hat{a}}
\tag{7.42}
$$

we have

$$
<\alpha^*|\hat{a}^\dagger = <\alpha^*|\alpha^*
\tag{7.43}
$$

For the inner product of coherent states, we obtain

$$
<\alpha^*|\alpha> = <0|e^{\alpha^* \hat{a}}|\alpha> = e^{\alpha^*\alpha} <0|\alpha> = e^{\alpha^*\alpha}
\tag{7.44}
$$

since $<0|\alpha> = <0|e^{\alpha \hat{a}^\dagger}|0> = <0|0> = 1$, as $<0|a^\dagger = 0$. We also have the completeness relation

$$
\mathbf{1} = \int \frac{d\alpha d\alpha^*}{2\pi i} e^{-\alpha\alpha^*}|\alpha><\alpha^*|
\tag{7.45}
$$

which is left as an exercise. (note that the constant depends on the definition of the complex integration measure $dzd\bar{z}$).

We will compute the transition amplitude between the Heisenberg states $|\alpha, t>_H$ and $_H< \alpha^*, t'|$,

$$F(\alpha^*, t'; \alpha, t) = {}_H< \alpha^*, t'|\alpha, t >_H = < \alpha^*|e^{-i\hat{H}(t'-t)}|\alpha > \tag{7.46}$$

where $|\alpha >$ and $< \alpha^*|$ are Schrödinger states.

The Hamiltonian in the presence of a driving force is

$$H(a^\dagger, a; t) = \omega a^\dagger a - \gamma(t)a^\dagger - \bar{\gamma}(t)a \tag{7.47}$$

The sources (driving forces) $\gamma, \bar{\gamma}$ are related to the J defined before in the Lagrangean formalism as

$$\gamma(t) = \frac{J(t)}{\sqrt{2\omega}}; \quad \bar{\gamma}(t) = \frac{\bar{J}(t)}{\sqrt{2\omega}} \tag{7.48}$$

Indeed, we can check that for real J, i.e. $J = \bar{J}$, the coupling is $\gamma(t)a^\dagger + \bar{\gamma}(t)a = J(t)q(t)$.

We need one more formula before we compute $F(\alpha^*, t'; \alpha, t)$, namely

$$< \alpha^*|\hat{H}(\hat{a}^\dagger, \hat{a}; t)|\beta >= H(\alpha^*, \beta; t) < \alpha^*|\beta >= H(\alpha^*, \beta; t)e^{\alpha^*\beta} \tag{7.49}$$

Now, like in the case of the phase space path integral, we divide the path in $N + 1$ small pieces, with $\epsilon \equiv (t' - t)/(N + 1)$, with times $t_0 = t, t_1, ..., t_n, t_{n+1} = t'$. We then insert the identity as the completeness relation (7.45) at each point t_i, $i = 1, ..., n$, dividing $e^{-i\hat{H}(t'-t)} = e^{-i\epsilon\hat{H}} \times ... \times e^{-i\epsilon\hat{H}}$. We obtain

$$F(\alpha^*, t'; \alpha, t) = \int \prod_i \left[\frac{d\alpha(t_i)d\alpha^*(t_i)}{2\pi i}e^{-\alpha^*(t_i)\alpha(t_i)}\right] < \alpha^*(t')|e^{-i\epsilon\hat{H}}|\alpha(t_n) > \times$$
$$\times < \alpha^*(t_n)|e^{-i\epsilon\hat{H}}|\alpha(t_{n-1}) >< \alpha^*(t_{n-1})|... < \alpha^*(t_1)|e^{-i\epsilon\hat{H}}|\alpha(t) > \tag{7.50}$$

Since

$$< \alpha^*(t_{i+1})|e^{-i\epsilon\hat{H}}|\alpha(t_i) >= e^{-i\epsilon H(\alpha^*(t_{i+1}),\alpha(t_i))}e^{\alpha^*(t_{i+1})\alpha(t_i)} \tag{7.51}$$

when we collect all the terms we obtain

$$\int \prod_i \left[\frac{d\alpha(t_i)d\alpha^*(t_i)}{2\pi i}\right] \exp \left[\alpha^*(t')\alpha(t_n) - \alpha^*(t_n)\alpha(t_n) + \alpha^*(t_n)\alpha(t_{n-1}) - \alpha^*(t_{n-1})\alpha(t_{n-1}) + ...\right.$$
$$\left.+... + \alpha^*(t_1)\alpha(t) - i\int_t^{t'} d\tau H(\alpha^*(\tau), \alpha(\tau); \tau)\right] \tag{7.52}$$

We see that we have pairs with alternating signs, with $\alpha(t_{i+1})\alpha(t_i) - \alpha^*(t_i)\alpha(t_i)$ being the discrete version of $d\tau\dot{\alpha}^*(\tau)\alpha(\tau)$, and the last term remaining uncoupled, giving finally

$$\int \prod_i \left[\frac{d\alpha(t_i)d\alpha^*(t_i)}{2\pi i}\right] \exp \left\{\int_t^{t'} d\tau[\dot{\alpha}^*(\tau)\alpha(\tau) - iH(\alpha^*(\tau), \alpha(\tau); \tau)] + \alpha^*(t)\alpha(t)\right\} \tag{7.53}$$

so that we obtain

$$F(\alpha^*, t'; \alpha, t) = \int \mathcal{D}\alpha\mathcal{D}\alpha^* \exp \left\{i\int_t^{t'} d\tau\left[\frac{\dot{\alpha}^*(\tau)}{i}\alpha(\tau) - H\right] + \alpha^*(t)\alpha(t)\right\} \tag{7.54}$$

Important concepts to remember

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- The source term J in the generating functional $Z[J]$ acts as a driving force, or souce, for the classical action.

- At the naive level, the Feynman propagator appears as the inversion of the kinetic operator, but we need to avoid the singularities.

- To avoid the singularities, we need to define boundary conditions for the space of functions. We find that the correct boundary conditions are $e^{-i\omega t}$ at $+\infty$ and $e^{+i\omega t}$ at $-\infty$, which means that the classical driven field has these boundary conditions, but the quantum fluctuations have zero boundary conditions. In this way, the zero modes do not appear in the space on which we invert Δ^{-1}.

- With the boundary conditions, we also get a term linear in J in the exponential, $Z[J] = \mathcal{N}'' e^{-1/2 J \cdot \Delta J + i q_{cl}(0) J}$.

- The correct treatment is in harmonic phase space, in terms of coherent states $|\alpha >$, which are eigenstates of $\hat{a}$.

- The transition amplitude between Heisenberg states of $|\alpha >$ is written as a path integral over $\alpha(t)$ and $\alpha^*(t)$.

Further reading: See chapters 1.4 in [4] and 2.3 in [3].

Exercises, Lecture 7

1) Using the $Z[J]$ for the free driven harmonic oscillator calculated in the lecture, calculate the 4-point function $G_4(t_1, t_2, t_3, t_4)$ for the free driven harmonic oscillator.

2) Prove that

$$1 = \int \frac{d\alpha d\alpha^*}{2\pi i} e^{-\alpha\alpha^*} |\alpha >< \alpha^*| \tag{7.55}$$

and calculate

$$e^{i[\hat{a}^\dagger \hat{a} + \lambda(\hat{a}+\hat{a}^\dagger)^3]}|\alpha > \tag{7.56}$$

8 Lecture 8. Euclidean formulation and finite temperature field theory

In this lecture we will define the Euclidean formulation of quantum mechanics, to be extended next lecture to quantum field theory, and the finite temperature version associated with it, making the connection to statistical mechanics.

But first, let us finish the discussion of the path integral in harmonic phase space. We have seen that the transition amplitude between Heisenberg states $|\alpha, t >$ and $< \alpha^*, t'|$ is given as a path integral,

$$F(\alpha^*, t'; \alpha, t) = \int \mathcal{D}\alpha \mathcal{D}\alpha^* \exp\left\{i \int_t^{t'} d\tau \left[\frac{\dot{\alpha}^*(\tau)}{i}\alpha(\tau) - H\right] + \alpha^*(t)\alpha(t)\right\} \qquad (8.1)$$

But we need to understand the boundary conditions for the path integral, related to the boundary conditions for the transition amplitude.

Classically, if we have two variables, like $\alpha(t)$ and $\alpha^*(t)$, obeying linear differential equations, or one variable obeying a quadratic differential equation, we could choose to impose their values (or the values of the function and its derivative in the second case) at a given initial time t, or the same at a final time t'. But in quantum mechanics, it is not so simple. We know that the eigenvalues of two operators that don't commute cannot be given (measured with infinite precision) at the same time (in the same state).

The precise statement is that if we have operators $\hat{A}$ and $\hat{B}$ such that $[\hat{A}, \hat{B}] \neq 0$, then defining ΔA by $(\Delta A)^2 = < \psi|(\hat{A} - A\mathbf{1})^2|\psi >$, where $|\psi >$ is some given state, we have (theorem in quantum mechanics) the generalized Heisenberg's uncertainty principle

$$(\Delta A)(\Delta B) \geq \frac{1}{2}| < \psi|i[\hat{A}, \hat{B}]|\psi > | \qquad (8.2)$$

For instance, since $[\hat{q}, \hat{p}] = i\hbar$, we get $(\Delta q)(\Delta p) \geq \hbar/2$ in a given state, for instance at a given time t for a single particle in its evolution.

That means that now, since $[\hat{a}, \hat{a}^\dagger] = 1$, we can't specify their eigenvalues α and α^* at the same time (with infinite precision). So the only possibility for boundary conditions is something that from the point of view of classical mechanics looks strange, but it's OK in quantum mechanis, namely to define

-at time t, $\alpha(t) = \alpha$ and $\alpha^*(t)$ is unspecified (free), and

-at time t', $\alpha^*(t') = \alpha^*$ and $\alpha(t)$ is unspecified (free).

The equations of motion, i.e. the equations obtained from the stationarity of the path integral, by varying the exponential with respect to $\alpha(\tau)$ and $\alpha^*(\tau)$, are

$$\dot{\alpha} + i\frac{\partial H}{\partial \alpha^*} = 0 \rightarrow \dot{\alpha} + i\omega\alpha - i\gamma = 0$$
$$\dot{\alpha}^* - i\frac{\partial H}{\partial \alpha} = 0 \rightarrow \dot{\alpha}^* - i\omega\alpha^* + i\bar{\gamma} = 0 \qquad (8.3)$$

The classical solutions of this system of equations is

$$\alpha_{cl}(\tau) = \alpha e^{i\omega(t-\tau)} + i\int_t^\tau e^{i\omega(s-\tau)}\gamma(s)ds$$

$$\alpha_{cl}^*(\tau) \quad = \quad \alpha^* e^{i\omega(\tau-t')} + i \int_\tau^{t'} e^{i\omega(\tau-s)}\bar{\gamma}(s)ds \tag{8.4}$$

On these solutions (i.e., using the above equations of motion), we can compute that the object in the exponent in the path integral is

$$\begin{aligned}
&= \quad \alpha^*(t)\alpha(t) + i \int_t^{t'} d\tau\gamma(\tau)\alpha^*(\tau) \\
&= \quad \alpha^*\alpha e^{-i\omega(t'-t)} + i \int_t^{t'} ds[\alpha e^{i\omega(t-s)}\bar{\gamma}(s) + \alpha^* e^{i\omega(s-t')}\gamma(s)] \\
&\quad -\frac{1}{2}\int_t^{t'} ds \int_t^{t'} ds'\gamma(s)\bar{\gamma}(s')e^{-i\omega|s'-s|}
\end{aligned} \tag{8.5}$$

In the above we have skipped some steps, which are left as an exercise.

Then, like we did in the last lecture for the configuration space path integral, we can do the gaussian integral by shifting $\alpha(t) = \alpha_{cl}(t) + \tilde{\alpha}(t)$ and $\alpha^*(t) = \alpha_{cl}^*(t) + \tilde{\alpha}^*(t)$, where $\alpha_{cl}(t), \alpha_{cl}^*(t)$ are the above classical solutions, and the tilde quantities are quantum fluctuations. Then, by shifting the path integral, we get that (due to the quadratic nature of the exponential) the object in the exponent,

$$E[\alpha(t), \alpha^*(t); \gamma, \bar{\gamma}] = E[\alpha_{cl}, \alpha_{cl}^*(t); \gamma, \bar{\gamma}] + E[\tilde{\alpha}(t), \tilde{\alpha}^*(t); 0] \tag{8.6}$$

resulting in the path integral being a constant times the exponential of the classical exponent,

$$\mathcal{N}e^{E[\alpha_{cl}(t), \alpha_{cl}^*(t); \gamma, \bar{\gamma}]} \tag{8.7}$$

We see that this object has the same structure as (7.33), since we can absorb the constant in (8.5) in $\mathcal{N}$, and then we are left with a term linear in J and a term quadratic in J in the exponent, like in the case of the configuration space path integral.

But we still need to relate the above harmonic phase space path integral with the configuration space path integral of the last lecture. First, we would need to have vacuum states at the two ends. Since $|\alpha > = e^{\alpha a^\dagger}|0 >$, choosing $\alpha = 0$ gives $|\alpha > = |0 >$, and choosing $\alpha^* = 0$ gives $< \alpha^*| = < 0|$. Next, we need to take $t \to -\infty$ and $t' \to \infty$, since this is the same that was required for the configuration space path integral. In this case, in the exponent in (8.5), the constant and linear terms disappear, and we are left with the object

$$Z[J] \equiv < 0, +\infty|0, -\infty >_J = \exp\left\{-\frac{1}{2}J \cdot \Delta_F \cdot J\right\} < 0|0 >_0 \tag{8.8}$$

where $|0, -\infty >$ means $|\alpha = 0, t = -\infty >$, and we have written the overall constant as $< 0|0 >_0$, since it is indeed what we obtain for the path integral if we put $J = 0$. Note that here we have

$$\Delta_F = \frac{1}{2\omega}e^{-i\omega|s-s'|} \tag{8.9}$$

in the quadratic piece in the critical exponent, as we can see by substituting $\gamma = J/\sqrt{2\omega}$. This is, as we saw, the Feynman propagator.

Now the boundary condition is $\alpha(t) = 0$ at $t = -\infty$, and $\alpha^*(t)$ free, in other words, pure creation part (α^* is eigenvalue of $a^\dagger$) at $t = -\infty$, and $\alpha^*(t) = 0$ at $t = +\infty$, and $\alpha(t)$ free, in other words, pure annihilation part (α is eigenvalue of a). Since

$$q(t) = \frac{1}{\sqrt{2\omega}}[ae^{-i\omega t} + a^\dagger e^{+i\omega t}] \tag{8.10}$$

we see that this is consistent with the previously defined (unrigorous) boundary condition, $q(t = -\infty) \sim e^{+i\omega t}$ and $q(t = +\infty) \sim e^{-i\omega t}$. But now we have finally defined rigorously the boundary condition and the resulting path integral.

Wick rotation to Euclidean time

But while the path integral is well defined now, its calculation in relevant cases of interest is not. A useful approximation for any path integral is what is known as "saddle point approximation", which is the gaussian integral around a classical solution, which is an extremum of the action,

$$S = S_{cl} + \frac{1}{2}\delta q_i S_{ij} \delta q_j + \mathcal{O}((\delta q)^3) \tag{8.11}$$

where $S_{cl} = S[q_{cl} : \delta S/\delta q(q_{cl}) = 0]$. If we have a free action, this is exact, and not an approximation. We have performed the gaussian integration as if it were correct, but in reality it is only correct for real integrals, which decay exponentially, $\int_{-\infty}^{+\infty} dx e^{-\alpha x^2}$, whereas for imaginary integrals, when the integral is over a phase (purely oscillatory), $\int dx e^{-i\alpha x^2}$ is much less well defined (since if we take $\int_{-\Lambda}^{+\Lambda}$ or $\int_{-\Lambda-C}^{+\Lambda+C}$ where $\Lambda \to \infty$ and C finite, the two integrals differ by a finite contribution that is highly oscillatory in the value of C, $\sim 2\int_0^C dx e^{-2i\alpha\Lambda x}$). And when it is a path integral instead of a single integral, it becomes even more obvious this is not so well defined.

So if we could have somehow instead of e^{iS}, e^{-S}, that would solve our problems. Luckily, this is what happens when we go to *Euclidean space*.

Consider a time independent Hamiltonian $\hat{H}$, with a complete set of eigenstates $\{|n>\}$ (so that $\mathbf{1} = \sum_n |n><n|$), and eigenvalues $E_n > 0$. Then the transition amplitude can be written as

$$\begin{aligned}
{}_H<q',t'|q,t>_H &= <q'|e^{-i\hat{H}(t'-t)}|q> = \sum_n \sum_m <q'|n><n|e^{-i\hat{H}(t'-t)}|m><m|q> \\
&= \sum_n <q'|n><n|q> e^{-iE_n(t'-t)} = \sum_n \psi_n(q')\psi_n^*(q)e^{-iE_n(t'-t)} \quad (8.12)
\end{aligned}$$

where we have used $<n|e^{-i\hat{H}(t'-t)}|m> = \delta_{nm}e^{-iE_n(t'-t)}$ and $<q|n> = \psi_n(q)$. This expression is analytic in $\Delta t = t' - t$.

Now consider the analytical continuation to Euclidean time, called *Wick rotation*, $\Delta t \to -i\beta$. Then we obtain

$$<q',\beta|q,0> = \sum_n \psi_n(q')\psi_n^*(q)e^{-\beta E_n} \tag{8.13}$$

Then we note that if we specialize to the case $q' = q$ and integrate over this value of q, we obtain the statistical mechanics partition function of a system at a temperature T, with $kT = 1/\beta$, thus obtaining a relation to statistical mechanics!

Indeed, then we obtain

$$\int dq <q, \beta | q, 0> = \int dq \sum_n |\psi_n(q)|^2 e^{-\beta E_n} = \text{Tr}\{e^{-\beta \hat{H}}\} = Z[\beta] \qquad (8.14)$$

This corresponds in the path integral to a taking closed paths of Euclidean time length $\beta = 1/(kT)$, since $q' \equiv q(t_E = \beta) = q(t_E = 0) \equiv q$.

Let's see how we write the path integral. The Minkowski space Lagrangean is

$$L(q, \dot{q}) = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 - V(q) \qquad (8.15)$$

meaning the exponent in the path integral becomes

$$iS[q] = i \int_0^{t_E=\beta}(-idt_E)\left[\frac{1}{2}\left(\frac{dq}{d(-it_E)}\right)^2 - V(q)\right] \equiv -S_E[q] \qquad (8.16)$$

where then by definition, the *Euclidean action* is

$$S_E[q] = \int_0^\beta dt_E\left[\frac{1}{2}\left(\frac{dq}{dt_E}\right)^2 + V(q)\right] = \int dt_E L_E(q, \dot{q}) \qquad (8.17)$$

We finally obtain the *Feynman-Kac formula*,

$$Z(\beta) = \text{Tr}\{e^{-\beta \hat{H}}\} = \int \mathcal{D}q e^{-S_E[q]}\Big|_{q(t_E+\beta)=q(t_E)} \qquad (8.18)$$

where the path integral is then taken over *all closed paths of Euclidean time length β.*

As we know, the partition function in statistical mechanics contains all the relevant information about the system, so using this formalism we can extract any statistical mechanics quantity of interest.

To do so, we can introduce currents $J(t)$ as usual and calculate the correlation functions. That is, we define

$$Z[\beta; J] = \int \mathcal{D}q e^{-S_E(\beta)+\int_0^\beta J_E(\tau)q_E(\tau)d\tau} \qquad (8.19)$$

Note that here, the current term is the usual we encountered before, since

$$i \int dt J(t)q(t) = i \int d(-it_E)J(-it_E)q(-it_E) \equiv \int dt_E J_E(t_E)q_E(t_E) \qquad (8.20)$$

Let's see a simple, but very useful quantity, the *propagator in imaginary (Euclidean) time*. We immediately find in the usual way

$$\frac{1}{Z(\beta)}\frac{\delta^2 Z[\beta; J]}{\delta J(\tau_1)\delta J(\tau_2)} = \frac{1}{Z(\beta)}\int \mathcal{D}q(\tau)q(\tau_1)q(\tau_2)e^{-S_E(\beta)} \qquad (8.21)$$

Note that here in $J(\tau_1), J(\tau_2), q(\tau_1), q(\tau_2)$ we have Euclidean time (τ), but since the formula for $Z[\beta; J]$ was obtained by analytical continuation, this is equal to

$$< \Omega|T\{\hat{q}(-i\tau_1)\hat{q}(-i\tau_2)\}|\Omega >_\beta = \frac{1}{Z(\beta)}\text{Tr}[e^{-\beta\hat{H}}T\{\hat{q}(-i\tau_1)\hat{q}(-i\tau_2)\}] \qquad (8.22)$$

Also note that here $Z(\beta)$ is a normalization constant, since it is J-independent (now we have a new parameter, β, but except for this dependence, it is constant). In the above, the Heisenberg operators are also Wick rotated, i.e.

$$\hat{q}(t) = e^{i\hat{H}t}\hat{q}e^{-i\hat{H}t} \Rightarrow$$
$$\hat{q}(-i\tau) = e^{\hat{H}\tau}\hat{q}e^{-\hat{H}\tau} \qquad (8.23)$$

We could ask: why is it not the Euclidean correlator the VEV of the time ordered Euclidean operators? The answer is that in Euclidean space, space and time are the same, so we can't define the "time ordering operator". In a certain sense, what we obtain is then just the VEV of the product of q's in Euclidean space, for the above $< \Omega|\hat{q}(\tau_1)\hat{q}(\tau_2)|\Omega >$, just that it is better defined as the continuation from Minkowski space of the VEV of the time-ordered product.

We can now extend the definition of the free propagator $\Delta(\tau)$ to the interval $[-\beta, \beta]$ by the periodicity of $q(\tau)$, $q(\tau + \beta) = q(\tau)$. We can then define

$$\Delta(\tau) = < \Omega|T\{\hat{q}(-i\tau)\hat{q}(0)\}|\Omega >_\beta \qquad (8.24)$$

such that

$$\Delta(\tau - \beta) = \Delta(\tau) \qquad (8.25)$$

We could of course compute it from the path integral in principle, but consider reversely, the following problem. The propagator equation

$$\left[-\frac{d^2}{dt^2} + \omega^2\right]K(\tau, \tau') = \delta(\tau, \tau') \qquad (8.26)$$

with the above periodicity, where $K(\tau, \tau') = \Delta_{free}(\tau - \tau')$, has a *unique* solution: if $\tau \in [0, \beta]$, the solution is

$$\Delta_{free}(\tau) = \frac{1}{2\omega}[(1 + n(\omega))e^{-\omega\tau} + n(\omega)e^{\omega\tau}] \qquad (8.27)$$

where

$$n(\omega) = \frac{1}{e^{\beta|\omega|} - 1} \qquad (8.28)$$

is the Bose-Einstein distribution. The proof of the above statement is left as an exercise.

We also can check that for $\beta \to \infty$,

$$\Delta_{free}(\tau > 0) \to \Delta_F(\tau > 0) \qquad (8.29)$$

So we obtain the Feynman propagator in the limit of zero temperature $(\beta = 1/(kT) \to \infty)$.

Moreover, in this zero temperature limit, in which we have an infinite period in Euclidean time, we have

$$< q', \beta | q, 0 > = \sum_n \psi_n(q') \psi_n^*(q) e^{-\beta E_n} \rightarrow \psi_0(q') \psi_0(q) e^{-\beta E_0} \qquad (8.30)$$

that is, we only get the *vacuum contribution*.

In conclusion, we can define the sources $J(t)$ to be nonzero on a finite time interval, and take infinitely long periodic Euclidean time. Then we obtain a definition of the *vacuum functional*, where the initial and final states are the vacuum $|\Omega>$, like we defined before in Minkowski space, and this functional is given by the path integral in Euclidean space.

Because of the statistical mechanics connection, we call $Z[J] = Z[\beta \rightarrow \infty; J]$ the *partition function*, and we will continue to call it like this from now on. We will also call later $-\ln Z[J] = W[J]$ the free energy, since the same happens in statistical mechanics.

Note that all we said here was for quantum mechanics, and the corresponding statistical mechanics by the Feynman-Kac formula. But we can generalize this formalism to quantum field theory, and we will do so next lecture. We can also generalize statistical mechanics to finite temperature field theory.

Driven harmonic oscillator

Let us now return to our basic example for everything, the harmonic oscillator. The Euclidean partition function is

$$Z_E[J] = \int \mathcal{D}q \exp\left\{ -\frac{1}{2} \int dt \left[\left(\frac{dq}{dt} \right)^2 + \omega^2 q^2 \right] + \int dt J(t) q(t) \right\} \qquad (8.31)$$

We saw that in Minkowski space the partial integration of the kinetic term can introduce problematic boundary terms related to the boundary conditions. But now, in Euclidean space, we have only closed paths, so there are no boundary terms!

Therefore we can write as before,

$$
\begin{aligned}
Z_E[J] &= \int \mathcal{D}q \exp\left\{ -\frac{1}{2} \int dt\, q(t) \left[-\frac{d^2}{dt^2} + \omega^2 \right] q(t) + \int dt J(t) q(t) \right\} \\
&= \mathcal{N} \exp\left\{ \frac{1}{2} \int ds \int ds' J(s) \Delta_E(s, s') J(s') \right\} \qquad (8.32)
\end{aligned}
$$

where the *Euclidean propagator* is defined by $-S_E = -1/2 q \Delta^{-1} q + ...$, giving

$$\Delta_E(s, s') = \left(-\frac{d^2}{ds^2} + \omega^2 \right)^{-1} (s, s') = \int \frac{dE_E}{2\pi} \frac{e^{-iE_E(s-s')}}{E_E^2 + \omega^2} \qquad (8.33)$$

Note first that the gaussian integration above is now well defined, as we explained, since we don't have oscillatory terms anymore, but we have the usual integral of a real decaying exponential, $\int dx e^{-\alpha x^2}$. Second, the Euclidean propagator is well defined, since in the integral, we don't have any singularities anymore, as $E_E^2 + \omega^2 > 0$, so we don't need to choose a particular integration contour avoiding them, as in the case of the Minkowski space propagator.

Let us summarize what we did: we Wick rotated the Minkowski space theory to Euclidean space in order to better define it. In Euclidean space we have no ambiguities anymore: the path integral is well defined, Gaussian integration and the propagators are also well defined. Then, in order to relate to physical quantities, we need to Wick rotate the calculations in Euclidean space back to Minkowski space.

Let's see that for the propagator. We have $t = -is$, where s is Euclidean time. But then we want to have the object Et be Wick rotation invariant, $Et = E_E s$, so $E_E = -iE$.

The Euclidean space propagator is well defined, but of course when we do the above Wick rotation to Minkowski space, we find the usual poles, since the Euclidean propagator has imaginary poles $E_E = \pm i\omega$, corresponding to poles at $E = \pm\omega$. Therefore we can't do the full Wick rotation, corresponding to a rotation of the integration contour with $\pi/2$ in the complex energy plane, since then we cross (or rather touch) the poles. To avoid that, we must rotate the contour only by $\pi/2 - \epsilon$,

$$E_E \to e^{-i(\frac{\pi}{2} - \epsilon)} E = -i(E + i\epsilon') \tag{8.34}$$

Doing this Wick rotation, we obtain

$$\Delta_E(s = it) = -i \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iEt}}{-E^2 + \omega^2 - i\epsilon} = \Delta_F(t) \tag{8.35}$$

that is, the Feynman propagator. We see that the $\pi/2 - \epsilon$ Wick rotation corresponds to the Feynman prescription for the integration contour to avoid the poles. It comes naturally from the Euclidean space construction (any other integration contour could not be obtained from any smooth deformation of the Euclidean contour), one more reason to consider the Feynman propagator as the relevant object for path integrals.

In conclusion, let us mention that the puristic view of quantum field theory is that the Minkowski space theory is not well defined, and to define it well, we must work in Euclidean space and then analytically continue back to Minkowski space.

Also, we saw that we can have a statistical mechanics interpretation of the path integral, which makes the formalism for quantum field theory and statistical mechanics the same. A number of books take this seriously, and treat the two subjects together, but we will continue with just quantum field theory.

Important concepts to remember

- Relating the harmonic phase space path integral with the configuration space path integral, we obtain the boundary conditions that we have a pure creation part at $t = -\infty$ and a pure annihilation part at $t = +\infty$, compatible with the configuration space boundary condition $q(t = -\infty) \sim e^{i\omega t}$, $q(t = +\infty) \sim e^{-i\omega t}$. This defines rigorously the path integral.

- The Feynman-Kac formula relates the partition function in statistical mechanics, $Z(\beta) = \text{Tr}\{e^{-\beta\hat{H}}\}$, with the Euclidean space path integral over all closed paths in the Euclidean time, with length β.

- The correlators in imaginary time (like the imaginary time propagator) are the analytical continuation of Minkowski space correlators.

- The free Euclidean propagator at periodicity β involves the Bose-Einstein distribution.

- The vacuum functional, or partition function $Z[J]$, is given by the Euclidean path integral of infinite periodicity.

- For the harmonic oscillator in Euclidean space, we can rigorously do the gaussian integration to obtain the exact solution for $Z[J]$, in terms of a well-defined Euclidean propagator.

- The analytical continuation (Wick rotation) of the Euclidean propagator, by smoothly deforming the integration contour to avoid the poles in the complex plane, uniquely selects the Feynman propagator.

Further reading: See chapters 1.5 in [4] and 3.7 in [3].

Exercises, Lecture 8

1) Complete the omitted steps in the proof for going from

$$F(\alpha^*, t'; \alpha, t) = \int \mathcal{D}\alpha \mathcal{D}\alpha^* \exp\left\{ i \int_t^{t'} d\tau \left[\frac{\dot{\alpha}^*(\tau)}{i}\alpha(\tau) - H\right] + \alpha^*(t)\alpha(t) \right\} \tag{8.36}$$

to

$$Z[J] \equiv < 0, +\infty | 0, -\infty >_J = \exp\left\{ -\frac{1}{2} J \cdot \Delta_F \cdot J \right\} < 0|0 >_0 \tag{8.37}$$

2) Prove that

$$\left[-\frac{d^2}{d\tau^2} + \omega^2 \right] K(\tau, \tau') = \delta(\tau - \tau'), \tag{8.38}$$

where $K(\tau, \tau') = \Delta_{free}(\tau - \tau')$, and $\Delta(\tau - \beta) = \Delta(\tau)$, has a *unique* solution: if $\tau \in [0, \beta]$, the solution is

$$\Delta_{free}(\tau) = \frac{1}{2\omega}[(1 + n(\omega))e^{-\omega\tau} + n(\omega)e^{\omega\tau}] \tag{8.39}$$

where

$$n(\omega) = \frac{1}{e^{\beta|\omega|} - 1} \tag{8.40}$$

9 Lecture 9. The Feynman path integral for a scalar field

In this lecture we generalize what we have learned from quantum mechanics to the case of quantum field theory.

We have seen that in order to better define the theory, we must do a Wick rotation to Euclidean space, $t = -it_E$. If we choose closed (periodic) paths, $q(t_E + \beta) = q(\beta)$ in Euclidean time, we obtain the statistical mechanics partition function, $Z(\beta) = \text{Tr}\{e^{-\beta \hat{H}}\}$, given as a path integral $\int \mathcal{D}q e^{-S_E[q]}|_{q(t_E+\beta)=q(t_E)}$ (Feynman-Kac formula). The Euclidean action is positive definite if the Minkowski space Hamiltonian was. To obtain the vacuum functional (transition between vacuum states), we take $\beta \to \infty$. In Euclidean space we obtain well defined path integrals; for instance, the driven harmonic oscillator can now be easily solved. Partial integration has no problems, since periodic paths have no boundary, and we obtain $Z_E[J] = \mathcal{N}e^{\frac{1}{2}J \cdot \Delta_E \cdot J}$, where the Euclidean propagator is well defined (has no poles). Wick rotation however will give the same problem: a full Wick rotation with $\pi/2$ rotation of the integration contour will touch the poles, so we must rotate only with $\pi/2 - \epsilon$, obtaining the Feynman propagator. No other propagator can arise from a *smooth* deformation of the integration contour.

To generalize to field theory, as usual we think of replacing $q_i(t) \to \phi_{\vec{x}}(t) \equiv \phi(\vec{x}, t) \equiv \phi(x)$. Of course, there are issues with the correct regularization involved in this (some type of discretization of space, we have discussed that a bit already), but we will ignore this, since we only want to see how to translate quantum mechanics results into quantum field theory results.

In *Minkowski space*, the action is

$$S[\phi] = \int d^4x \left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi) \right] \tag{9.1}$$

and the Minkowski space n-point functions, or Green's functions are

$$G_n(x_1, ..., x_n) = <0|T\{\hat{\phi}(x_1)...\hat{\phi}(x_n)\}|0> = \int \mathcal{D}\phi e^{iS[\phi]}\phi(x_1)...\phi(x_n) \tag{9.2}$$

To define the theory better, we go to *Euclidean space*, where we take only periodic paths with infinite period, obtaining the *vacuum functional*. The Euclidean action is

$$S_E[\phi] = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi) \right] \tag{9.3}$$

where, since we are in Euclidean space, $a_\mu b_\mu = a_\mu b^\mu = a_\mu b_\nu \delta^{\mu\nu}$, and time is defined as $t_M \equiv x^0 = -x_0 = -it_E$, $t_E = x_4 = x^4$, and so $x^4 = ix^0$.

The Euclidean space Green's functions are

$$G_n^{(E)}(x_1, ..., x_n) = \int \mathcal{D}\phi e^{-S_E[\phi]}\phi(x_1)...\phi(x_n) \tag{9.4}$$

We can write their generating functional, the *partition function*, so called because of the connection with statistical mechanics which stays the same: at finite periodicity β in Euclidean time,

$$Z[\beta, J] = \text{Tr}\{e^{-\beta \hat{H}_J}\} = \int \mathcal{D}\phi e^{-S_E[\phi]+J\cdot\phi}|_{\phi(\vec{x},t_E+\beta)=\phi(\vec{x},t_E)}, \tag{9.5}$$

so in the vacuum,

$$Z[J] = \int \mathcal{D}\phi e^{-S_E[\phi]+J\cdot\phi} \equiv {}_J<0|0>_J \tag{9.6}$$

where in d dimensions

$$J \cdot \phi \equiv \int d^d x J(x)\phi(x) \tag{9.7}$$

So the Green's functions are obtained from derivatives as usual,

$$G_n(x_1, ..., x_n) = \frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)}\int \mathcal{D}\phi e^{-S_E+J\cdot\phi}|_{J=0} \tag{9.8}$$

Note however the absence of the factors of i in the denominators, since we now have $J \cdot \phi$ instead of $iJ \cdot \phi$. The partition function sums the Green's functions as usual,

$$Z[J] = \sum_{n\geq 0}\frac{1}{n!}\int \prod_{i=1}^{n} d^d x_i G(x_1, ..., x_n)J(x_1)...J(x_n) \tag{9.9}$$

Perturbation theory

We now move to analyzing the perturbation theory, i.e. when $S[\phi] = S_0[\phi] + S_I[\phi]$, where S_0 is a free (quadratic) part and S_I is an interaction piece. As before, we will try to perturb in the interaction piece. We can generalize the harmonic oscillator case and calculate transition amplitudes between states. But even though in physical situations we need to calculate transitions between physical states, which are usually wave functions over particle states of given momentum, it is much easier from a theoretical point of view to calculate the Green's functions (transitions between vacuum states).

We will see however that the two are related. Namely, we can consider the Green's functions in momentum space, and the S-matrix elements, of the general type $S_{fi} =< f|S|i >$, where $S = U_I(+\infty, -\infty)$. Then the S-matrix elements will be related in a more precise manner later with the *residues at all the mass-shell poles* ($p_i^2 = -m_i^2$ for external lines) of the momentum space Green's functions. The proof of this formula (which will be just written down later in the course), called the *LSZ formula*, will be given in QFT II, since the proof requires the use of renormalization.

The momentum space Green's functions are

$$\tilde{G}_n(p_1, ..., p_n) = \int d^d x_1...d^d x_n e^{i(p_1 x_1 +....p_n x_n)}G_n(x_1, ..., x_n) \tag{9.10}$$

But due to translational invariance (any good theory must be invariant under translations, $x_i \to x_i + a$),

$$G_n(x_1, ..., x_n) = G_n(x_1 - X, x_2 - X, ..., x_n - X) \tag{9.11}$$

Choosing $X = x_1$ and changing integration variables $x_i \to x_i + X$, for $i = 2, ..., n$, we get

$$
\begin{aligned}
\tilde{G}_n(p_1, ..., p_n) &= \left[\int d^d x_1 e^{i x_1 (p_1 + ... + p_n)} \right] \int d^d x_2 ... d^d x_n e^{i(x_2 p_2 + ... x_n p_n)} G_n(0, x_2, ..., x_n) \\
&= (2\pi)^d \delta^d(p_1 + ... + p_n) G_n(p_1, ..., p_n)
\end{aligned}
\tag{9.12}
$$

and we usually calculate $G_n(p_1, ..., p_n)$, where these external momenta already satisfy momentum conservation.

As we said, it will be much easier theoretically to work with Green's functions rather than S-matrices, and the two are related.

Dyson's formula

This is a formula that was initially derived in the operator formalism, but in the path integral formalism it is kind of trivial. It is useful in order to define the perturbation theory.

Working in the Euclidean case, we define $|0>$ as the vacuum of the *free theory*, and we consider VEVs in the free theory of some operators, which as usual are written as path integrals,

$$
< 0|\mathcal{O}[\{\hat{\phi}\}]|0 > = \int \mathcal{D}\phi e^{-S_0[\phi]} \mathcal{O}[\{\phi\}]
\tag{9.13}
$$

Now, considering the particular case of $\mathcal{O} = e^{-S_I[\phi]}$, we obtain

$$
\int \mathcal{D}\phi e^{-S_0[\phi] - S_I[\phi]} = < 0|e^{-S_I[\hat{\phi}]}|0 >
\tag{9.14}
$$

If the operator $\mathcal{O}$ contains also a product of fields, we obtain the Green's functions, i.e.

$$
G_n(x_1, ..., x_n) = < 0|\hat{\phi}(x_1)...\hat{\phi}(x_n) e^{-S_I[\hat{\phi}]}|0 > = \int \mathcal{D}\phi e^{-S_0[\phi]} \phi(x_1)...\phi(x_n) e^{-S_I[\phi]}
\tag{9.15}
$$

We note here again that in Euclidean space, there is no distinction between time and space, so we cannot define an "euclidean time ordering", so in a certain sense we have the usual product. Just when we analytically continue (Wick rotate) from Minkowski space we can define the time ordering implicitly (via the Minkowski time ordering).

Finally, for the generating functional of the Green's functions, the partition function, we can similarly write, by summing the Green's functions,

$$
Z[J] = < 0|e^{-S_I[\hat{\phi}]} e^{\int d^d x J(x) \hat{\phi}(x)}|0 > = \int \mathcal{D}\phi e^{-S_0[\phi] + J \cdot \phi} e^{-S_I[\phi]}
\tag{9.16}
$$

which is called *Dyson's formula*, and as we can see, it is rather trivial in this path integral formulation, but for S-matrices in the operator formalism in Minkowski space, where it was originally derived, it is less so.

Solution of the free field theory

We now solve for the free field theory, $S_I[\phi] = 0$. We have

$$
Z_0[J] = \int \mathcal{D}\phi e^{-S_0[\phi] + J \cdot \phi} = \int \mathcal{D}\phi \left\{ -\frac{1}{2} \int d^d x [\partial_\mu \phi \partial_\mu \phi + m^2 \phi^2] + J \cdot \phi \right\}
\tag{9.17}
$$

In Euclidean space we have no boundary terms, so we can partially integrate the kinetic term to obtain

$$Z_0[J] = \int \mathcal{D}\phi \exp\left\{ -\frac{1}{2}\int d^dx\,\phi[-\partial_\mu\partial_\mu + m^2]\phi + J\cdot\phi \right\}, \tag{9.18}$$

where $[-\partial_\mu\partial_\mu + m^2] \equiv \Delta^{-1}$ and we can then shift the integration variable by first writing

$$
\begin{aligned}
Z_0[J] &= \int \mathcal{D}\phi \exp\left\{ -\frac{1}{2}[\phi - J\cdot\Delta]\Delta^{-1}[\phi - \Delta\cdot J] + \frac{1}{2}J\cdot\Delta\cdot J \right\} \\
&= e^{\frac{1}{2}J\cdot\Delta\cdot J}\int \mathcal{D}\phi' e^{-\frac{1}{2}\phi'\cdot\Delta^{-1}\phi'}
\end{aligned}
\tag{9.19}
$$

and now the path integral that remains is just a normalization constant, giving finally

$$Z_0[J] = e^{\frac{1}{2}J\cdot\Delta\cdot J} < 0|0 >_0 \tag{9.20}$$

and we can put $Z_0[0] = < 0|0 >_0$ to 1. Note that here the propagator comes from $\Delta^{-1} = -\partial_\mu\partial_\mu + m^2$ and is given by

$$\Delta(x,y) = \int \frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot(x-y)}}{p^2 + m^2} \tag{9.21}$$

and now again it is nonsingular, since in the denominator $p^2 + m^2 > 0$. But again, analytical continuation back to Minkowski space (Wick rotating back) can lead to the usual poles at $|p| = \pm im$, so we need to rotate the integration contour in the complex p^0 plane with $\pi/2 - \epsilon$ instead of the $\pi/2$. Then we obtain the Feynman propagator, since $p_E^2 + m^2 \to p_M^2 + m^2 - i\epsilon$. Note that here, $p_E^2 = p_\mu^E p_\nu^E \delta^{\mu\nu}$, but $p_M^2 = p_\mu^M p_\nu^M \eta^{\mu\nu} = -(p^0)^2 + \vec{p}^2$. And again, after the rotation (smooth deformation of the contour),

$$\Delta(s = it, \vec{x}) = \Delta_F(t, \vec{x}) \tag{9.22}$$

uniquely chosen.

Wick's theorem

In the path integral formalism, Wick's theorem is both simple to state, and easy to prove, though it will take us some examples (next class) to figure out why it is the same one as in the operator formalism.

Let's start out with an example. Consider, e.g., the function $F[\{\phi\}] = \phi^2(x_1)\phi(x_2)\phi^4(x_3)$. Then its VEV is a path integral as we saw, which we can write as

$$
\begin{aligned}
< 0|F[\{\phi\}]|0 > &= \int \mathcal{D}\phi\, e^{-S_0[\phi]}\phi^2(x_1)\phi(x_2)\phi^4(x_3) \\
&= \left(\frac{\delta}{\delta J(x_1)}\right)^2 \frac{\delta}{\delta J(x_2)}\left(\frac{\delta}{\delta J(x_3)}\right)^4 \int \mathcal{D}\phi\, e^{-S_0 + J\cdot\phi}|_{J=0}
\end{aligned}
\tag{9.23}
$$

or (as we can easily see) in general, for an arbitrary function F, and in the presence of an arbitrary source J,

$$_J < 0|F[\{\phi\}]|0 >_J = F\left[\left\{\frac{\delta}{\delta J}\right\}\right] Z_0[J]|_J \tag{9.24}$$

In particular, we can use Dyson's formula, i.e. apply it for

$$F[\{\phi\}] = e^{-\int d^d x V(\phi(x))} \tag{9.25}$$

to obtain

$$Z[J] = e^{-\int d^d x V\left(\frac{\delta}{\delta J(x)}\right)} Z_0[J] = e^{-\int d^d x V\left(\frac{\delta}{\delta J(x)}\right)} e^{\frac{1}{2} J \cdot \Delta \cdot J} \tag{9.26}$$

which is the form of Wick's theorem in the path integral formalism.

This formula seems very powerful. Indeed, we seem to have solved completely the interacting theory, as we have a closed formula for the partition function, from which we can derive everything.

Of course, we can't get something for free: if it was tricky in the operator formalism, it means here also is tricky. The point is that this formal expression is in general not well defined: there are divergences (infinite integrals that will appear), and apparent singularities (like what happens for instance when several derivatives at the same point act at the same time). The above formula then has to be understood as a *perturbative* expansion: we expand the exponential, and then calculate the derivatives term by term. We will do some examples first in the next class, then formulate Feynman rules, which will be the same ones as we have found in the operator formalism.

Important concepts to remember

- The generalization from quantum mechanics to quantum field theory is obvious: the vacuum functional is the path integral in euclidean space over periodic paths of infinite period, with a positive definite euclidean action.

- Green's functions are obtained from a partition function with sources.

- In perturbation theory, we will study Green's functions, because they are simpler, but one can relate them to the S-matrix elements (real scattering amplitudes) via the LSZ formula, whose precise form will be given later.

- For momentum space Green's functions, one usually factorizes the overall momentum conservation, and talks about the Green's functions with momenta that already satisfy momentum conservation.

- Dyson's formula relates $< 0|e^{-S_I[\hat{\phi}]} e^{J \cdot \hat{\phi}}|0 >$ with the partition function (path integral over the full action, with sources).

- In free field theory, $Z_0[J] = e^{\frac{1}{2} J \cdot \Delta \cdot J}$.

- Wick's theorem gives the partition function of the interacting theory as the formal expression $Z[J] = e^{-\int V[\delta/\delta J]} Z_0[J]$.

Further reading: See chapters 2.1,2.2 in [4] and 3.1,3.2 in [3].

Exercises, Lecture 9

1) Using the formulas given in this class, compute the 4-point function in momentum space, in free field theory.

2) If $S_I = \int \lambda \phi^4/4!$, write down a path integral expression for the 4-point function $G_4(x_1, ..., x_4)$ up to (including) order λ^2.

10 Lecture 10. Wick theorem for path integrals and Feynman rules part I

In the previous class we have seen the Euclidean formulation of scalar field theories. We have written Dyson's formula for the partition function,

$$Z[J] = <0|e^{-S_I[\hat{\phi}]}e^{\int d^dx J(x)\hat{\phi}(x)}|0> = \int \mathcal{D}\phi e^{-S_0[\phi]+J\cdot\phi}e^{-S_I[\phi]} \qquad (10.1)$$

calculated the partition function of the free theory,

$$Z_0[J] = e^{\frac{1}{2}J\cdot\Delta\cdot J} <0|0> = e^{\frac{1}{2}J\cdot\Delta\cdot J} \qquad (10.2)$$

where the Euclidean propagator is

$$\Delta(x,y) = \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip\cdot(x-y)}}{p^2+m^2} \qquad (10.3)$$

And finally we wrote Wick's theorem for path integrals, which is

$$Z[J] = e^{-\int d^dx V\left(\frac{\delta}{\delta J(x)}\right)} Z_0[J] = e^{-\int d^dx V\left(\frac{\delta}{\delta J(x)}\right)} e^{J\cdot\Delta\cdot J} \qquad (10.4)$$

It is not completely obvious why this is the same Wick's theorem from the operator formalism, so we will see this by doing explicit examples.

Let's consider the theory at zeroth order in the coupling constant, i.e. the free theory, $Z_0[J]$.

We will denote the Green's functions as $G_n^{(p)}(x_1, ..., x_n)$ for the n-point function at order p in the coupling.

We start with the one-point function. At nonzero J,

$$G_1^{(0)}(x_1)_J = \frac{\delta}{\delta J(x_1)}e^{\frac{1}{2}J\cdot\Delta\cdot J} = \Delta\cdot J(x_1)e^{\frac{1}{2}J\cdot\Delta\cdot J} \qquad (10.5)$$

where $\Delta\cdot J(x_1) = \int d^dx\Delta(x_1,x)J(x)$. Putting $J = 0$, the one-point function is zero,

$$G_1^{(0)}(x_1) = 0 \qquad (10.6)$$

We can easily see that this generalizes for all the odd n-point functions, since we have even numbers of J's in $Z_0[J]$, so by taking an odd number of derivatives and then putting $J = 0$ we get zero,

$$G_{2k+1}^{(0)}(x_1, ..., x_{2k+1}) = 0 \qquad (10.7)$$

The next is the 2-point function,

$$\begin{aligned} G_2^{(0)}(x_1, x_2)_J &= \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}e^{\frac{1}{2}J\cdot\Delta\cdot J} = \frac{\delta}{\delta J(x_1)}[\Delta\cdot J(x_2)e^{\frac{1}{2}J\cdot\Delta\cdot J}] \\ &= [\Delta(x_1,x_2)+(\Delta\cdot J(x_2))(\Delta\cdot J(x_1))]e^{\frac{1}{2}J\cdot\Delta\cdot J} \end{aligned} \qquad (10.8)$$

and by putting $J = 0$ we find

$$G_2^{(0)}(x_1, x_2) = \Delta(x_1, x_2) \tag{10.9}$$

The corresponding Feynman diagram is obtained by drawing a line connecting the external points x_1 and x_2.

For the next nontrivial case, the 4-point function is

$$
\begin{aligned}
G_4^{(0)}(x_1, x_2, x_3, x_4) &= \prod_{i=1}^4 \frac{\delta}{\delta J(x_1)} e^{\frac{1}{2}J \cdot \Delta \cdot J}|_{J=0} \\
&= \Delta(x_1, x_2)\Delta(x_3, x_4) + \Delta(x_1, x_3)\Delta(x_2, x_4) + \Delta(x_1, x_4)\Delta(x_2, x_3)
\end{aligned}
\tag{10.10}
$$

which can be represented as the sum of Feynman diagrams with lines connecting (12) and (34); (13) and (24); (14) and (23), as in Fig.22. The (simple) details are left as an exercise.
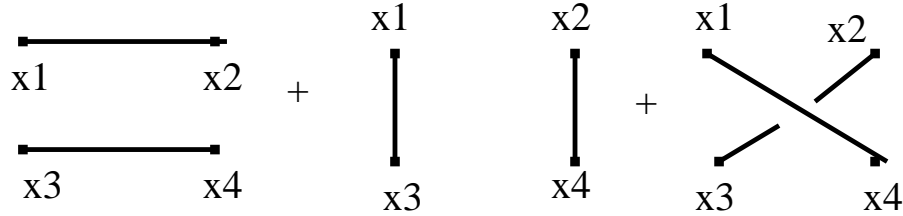
Figure 22: Free contractions for the 4-point function.

We then get the general rule for computing $G_n^{(0)}(x_1, ..., x_n)$. Write all the Feynman diagrams by connecting pairwise all external points in all possible ways.

We now move to the first nontrivial example of interaction. Consider the theory with $V(\phi) = \lambda\phi^3/3!$. Of course, such a theory is not so good: the Hamiltonian is unbounded from below (the energy becomes arbitrarily negative for large enough negative ϕ), so the system is unstable. But we just want to use this as a simple example of how to calculate Feynman diagrams.

Consider the theory at first order in λ, i.e. we replace

$$e^{-\int d^d x \frac{\lambda}{3!} \left(\frac{\delta}{\delta J(x)}\right)^3} \rightarrow -\int d^d x \frac{\lambda}{3!} \left(\frac{\delta}{\delta J(x)}\right)^3 \tag{10.11}$$

Then we have

$$
\begin{aligned}
Z^{(1)}[J] &= -\int d^d x \frac{\lambda}{3!} \left(\frac{\delta}{\delta J(x)}\right)^3 e^{\frac{1}{2}J \cdot \Delta \cdot J} = -\frac{\lambda}{3!} \int d^d x \left(\frac{\delta}{\delta J(x)}\right)^2 [\Delta \cdot J(x) e^{\frac{1}{2}J \cdot \Delta \cdot J}] \\
&= -\frac{\lambda}{3!} \int d^d x \frac{\delta}{\delta J(x)} [\Delta(x, x) + (\Delta \cdot J(x))^2] e^{\frac{1}{2}J \cdot \Delta \cdot J}
\end{aligned}
$$

$$= -\frac{\lambda}{3!} \int d^d x [3\Delta(x,x)(\Delta \cdot J)(x) + (\Delta \cdot J(x))^3] e^{\frac{1}{2}J\cdot\Delta\cdot J} \tag{10.12}$$

From this expression, which contains only odd powers of J, we see that the even n-point functions at order 1 are zero,

$$G^{(1)}_{2k}(x_1, ..., x_{2k}) = 0 \tag{10.13}$$

The one-point function is

$$G^{(1)}_1(x_1) = \frac{\delta}{\delta J(x_1)} Z^{(1)}[J]|_{J=0} = -\frac{\lambda}{2} \int d^d x \Delta(x,x)\Delta(x,x_1) \tag{10.14}$$

which diverges due to $\Delta(x,x)$, and has as Feynman diagram a propagator from the external point x_1 to the point x to be integrated over, followed by a loop starting and ending at x, as in Fig.23.
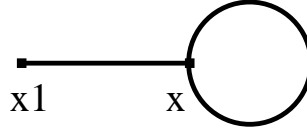
Figure 23: One point function tadpole diagram.

Next, we calculate the 3-point function,

$$
\begin{aligned}
G^{(1)}_3(x_1, x_2, x_3) &= \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \frac{\delta}{\delta J(x_3)} Z^{(1)}[J]|_{J=0} \\
&= -\lambda \int d^d x \{\Delta(x,x_1)\Delta(x,x_2)\Delta(x,x_3) + \frac{1}{2}\Delta(x,x)[\Delta(x,x_1)\Delta(x_2,x_3) \\
&\quad + \Delta(x,x_2)\Delta(x_1,x_3) + \Delta(x,x_3)\Delta(x_1,x_2)]\}
\end{aligned} \tag{10.15}
$$

which can be represented as 4 Feynman diagrams: one with a 3-vertex connected with the 3 external points, one with x_2 connected with x_3 and a $G^{(1)}_1(x_1)$ contribution, i.e. a line with a loop at the end connected to x_1, and the 2 permutations for it, as in Fig.24.
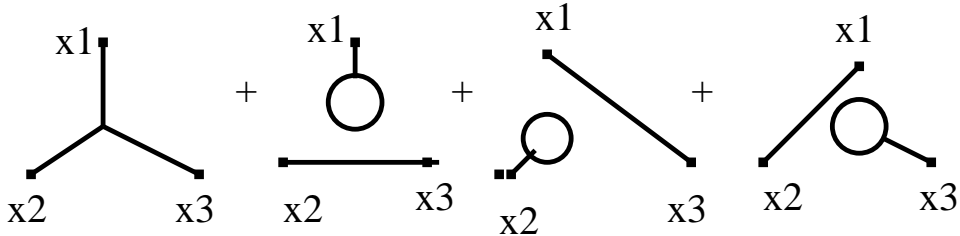
Figure 24: Diagrams for the 3-point function at order λ.

We now exemplify the diagrams at second order with the 0-point function (vacuum bubble), at second order $\mathcal{O}(\lambda^2)$, i.e. $Z^{(2)}[J = 0]$. We have

$$Z^{(2)}[J] = \frac{1}{2!}\int d^dx \frac{\lambda}{3!}\left(\frac{\delta}{\delta J(x)}\right)^3 \int d^dy \frac{\lambda}{3!}\left(\frac{\delta}{\delta J(y)}\right)^3 e^{\frac{1}{2}J\cdot\Delta\cdot J} \qquad (10.16)$$

But when we put $J = 0$, only terms with 6 J's contribute, so we have

$$Z^{(2)}[J = 0] = \frac{1}{2!}\int d^dx \frac{\lambda}{3!}\left(\frac{\delta}{\delta J(x)}\right)^3 \int d^dy \frac{\lambda}{3!}\left(\frac{\delta}{\delta J(y)}\right)^3 \frac{1}{3!}\left(\frac{1}{2}\right)^3 \int d^dz_1 d^dz_2 d^dz_3 d^dz_1' d^dz_2' d^dz_3'$$
$$J(z_1)\Delta(z_1, z_1')J(z_1')J(z_2)\Delta(z_2, z_2')J(z_2')J(z_3)\Delta(z_3, z_3')J(z_3')|_{J=0} \qquad (10.17)$$

When we do the derivatives, we obtain a result that is the sum of two Feynman diagrams,

$$Z^{(2)}[J = 0] = \frac{\lambda^2}{2^3}\int d^dx d^dy \Delta(x, x)\Delta(x, y)\Delta(y, y) + \frac{\lambda^2}{2\cdot 3!}\int d^dx d^dy \Delta^3(x, y) \qquad (10.18)$$

the two Feynman diagrams being one with two loops (circles) connected by a line, and one with 3 propagators connecting the same two points (integrated over), x and y, as in Fig.25. The details (which derivatives give which diagram) are left as an exercise.
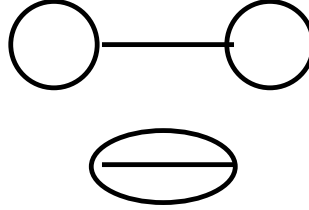
Figure 25: Diagrams for the vacuum bubble (zero point function) at order λ^2.

Wick's theorem: second form
We now formulate a second form for the Wick theorem,

$$Z[J] = e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\left\{e^{-\int d^dx V(\phi)+J\cdot\phi}\right\}|_{\phi=0}$$
$$= \exp\left[\frac{1}{2}\int d^dx d^dy \Delta(x - y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(x)}\right]\left\{e^{-\int d^dx V(\phi)+J\cdot\phi}\right\}|_{\phi=0} \qquad (10.19)$$

To prove it, we use the previous form of the Wick theorem, and the following
Lemma (Coleman)
Consider two functions of multi-variables,

$$F(x) = F(x_1, ..., x_n); \quad G(y) = G(y_1, ..., y_n) \qquad (10.20)$$

Then we have

$$F\left(\frac{\partial}{\partial x}\right)G(x) = G\left(\frac{\partial}{\partial y}\right)\left\{F(y)e^{x\cdot y}\right\}_{y=0} \qquad (10.21)$$

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To prove it, because of the Fourier decomposition theorem, we only have to prove the lemma for

$$F(x) = e^{a \cdot x}; \quad G(y) = e^{b \cdot y} \tag{10.22}$$

Let's do that, first on the left hand side of the lemma,

$$F\left(\frac{\partial}{\partial x}\right) G(x) = e^{a \cdot \frac{\partial}{\partial x}} e^{b \cdot x} = \sum_m \frac{1}{m!} \left(a \cdot \frac{\partial}{\partial x}\right)^m e^{b \cdot x} = \sum_m \frac{1}{m!} (a \cdot b)^m e^{b \cdot x} = e^{a \cdot b} e^{b \cdot x} = e^{b \cdot (a+x)} \tag{10.23}$$

and then on the right hand side of the lemma,

$$G\left(\frac{\partial}{\partial y}\right) \left\{ F(y) e^{x \cdot y} \right\}_{y=0} = e^{b \cdot \frac{\partial}{\partial y}} \left\{ e^{a \cdot y} e^{x \cdot y} \right\}_{y=0} \tag{10.24}$$

Using the result for the lhs of the lemma, we have

$$G\left(\frac{\partial}{\partial y}\right) \left\{ F(y) e^{x \cdot y} \right\}_{y=0} = e^{b \cdot (a+x)} e^{(a+x) \cdot y}|_{y=0} = e^{b \cdot (a+x)} \tag{10.25}$$

which is the same as the left hand side of the lemma. Q.E.D.

We can now apply this lemma on the previous form of the Wick's theorem, with $x \to J(x)$ and $y \to \phi(x)$, so we generalize between a discrete and finite number of variables in the above to a continuum of variables. Then we obtain

$$Z[J] = e^{-\int d^d x V \left[\frac{\delta}{\delta J}\right]} e^{\frac{1}{2} J \cdot \Delta \cdot J} = e^{\frac{1}{2} \frac{\delta}{\delta \phi} \cdot \Delta \cdot \frac{\delta}{\delta \phi}} \left\{ e^{-\int d^d x V(\phi) + J \cdot \phi} \right\}_{\phi=0} \tag{10.26}$$

Q.E.D.

Feynman rules in x-space

Consider the general polynomial potential

$$V(\phi) = \lambda \phi^p \tag{10.27}$$

We write down the Feynman rules for the n-point functions, in order λ^N. But note that now we will consider Feynman diagrams that give the same result as different. This is an algorithmic method, so when we can't compute the symmetry factors for diagrams, we can use this construction. We will return to the usual Feynman rules in the next class.

The rules are: we write down the n external points, $x_1, ..., x_n$, each with a line (leg) sticking out, and then N vertices, i.e. internal (to be integrated over) points $y_1, ..., y_N$, each with p legs sticking out. We then connect all the lines sticking out in all possible ways by propagators, thus constructing all the Feynman diagrams. Then

-For each p-legged vertex, we write a factor of $-\lambda$, see Fig.26. (*Note the missing factor of i with respect to Minkowski space. But now we are in Euclidean space.*)

-For each propagator from z to w, we write a $\Delta(z - w)$.

-The resulting expression $I_D(x_1, ..., x_n; y_1, ..., y_N)$ is integrated over the vertices to obtain the result for the Feynman diagram,

$$F_D^{(N)}(x_1, ..., x_n) = \int d^d y_1 ... d^d y_N I_D(x_1, ..., x_n; y_1, ..., y_N) \tag{10.28}$$
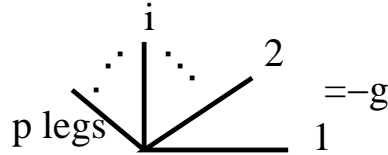
Figure 26: Feynman diagram for the p-vertex.

-Then the n-point function is given by

$$G_n(x_1, ..., x_n) = \sum_{N \geq 0} G_n^{(N)}(x_1, ..., x_n) = \sum_{N \geq 0} \frac{1}{N!} \sum_D F_D^{(N)}(x_1, ..., x_n) \qquad (10.29)$$

As an **example**, we take the 2-point function to order λ^1 for a potential with $p = 4$. Then we draw the 2 points x_1, x_2 with a line sticking out and a vertex y with 4 legs sticking out, as in Fig.27a.
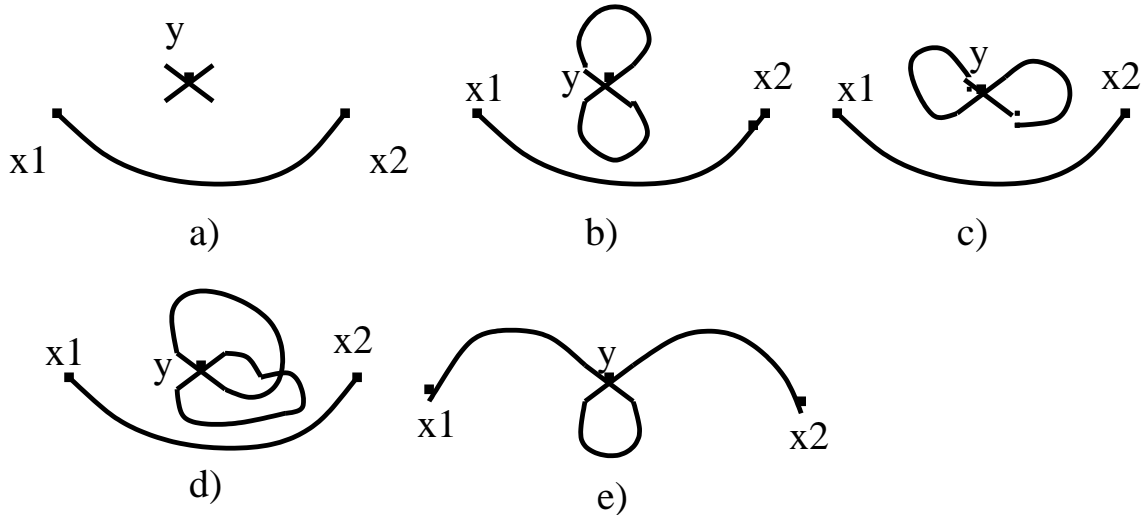
Figure 27: We can draw all the possible Feynman diagrams at order λ in the 2-point funtion by drawing the vertex and the external lines (a). Then we can connect them in all possible ways, obtaining diagrams b,c,d,e.

We can connect x_1 with x_2, and we have 3 ways of connecting the 4 legs of the vertex to each other, as in Figs.27b,c,d, leading to

$$F_{D1} = F_{D2} = F_{D3} = -\lambda \int d^d y \Delta(x_1 - x_2) \Delta(y - y) \Delta(y - y) \qquad (10.30)$$

We can also connect each of the x_1 and x_2 with one of the legs of the vertex, and the remaining two legs of the vertex to each other, as in Fig.27e. That gives 12 diagrams, since

we have 6 ways of choosing 2 lines out of the 4, and 2 ways of choosing which of the two to connect with x_1. Thus we have

$$F_{D4} = ... = F_{D15} = -\lambda \int d^d y \Delta(x_1 - y)\Delta(x_2 - y)\Delta(y - y) \qquad (10.31)$$

Important concepts to remember

- In free field theory (zeroth order in perturbation), the odd n-point functions are zero, $G^{(0)}_{2k+1}(x_1, ..., x_{2k+1}) = 0$.

- For $G^{(0)}_{2k}(x_1, ..., x_{2k})$, we write all the Feynman diagrams by connecting pairwise all external legs in all possible ways.

- We can write down a second form of the Wick theorem, using Coleman's lemma, as $Z[J] = e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\left\{e^{-\int V(\phi)+J\cdot\phi}\right\}|_{\phi=0}$.

- The Feynman rules in x-space, for $G_n^{(N)}$ in $\lambda\phi^p$ theory, the long form, are: write n external points, with a leg sticking out, and N vertices, with p legs sticking out, then connect all the legs in all possible ways. For a vertex write $-\lambda$, for a propagator write $\Delta(x, y)$. Then integrate over the vertices, sum over diagrams (now we will have many diagrams giving the same result) and sum over N with $1/N!$.

- This prescription is equivalent to the usual one of writing only inequivalent diagrams, and writing a symmetry factor S.

Further reading: See chapters 2.3 and 4.1 in [4].

Exercises, Lecture 10

1) *Prove* that $G_4^{(0)}(x_1, ..., x_4)$ is the sum of the three diagrams with connections of (12),(34); (13),(24); (14),(23) in Fig.22 and then generalize to *write down* $G_6^{(0)}(x_1, ..., x_6)$.

2) Explain which derivatives give which Feynman diagram in the calculation of the 0-point function to $\mathcal{O}(\lambda^2)$ in $\lambda \phi^3$ theory (two circles connected by a line; and two points connected by 3 propagators as in Fig.25), and then write down the Feynman diagrams for $G_4^{(2)}(x_1, ..., x_4)$ in $\lambda \phi^4$ theory.

11 Lecture 11. Feynman rules in x-space and p-space

Proof of the Feynman rules

Let's compute now the Green's functions,

$$G(x_1, ..., x_n) = \frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)}\Big|_{J=0} = e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\{\phi(x_1)...\phi(x_n)e^{-\int d^d x[V(\phi(x))-J(x)\phi(x)]}\}\Big|_{\phi=0}\Big|_{J=0}$$

(11.1)

Note that these Green's functions are the Green's functions corresponding to
$< 0|T\{\phi(x_1)...\phi(x_n)e^{i\int H_{int}}\}|0 >$ (in the *free* vacuum of the theory) in Minkowski space, and as a result they still contain vacuum bubbles. The vacuum bubbles will be seen later to factorize exactly as in the operator formalism, and then we will calculate the Green's functions corresponding to $< \Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega >$ from $W = -\ln Z[J]$.

We next note that in the above we can drop the $J \cdot \phi$ term, since by taking all the derivatives we will have only terms linear in J from this contribution, and putting $J = 0$ these terms vanish. The result is then

$$G(x_1, ..., x_n) = e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\{\phi(x_1)...\phi(x_n)e^{-\int d^d x V(\phi)}\}\Big|_{\phi=0}$$

(11.2)

This means that in order N (i.e.,$\mathcal{O}(\lambda^N)$) we find

$$G^{(N)}(x_1, ..., x_n) = e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\{\phi(x_1)...\phi(x_n)\frac{(-\lambda)^N}{N!}\int d^d y_1...d^d y_N \phi^p(y_1)...\phi^p(y_N)\}\Big|_{\phi=0}$$

(11.3)

Consider now that there are $Q = n + pN$ ϕ's, on which we act with the derivatives on the exponential. If we have more derivatives than fields, we obviously get zero, but we also get zero if we have fewer derivatives, because at the end we must put $\phi = 0$, so if we have ϕ's left over, we also get zero. Therefore we only get a nonzero result if $Q = 2q$ (even), and then the result is

$$\begin{aligned} G^{(N)}(x_1, ..., x_n) &= \frac{1}{q!2^q}\int d^d z_1 d^d w_1...d^d z_q d^d w_q \times \\ &\times \frac{\delta}{\delta\phi(z_1)}\Delta(z_1 - w_1)\frac{\delta}{\delta\phi(w_1)}...\frac{\delta}{\delta\phi(z_q)}\Delta(z_q - w_q)\frac{\delta}{\delta\phi(w_q)} \times \\ &\times \{\phi(x_1)...\phi(x_n)\frac{(-\lambda)^N}{N!}\int d^d y_1...d^d y_N \phi^p(y_1)...\phi^p(y_N)\} \end{aligned}$$

(11.4)

Then when acting with a factor of $\int d^d z d^d w \frac{\delta}{\delta\phi(z)}\Delta(z - w)\frac{\delta}{\delta\phi(w)}$ on some $\phi(x)\phi(z)$ we obtain a Wick contraction $\overline{\phi(x)\phi(y)}$, i.e. we replace it with $\Delta(x - y)$. But with a factor of 2 in front, since it comes from $\int d^d z d^d w[\delta(z-x)\delta(w-y)+\delta(z-y)\delta(w-x)]\Delta(z-w)$. Since all the q factors above give the same factor of 2, in total we get a 2^q cancelling the $1/2^q$. Also, we have q factors as above with which we can act on the $Q = 2q$ ϕ's, and so by permuting them, we get the same contribution. Then the resulting factor of $q!$ will cancel the $1/q!$ in front.

Statistical weight factor (symmetry factor)

As explained before, from most diagrams, we get a lot of contributions that are identical, almost giving an $p!$ factor for each vertex, and an overall $N!$, so it is customary to redefine $\lambda = \lambda_p/p!$, so as to (almost) cancel this factor. Then, we have in front of the result for a Feynman diagram a factor of $1/S$, where by definition the *statistical weight factor, or symmetry factor S* is

$$S = \frac{N!(p!)^N}{\#\text{of equivalent diagrams}} \qquad (11.5)$$

Then we can construct only the topologically inequivalent diagrams, and associate a statistical weight factor, or symmetry factor S, corresponding to all possible symmetries of the diagram, that leave it invariant. If there is no symmetry, then $S = 1$.

For tree digrams, always $S_{tree} = 1$. For instance, we can write a Feynman diagram in ϕ^4 theory, like in Fig.28.
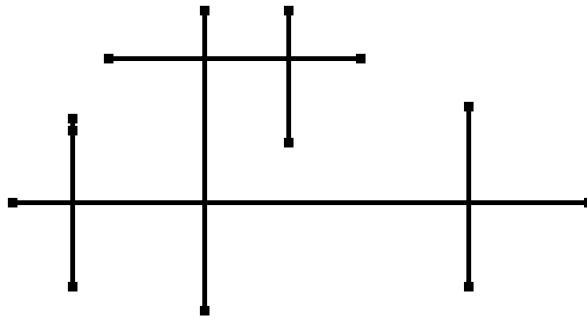
Figure 28: Example of a tree diagram. It has $S = 1$.

There are $N!$ ways to label the vertices as $y_1, ..., y_N$. Many topologically equivalent diagrams obtained by connecting the $p = 4$ legs on the vertices with other legs from vertices or from the external lines, can be thought of as just the above relabelling the vertices. Then there are $p! = 4!$ ways of attaching given propagators at the $p = 4$ legs of the vertex. In total, we cancel the $1/(N!(p!)^N)$ factor in front, giving $S_{tree} = 1$.

We can do more symmetry factors. The figure 8 vacuum bubble diagram of Fig.29a has a symmetry factor of 8, which matches with the number of diagrams. One vertex with 4 legs has 3 ways of connecting them 2 by 2, giving $3 \times 1/(1!(4!)^1) = 1/8$.

The propagator from x_1 to x_2 with a loop at an intermediate point x, as in Fig.29b has a symmetry factor of 2. From the vertex there are 4 ways to connect one leg to x_1, then 3 ways to connect to x_2, and one way to connect the remaining 2 legs between themselves, for a total of $4 \cdot 3/(1!(4!)^1) = 1/2$.

The "setting sun" diagram, propagating from x_1 to x_2, with 3 propagators between two intermediate points x and y, as in Fig.29c has a symmetry factor of 3!, from permuting the 3 propagators between x and y. On the other hand, we draw 2 vertices with 4 legs, and two external lines. For the first contraction there are 8 ways to connect x to a vertex, then 4 ways to connect y with the other vertex, and then $3! = 3 \times 2 \times 1$ to connect the 3 internal propagators, for a total of $8 \cdot 4 \cdot 3!/(2!(4!)^2) = 1/3!$, as expected.

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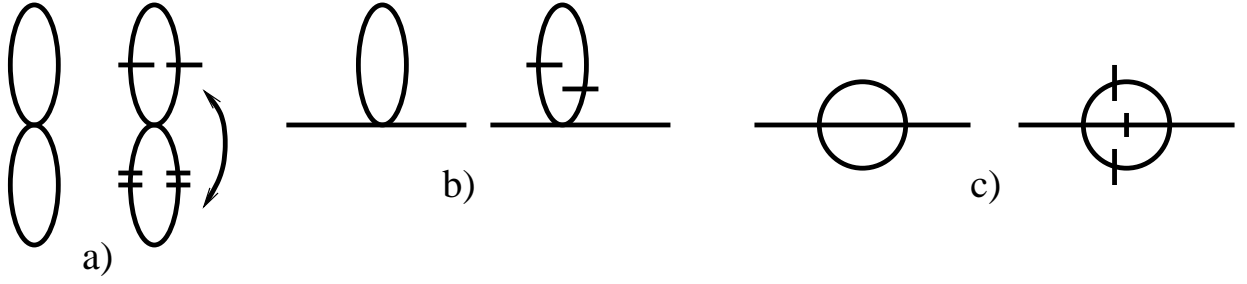
Figure 29: Computing the symmetry factors for 3 loop diagrams (a,b,c).

Feynman rules in p space

As we already explained, we can define the Fourier transform of the x-space Green's function, but by translational invariance we can always factor out a delta function, with the remaining Green's function, already satisfying momentum conservation, being the one we usually calculate,

$$\tilde{G}(p_1, ..., p_n) = \int \prod_i d^d x_i e^{i \sum x_j p_j} G(x_1, ..., x_n) = (2\pi)^d \delta^d(p_1 + ... + p_n) G(p_1, ..., p_n) \quad (11.6)$$

where in $G(p_1, ..., p_n)$ the momenta already satisfy momentum conservation.

Since the Euclidean propagator is

$$\Delta(x - y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y)}}{p^2 + m^2} \quad (11.7)$$

in momentum space we have

$$\Delta(p) = \frac{1}{p^2 + m^2} \quad (11.8)$$

By convention, a momentum p going out of a point x means a factor of e^{ipx} and going in means a factor of e^{-ipx}, as in Fig.30.
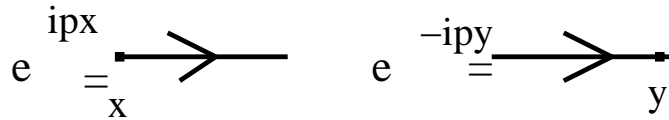
Figure 30: Momentum convention for external lines.

We write a version of the Feynman diagrams where we already include the momentum conservation. But for that, we need to know the number of *loops*, i.e. independent loops

(circles) in our diagrams, or the number of integrations left over after we use the momentum conservation delta functions.

We denote by V the number of vertices, I the number of internal lines, E the number of external lines, and L the number of loops. There are I momentum variables (one for each internal lines), but there are V constraints on them = momentum conservation delta functions. However, one of these is the overall one which we put outside G to form $\tilde{G}$. So for the calculation of G, we have $L = I - V + 1$ loops (independent integrations). But we must express this in terms of things that are independent of the diagram, and depend only on having $G^{(N)}(p_1, ..., p_n)$, so we must find another formula. If we cut all propagators (or equivalently, consider the diagram before we make the connections of the p legs on vertices with other legs on them and on external lines), the legs on the vertices come as follows. For each external line, one of the two pieces of propagator stays on the external point, but one comes from a vertex. For internal lines however, both come from the vertex, so all in all, $E + 2I = pV$. Replacing this formula in the above for L, we finally find

$$L = V \left(\frac{p}{2} - 1 \right) - \frac{E}{2} + 1 \tag{11.9}$$

So after using all the momentum conservation delta functions, we are left with L independent integrations, and the Feynman rules become:

-draw all the topologically inequivalent Feynman diagrams with n external lines and N vertices.

-label external momenta $p_1, ..., p_n$ with arrows for them, introduce $l_1, ..., l_L$ independent loop momenta, integrated over with $\int d^d l_1/(2\pi)^d ... d^d l_L/(2\pi)^d$, and label the propagator lines by q_j, using the l_i and momentum conservation for them.

-for an external line (between two points), put a propagator $1/(p_i^2 + m^2)$.

-for an internal line (between two points), put a propagator $1/(q_j^2 + m^2)$.

-for a vertex, we have a factor of $-\lambda_p$

-calculate the statistical weight factor (symmetry factor) S and multiply the diagram by $1/S$.

-sum over Feynman diagrams.

Let's understand better the p space rules. Consider the x-space "setting sun" diagram consider above, and Fourier transform it to p space. This gives $\tilde{G}(p_1, p_2)$, with two external points, but without x labels, so it has a $\Delta(p_1)$ and a $\Delta(p_2)$ factor. The external line rule for a momentum p going out of an external point labelled by x, giving a factor of e^{ipx} is just a wavefunction for a more physical case: the only way to introduce an x dependence into a p space object is an explicit wavefunction for the external states of momenta p_i. If there is no label x, we have no wavefunction. But also, if there is no external point (what is known as a *amputated diagram*, to be discussed later in the course), we have no corresponding external propagator. The final thing to observe is of course that in general, we are not interested in $\tilde{G}(p_1, p_2)$, but in $G(p_1, p_2) = G(p_1, -p_1) \equiv G(p_1)$, in which momentum conservation is already taken into account.

For instance, consider the free propagator, a line between the two external points. We could in principle consider momentum p_1 going in at one point, and p_2 going in at the other.

But then

$$
\begin{aligned}
\tilde{G}(p_1, p_2) &= \int d^d x_1 d^d x_2 e^{i(p_1 x_1 + p_2 x_2)} G(x_1, x_2) = \int d^d x_1 d^d x_2 e^{i(p_1 x_1 + p_2 x_2)} \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x_1 - x_2)}}{p^2 + m^2} \\
&= \int d^d x_1 e^{ip_1 x_1} \frac{e^{ip_2 x_1}}{p_2^2 + m^2} = (2\pi)^d \delta^d(p_1 + p_2) \frac{1}{p_2^2 + m^2} = (2\pi)^d \delta^d(p_1 + p^2) G(p_2)
\end{aligned}
$$
(11.10)

Most general bosonic field theory

We now deal with a general bosonic field theory. Consider fields $\phi_r(x)$, where r is some general label, which could signify some Lorentz index, like on a vector field $A_\mu(x)$, or some label for several scalar fields, or a combination of both.

Consider the kinetic term

$$
S_0 = \frac{1}{2} \int d^d x \sum_{r,s} \phi_r(x) \Delta_{r,s}^{-1} \phi_s(x)
$$
(11.11)

where $\Delta_{r,s}^{-1}$ is some differential operator. We invert this object, obtaining the propagator $\Delta_{r,s}$. If it is not possible, it means that there are redundancies, and we must find the independent fields. For instance, for a gauge field $A_\mu(x)$, there is a gauge invariance, $\delta A_\mu = \partial_\mu \lambda(x)$, which means that we must introduce a gauge fixing (and also introduce "ghosts", to be defined later) in order to define independent fields. Then we can invert the propagator.

Then we can derive a generalized Wick's theorem, for a general interaction term

$$
S_{r_1 \ldots r_p} = \int d^d x \mathcal{A}_{r_1 \ldots r_p} \phi_{r_1}(x) \ldots \phi_{r_p}(x)
$$
(11.12)

where $\mathcal{A}_{r_1 \ldots r_p}$ can contain couplings and derivatives, so it's more general than the construction of ϕ^p theory above.

Then, the vertex is

$$
-\int d^d x_1 \ldots d^d x_p e^{i(k_1 x_1 + \ldots k_p x_p)} \frac{\delta}{\delta \phi_{r_1}(x_1)} \ldots \frac{\delta}{\delta \phi_{r_p}(x_p)} S_{r_1 \ldots r_p}
$$
(11.13)

(Note that we don't really need to define $\mathcal{A}_{r_1 \ldots r_p}$, all we need is the form of the interaction term in the action, $S_{r_1 \ldots r_p}$.) Let's check that this gives the correct result for the $\lambda_4 / 4! \phi^4$ theory. Indeed, then we have the interaction term

$$
S_I = \frac{\lambda_4}{4!} \int d^d x \phi^4(x)
$$
(11.14)

We easily find that applying the above formula we get

$$
-(2\pi)^d \delta^d(k_1 + \ldots k_4) \lambda_4
$$
(11.15)

which is the correct vertex (it has the $-\lambda$ factor, as well as the momentum conservation at the vertex that we need to take into account).

Important concepts to remember

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- We can write the Feynman rules in p space directly in terms of independent integrations = loop momenta. The number of loops or independent integrations is $L = V(p/2 - 1) - E/2 + 1$.

- The Feynman rules then consist of labelling the internal lines using the loop momenta and momentum conservation, with $\Delta(p_i)$ for external line (ending at an unlabelled point) and $\Delta(q_j)$ for internal line, and $-\lambda_p$ for vertices. Then we must divide by the statistical weight factor (symmetry factor for the diagram).

- For the most general bosonic theory, we need to invert the kinetic term to find the propagator (if there are redundancies, like for gauge fields, we must first find the correct kinetic term for the independent fields). For the interaction term $S_{r_1...r_p}$, we find the vertex as the Fourier transform of $-\delta^n/\delta\phi_{r_1}...\phi_{r_p}S_{r_1...r_p}$.

Further reading: See chapters 2.4.2, 2.4.3, 2.7 in [4], 4.1 in [3] and 3.4 in [1].

Exercises, Lecture 11

1) Write down the statistical weight factor (symmetry factor) for the Feynman diagram in Fig.31 in x-space, and then write down an integral expression for it, applying the Feynman rules.
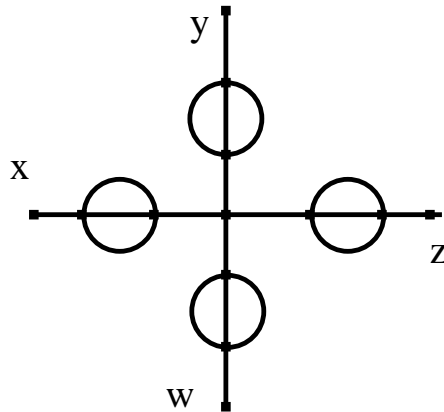

Figure 31: x-space Feynman diagram.

2) Consider the interaction term

$$S_I = \int d^d x \sum_{i_1, i_2 = 1}^{N} (\partial^2 \phi^{i_1})(\partial_\mu \phi^{i_2}) \phi_{i_1} (\partial^\mu \phi_{i_2}) \tag{11.16}$$

for N scalars ϕ_i, $i = 1, ..., N$, each with the usual massless kinetic term. Write down the Feynman rules for this model.

12 Lecture 12. Quantization of the Dirac field and fermionic path integral

The Dirac equation

We start by reviewing the classical Dirac field.

Spinors are fields in a Hilbert space acted on by gamma matrices. The gamma matrices are objects satisfying the *Clifford algebra*

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1} \tag{12.1}$$

Therefore the spinors are representations of the Clifford algebra (Hilbert spaces on which we represent the abstract algebra). Since then

$$S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \tag{12.2}$$

satisfy the Lorentz algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i[g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}], \tag{12.3}$$

the spinors are also a representation of the Lorentz algebra, called spinor representation.

For instance, in 3 Euclidean dimensions, the gamma matrices are the same as the Pauli matrices, $\gamma^i = \sigma^i$, since indeed then $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. Remember that the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12.4}$$

satisfy the relation

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k, \tag{12.5}$$

from which follows that they indeed satisfy the Clifford algebra in 3 Euclidean dimensions.

We can make linear transformations on the spinor space (Hilbert space for the Clifford algebra), via $\psi \to S\psi$ and $\gamma^\mu \to S\gamma^\mu S^{-1}$, and thus choose various representations for the gamma matrices. A particularly useful one such representation is called the Weyl (or chiral) representation, which is defined by

$$\gamma^0 = -i\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}; \quad \gamma^i = -i\begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}; \quad i = 1, 2, 3, \tag{12.6}$$

where $\mathbf{1}$ and $\mathbf{0}$ are 2×2 matrices. For this representation, we can immediately check that $(\gamma^0)^\dagger = -\gamma^0$ and $(\gamma^i)^\dagger = \gamma^i$. We can also define the 4-vector 2×2 matrices

$$\sigma^\mu = (\mathbf{1}, \sigma^i); \quad \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^i), \tag{12.7}$$

such that in the Weyl representation we can write

$$\gamma^\mu = -i\begin{pmatrix} \mathbf{0} & \sigma^\mu \\ \bar{\sigma}^\mu & \mathbf{0} \end{pmatrix} \tag{12.8}$$

We can also define the matrix

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{12.9}$$

which in the Weyl representation becomes just

$$\gamma_5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}. \tag{12.10}$$

This is not a coincidence, in reality the Weyl representation was chosen such that γ_5, the product of gamma matrices, has this form. Also, the notation γ_5 is not coincidental: if we go to 5 dimensions, the first 4 gamma matrices are the same, and the 5th is γ_5.

Also, as a side remark, the pattern of obtaining the gamma matrices as tensor products of σ^i and $\mathbf{1}$ continues to be valid in all dimensions. The gamma matrices in 2 Euclidean dimensions can be chosen to be σ^1, σ^2 (note that then $\sigma^3 = -i\sigma^1\sigma^2$) and in higher dimensions we can always build the gamma matrices in terms of them.

We now write down the Dirac equation for a Dirac spinor,

$$(\gamma^\mu\partial_\mu + m)\psi = 0. \tag{12.11}$$

We can define the *Dirac conjugate*,

$$\bar{\psi} = \psi^\dagger\beta; \quad \beta = i\gamma^0 \tag{12.12}$$

which is defined such that $\bar{\psi}\psi$ is a Lorentz invariant. Note that there are several conventions possible for β (for instance, others use $\beta = i\gamma_0 = -i\gamma^0$, or $\beta = \gamma^0$ or γ_0). With these conventions, the Dirac action is written as

$$S_\psi = -\int d^4x\,\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi. \tag{12.13}$$

We will use the notation $\slashed{\partial} \equiv \gamma^\mu\partial_\mu$, which is common. Since the Dirac field ψ is complex, by varying the action with respect to $\bar{\psi}$ (considered as independent from ψ), we get the Dirac equation. Note that shortly we will find for Majorana (real) spinors, for which $\bar{\psi}$ is related to ψ, there is a factor of 1/2 in front of the action, since by varying with respect to ψ we get two terms (one where we vary ψ proper, and one where we vary the ψ from $\bar{\psi}$, giving the same result).

Weyl spinors

In 4d Minkowski dimensions, the Dirac representation is reducible as a representation of the Lorentz algebra (or of the Clifford algebra). The irreducible representation (irrep) is found by immposing a constraint on it, and the spinors are called Weyl (or chiral) spinors. In the Weyl representation for the gamma matrices, the Dirac spinors splits simply as

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{12.14}$$

which is why the Weyl representation for gamma matrices was chosen such. In general, we have

$$\psi_L = \frac{\mathbf{1} + \gamma_5}{2}\psi_D \Rightarrow \frac{\mathbf{1} - \gamma_5}{2}\psi_L = 0;$$

$$\psi_R = \frac{1 - \gamma_5}{2}\psi_D \Rightarrow \frac{1 + \gamma_5}{2}\psi_R = 0 \tag{12.15}$$

and we note that we chose the Weyl representation for gamma matrices such that

$$\frac{1 + \gamma_5}{2} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}; \quad \frac{1 - \gamma_5}{2} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{12.16}$$

Another possible choice for irreducible representation, completely equivalent (in 4 Minkowski dimensions) to the Weyl representation, is the *Majorana representation*. Majorana spinors satisfy the reality condition

$$\bar{\psi} = \psi^C \equiv \psi^T C \tag{12.17}$$

where ψ^C is called Majorana conjugate (thus the reality condition is "Dirac conjugate equals Majorana conjugate"), and C is a matrix called *charge conjugation matrix*. Note that since ψ^T is just another ordering of ψ, whereas $\bar{\psi}$ contains $\psi^\dagger = (\psi^*)^T$, this is indeed a reality condition $\psi^* = (...)\psi$.

The charge conjugation matrix in 4 Minkowski dimensions satisfies

$$C^T = -C; \quad C\gamma^\mu C^{-1} = -(\gamma^\mu)^T \tag{12.18}$$

In other dimensions and/or signatures, the definition is more complicated, and it can involve other signs on the right hand side of the two equations above.

In the Weyl representation, we can choose

$$C = \begin{pmatrix} -\epsilon^{\alpha\beta} & \mathbf{0} \\ \mathbf{0} & \epsilon^{\alpha\beta} \end{pmatrix} \tag{12.19}$$

where we have

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2 \tag{12.20}$$

and so

$$C = \begin{pmatrix} -i\sigma^2 & \mathbf{0} \\ \mathbf{0} & i\sigma^2 \end{pmatrix} = -i\gamma^0\gamma^2 \tag{12.21}$$

Then we have also

$$C^{-1} = \begin{pmatrix} i\sigma^2 & \mathbf{0} \\ \mathbf{0} & -i\sigma^2 \end{pmatrix} = -C \tag{12.22}$$

We can now check explicitly that this C is indeed a representation for the C-matrix, i.e. that it satisfies (12.18).

As we mentioned, the action for Majorana fields, with $\bar{\psi}$ related to ψ, is

$$S_\psi = -\frac{1}{2}\int d^4x\,\bar{\psi}(\slashed{\partial} + m)\psi \tag{12.23}$$

The Dirac equation implies the Klein-Gordon (KG) equation; more precisely, the Dirac equation is a sort of square root of the KG equation, which is roughly how Dirac thought about deriving it. Let's see this. First, we note that

$$\slashed{\partial}\slashed{\partial} = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\nu\partial_\nu = \Box \tag{12.24}$$

Then, it follows that

$$(\not{\partial} - m)(\not{\partial} + m)\psi = (\partial^2 - m^2)\psi, \tag{12.25}$$

so indeed, the KG operator is the product of the Dirac operator with $+m$ and the Dirac operator with $-m$.

Solutions of the free Dirac equation

Since a solution of Dirac is also a solution of KG, solutions of Dirac have to be solutions of KG, i.e. $e^{\pm ip \cdot x}$, with $p^2 + m^2 = 0$, times some matrices (column vectors) depending on p.

One set of solutions can then be written as

$$\psi(x) = u(p)e^{ip \cdot x} \tag{12.26}$$

where $p^2 + m^2 = 0$ and $p^0 > 0$. Since this is basically a Fourier transform, $u(p)$ satisfies the Fourier transformed equation,

$$(i\not{p} + m)u(p) = 0 \tag{12.27}$$

where $\not{p} \equiv \gamma^\mu p_\mu$ (note that $p_\mu = g_{\mu\nu}p^\nu$, and for the same p^μ, the p_μ changes by a sign between our "mostly plus" metric convention and the "mostly minus" one). The two solutions for the above equation for $u(p)$ can be written compactly and formally as

$$u^s(p) = \begin{pmatrix} \sqrt{-p \cdot \sigma}\,\xi^s \\ \sqrt{-p \cdot \bar{\sigma}}\,\xi^s \end{pmatrix} \tag{12.28}$$

where $s = 1, 2$ and $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Here $\sqrt{-p \cdot \sigma} = \sqrt{-p_\mu \sigma^\mu}$ is understood in matrix sense (if $A^2 = B$, then $A = \sqrt{B}$). In the case the matrix is diagonal, we can take square root of the diagonal elements, but in general we can't. For instance, in the rest frame, where $\vec{p} = 0$ and $p^0 = m$,

$$u^s(p) \propto \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{12.29}$$

The other two solutions are of type

$$\psi(x) = v(p)e^{-ip \cdot x} \tag{12.30}$$

where $p^2 + m^2 = 0$, $p^0 > 0$ and $v(p)$ satisfies

$$(-i\not{p} + m)v(p) = 0 \tag{12.31}$$

The two $v^s(p)$ can be similarly written as

$$v^s(p) = \begin{pmatrix} \sqrt{-p \cdot \sigma}\,\xi^s \\ -\sqrt{-p \cdot \bar{\sigma}}\,\xi^s \end{pmatrix} \tag{12.32}$$

and in the rest frame we have

$$v^s(p) \propto \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \qquad (12.33)$$

The normalization conditions for the $u(p)$ and $v(p)$ are written in Lorentz invariant form as

$$\begin{aligned} \bar{u}^r(p)u^s(p) &= 2m\delta^{rs} \\ \bar{v}^r(p)v^s(p) &= -2m\delta^{rs} \end{aligned} \qquad (12.34)$$

or using $u^\dagger$ and $v^\dagger$ as

$$\begin{aligned} u^{r\dagger}(p)u^s(p) &= 2E_p\delta^{rs} \\ v^{r\dagger}(p)v^s(p) &= 2E_p\delta^{rs} \end{aligned} \qquad (12.35)$$

The u and v solutions are orthogonal, i.e.

$$\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0 \qquad (12.36)$$

Note that now

$$u^{r\dagger}(p)v^s(p) \neq 0; \quad v^{r\dagger}(p)u^s(p) \neq 0 \qquad (12.37)$$

however, we have also

$$u^{r\dagger}(\vec{p})v^s(-\vec{p}) = v^{r\dagger}(-\vec{p})u^s(\vec{p}) = 0 \qquad (12.38)$$

Quantization of the Dirac field

The spin-statistics theorem says that fields of spin $S = (2k+1)/2$ ($k \in \mathbb{N}$) obey Fermi statistics, which means that in their quantization we must use anticommutators, $\{,\}$, instead of commutators $[,]$. On the other hand, for $S = k$, we have the Bose-Einstein statistics, with commutators $[,]$ for their quantization.

The Lagrangean for the Dirac field is

$$\mathcal{L} = -\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + ... = -\psi^\dagger i\gamma^0\gamma^\mu\partial_\mu\psi + ... = +i\psi^\dagger\partial_0\psi + ... \qquad (12.39)$$

(since $(\gamma^0)^2 = -1$). Then the canonical conjugate to ψ is

$$p_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger \qquad (12.40)$$

Then the Hamiltonian is

$$H = p\dot{q} - L = i\int d^3x\,\psi^\dagger(+\gamma^0\gamma^i\partial_i + m\gamma^0)\psi \qquad (12.41)$$

When we quantize, we write anticommutation relations at equal time, $\{,\}_{P.B.} \to 1/i\hbar\{,\}$, namely (after cancelling the i)

$$\{\psi_\alpha(\vec{x},t), \psi_\beta^\dagger(\vec{y},t)\} = \delta^3(\vec{x} - \vec{y})\delta_{\alpha\beta}$$

$$\{\psi_\alpha(\vec{x}, t), \psi_\beta(\vec{y}, t)\} = \{\psi_\alpha^\dagger(\vec{x}, t), \psi_\beta^\dagger(\vec{y}, t)\} = 0 \tag{12.42}$$

are the *equal time anticommutation relations*.

The quantization proceeds exactly as for the complex scalar field, just that instead of the properly normalized solution of the KG equation, $e^{\pm ip \cdot x}$, we have the above solutions of the Dirac equation, and again, their coefficients are harmonic oscillator operators. Therefore we have

$$
\begin{aligned}
\psi(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a_{\vec{p}}^s u^s(p) e^{ip \cdot x} + b_{\vec{p}}^{s\dagger} v^s(p) e^{-ip \cdot x}) \\
\bar{\psi}(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (b_{\vec{p}}^s \bar{v}^s(p) e^{ip \cdot x} + a_{\vec{p}}^{s\dagger} \bar{u}^s(p) e^{-ip \cdot x})
\end{aligned} \tag{12.43}
$$

where the a's, $a^\dagger$'s, and b's and $b^\dagger$'s are the annihilation/creation operators obeying the anticommutation relations of the *fermionic* harmonic oscillator, i.e.

$$\{a_{\vec{p}}^r, a_{\vec{q}}^{\dagger s}\} = \{b_{\vec{p}}^r, b_{\vec{q}}^{\dagger s}\} = (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \delta_{rs} \tag{12.44}$$

and the rest of anticommutators are zero. There is a Fock vacuum $|0>$, satisfying

$$a_{\vec{p}}^s |0> = b_{\vec{p}}^s |0> = 0 \tag{12.45}$$

and then Fock states created by acting with $a_{\vec{p}}^{s\dagger}$ and $b_{\vec{p}}^{s\dagger}$ on it. But, due to the anticommutation relations, which mean in particular that $(a_{\vec{p}}^{s\dagger})^2 = (b_{\vec{p}}^{s\dagger})^2 = 0$, we can't have two excitations in the same Fock state,

$$(a_{\vec{p}}^{s\dagger})^2 |\psi> = (b_{\vec{p}}^{s\dagger})^2 |\psi> = 0 \tag{12.46}$$

which is a manifestation of the Pauli exclusion principle.

The Hamiltonian is then written as

$$
\begin{aligned}
H &= \int \frac{d^3 p}{(2\pi)^3} \sum_s E_p (a_{\vec{p}}^{\dagger s} a_{\vec{p}}^s - b_{\vec{p}}^s b_{\vec{p}}^{\dagger s}) \\
&= \int \frac{d^3 p}{(2\pi)^3} \sum_s E_p (a_{\vec{p}}^{\dagger s} a_{\vec{p}}^s + b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s - \{b_{\vec{p}}^{\dagger s}, b_{\vec{p}}^s\})
\end{aligned} \tag{12.47}
$$

where the anticommutator in the integral in $= 1$, giving a negative infinite constant. As in the bosonic case, the infinite constant is removed by *normal ordering*. The difference is that now, since $a_{\vec{p}}^r a_{\vec{q}}^s = -a_{\vec{q}}^s a_{\vec{p}}^r$, $a_{\vec{p}}^{r\dagger} a_{\vec{q}}^{s\dagger} = -a_{\vec{q}}^{s\dagger} a_{\vec{p}}^{r\dagger}$ and similar for b's, we have

$$: a_{\vec{p}}^r a_{\vec{q}}^s := a_{\vec{p}}^r a_{\vec{q}}^s = -a_{\vec{q}}^s a_{\vec{p}}^r, \quad \text{etc.} \tag{12.48}$$

and so also

$$: a_{\vec{p}}^r a_{\vec{q}}^{s\dagger} := -a_{\vec{q}}^{s\dagger} a_{\vec{p}}^r \tag{12.49}$$

So the net effect of normal ordering is to remove the anticommutator terms, moving all the creation operators to the right with the anticommutation rules. Note that since the fermions

have a negative infinite constant, whereas the bosons have a positive infinite constant, the infinite constant can cancel in a theory with an equal number of bosonic and fermionic modes, which is called a *supersymmetric theory*. These supersymmetric theories therefore possess a better behaviour with respect to unphysical infinities.

Also, note that we can interpret $a_{\vec{p}}^{\dagger}$ as creating positive energy particles and $b_{\vec{p}}$ as creation operator for a negative energy particle, as the second term in the first line in (12.47) shows (compare with the first). Dirac introduced the concept of Dirac sea, which is that there is a full "sea" of occupied states of negative energy (remember that for fermions a state can only be occupied by a single particle). Then $b_{\vec{p}}^{\dagger}$ destroys a particle of negative energy, which is equivalent to creating a "hole" in the Dirac sea (unoccupied state in a sea of occupied states), which acts as a state of positive energy (consider the second line in (12.47)).

The fermionic path integral

How do we write a path integral? The path integral is used to describe quantum mechanics, but it involves classical objects (functions), just going over all possible paths instead of just the classical path. These classical objects are obtained in the classical limit of quantum operators, i.e. when $\hbar \to 0$. For bosons, for instance for the harmonic oscillators $[a, a^{\dagger}] = \hbar \to 0$, so in the path integral over the harmonic phase space we used the classical objects α and α^{*}, for the quantum objects a and $a^{\dagger}$.

But now, we have anticommutation relations, in particular for the fermionic harmonic oscillators, $\{\hat{a}, \hat{a}^{\dagger}\} = \hbar \to 0$. So in the classical limit for these operators we don't obtain the usual functions, but we obtain objects that anticommute, forming what is known as a *Grassmann algebra*. Considering the $a, a^{\dagger}$ of a fermionic harmonic oscillator, we have

$$\{a, a^{\dagger}\} = \{a, a\} = \{a^{\dagger}, a^{\dagger}\} = 0 \tag{12.50}$$

The objects $a, a^{\dagger}$ are known as the "odd" part of the algebra, since they anticommute, whereas products of odd objects are called "even", since they commute. For instance, we see that $[aa^{\dagger}, aa^{\dagger}] = 0$. Thus in general we have $bose \times bose = bose$, $fermi \times fermi = bose$ and $bose \times fermi = fermi$, where $bose$ stands for even and $fermi$ stands for odd.

However, what is the meaning of a "classical" Grassmann field if even in the classical $\hbar \to 0$ limit we can't put more than one particle in the same state: $\{a^{\dagger}, a^{\dagger}\} = 0 \Rightarrow a^{\dagger}a^{\dagger}|0> = 0$, where $a^{\dagger}$ is an object that appears in the expansion of the "classical" Grassmann field $\psi(x)$. The meaning is not clear, but this is one of those instances where in quantum mechanics there are questions which are meaningless. The point is that this is a formalism that works for the path integral (i.e., we obtain the right results, which agree with experiments and also with the operator formalism of quantum field theory), and the path integral is quantum mechanical in any case, even if it is written as an integral over "classical" Grassmann fields.

Definitions

Let's now define better the calculus with Grassmann fields. The general Grassmann algebra of N objects x_i, $i = 1, ..., N$ is $\{x_i, x_j\} = 0$, together with the element **1**, which commutes with the rest, $[x_i, \mathbf{1}] = 0$, and with complex number coefficients (i.e., it is an algebra over $\mathbb{C}$).

Since $(x_i)^2 = 0$, the Fourier expansion of a general function stops after a finite number

of terms, specifically after N terms:

$$F(\{x_i\}) = F(x_0) + \sum_i f_i^{(1)} x_i + \sum_{i<j} f_{ij}^{(2)} x_i x_j + \sum_{i<j<k} f_{ijk}^{(3)} x_i x_j x_k + ... + f_{12...N}^{(N)} x_1...x_N \quad (12.51)$$

Only for an infinite number of generators x_i is the Taylor expansion infinite in extent. Note that here, since we cannot add bosons (even objects) with fermions (odd objects), the functions must be either even, in which case there is always an even number of x's in the expansion, or odd, in which case there is an odd number of x's in the expansion.

However, we will often think of only a subset of the x's in making the expansion. For instance, we can expand in terms of a single x (even though there are more objects in the Grassmann algebra), and write for a general even function of x the expansion

$$f(x) = a + bx \quad (12.52)$$

where, since x is odd, a is even and b is odd (fermionic). This is possible, since there is at least one other Grassmann object, for instance called y, so we could have $b = cy$, i.e. $f(x) = a + cyx$.

In fact, we will often consider (at least) an even number of x's, half of which are used for the expansion, and half for the coefficients, allowing us to write the general expansion (12.51) with coefficients $f^{(2k)}$ being even (bosonic) but $f^{(2k+1)}$ being odd (fermionic). (This is what one does in the case of supersymmetry, where one expands a "superfield" Φ in terms of auxiliary θ's like the x's above, with odd coefficients which are still Grassmann functions corresponding to spinor fields.)

We define differentiation by first writing

$$\left(\frac{\partial}{\partial x_i} x_j \right) = \delta_{ij} \quad (12.53)$$

but since differential operators must also be Grassmann objects, we have

$$\frac{\partial}{\partial x_i}(x_j...) = \delta_{ij}(...) - x_j \frac{\partial}{\partial x_i}(...) \quad (12.54)$$

Note that for bosons we would have a plus sign in the second term, but now $\partial/\partial x_i$ anticommutes past x_j.

As an example, we have

$$\frac{\partial}{\partial x_i}(x_1 x_2...x_n) = \delta_{i1} x_2 x_3...x_n - \delta_{i2} x_1 x_3...x_n + \delta_{i3} x_1 x_2 x_4...x_n + ... + (-)^{n-1} \delta_{in} x_1 x_2...x_{n-1}$$
$$(12.55)$$

Then, for instance, differentiation of a product of even functions $f(x)$ and $g(x)$ is done in the usual way,

$$\frac{\partial}{\partial x_i}(f(x)g(x)) = \left(\frac{\partial}{\partial x_i} f(x) \right) g(x) + f(x) \left(\frac{\partial}{\partial x_i} g(x) \right) \quad (12.56)$$

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but if $f(x)$ is an odd function, we have the modified rule

$$\frac{\partial}{\partial x_i}(f(x)g(x)) = \left(\frac{\partial}{\partial x_i}f(x)\right)g(x) - f(x)\left(\frac{\partial}{\partial x_i}g(x)\right) \tag{12.57}$$

As another example, consider the function $e^{\sum_i x_i}$, which is defined in the usual way, just that now we substitute $(x_i)^2 = 0$ in the Taylor expansion of the exponential. Then we have

$$\frac{\partial}{\partial x_k}e^{\sum_i x_i y_i} = y_k e^{\sum_i x_i y_i}$$
$$\frac{\partial}{\partial y_k}e^{\sum_i x_i y_i} = -x_k e^{\sum_i x_i y_i} \tag{12.58}$$

Next, we define integration, which is not defined as a Riemann sum in the usual way for functions over complex numbers. Indeed, due to the Grassmann properties, we cannot, but rather, we must define it as a linear operator, and in particular we can only define the indefinite integral, not a definite one.

Since the two basic elements of the algebra with an element x is $\mathbf{1}$ and x, we must define what happens for these two elements. We define

$$\int dx\mathbf{1} = 0; \quad \int dx\, x = 1 \tag{12.59}$$

For several x_i's, we define the anticommutation relations for $i \neq j$,

$$\{dx_i, dx_j\} = 0; \quad \{x_i, dx_j\} = 0 \tag{12.60}$$

For instance, we have

$$\int dx_1 dx_2 x_1 x_2 = -\int dx_1 x_1 \int dx_2 x_2 = -1 \tag{12.61}$$

Then the integral operation is translational invariant, since

$$\int dx f(x+a) = \int dx[f^0 + f^1(x+a)] = \int dx f^1 x = \int dx f(x) \tag{12.62}$$

In view of the above relations, we see that *integration is the same as differentiation* (satisfies the same rules). For instance,

$$\int dx_2 x_1 x_2 x_3 = -\int dx_2 x_2 x_1 x_3 = -x_1 x_3 \tag{12.63}$$

the same as

$$\frac{\partial}{\partial x_2}x_1 x_2 x_3 = -\frac{\partial}{\partial x_2}x_2 x_1 x_3 = -x_1 x_3 \tag{12.64}$$

We now define the delta function on the Grassmann space. We have

$$\delta(x) = x \tag{12.65}$$

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To prove this, consider a general even function of x, $f(x) = f^0 + f^1 x$, where f^0 is even and f^1 is odd. Then

$$\int dx \delta(x - y) f(x) = \int dx (x - y)(f^0 + f^1 x) = \int dx (x f^0 - y f^1 x)$$

$$= f^0 + y \int dx f^1 x = f^0 - y f^1 = f^0 + f^1 y = f(y) \tag{12.66}$$

Finally, let us consider the change of variables from the Grassman variable x to the variable $y = ax$, where $a \in \mathbb{C}$. But then we must have

$$1 = \int dx x = \int dy y = a \int dy x \tag{12.67}$$

so it follows that

$$dy = \frac{1}{a} dx \tag{12.68}$$

which is not like the usual integration, but is rather like differentiation.

Important concepts to remember

- Spinor fields are representations of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and spinorial representations of the Lorentz algebra.

- In the Weyl (chiral) representation, γ_5 is diagonal and has $\pm\mathbf{1}$ on the diagonal.

- The Dirac action is $-\int \bar{\psi}(\slashed{\partial} + m)\psi$, with $\bar{\psi} = \psi^\dagger i \gamma^0$.

- The irreducible spinor representations are, in 4d Minkowski space, either Weyl spinors, or Majorana spinors. Weyl spinors satisfy $(\mathbf{1} \pm \gamma_5)/2\psi = 0$ and Majorana spinors satisfy $\bar{\psi} = \psi^T C$, with C the charge conjugation matrix.

- The Dirac operator is a kind of square root of the KG operator, since $(\slashed{\partial} - m)(\slashed{\partial} + m) = \Box - m^2$.

- The solutions of the Dirac equation are $u(p)e^{ip \cdot x}$ and $v(p)e^{-ip \cdot x}$, with $u(p)$ and $v(p)$ are orthornormal, and in canonical quantization we expand in these solutions, with fermionic harmonic oscillator coefficients.

- The momentum conjugate to ψ is $i\psi^\dagger$, giving the nontrivial anticommutator $\{\psi_\alpha(\vec{x}, t), \psi_\beta^\dagger(\vec{y}, t)\} = \delta(\vec{x} - \vec{y})\delta_{\alpha\beta}$.

- The infinite constant (zero point energy) in the Hamiltonian is negative.

- We can interpret as: $a^\dagger$ creates positive energy states, b creates negative energy states, and $b^\dagger$ destroys negative energy states, thus effectively creating "holes" in the Dirac sea of positive energy.

- For the fermionic path integral, we use Grassmann algebra-valued objects.

- The Taylor expansion of a function of Grassmann-algebra objects ends at order N, the number of x_i's.

- Grassmann differentiation is also anticommuting with others and with x's.

- Grassmann integration is the same as Grassmann differentiation and $\delta(x) = x$ for Grassmann variables.

Further reading: See chapters 3.1.1, 3.1.2, 3.3 in [4], 7.1 in [1], 3.2 and 3.3 in [2] and 5.2 and 5.3 in [3].

Exercises, Lecture 12

1) Consider the Rarita-Schwinger action for a vector-spinor field $\psi_{\mu\alpha}$ in Minkowski space,

$$S = -\frac{1}{2}\int d^4x\,\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho \tag{12.69}$$

where $(\psi_\mu)_\alpha$ is a *Majorana* spinor and $\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$. Calculate the variation δS under a variation $\delta\psi$, and then the equations of motion, in x and p space.

2) Consider the Lagrangean for a *Dirac* field in Minkowski space,

$$S = -\int d^4x[\bar{\psi}(\slashed{\partial} + m)\psi + \alpha(\bar{\psi}\psi)^2] \tag{12.70}$$

and use the free field quantization in the lecture. Compute the quantum Hamiltonian in terms of $a, a^\dagger, b, b^\dagger$ and then the normal ordered Hamiltonian.

13 Lecture 13. Wick theorem, gaussian integration and Feynman rules for fermions

Last lecture we saw that the path integral for fermions is done using integrals of objects in a Grassmann algebra. We defined the calculus on the Grassmann algebra, defining an anticommuting differential operator, and an integral operator (defined as a linear operation, not a Riemann sum) which turned out to have the same properties as the differential operator.

We then saw that changing variables in the Grassmann algebra, between x and $y = ax$, where a is a c-number, give us $dy = dx/a$, like for the differential operator. The obvious generalization to a n-dimensional Grassmann algebra is: $y = A \cdot x$ implies $d^n y = d^n x / \det A$.

Gaussian integration - the real case.

We now define Gaussian integration over a Grassmann algebra over real numbers.

Theorem 1

Consider a real $n \times n$ antisymmetric matrix $\{A_{ij}\}$, such that the (symmetric) matrix A^2 has negative, nonzero eigenvalues. Then $n = 2m$, and for $x_1, ..., x_n$ Grassmann variables,

$$\int d^n x e^{x^T A x} = 2^m \sqrt{\det A} \tag{13.1}$$

Proof:

Consider first the case $m = 1$, i.e. $n = 2$. Then

$$A = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} -\lambda^2 & 0 \\ 0 & -\lambda^2 \end{pmatrix} \tag{13.2}$$

and $\det A = \lambda^2 > 0$, whereas $x^T A x = \lambda(x_1 x_2 - x_2 x_1) = 2\lambda x_1 x_2$. Since $(x_i)^2 = 0$, we then have

$$e^{x^T A x} = 1 + 2\lambda x_1 x_2 \Rightarrow \int d^2 x e^{x^T A x} = \int dx_2 dx_1 [1 + 2\lambda x_1 x_2] = 2\lambda \tag{13.3}$$

Note that here we had to choose a specific order for the integrals in $d^2 x$, namely $dx_2 dx_1$, in order to get a plus sign.

Next, we consider a version of the

Jordan lemma for the Grassmann algebra, namely that there is a transformation matrix B (real, orthogonal) acting on the Grassmann algebra space such that $B^T A B$ is block-diagonal in Jordan blocks, i.e.

$$B^T A B = \begin{pmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & & \\ & & 0 & \lambda_2 & & & \\ & & -\lambda_2 & 0 & & & \\ & & & & ... & & \\ & & & & & 0 & \lambda_m \\ & & & & & -\lambda_m & 0 \end{pmatrix} \tag{13.4}$$

which gives then $\det A = \det(B^T A B) = \lambda_1^2 ... \lambda_m^2$.

Proof of lemma:

We need to show that there is a basis of vectors $\{\vec{e}_1, \vec{e}_{-1}, \vec{e}_2, \vec{e}_{-2}, ..., \vec{e}_m, \vec{e}_{-m}\}$ such that

$$A\vec{e}_k = \lambda_k \vec{e}_k; \quad A\vec{e}_{-k} = -\lambda_k \vec{e}_k \tag{13.5}$$

Then we have $A^2 \vec{e}_{\pm k} = -\lambda_k^2 \vec{e}_{\pm k}$. We see then that it is sufficient to take a basis of eigenvectors of A^2, since we already know A^2 has nonzero negative eigenvalues. We need to check that there is a two-fold degeneracy for the basis, which is true, since if $\vec{e}$ has eigenvalue $-\lambda^2$, so does $\vec{e}' = A\vec{e}/\lambda$, as we can easily check: $A^2\vec{e}' = -\lambda A\vec{e} = -\lambda^2 \vec{e}'$. The basis of vectors is linearly independent, since if there would be a number f such that $\vec{e}' = f\vec{e}$, on one hand, we could write what $\vec{e}'$ is and obtain $A\vec{e} = \lambda f\vec{e}$, and applying twice we obtain $A^2\vec{e} = \lambda^2 f^2 \vec{e}$. On the other hand, we know that $A^2\vec{e} = -\lambda^2 \vec{e}$, but since f is real, $f^2 > 0$, giving a contradiction. Therefore $\vec{e}$ and $\vec{e}'$ are linearly independent, and we have found the Jordan basis for the matrix A. Finally, $\vec{e} \cdot \vec{e}' = \frac{1}{\lambda}\vec{e}A\vec{e} = 0$, since A is antisymmetric, hence the eigenvectors are also orthogonal. q.e.d. lemma.

Then we can write

$$\int d^n x e^{x^T A x} = \int d^n x' e^{(B^T x)^T B^T A B (B^T x)} = \prod_{i=1}^m [m = 1 \ \text{case}]_i = 2^m \sqrt{\det A} \tag{13.6}$$

where $x' = B^T x$. q.e.d. theorem 1.

Theorem 2. Complex gaussian integration.

We write independent complex variables x_i, y_i, without putting $y_i = x_i^*$, and an $n \times n$ antisymmetric matrix A, and then we have

$$\int d^n x d^n y e^{y^T A x} = \det A \tag{13.7}$$

We consider this since in the path integral we have something exactly of this form, for instance $\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\bar{\psi}(\slashed{\partial}+m)\psi}$.

Proof:

In $e^{y^T A x}$, the terms that are nonzero under the integration are only terms of order n (since $(x_i)^2 = 0$, so for $n + 1$ terms necessarily one of the x's is repeated, and for $< n$ terms we get some $\int dx_i 1 = 0$). Specifically, these terms will be:

$$\frac{1}{n!} \sum_P \sum_Q (y_{P(1)} A_{P(1)Q(1)} x_{Q(1)})...(y_{P(n)} A_{P(n)Q(n)} x_{Q(n)})$$

$$= \frac{1}{n!} \sum_P \sum_Q (y_1 A_{1,QP^{-1}(q)} x_{QP^{-1}(1)})...(y_n A_{n,QP^{-1}(n)} x_{QP^{-1}(n)})$$

$$= 1 \times \sum_{Q'} (y_1 A_{1,Q'(q)} x_{Q'(1)})...(y_n A_{n,Q'(n)} x_{Q'(n)})$$

$$= \epsilon(y_1...y_n)(x_1...x_n) \sum_Q \epsilon_Q A_{1Q(1)}...A_{nQ(n)}$$

$$= \epsilon(y_1...y_n)(x_1...x_n) \det A \tag{13.8}$$

where in the first line we wrote the sums over permutations P, Q of $1, 2, ..., n$, in the second we have rewritten the sum over permutations, such that the $\sum_P$ is done trivially as a factor of $n!$, then we commuted the x's and the y's, obtaining $y_1 x_1 y_2 x_2 ... y_n x_n = \epsilon y_1 ... y_n x_1 ... x_n$, with ϵ a sign that depends on n, but not on Q. We can fix the sign of the integral $d^n x d^n y (y_1 ... y_n)(x_1 ... x_n)$ to be ϵ by properly defining the order of dx's and dy's, in which case we obtain just $\det A$ for the gaussian integral. q.e.d. theorem 2.

Real vs. complex integration

We can also check that complex Gaussian integration is consistent with the real integration case. If the complex objects x_i and y_i are written as $x_i = a_i + ib_i$ and $y_i = a_i - ib_i$, then $y^T A x = a^T A a + b^T A b$. The Jacobian of the transformation $d^n x d^n y = J d^n a d^n b$ is $J = 2^{-m}$ (for usual complex numbers we can check that it would be 2^m, and the Grassmann integral works as the inverse). Then the result of the integral done as the double real integral over $d^n a d^n b$ is $J 2^m (\sqrt{\det A})^2 = \det A$, as we want.

The fermionic harmonic oscillator

For the fermionic harmonic oscillator, the quantum hamiltonian is

$$\hat{H}_F = \omega \left(\hat{b}^\dagger \hat{b} - \frac{1}{2} \right) \tag{13.9}$$

We consider then as in the bosonic case coherent states $|\beta>$ defined by (β is a Grassmann valued number)

$$|\beta> = e^{\hat{b}^\dagger \beta}|0> = (1 + \hat{b}^\dagger \beta)|0> = (1 - \beta \hat{b}^\dagger)|0> \Rightarrow \hat{b}|\beta> = \beta|0> = \beta(1 - \beta \hat{b}^\dagger)|0> = \beta|\beta> \tag{13.10}$$

that satisfy the completeness relation

$$\mathbf{1} = \int d\beta^* d\beta |\beta> < \beta^*| e^{-\beta^* \beta} \tag{13.11}$$

Then consider the Hamiltonian with currents

$$H(b^\dagger, b; t) = \omega b^\dagger b - b^\dagger \eta(t) - \bar{\eta}(t) b \tag{13.12}$$

and define as in the bosonic case the transition amplitude

$$F(\beta^*, t'; \beta, t) = <\beta^*, t'|\beta, t> \tag{13.13}$$

for which we do the same steps as for the bosonic case and find the path integral

$$F(\beta^*, t'; \beta, t) = \int \mathcal{D}\beta^* \mathcal{D}\beta \exp \left\{ i \int_t^{t'} d\tau [-i\dot{\beta}^*(\tau)\beta(\tau) - H] + \beta^*(t)\beta(t) \right\} \tag{13.14}$$

The equations of motion of the Hamiltonian with sources are

$$\dot{\beta}^* - i\omega\beta^* + i\bar{\eta} = 0$$
$$\dot{\beta} + i\omega\beta - i\eta = 0 \tag{13.15}$$

with solutions

$$\begin{aligned}
\beta(\tau) &= \beta e^{i\omega(t-\tau)} + i \int_t^\tau e^{i\omega(s-\tau)}\eta(s)ds \\
\beta^*(\tau) &= \beta^* e^{i\omega(\tau-t')} + i \int_\tau^{t'} e^{i\omega(\tau-s)}\bar{\eta}(s)ds
\end{aligned} \tag{13.16}$$

For the case $\beta = \beta^* = 0$, and $t \to -\infty$ and $t' \to +\infty$ we obtain the vacuum functional (vacuum expectation value).

In the usual manner, we obtain

$$Z[\eta, \bar{\eta}] = <0|0> \exp\left\{ -\int_{-\infty}^{+\infty} d\tau \int_\tau^{+\infty} ds\, e^{i\omega(\tau-s)}\bar{\eta}(s)\eta(\tau) \right\} \tag{13.17}$$

The details are left as an exercise. Renaming $\beta \to \psi$ and $\beta^* \to \bar{\psi}$ for easy generalization to the field theory case, and writing $\int \dot{\beta}^*\beta \to \int \partial_t\bar{\psi}\psi = -\int \bar{\psi}\partial_t\psi$, the path integral becomes

$$\begin{aligned}
Z[\eta, \bar{\eta}] &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ i\int dt[\bar{\psi}(i\partial_t - \omega)\psi + \bar{\psi}\eta + \bar{\eta}\psi] \right\} \\
&= Z[0,0] \exp\left\{ -\int ds d\tau \bar{\eta}(s)D_F(s,\tau)\eta(\tau) \right\}
\end{aligned} \tag{13.18}$$

Here we have defined the Feynman propagator such that (as usual) $iS^M = -\bar{\psi}\Delta^{-1}\psi$, i.e.

$$D_F(s,\tau) = (-i(i\partial_t - \omega))^{-1}(s,\tau) = i\int \frac{dE}{2\pi} \frac{e^{-iE(s-\tau)}}{E - \omega + i\epsilon} = \theta(s-\tau)e^{-i\omega(s-\tau)} \tag{13.19}$$

where we have first written the Fourier transform, then we have done the complex integral with the residue theorem: there is now a single pole at $E = \omega - i\epsilon$. We get a nonzero result only if we close the contour below (in the lower half plane), which we can do only if $s - \tau > 0$, so that $-iE(s-\tau) = -|\text{Im}E|(s-\tau) + \text{imaginary}$, otherwise the result is zero. Substituting this $D_F(s,\tau)$ we indeed obtain (13.17).

In the Euclidean case, we write $t_M = -it_E$ as usual, and substituting in $Z[\eta, \bar{\eta}]$ we obtain

$$\begin{aligned}
Z_E[\eta, \bar{\eta}] &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[-\int dt_E \bar{\psi}\left(\frac{\partial}{\partial t_E} + \omega\right)\psi + \int dt_E(\bar{\psi}\eta + \bar{\eta}\psi) \right] \\
&= Z[0,0] \exp\left\{ \int d\tau ds \bar{\eta}(s)D(s,\tau)\eta(\tau) \right\}
\end{aligned} \tag{13.20}$$

where we define $-S_E = -\bar{\psi}D^{-1}\psi + ...$, i.e.

$$D(s,\tau) = (\partial_{\tau_E} + \omega)^{-1} = i\int \frac{dE}{2\pi} \frac{e^{-iE(s-\tau)}}{E + i\omega} \tag{13.21}$$

and as usual this Euclidean propagator is well defined.

The Wick rotation to Minkowski space also proceeds as usual, namely $\tau_E \to i\tau_M$ and in order to have the same $E\tau$ product, $E_E = (-i + \epsilon)E_M$, where we only rotate with $\pi/2 - \epsilon$ to avoid crossing the Minkowski space pole. Then we obtain the Feynman propagator as usual,

$$D(is_M, i\tau_M) = i \int \frac{dE_M}{2\pi} \frac{e^{-iE_M(s_M - \tau_M)}}{E_M - \omega + i\epsilon'} = D_F(s_M, \tau_M) \tag{13.22}$$

We are now ready to generalize to field theory. The Minkowski space Lagrangean is

$$\mathcal{L}_F^{(M)} = -\bar{\psi}(\slashed{\partial} + m)\psi \tag{13.23}$$

The Euclidean action is defined by the Wick rotation $t_E = it_M$, with $iS^M = -S^E$ as usual, with $\int dt_M = -i \int dt_E$, thus giving

$$\mathcal{L}_F^{(E)} = +\bar{\psi}(\slashed{\partial} + m)\psi \tag{13.24}$$

Note however that since the gamma matrices must satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, in the Minkowski case we had $(\gamma^0)^2 = -1$, but in the Euclidean case we have $(\gamma^4)^2 = +1$, which is satisfied by choosing the Wick rotation $\gamma^4 = i\gamma^0$ (same rotation as for t), and as a result we also write

$$\bar{\psi} = \psi^\dagger i\gamma^0 = \psi^\dagger \gamma^4 \tag{13.25}$$

Also, in Euclidean space with $\gamma^4 = i\gamma^0$ we can check that $\gamma^\mu = \gamma_\mu = (\gamma^\mu)^\dagger$.

Then the free fermionic partition function is

$$\begin{aligned} Z_F^{(0)}[\bar{\eta}, \eta] &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ -\int d^4x \bar{\psi}(\slashed{\partial} + m)\psi + \int d^4x(\bar{\eta}\psi + \bar{\psi}\eta) \right\} \\ &= Z[0,0]e^{\bar{\eta}(\slashed{\partial} + m)^{-1}\eta} \end{aligned} \tag{13.26}$$

The euclidean propagator is defined by $-S_E = -\bar{\psi}\Delta^{-1}\psi$, giving

$$S_F(x,y) = (\slashed{\partial} + m)^{-1} = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{-\slashed{p} + im} \tag{13.27}$$

Note that then

$$(\slashed{\partial} + m)_x S_F = \int \frac{d^4p}{(2\pi)^4}(i\slashed{p} + m)e^{ip\cdot(x-y)} \frac{i}{-\slashed{p} + im} = \delta^4(x-y) \tag{13.28}$$

and note also that

$$\frac{1}{-\slashed{p} + im} = \frac{-\slashed{p} - im}{p^2 + m^2} \tag{13.29}$$

The relation to Minkowski space operators is also as in the bosonic case, namely consider the general $N + M$ point function

$$< 0|T\{\hat{\psi}_{\alpha_1}(x_1)...\hat{\psi}_{\alpha_N}(x_N)\hat{\bar{\psi}}_{\beta_1}(y_1)...\hat{\bar{\psi}}_{\beta_M}(y_M)\}|0 >$$
$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS_F^{(M)}}\{\psi_{\alpha_1}(x_1)...\psi_{\alpha_N}(x_N)\bar{\psi}_{\beta_1}(y_1)...\bar{\psi}_{\beta_N}(y_N)\} \tag{13.30}$$

In particular, the propagator in the free theory is

$$S_F^{(M)}(x-y) = <0|T\{\psi(x)\bar\psi(y)\}|0> = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{-i\slashed{p}-m+i\epsilon} = \int \frac{d^4p}{(2\pi)^4}\frac{-\slashed{p}-im}{p^2+m^2-i\epsilon}e^{ip\cdot(x-y)}$$

(13.31)

As usual, by Wick rotation $(t_E = it_M, E_E = (-i+\epsilon)E_M)$, we get

$$S_F^{(E)}(it_M - it'_M; \vec{x}-\vec{y}) = iS_F^{(M)}(x-y)$$

(13.32)

The **Wick theorem for Fermi fields**, for instance for some interaction with a scalar ϕ with source J and interaction term $S_I[\bar\psi, \psi, \phi]$, is

$$Z[\bar\eta, \eta, J] = e^{-S_I\left(-\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar\eta}, \frac{\delta}{\delta J}\right)}Z_F^{(0)}[\bar\eta, \eta]Z_\phi^{(0)}[J]$$

(13.33)

Note that the only difference from the bosonic case is the minus sign for $\delta/\delta\eta$, due to the fact that we have the source term $\int d^4x[\bar\eta\psi + \bar\psi\eta + \phi J]$, so we need to commute the $\delta/\delta\eta$ past $\bar\psi$ to act on η.

Coleman's lemma for fermions is also the same, except for the same minus sign, i.e.

$$F\left(-\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar\eta}\right)Z[\bar\eta, \eta] = Z\left[-\frac{\delta}{\delta\psi}, \frac{\delta}{\delta\bar\psi}\right]\left(F(\bar\psi, \psi)e^{\bar\psi\eta+\bar\eta\psi}\right)|_{\bar\psi=\psi=0}$$

(13.34)

Then applying it to the first form of the Wick theorem we get the second form of the Wick theorem,

$$Z[\bar\eta, \eta, J] = e^{-S_I\left(-\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar\eta}, \frac{\delta}{\delta J}\right)}e^{\bar\eta S_F\eta}Z_\phi^{(0)}[J] = e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar\psi}}e^{-S_I\left(\bar\psi, \psi, \frac{\delta}{\delta J}\right)+\int(\bar\psi\eta+\bar\eta\psi)}Z_\phi^{(0)}[J]|_{\psi=\bar\psi=\phi=0}$$

(13.35)

Feynman rules for Yukawa interaction

We are now ready to write down the Feynman rules for the Yukawa interaction between two fermions and a scalar,

$$\mathcal{L}_Y = g\bar\psi\psi\phi$$

(13.36)

Then the second form of the Wick theorem is

$$Z[\bar\eta, \eta, J] = e^{\frac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}}e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar\psi}}e^{-g\int d^4x\bar\psi\psi\phi+\int d^4x[\bar\psi\eta+\bar\eta\psi+J\phi]}|_{\psi=\bar\psi=\phi=0}$$

(13.37)

For the free two point function, we must take two derivatives, and consider only the order $1 = g^0$ term, i.e.

$$<0|\psi(x)\bar\psi(y)|0> = \frac{\delta}{\delta\bar\eta(x)}\left(-\frac{\delta}{\delta\eta(y)}\right)Z[\bar\eta, \eta, J]|_{\eta=\bar\eta=J=0}$$
$$= -\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar\psi}\{\psi(x)\bar\psi(y)\}|_{\psi=\bar\psi=\phi=0} = S_F(x-y)$$

(13.38)

Therefore the first Feynman rule is that for propagator we have a $S_F(x-y)$, represented by a solid line with an arrow between y ($\bar\psi(y)$) and x ($\psi(x)$), see Fig.32. Note that now,
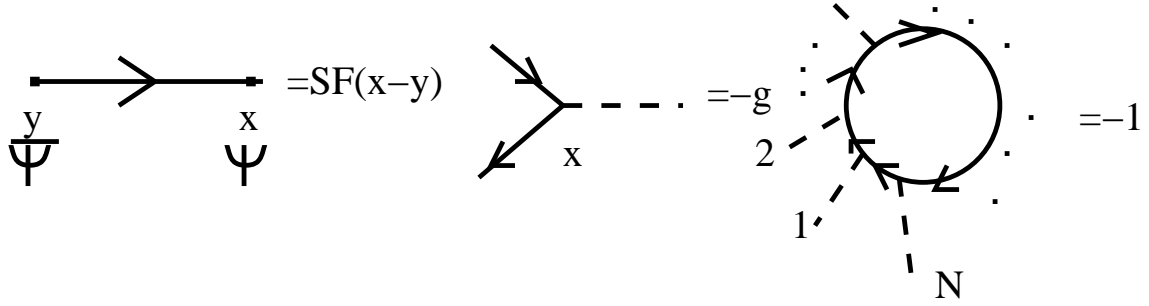
Figure 32: x-space Feynman rules for Yukawa interaction.

because fermions anticommute, the order matters, and if we interchange two fermions, for instance these ψ and $\bar{\psi}$ (interchange the order of the arrow), then we have a minus sign.

A scalar is now represented by a dotted line without an arrow on it.

Also, the vertex is standard. Since all the fields in the interaction term are different (no field is repeated), we have just a factor of $-g$ for the vertex (in the pure scalar case, for a $\phi^n/n!$ interaction we have a factor of $n!$ in front, which comes from permuting various ϕ's, but now we don't have this).

The interchange of two fermions means also the only real difference from the bosonic rules. If we have a fermion loops with N lines sticking out of it (in this case, ϕ lines), it means that we must act with $-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}$ on factors of $\psi\bar{\psi}$, obtaining S_F as above. But the factors of $\psi\bar{\psi}$ come from expanding the $e^{-g\int\bar{\psi}\psi\phi}$, and since the $-g$ is part of vertex, the important part is

$$\left(-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}\right)...\left(-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}\right)[(\bar{\psi}\psi)...(\bar{\psi}\psi)] \tag{13.39}$$

But in order to form the $\psi\bar{\psi}$ combinations giving us the S_F propagators, which by our assumption are cyclically linked (form a loop) as $S_F(x_N - x_1)S_F(x_1 - x_2)...S_F(x_{N-1} - x_N)$, we must commute the last $\psi(x_N)$ past the other $2(N-1)+1$ fermions, to put it in front, giving a minus sign.

Therefore *a fermion loop gives a minus sign.*

The **p-space Feynman rules** are then

-the scalar propagator is as usual

$$\frac{1}{p^2 + M^2} \tag{13.40}$$

-the fermion propagator is

$$\frac{1}{i\not{p} + m} = \frac{i}{-\not{p} + im} = \frac{-i\not{p} + m}{p^2 + m^2} \tag{13.41}$$

-the vertex between two fermions and a scalar is $-g$.

-there is a minus sign for a fermion loop, and for interchanging a ψ with a $\bar{\psi}$.

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Then for a **general interaction between fermions and a number of scalars**, with interaction term

$$S_I(\bar{\psi}, \psi, \phi_1, ..., \phi_n) \tag{13.42}$$

the vertex is found exactly as in the pure bosonic case, with the exception of the usual minus sign in the functional derivative for fermions, namely

$$\int d^4x d^4y d^4z_1...d^4z_n e^{-i(px+p'y+q_1z_1+...q_nz_n)} \times$$

$$\times \frac{\delta}{\delta\phi_1(z_1)}...\frac{\delta}{\delta\phi_n(z_n)} \left(-\frac{\delta}{\delta\bar{\psi}(y)}\frac{\delta}{\delta\psi(x)} \right) \{-S_I[\bar{\psi}, \psi, \phi_1, ..., \phi_n]\} \tag{13.43}$$
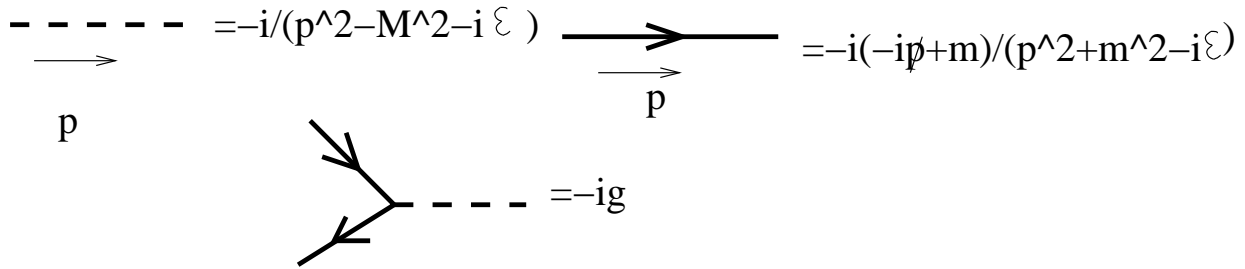
Figure 33: p-space Feynman rules in Minkowski space.

The Minkowski space Feynman rules in p-space are (see Fig.33)
-The scalar propagator is

$$\frac{-i}{p^2 + M^2 - i\epsilon} \tag{13.44}$$

-The fermion propagator

$$-i\frac{-i\not{p} + m}{p^2 + m^2 - i\epsilon} \tag{13.45}$$

-The vertex is $-ig$.
-There is a minus sign for a fermion loop or for changing a ψ with $\bar{\psi}$.

Important concepts to remember

- Real gaussian integration over n Grassmann variables gives $2^{n/2}\sqrt{\det A}$, and complex gaussian integration, $\int d^n x d^n y e^{y^T Ax}$, gives $\det A$.

- For the fermionic harmonic oscillator, the vacuum functional (vacuum expectation value) gives the same expression as in the bosonic case, just in terms of the fermion propagator $D_F(s, \tau)$.

- Again the Euclidean propagator is Wick rotated to the Minkowski space propagator.

- The generalization to field theory contains no new facts.

- We can now have $N + M$ point functions of ψ's and $\bar{\psi}$'s, again given by the usual path integrals.

- In the Wick theorem for fermions the only new thing is that we replace $\bar{\psi}$ by $-\delta/\delta\eta$, due to the anticommuting nature of the sources and fields.

- In the Feynman rules for Yukawa theory, the only new thing is the fact that for a fermion loop we have a minus sign, as we have for interchanging a ψ with a $\bar{\psi}$, or equivalently, the direction of the arrow. The vertex factor for $-g \int \bar{\psi}\psi\phi$ is $-g$.

Further reading: See chapters 3.1.3, 3.1.4, 3.3.2, 3.3.3 in [4], 7.2 in [1], 5.3 in [3] and 4.7 in [2].

Exercises, Lecture 13

1) a) Consider θ^α, $\alpha = 1, 2$ Grassmann variables and the even function

$$\Phi(x, \theta) = \phi(x) + \sqrt{2}\theta^\alpha \psi_\alpha(x) + \theta^2 F(x) \tag{13.46}$$

where $\theta^2 \equiv \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta$. Calculate

$$\int d^2\theta(a_1\Phi + a_2\Phi^2 + a_3\Phi^3), \tag{13.47}$$

where

$$d^2\theta = -\frac{1}{4}\int d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \tag{13.48}$$

and $x, a_1, a_2, a_3 \in \mathbb{R}$.

b) Fill in the details missed in the text for the calculation of

$$Z[\bar{\eta}, \eta] = Z[0, 0] \exp\left\{-i\int d\tau ds\bar{\eta}(s)D_F(s, \tau)\eta(\tau)\right\} \tag{13.49}$$

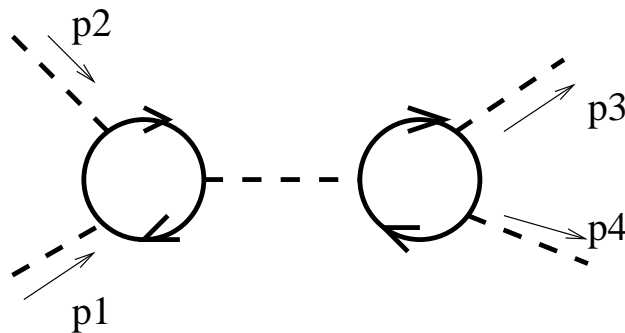2) Write down an expression for the Feynman diagram in Yukawa theory in Fig.34.

Figure 34: Feynman diagram in Yukawa theory.

Note: there are no external points on the p_i lines.

14 Lecture 14. Spin sums, Dirac field bilinears and C,P,T symmetries for fermions

In the previous 2 lectures we have defined the quantization, path integrals and Feynman rules for fermions. This lecture is dedicated to some technical details for fermions that we will use later on.

Often when doing Feynman diagrams we will need to do sums over polarizations of fermions. For instance, if we do an experiment where we can't measure the spin of external particles, we should sum over this spin. It is therefore important to compute the relevant spin sums.

The first one is

$$
\begin{aligned}
\sum_{s=1,2} u^s(p)\bar{u}^s(p) &= \sum_s \begin{pmatrix} \sqrt{-p\cdot\sigma}\xi^s \\ \sqrt{-p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{-p\cdot\bar{\sigma}} & \xi^{s\dagger}\sqrt{-p\cdot\sigma} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{-p\cdot\sigma}\sqrt{-p\cdot\bar{\sigma}} & \sqrt{-p\cdot\sigma}\sqrt{-p\cdot\sigma} \\ \sqrt{-p\cdot\bar{\sigma}}\sqrt{-p\cdot\bar{\sigma}} & \sqrt{-p\cdot\bar{\sigma}}\sqrt{-p\cdot\sigma} \end{pmatrix} \\
&= \begin{pmatrix} m & -p\cdot\sigma \\ -p\cdot\bar{\sigma} & m \end{pmatrix} = -i\slashed{p} + m
\end{aligned}
\tag{14.1}
$$

where we have used $\bar{u} = u^\dagger i\gamma^0$, $\sum_{s=1,2}\xi^s\xi^{s\dagger} = \mathbf{1}$ and $\{\sigma_i,\sigma_j\} = 2\delta_{ij}$ implying $\{\sigma^{(\mu},\bar{\sigma}^{\nu)}\} = -2\eta^{\mu\nu}$, and so

$$
(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_\mu p_\nu \frac{1}{2}\{\sigma^\mu,\bar{\sigma}^\nu\} = -p^2 = m^2
\tag{14.2}
$$

The second sum is over v's, giving similarly

$$
\sum_s v^s(p)\bar{v}^s(p) = -i\slashed{p} - m
\tag{14.3}
$$

Dirac field bilinears

A single fermion is classically unobservable (it is a Grassmann object, which cannot be measured experimentally). Classically observable objects are fermion bilinears, which are commuting variables, and can have a vacuum expectation value (VEV) that one can measure experimentally. Even in the quantum theory, the fermion bilinears are objects that appear very often, so it is important to understand their properties, in particular their properties under Lorentz transformations.

We have defined $\bar{\psi}$ by the condition that $\bar{\psi}\psi$ is a Lorentz scalar, so that is given. The spinorial representations of spin $1/2$ are the fundamental, or 1/2, to which ψ belongs, and the conjugate representation, or $\overline{1/2}$, to which $\bar{\psi}$ belongs, such that $\bar{\psi}\psi$ is a scalar. The Lorentz transformation of ψ is

$$
x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow \psi'^\alpha = S^\alpha{}_\beta(\Lambda)\psi^\beta
\tag{14.4}
$$

The next bilinear is $\bar{\psi}\gamma^\mu\psi$. γ^μ is a vector under Lorentz transformations, meaning that

$$
S^\alpha{}_\gamma(\gamma^\mu)^\gamma{}_\delta(S^{-1})^\delta{}_\beta = \Lambda^\mu{}_\nu(\gamma^\nu)^\alpha{}_\beta
\tag{14.5}
$$

and therefore $\bar{\psi}\gamma^\mu\psi$ is a vector, i.e.

$$\bar{\psi}\gamma^\mu\psi \to \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi \tag{14.6}$$

What other bilinears can we form? The most general would be $\bar{\psi}\Gamma\psi$, where Γ is a general 4×4 matrix, which can be decomposed in a basis of 16 4×4 matrices. Such a basis is given by

$$\mathcal{O}_i = \{\mathbf{1}, \gamma^\mu, \gamma_5, \gamma^\mu\gamma_5, \gamma^{\mu\nu} = \frac{1}{2}\gamma^{[\mu}\gamma^{\nu]}\} \tag{14.7}$$

which is a total number of $1 + 4 + 1 + 4 + 4\cdot 3/2 = 16$ matrices. Then this basis is normalized as

$$\text{Tr}\{\mathcal{O}^i\mathcal{O}_j\} = 4\delta^i_j \tag{14.8}$$

since for instance $\text{Tr}\{\mathbf{1}\times\mathbf{1}\} = 4$ and

$$\text{Tr}\{\gamma^\mu\gamma_\nu\} = \frac{1}{2}\text{Tr}\{\gamma^\mu, \gamma_\nu\} = 4\delta^\mu_\nu \tag{14.9}$$

Moreover, this basis is complete, satisfying the completeness relation

$$\delta^\beta_\alpha\delta^\delta_\gamma = \frac{1}{4}(\mathcal{O}_i)^\delta_\alpha(\mathcal{O}_i)^\beta_\gamma \tag{14.10}$$

which is indeed a completeness relation, since by multiplying by M^γ_β we get

$$M^\delta{}_\alpha = \frac{1}{4}\text{Tr}(M\mathcal{O}_i)(\mathcal{O}_i)^\delta{}_\alpha \tag{14.11}$$

which is indeed the expansion in a complete basis. The coefficient is correct, since by multiplying again with $\mathcal{O}_{j\delta}^\alpha$ we get an identity.

Now by multiplying (14.10) by arbitrary spinors $\chi_\beta, \bar{\psi}^\gamma$ and $N\phi_\delta$, where N is a matrix, we obtain the relation (after multiplying it by yet another matrix M)

$$M\chi(\bar{\psi}N\phi) = -\frac{1}{4}\sum_j M\mathcal{O}_j N\phi(\bar{\psi}\mathcal{O}_j\chi) \tag{14.12}$$

which is called *Fierz identity* or *Fierz recoupling*. Note the minus sign which appears because we interchanged the order of the fermions. The Fierz identity is useful for simplifying fermionic terms, allowing us to "recouple" the fermions and hopefully obtain something simpler.

We might think that we could introduce other Lorentz structures made up from gamma matrices, but for instance

$$\gamma_{[\mu\nu\rho\sigma]} = \gamma_{[\mu}\gamma_\nu\gamma_\rho\gamma_{\sigma]} \propto \epsilon_{\mu\nu\rho\sigma}\gamma_5 \tag{14.13}$$

since $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, and the gamma matrix products are antisymmetric by construction, since gamma matrices anticommute. Then we also have

$$\gamma_{[\mu\nu\rho]} \propto \epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 \tag{14.14}$$

and there are no higher antisymmetric bilinears, since the antisymmetric products of 5 γ_μ is zero.

As terminology, $\mathbf{1}$ is a scalar, γ_5 is a pseudoscalar, γ^μ is a vector and $\gamma^\mu\gamma_5$ is a pseudovector, and $\gamma^{\mu\nu}$ is an antisymmetric tensor. Here for instance pseudoscalar means that it transforms like a scalar under usual Lorentz transformations, except under parity gets an extra minus. Similarly, pseudo-something means it transforms usually under Lorentz transformations, except under parity gets an extra minus sign.

If $\psi(x)$ satisfies the Dirac equation, then the *vector current* $j^\mu = \bar\psi(x)\gamma^\mu\psi(x)$ is conserved:

$$\partial_\mu j^\mu = (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu\partial_\mu\psi = (m\bar\psi)\psi + \bar\psi(-m\psi) = 0 \tag{14.15}$$

and the *axial vector current* $j^{\mu 5} = \bar\psi(x)\gamma^\mu\gamma_5\psi(x)$

$$\partial_\mu j^{\mu 5} = (\partial_\mu\bar\psi)\gamma^\mu\gamma_5\psi + \bar\psi\gamma^\mu\gamma_5\partial_\mu\psi = m\bar\psi\gamma_5\psi - \bar\psi\gamma_5\slashed\partial\psi = 2m\bar\psi\gamma_5\psi \tag{14.16}$$

is conserved $\partial_\mu j^{\mu 5} = 0$ if $m = 0$ only.

C,P,T symmetries for fermions

The continuous Lorentz group $L_+^\uparrow$, the proper orthochronous group, continuously connected with the identity, is a symmetry. But we have also the parity and time reversal transformations

$$\begin{array}{rcl} P &:& (t, \vec{x}) \to (t, -\vec{x}) \\ T &:& (t, \vec{x}) \to (-t, \vec{x}) \end{array} \tag{14.17}$$

which can also be symmetries, but not necessarily (there is no physical principle which demands it).

Besides $L_+^\uparrow$, there are 3 other disconnected groups, $L_-^\uparrow$, $L_+^\downarrow$ and $L_+^\downarrow$. L_+ is called proper group, and L_- improper, and the two are related by P, whereas $L^\uparrow$ is called orthocronous and $L^\downarrow$ nonorthochronous, and the two are related by T.

Besides these transformations, we also have C, the charge conjugation transformation, which transforms particles into antiparticles.

For a long time, it was thought that C, P, T must be symmetries of physics, for instance since:

-gravitational, electromagnetic and strong interactions respect C, P, T.

-weak interactions however break C and P separately, but preserve CP. The Nobel prize was awarded for the discovery of this breaking of parity, thought to have been fundamental.

-by now there is good experimental evidence for CP breaking also, which would point to interactions beyond the Standard Model.

-However, there is a theorem, that under very general conditions (like locality and unitarity), CPT must be preserved always, so CP breaking implies T breaking.

In the quantum theory, the action of the C, P, T symmetries is encoded in operators C, P, T that act on other operators. The c-numbers are generally not modified (though see T below), even if they contain $t, \vec{x}$.

Parity

Parity is defined as being the same as reflection in a mirror. If we look in a mirror at a momentum vector with a spin, or helicity, represented by a rotation around the momentum axis, we see the momentum inverted, but not the spin, hence parity flips the momentum $\vec{p}$, but not the spin s, see Fig.35.
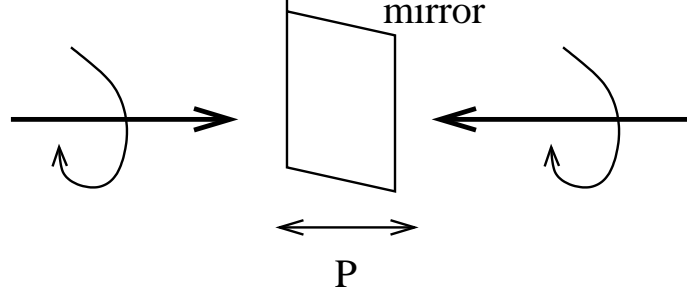

Figure 35: Parity is reflexion in a mirror. The momentum gets inverted, but not the spin.

That means that the action of P on the annihilation/creation operators in the expansion of the quantum field ψ is

$$Pa^s_{\vec{p}}P^{-1} = \eta_a a^s_{-\vec{p}}$$
$$Pb^s_{\vec{p}}P^{-1} = \eta_b b^s_{-\vec{p}} \qquad (14.18)$$

where η_a and η_b are possible phases. Since two parity operations revert the system to its original, we have $P^2 = \mathbf{1}$, so $P^{-1} = P$, and then also $\eta_a^2 = \eta_b^2 = \pm 1$, since observables are fermion bilinears, so a $\pm$ sign doesn't make a difference.

Defining the momenta $\tilde{p} = (p^0, -\vec{p})$, we have

$$
\begin{aligned}
u(p) &= \begin{pmatrix} \sqrt{-p \cdot \sigma}\xi \\ \sqrt{-p \cdot \bar{\sigma}}\xi \end{pmatrix} = \begin{pmatrix} \sqrt{-\tilde{p} \cdot \bar{\sigma}}\xi \\ \sqrt{-\tilde{p} \cdot \sigma}\xi \end{pmatrix} = i\gamma^0 u(\tilde{p}) \\
v(p) &= \begin{pmatrix} \sqrt{-p \cdot \sigma}\xi \\ -\sqrt{-p \cdot \bar{\sigma}}\xi \end{pmatrix} = \begin{pmatrix} \sqrt{-\tilde{p} \cdot \bar{\sigma}}\xi \\ -\sqrt{-\tilde{p} \cdot \sigma}\xi \end{pmatrix} = -i\gamma^0 v(\tilde{p})
\end{aligned} \qquad (14.19)
$$

where

$$\gamma^0 = -i \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \qquad (14.20)$$

Then

$$
\begin{aligned}
P\psi(x)P &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s (\eta_a a^s_{-\vec{p}} u^s(p)e^{ip\cdot x} + \eta_b^* b^{s\dagger}_{-\vec{p}} v^s(p)e^{-ip\cdot x}) \\
&= \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_s (\eta_a a^s_{\tilde{p}} i\gamma^0 u^s(\tilde{p})e^{i\tilde{p}(t,-\vec{x})} - \eta_b^* b^{s\dagger}_{\tilde{p}} i\gamma^0 v^s(\tilde{p})e^{-i\tilde{p}(t,-\vec{x})})
\end{aligned} \qquad (14.21)
$$

where in the second line we have changed the integration variable from p to $\tilde{p}$, and then used the relation between $u(p)$ and $u(\tilde{p})$ and similarly for v. Finally, we see that we need to have $\eta_b^* = -\eta_a$ to relate to another ψ field. In that case, we obtain

$$P\psi(x)P = \eta_a i\gamma^0 \psi(t, -\vec{x}) \qquad (14.22)$$

Then we have also

$$(P\psi(x)P)^\dagger = -i\eta_a^*\psi^\dagger(t,-\vec{x})\gamma^{0\dagger} = i\eta_a^*\psi^\dagger(t,-\vec{x})\gamma^0 \tag{14.23}$$

and so

$$P\bar{\psi}(x)P = (P\psi(t,\vec{x})P)^\dagger i\gamma^0 = i\eta_a^*\bar{\psi}(t,-\vec{x})\gamma^0 \tag{14.24}$$

We can now also compute the behaviour of fermion bilinears under parity. We obtain

$$P\bar{\psi}\psi P = |\eta_a|^2\bar{\psi}(t,-\vec{x})\gamma^0(-\gamma^0)\psi(t,-\vec{x}) = \bar{\psi}\psi(t,-\vec{x}) \tag{14.25}$$

i.e., it transforms as a scalar. Here we have used that η_a is a phase, so $|\eta_a|^2 = 1$, and $-(\gamma^0)^2 = 1$.

Next, we have

$$P\bar{\psi}\gamma_5\psi P = \bar{\psi}(t,-\vec{x})\gamma^0\gamma_5(-\gamma^0)\psi(t,-\vec{x}) = -\bar{\psi}\gamma_5\psi(t,-\vec{x}) \tag{14.26}$$

so is a pseudoscalar. Also,

$$P\bar{\psi}\gamma^\mu\psi P = \bar{\psi}\gamma^0\gamma^\mu(-\gamma^0)\psi(t,-\vec{x}) = (-1)^\mu\bar{\psi}\gamma^\mu\psi(t,-\vec{x}) \tag{14.27}$$

where $(-1)^\mu = +1$ for $\mu = 0$ and $= -1$ for $\mu \neq 0$, so it transforms like a vector. Further,

$$P\bar{\psi}\gamma^\mu\gamma_5\psi P = \bar{\psi}\gamma^0\gamma^\mu\gamma_5(-\gamma^0)\psi(t,-\vec{x}) = -(-1)^\mu\bar{\psi}\gamma^\mu\gamma_5\psi(t,-\vec{x}) \tag{14.28}$$

so it transforms as a pseudovector.

Time reversal

Consider now the time reversal transformation. Under it, the momentum changes direction (time flows opposite), and also the spin changes, since it rotates back, see Fig.36. However, time reversal cannot be made into a linear operator like P, but it can be made into an *antilinear operator*,

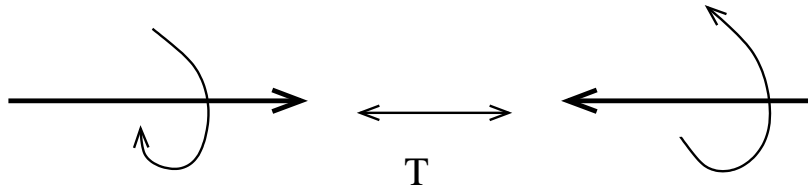$$T(\text{c}-\text{number}) = (\text{c}-\text{number})^*T \tag{14.29}$$

Figure 36: Time reversal inverts both momentum and spin.

Then we have

$$\begin{aligned} Ta_{\vec{p}}^s T &= a_{-\vec{p}}^{-s} \\ Tb_{\vec{p}}^s T &= b_{-\vec{p}}^{-s} \end{aligned} \tag{14.30}$$

and since the T operator is antilinear, for the full Dirac equation modes we have

$$Ta^s_{\vec{p}}u^s(p)e^{ip\cdot x}T = a^{-s}_{-\vec{p}}[u^s(p)]^*e^{-ip\cdot x}$$
$$Tb^{s\dagger}_p v^s(p)e^{-ip\cdot x}T = b^{-s\dagger}_{-\vec{p}}[v^s(p)]^*e^{ip\cdot x} \tag{14.31}$$

We now consider a general spin basis, instead of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, corresponding to spin oriented along the z direction, we take a spin direction of arbitrary spherical angles θ, ϕ, giving

$$\xi(\uparrow) = R(\theta, \phi)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta/2\\e^{i\phi}\sin\theta/2\end{pmatrix}$$
$$\xi(\downarrow) = R(\theta, \phi)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-e^{-i\phi}\sin\theta/2\\\cos\theta/2\end{pmatrix} \tag{14.32}$$

and we have $\xi^s = (\xi(\uparrow), \xi(\downarrow))$, and then $(\vec{\sigma}\sigma^2 = \sigma^2(-\vec{\sigma}^*) \Rightarrow$ if $\vec{n}\cdot\vec{\sigma}\xi = +\xi$, then $(\vec{n}\cdot\vec{\sigma})(-i\sigma^2\xi^*) = -(-i\sigma^2\xi^*))$

$$\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow)) = -i\sigma^2(\xi^s)^* \tag{14.33}$$

Analogously, we write for the annihilation operators $a^{-s}_{\vec{p}} = (a^2_{\vec{p}}, -a^1_{\vec{p}})$ and $b^{-s}_{\vec{p}} = (b^2_{\vec{p}}, -b^1_{\vec{p}})$.

For the column vectors we have

$$u^{-s}(\tilde{p}) = \begin{pmatrix}\sqrt{-\tilde{p}\cdot\sigma}(-i\sigma^2\xi^{s*})\\\sqrt{-\tilde{p}\cdot\bar{\sigma}}(-i\sigma^2\xi^{s*})\end{pmatrix} = \begin{pmatrix}-i\sigma^2\sqrt{-p\cdot\sigma^*}\xi^{s*}\\-i\sigma^2\sqrt{-p\cdot\bar{\sigma}^*}\xi^{s*}\end{pmatrix} = -i\begin{pmatrix}\sigma^2 & 0\\0 & \sigma^2\end{pmatrix}[u^s(p)]^* = \gamma^1\gamma^3[u^s(p)]^* \tag{14.34}$$

Here we have used the relation $\sqrt{\tilde{p}\cdot\sigma}\sigma^2 = \sigma^2\sqrt{p\cdot\sigma^*}$, which we can prove by expanding the square root for small $\vec{p}$, and then $(\vec{p}\cdot\vec{\sigma})\sigma_2 = -\sigma_2(\vec{p}\cdot\vec{\sigma}^*)$, true since $\sigma_2^* = -\sigma_2, \sigma_{1,3}^* = \sigma_{1,3}$.

Then we can compute the effect on the quantum field operator, and obtain

$$T\psi(t, \vec{x})T = -\gamma^1\gamma^3\psi(-t, \vec{x}) \tag{14.35}$$

For the fermion bilinears, we then obtain

$$T\bar{\psi}\psi T = +\bar{\psi}\psi(-t, \vec{x})$$
$$T\bar{\psi}\gamma_5\psi T = -\bar{\psi}\gamma_5\psi(-t, \vec{x})$$
$$T\bar{\psi}\gamma^\mu\psi T = (-1)^\mu\bar{\psi}\gamma^\mu\psi(-t, \vec{x}) \tag{14.36}$$

Charge conjugation

Charge conjugation is the operation that takes particles to antiparticles. On the classical spinor space, it acts not on the classical $(t, \vec{x})$, but by the C-matrix (charge conjugation matrix) defined before. We need to define how it acts on the quantum field operators.

As we said before, due to the fact that $\psi \sim a^s_{\vec{p}}u^s(p)e^{ip\cdot x} + b^{s\dagger}_{\vec{p}}v^s(p)e^{-ip\cdot x}$, and in the energy we have $\int \omega(a^\dagger a - bb^\dagger)$, we can understand $a^\dagger$ as creation operator for fermions of positive energy (e.g. electrons e^-) and $b^\dagger$ as an annihilation operator for fermions of negative energy

in the fully occupied "Dirac sea", equivalent to a creation operator for a "hole" of effective positive energy, or *antiparticle* (e.g. positron e^+).

Therefore the C operator maps a's into b's,

$$
\begin{aligned}
Ca_{\vec{p}}^s C^{-1} &= b_{\vec{p}}^s \\
Cb_{\vec{p}}^s C^{-1} &= a_{\vec{p}}^s
\end{aligned}
\tag{14.37}
$$

Note that we have ignored possible phases here. Then we use that

$$
(v^s(p))^* = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{-p\cdot\sigma}\,\xi^s \\ \sqrt{-p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} = \gamma^2 \begin{pmatrix} \sqrt{-p\cdot\sigma}\,\xi^s \\ \sqrt{-p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}
\tag{14.38}
$$

where again we used $\sqrt{-p\cdot\sigma}\,\sigma^2 = \sigma^2\sqrt{-p\cdot\bar{\sigma}^*}$. Then we have for the column vectors

$$
\begin{aligned}
u^s(p) &= \gamma^2(v^s(p))^* \\
v^s(p) &= \gamma^2(u^s(p))^*
\end{aligned}
\tag{14.39}
$$

Finally, using the linearity property of the C operator and the above properties, we can calculate the effect on the quantum field operator,

$$
C\psi(x)C^{-1} = \gamma^2\psi^*(x) = \gamma^2(\psi^\dagger)^T
\tag{14.40}
$$

Therefore the charge conjugation operator interchanges ψ and $\psi^\dagger$ up to some matrix.

For the fermion bilinears we find

$$
\begin{aligned}
C\bar{\psi}\psi C^{-1} &= \bar{\psi}\psi \\
C\bar{\psi}\gamma_5\psi C^{-1} &= \bar{\psi}\gamma_5\psi \\
C\bar{\psi}\gamma^\mu\psi C^{-1} &= -\bar{\psi}\gamma^\mu\psi \\
C\bar{\psi}\gamma^\mu\gamma_5\psi C^{-1} &= \bar{\psi}\gamma^\mu\gamma_5\psi
\end{aligned}
\tag{14.41}
$$

The full transformations under P, T, C and CPT, for various bilinears and for the derivatives ∂_μ, are

$$
\begin{aligned}
\bar{\psi}\psi &: \quad +1, +1, +1, +1 \\
\bar{\psi}\gamma_5\psi &: \quad -1, -1, +1, +1 \\
\bar{\psi}\gamma^\mu\psi &: \quad (-1)^\mu, (-1)^\mu, -1, -1 \\
\bar{\psi}\gamma^\mu\gamma_5\psi &: \quad -(-1)^\mu, (-1)^\mu, +1, -1 \\
\bar{\psi}\gamma^{\mu\nu}\psi &: \quad (-1)^\mu(-1)^\nu, -(-1)^\mu(-1)^\nu, -1, +1 \\
\partial_\mu &: \quad (-1)^\mu, -(-1)^\mu, +1, -1
\end{aligned}
\tag{14.42}
$$

where of course under P and T we also transform the position for the fields.

Important concepts to remember

- The spin sums are $\sum_s u^s(p)\bar{u}^s(p) = -i\slashed{p} + m$ and $\sum_s v^s(p)\bar{v}^s(p) = -i\slashed{p} - m$.

- The complete basis of 4×4 matrices is $\mathcal{O}_i = \{\mathbf{1}, \gamma^\mu, \gamma_5, \gamma^\mu \gamma_5, \gamma^{\mu\nu}\}$, and fermion bilinears are $\bar{\psi} \mathcal{O}_i \psi$.

- The presence of γ_5 means a pseudoscalar/pseudovector,etc., i.e. an extra minus under parity.

- Fierz recoupling follows from the completeness of the $\mathcal{O}_i$ basis.

- Vector currents are conserved, and axial vector currents are conserved only if $m = 0$.

- T,C,P act on quantum operators.

- Time reversal T is antilinear and charge conjugation C changes particles to antiparticles.

Further reading: See chaptes 3.4 and 3.6 in [2].

Exercises, Lecture 14

1) Using the Fierz identity, prove that we have for Majorana spinors λ^a

$$(\bar{\lambda}^a \gamma_\mu \lambda^c)(\bar{\epsilon} \gamma^\mu \lambda^b) f_{abc} = 0 \qquad (14.43)$$

where f_{abc} is totally antisymmetric, and also using

$$\begin{aligned}
\gamma_\mu \gamma_\rho \gamma^\mu &= -2\gamma_\rho \\
\gamma_\mu \gamma_{\rho\sigma} \gamma^\mu &= 0
\end{aligned} \qquad (14.44)$$

and that Majorana spinors satisfy

$$\begin{aligned}
\bar{\epsilon}\chi &= \bar{\chi}\epsilon \\
\bar{\epsilon}\gamma_\mu \chi &= -\bar{\chi}\gamma_\mu \epsilon \\
\bar{\epsilon}\gamma_5 \chi &= \bar{\chi}\gamma_5 \epsilon \\
\bar{\epsilon}\gamma_\mu \gamma_5 \chi &= \bar{\chi}\gamma_\mu \gamma_5 \epsilon
\end{aligned} \qquad (14.45)$$

2) Prove that the transformations of $\bar{\psi}\gamma^\mu \gamma_5 \psi$ under T, C and CPT are with $(-1)^\mu$, $+1$ and -1 respectively.

15 Lecture 15. Dirac quantization of constrained systems

In this lecture we will describe the quantization of constrained systems developed by Dirac. I will largely follow the book by P.A.M. Dirac, "Lectures on Quantum Mechanics", which is a series of 4 lectures given at Yeshiva University, published in 1964, and is possibly one of the most influential books in physics, per number of pages. The first 2 lectures describe the procedure now known as Dirac quantization, which is very important for modern theoretical physics, and even now, there is little that needs to be changed in the presentation of Dirac.

So what is the problem that we want to solve? We want to quantize gauge fields (we will do so next lecture), which have an action

$$S = -\frac{1}{4} \int d^4x F_{\mu\nu}^2 \tag{15.1}$$

where the field strength is written in terms of the gauge field A_μ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{15.2}$$

therefore we have a gauge invariance

$$\delta A_\mu = \partial_\mu \Lambda \tag{15.3}$$

which leaves $F_{\mu\nu}$, and therefore the action, invariant. This means that there is a redundancy in the system, there are less degrees of freedom in the system than variables used to describe it (we can use Λ to put one component, for instance A_0, to zero). To quantize only the physical degrees of freedom, we must fix a gauge, for instance the Lorentz, or covariant, gauge $\partial^\mu A_\mu = 0$. That imposes a constraint on the system, so we must learn how to quantize in the presence of constraints, which is the subject that Dirac tackled. We can deal with this in the Hamiltonian formalism, leading to the Dirac quantization of constrained systems. Even though we will not use much of the Dirac formalism later, it is important theoretically, to understand the concepts involved better, hence we started with it.

In the Lagrangean formulation of classical mechanics, constraints are introduced in the Lagrangean by multiplying them with Lagrange multipliers. We will see that in the Hamiltonian formulation, we also add the constraints to the Hamiltonian with some coefficients, with some notable differences.

In the Hamiltonian formalism, we first define conjugate momenta as

$$p_n = \frac{\partial L}{\partial \dot{q}_n} \tag{15.4}$$

and then the Hamiltonian is

$$H = \sum_n (p_n \dot{q}_n) - L \tag{15.5}$$

The Hamilton equations of motion are

$$\dot{q}_n = \frac{\partial H}{\partial p_n}; \quad \dot{p}_n = -\frac{\partial H}{\partial q_n} \tag{15.6}$$

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We define the Poisson brackets of two functions of p and q, $f(p,q)$ and $g(p,q)$, as

$$\{f,g\}_{P.B.} = \sum_n \left[\frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} \right] \qquad (15.7)$$

Then the Hamilton equations of motion are

$$\dot{q}_n = \{q_n, H\}_{P.B.}; \quad \dot{p}_n = \{p_n, H\}_{P.B.} \qquad (15.8)$$

and more generally, we can write for time evolution (equation of motion) of some function on phase space, $g(q,p)$,

$$\dot{g} = \{g, H\}_{P.B.} \qquad (15.9)$$

But now assume that there are M constraints on phase space,

$$\phi_m(q,p) = 0, \quad m = 1, ..., M \qquad (15.10)$$

We will call these the *primary constraints* of the Hamiltonian formalism, we will see shortly why.

We will define $\approx$, called *weak equality*, as equality only after using the constraints $\phi_m = 0$. Note that we must use the constraints only *at the end of a calculation*, for instance when using Poisson brackets, since the Poisson brackets are defined by assuming a set of independent variables q_n, p_n (so that partial derivatives mean when we keep all the other variables fixed; this would not be possible if there is a relation between all the q's and p's). So only when all the dynamical calculations are done, can we use $\phi_m = 0$.

So the weak equality means that by definition

$$\phi_m \approx 0 \qquad (15.11)$$

since otherwise ϕ_m is some function of q's and p's, for which we can calculate Poisson brackets, and is not *identically* zero.

Note that as far as the time evolution is concerned, there is no difference between H and $H + u_m \phi_m$ (indistinguishable), since $\phi_m \approx 0$. Then the equations of motion are

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + u_m \frac{\partial \phi_m}{\partial p_n} \approx \{q_n, H + u_m \phi_m\}_{P.B.}$$
$$\dot{p}_n = -\frac{\partial H}{\partial q_n} - u_m \frac{\partial \phi_m}{\partial q_n} \approx \{p_n, H + u_m \phi_m\}_{P.B.} \qquad (15.12)$$

and in general we write

$$\dot{g} \approx \{g, H_T\}_{P.B.} \qquad (15.13)$$

where the *total Hamiltonian* H_T is

$$H_T = H + u_m \phi_m \qquad (15.14)$$

Note that in the above we have used the fact that the Poisson bracket of the coefficient u_m (which is in principle a function of (q,p)) does not contribute, since it is multiplied by ϕ_m, so $\{g, u_m\}\phi_m \approx 0$.

Let us now apply the above time evolution to the particular case of the constraints ϕ_m themselves. Physically, we know that if ϕ_m are good constraints, the time evolution should keep us within the constraint hypersurface, so we must have $\dot{\phi}_m \approx 0$ (which means that the time variation of ϕ_m must be proportional to some ϕ's itself). That gives the equations

$$\{\phi_m, H\}_{P.B.} + u_n\{\phi_m, \phi_n\}_{P.B.} \approx 0 \tag{15.15}$$

This in turn will give other, potentially independent, constraints, called *secondary constraints*. We can then apply the time variation again, and gain new constraints, etc., until we get nothing new, finally obtaining the full set of secondary constraints, that we will call ϕ_k, $k = M + 1, ..., M + K$. Together, all the constraints, primary and secondary, form the set

$$\phi_j \approx 0, \quad j = 1, ..., M + K \tag{15.16}$$

While there is not much difference between primary and secondary constraints, as we will see, they act both in the same way, there is one distinction that is useful. We will call a function $R(p, q)$ on phase space *first class* if

$$\{R, \phi_j\}_{P.B.} \approx 0 (\text{i.e.}, = r_{jj'}\phi_{j'}) \tag{15.17}$$

for all $j = 1, ..., M + K$, and *second class* if $\{R, \phi_j\}_{P.B.}$ is not ≈ 0 for some ϕ_j. In particular, we have for the time evolution of all the constraints, by definition (all the set of constraints satisfies that their time evolution is also a constraint)

$$\{\phi_j, H_T\}_{P.B.} \approx \{\phi_j, H\}_{P.B.} + u_m\{\phi_j, \phi_m\}_{P.B.} \approx 0 \tag{15.18}$$

(here again we dropped the term $\{u_m, \phi_j\}\phi_m \approx 0$). These equations tell us first that H_T is first class, and second, give us a set of $M + K$ equations for the M coefficients u_m, which are in principle functions of q and p, i.e. $u_m(q, p)$. Even if the system looks overconstrained, these equations need to have at least a solution from physics consistency, since if not, it would mean that there is no possible consistent time evolution of constraints, which is clearly wrong. But in general the solution is not unique. If U_m is a particular solution of (15.18), then the general solution is

$$u_m = U_m + v_a V_{am} \tag{15.19}$$

where v_a are arbitrary numbers, and V_{am} satisfy the equation

$$V_{am}\{\phi_j, \phi_m\}_{P.B.} \approx 0 \tag{15.20}$$

Then we can split the total Hamiltonian, which is as we saw first class, as

$$\begin{aligned} H_T &= H + U_m\phi_m + v_a V_{am}\phi_m \\ &= H' + v_a\phi_a \end{aligned} \tag{15.21}$$

where

$$H' = H + U_m\phi_m \tag{15.22}$$

is first class, by the definition of U_m as a particular solution of (15.18), and

$$\phi_a = V_{am}\phi_m \qquad (15.23)$$

are first class primary constraints, because of the definition of V_{am} as satisfying (15.20).

Theorem. If R and S are first class, i.e. $\{R, \phi_j\}_{P.B.} = r_{jj'}\phi_{j'}$, $\{S, \phi_j\}_{P.B.} = s_{jj'}\phi_{j'}$, then $\{R, S\}_{P.B.}$ is first class.

Proof: We first use the Jacobi identity for antisymmetric brackets (like the commutator and the Poisson brackets), which is an identity (type 0=0) proved by writing explicitly the brackets,

$$\{\{R, S\}, P\} + \{\{P, R\}, S\} + \{\{S, P\}, R\} = 0 \qquad (15.24)$$

to write

$$
\begin{aligned}
\{\{R, S\}_{P.B.}, \phi_j\}_{P.B.} &= \{\{R, \phi_j\}_{P.B.}, S\}_{P.B.} - \{\{S, \phi_j\}_{P.B.}, R\}_{P.B.} \\
&= \{r_{jj'}\phi_{j'}, S\}_{P.B.} - \{s_{jj'}\phi_{j'}, R\}_{P.B.} \\
&= -r_{jj'}s_{j'j''}\phi_{j''} + \{r_{jj'}, S\}_{P.B.}\phi_{j'} \\
&\quad -s_{jj'}(-r_{j'j''}\phi_{j''}) - \{s_{jj'}, R\}_{P.B.}\phi_{j'} \approx 0 \qquad (15.25)
\end{aligned}
$$

q.e.d.

Finally, we can add the first class secondary constraints $\phi_{a'}$ as well to the Hamiltonian, since there is no difference between the first class primary and first class secondary constraints, obtaining the *extended Hamiltonian* H_E,

$$H_E = H_T + v'_{a'}\phi_{a'} \qquad (15.26)$$

However, the second class constraints ϕ_w are different. We cannot add them to the Hamiltonian, since $\{\phi_w, \phi_j\}_{P.B}$ is not ≈ 0, so the time evolution of ϕ_j, $\dot{\phi}_j = \{\phi_j, H\}_{P.B.}$, would be modified.

Therefore, we have the physical interpretation that first class constraints generate motion tangent to the constraint hypersurface (adding them to the Hamiltonian, the new term is ≈ 0, i.e. in the constraint hypersurface $\phi_m = 0$). On the other hand, second class constraints generate motion away from the constraint hypersurface, since adding them to the Hamiltonian adds terms which are not ≈ 0 to the time evolution of ϕ_j.

Quantization

Now we finally come to the issue of quantization. Normally, we substitute the Poisson bracket $\{,\}_{P.B.}$ with $1/i\hbar[,]$ when quantizing. But now there is a subtlety, namely the constraints.

The simplest possibility is to impose the constraints on states (wavefunctions), i.e. to put

$$\hat{\phi}_j|\psi> = 0. \qquad (15.27)$$

But that leads to a potential problem. For consistency, applying this twice we get $[\hat{\phi}_j, \hat{\phi}_{j'}]|\psi> = 0$. But since the ϕ_j are supposed to be ALL the constraints, the commutator must be a linear combination of the constraints themselves, i.e.

$$[\hat{\phi}_j, \hat{\phi}_{j'}] = c_{jj'j''}\hat{\phi}_{j''} \qquad (15.28)$$

Note the fact that in quantum mechanics the c's could in principle not commute with the $\hat{\phi}_j$, so we must put the c's to the left of $\hat{\phi}_j$'s, to have the commutator on the left hand side equal zero.

Then the time evoluton of the constraints should also be a constraint, so

$$[\hat{\phi}_j, H] = b_{jj'}\hat{\phi}_{j'} \tag{15.29}$$

If there are no second class constraints, then $\{\phi_j, \phi_{j'}\}_{P.B.} \approx 0$ (the condition for all the constraints to be first class), which in the quantum case turns into the algebra of constraints (15.28), and we have no problem.

But if there are second class constraints, we have a problem, since then the fact that $\{\phi_w, \phi_{j'}\}_{P.B.}$ is not ≈ 0 contradicts the consistency condition (15.28).

Let's see what happens in an example. Consider a system with several degrees of freedom, and the constraints $q_1 \approx 0$ and $p_1 \approx 0$. But then their Poisson brackets give $\{q_1, p_1\}_{P.B.} = 1 \neq 0$, which means that they are not first class. But then we can't quantize the constraints by imposing them on states as

$$\hat{q}_1|\psi> = 0 = \hat{p}_1|\psi> \tag{15.30}$$

since then on one hand applying the constraints twice we get $[\hat{q}_1, \hat{p}_1]|\psi> = 0$, but on the other hand the commutator gives

$$[\hat{q}_1, \hat{p}_1]\psi = i\hbar|\psi> \tag{15.31}$$

which is a contradiction. Therefore one of the two assumptions we used is invalid: either $\hat{\phi}_j|\psi> = 0$ is not a good way to impose the constraint, or the quantization map $\{,\}_{P.B.} \rightarrow 1/i\hbar[,]$ is invalid. It turns out that the latter is the case. We need to modify the Poisson bracket in order to map to $1/i\hbar[,]$. In the example above, the solution is obvious. Since the constraints are $q_1 = p_1 = 0$, we can just drop them from the phase space, and write the modified Poisson bracket

$$\{f, g\}_{P.B.'} = \sum_{n=2}^{N} \left(\frac{\partial f}{\partial q_n} \frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n} \frac{\partial g}{\partial q_n} \right) \tag{15.32}$$

But what do we do in general?

In general we must introduce the *Dirac brackets* as follows. We consider the *independent* second class constraints (i.e., taking out the linear combinations which are first class, remaining only with constraints which have all the linear combinations being second class), called χ_s, and we define $c_{ss'}$ by

$$c_{ss'}\{\chi_{s'}, \chi_{s''}\}_{P.B.} = \delta_{ss''}, \tag{15.33}$$

i.e., we have the inverse matrix of Poisson brackets, $c_{ss'} = \{\chi_s, \chi_{s'}\}_{P.B.}^{-1}$. Then we define the Dirac brackets as

$$[f, g]_{D.B.} = \{f, g\}_{P.B.} - \{f, \chi_s\}_{P.B.}c_{ss'}\{\chi_{s'}, g\}_{P.B.} \tag{15.34}$$

Then if we use the Dirac brackets instead of the Poisson brackets, the time evolution is not modified, since

$$[g, H_T]_{D.B.} = \{g, H_T\}_{P.B.} - \{g, \chi_s\} c_{ss'} \{\chi_{s'}, H_T\}_{P.B.} \approx \{g, H_T\}_{P.B.} \approx \dot{g} \qquad (15.35)$$

where we have used the fact that H_T is first class, so $\{\chi_{s'}, H_T\}_{P.B.} \approx 0$.

On the other hand, the Dirac bracket of the second class constraints with any function on the phase space is zero, since

$$[f, \chi_{s''}]_{D.B.} = \{f, \chi_{s''}\}_{P.B.} - \{f, \chi_s\}_{P.B} c_{ss'} \{\chi_{s'}, \chi_{s''}\}_{P.B.} = 0 \qquad (15.36)$$

(equal to 0 strongly!) where we have used the definition of $c_{ss'}$ as the inverse matrix of constraints. That means that in classical mechanics we can put $\chi_s = 0$ *strongly* if we use Dirac brackets. In quantum mechanics, we again impose it on states as

$$\hat{\chi}_s |\psi> = 0 \qquad (15.37)$$

but now we don't have any more contradictions, since the Dirac bracket, turning into a commutator in quantum mechanics, is zero for the second class constraints with anything. Therefore $[\hat{f}, \hat{\chi}_s]|\psi> = 0$ gives no contradictions now.

Finally, note that in quantum mechanics the difference between the primary and secondary constraints is irrelevant, whereas the difference between first and second class constraints is important, since second class constraints modify the Dirac bracket.

Example: electromagnetic field

We now return to the example which started the discussion, the electromagnetic field with action

$$S = -\int d^4x \frac{F_{\mu\nu}^2}{4} = \int \left[-\frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} F_{0i} F^{0i} \right] \qquad (15.38)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We also denote derivatives as $\partial_\mu B = B_{,\mu}$. Then the momenta conjugate to the fields A_μ are

$$P^\mu = \frac{\delta \mathcal{L}}{\delta A_{\mu,0}} = F^{\mu 0} \qquad (15.39)$$

Since $F^{00} = 0$ by antisymmetry, it means we have the primary constraint

$$P^0 \approx 0 \qquad (15.40)$$

The basic Poisson brackets of the fields A_μ with their conjugate momenta are as usual

$$\{A_\mu(\vec{x}, t), P^\nu(\vec{x}', t)\}_{P.B.} = \delta_\mu^\nu \delta^3(\vec{x} - \vec{x}') \qquad (15.41)$$

The Hamiltonian is then

$$
\begin{aligned}
H &= \int d^3x P^\mu A_{\mu,0} - L = \int d^3x \left[F^{i0} A_{i,0} + \frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} F^{i0} F_{i0} \right] \\
&= \int d^3x \left[F^{i0} A_{0,i} + \frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} F^{i0} F_{i0} \right]
\end{aligned}
$$

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$$= \int d^3x \left[\frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} P^i P^i - A_0 P^i_{,i} \right] \tag{15.42}$$

where in the second equality we used $F_{i0} = A_{0,i} - A_{i,0}$ and in the second we used $P^i = F^{0i}$ and partial integration.

Next we compute the secondary constraints. First we compute the time evolution of the primary constraint, giving a secondary constraint

$$\{P^0, H\}_{P.B.} = P^i_{,i} \approx 0 \tag{15.43}$$

The time evolution of the secondary constraint is in turn trivial,

$$\{P^i_{,i}, H\}_{P.B.} = 0 \tag{15.44}$$

so there are no other secondary constraints.

All the constraints are first class, since we have

$$\{P^0, P^0\}_{P.B.} = 0 = \{P^0, P^i_{,i}\}_{P.B.} = \{P^i_{,i}, P^j_{,j}\}_{P.B.} = 0 \tag{15.45}$$

That means that there is no need for Dirac brackets. We have $H = H'$, and we add the first class primary constraint P^0 with an arbitrary coefficient, giving the total Hamiltonian

$$H_T = H' + \int v P^0 = \int \left(\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} P^i P^i \right) - \int A_0 P^i_{,i} + \int v(x) P^0(x) \tag{15.46}$$

The extended Hamiltonian is found by adding also the secondary first class constraint with an arbitrary coefficient,

$$H_E = H_T + \int d^3x u(x) P^i_{,i}(x) \tag{15.47}$$

But A_0 and P^0 contain no relevant information. We can put $P^0 = 0$ in H_T since that only makes the time evolution of A_0 trivial, $\dot{A}_0 = 0$, and then we can get rid of A_0 by redefining $u'(x) = u(x) - A_0(x)$, obtaining the Hamiltonian

$$H_E = \int d^3x \left(\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} P^i P^i \right) + \int d^3x u'(x) P^i_{,i}(x) \tag{15.48}$$

Finally, we should note that the Dirac formalism is very important in modern theoretical physics. It is the beginning of more powerful formalisms: BRST quantization, Batalin-Vilkovitsky (BV) formalism, field-antifield formalism, used for gauge theories and string theory, among others. However, we will not explain them here.

Important concepts to remember

- Dirac quantization of constrained system starts in the Hamiltonian formalism with the primary constraints $\phi_m(p,q) = 0$.

- We write weak equality $F(q, p) \approx 0$ if at the end of the calculation we use $\phi_m = 0$.

- The primary constraints are added to the Hamiltonian, forming the total Hamiltonian $H_T = H + u_m \phi_m$.

- The secondary constraints are obtained from the time evolution of the primary constraints, $\{\phi_m, H_T\}_{P.B.}$ and iterated until nothing new is found. In total, ϕ_j, $j = 1, ..., M + K$ are all the constraints.

- First class quantities R satisfy $\{R, \phi_j\} \approx 0$, and second class quantities don't.

- H_T is written as $H_T = H' + v_a \phi_a$ where H' is first class, ϕ_a are first class primary constraints, and v_a are numbers.

- The extended Hamiltonian is obtained by adding the first class secondary constraints $H_E = H_T + v'_{a'} \phi_{a'}$.

- When quantizing, we must impose the constraints on states, $\hat{\phi}_j | \psi >= 0$, but to avoid inconsistencies in the presence of second class constraints χ_s, we must introduce Dirac brackets (to be replaced by the quantum commutators), by subtracting $\{f, \chi_s\}_{P.B.} \{\chi_s, \chi_{s'}\}^{-1}_{P.B.} \{\chi_{s'}, g\}_{P.B.}$.

- The electromagnetic field has primary constraint $P^0 \approx 0$ and secondary constraint $P^i{}_{,i} = F^{0i}{}_{,i} \approx 0$, but in the extended Hamiltonian we can drop A_0 and P^0.

Further reading: See Dirac's book [5].

Exercises, Lecture 15

Dirac spinor

1) Consider the Dirac action for a Dirac spinor in Minkowski space,

$$S_{cl} = \int d^4x(-\psi^\dagger i\gamma^0\gamma^\mu\partial_\mu\psi) \tag{15.49}$$

qith 8 independent variables, ψ^A and $\psi_A^* = (\psi^A)^*$, where $A = 1, ..., 4$ is a spinor index. Calculate p_A and p^{*A} and the 8 resulting primary constraints. Note that classical fermions are anticommuting, so we define p by taking the derivatives from the left, e.g. $\frac{\partial}{\partial\psi}(\psi\chi) = \chi$, so that $\{p^A, q_B\} = -\delta_B^A$. In general, we must define

$$\{f, g\}_{P.B.} = -(\partial f/\partial p^\alpha)\left(\frac{\partial}{\partial q_\alpha}g\right) + (-)^{fg}(\partial g/\partial p^\alpha)\left(\frac{\partial}{\partial q_\alpha}f\right) \tag{15.50}$$

where $\partial f/\partial p^\alpha$ is the right derivative, e.g. $\partial/\partial\psi(\chi\psi) = \chi$, and $(-)^f g = -1$ for f and g being both fermionic and $+1$ otherwise (if f and/or g is bosonic). This bracket is antisymmetric for f, g being bose-bose or bose-fermi, and symmetric if f and g are both fermionic. Then compute H_T by adding to the classical Hamiltonian H the 8 primary constraints with coefficients u_m. Check that there are no secondary constraints, and then from

$$\{\phi_m, H_T\}_{P.B.} \approx 0 \tag{15.51}$$

solve for u^A, v_A.

2) (continuation) Show that all constraints are second class, thus finding that

$$H_T = H' = H_E \tag{15.52}$$

Write Dirac brackets. Using the Dirac brackets we can now put $p^* = 0$ and replace $i\psi^*$ by $-p$. Show that finally, we have

$$H_T = -\int d^3x(p\gamma^0\gamma^k\partial_k\psi) \tag{15.53}$$

and

$$[\psi^A, p_B]_{D.B.} = \{\psi^A, p_B\}_{P.B.} = -\delta_A^B\delta^3(\vec{x} - \vec{y}) \tag{15.54}$$

(*Observation.* Note that the analysis for Majorana (real) spinors is somewhat different, and there one finds

$$[\psi^A, p_B]_{D.B.} = -\frac{1}{2}\delta_B^A\delta^4(\vec{x} - \vec{y}).) \tag{15.55}$$

16 Lecture 16. Quantization of gauge fields, their path integral, and the photon propagator

As we saw last lecture, for gauge fields there are redundancies in the description due to the gauge invariance $\delta A_\mu = \partial_\mu \lambda$, which means that there are components which are not degrees of freedom. For instance, since $\lambda(x)$ is an arbitrary function, we could put $A_0 = 0$ by a gauge transformation. So we must impose a gauge in order to quantize.

-The simplest choice perhaps would be to impose a **physical gauge** condition, like the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, where only the physical modes propagate (no redundant modes present), and impose quantization of only these modes.

-But we will see it is better to impose a **covariant gauge** condition, like the Lorentz gauge $\partial^\mu A_\mu = 0$, and quantize all the modes, imposing some constraints. This will be called covariant quantization.

Physical gauge

Let's start with the discussion of the quantization in Coulomb, or radiation gauge. The equation of motion for the action

$$S = -\frac{1}{4} \int d^4x\, F_{\mu\nu}^2 = -\frac{1}{2} \int [(\partial_\mu A_\nu)\partial^\mu A^\nu - (\partial_\mu A_\nu)\partial^\nu A^\mu] \tag{16.1}$$

is

$$\Box A^\nu - \partial^\nu(\partial^\mu A_\mu) = 0 \tag{16.2}$$

and is difficult to solve. However, we see that in the *Lorenz, or covariant gauge* $\partial^\mu A_\mu = 0$, the equation simplifies to just the Klein-Gordon (KG) equation,

$$\Box A_\mu = 0 \tag{16.3}$$

which has solution of the type $A_\mu \sim \epsilon_\mu(k)e^{\pm ik\cdot x}$, with $k^2 = 0$ and $k^\mu \epsilon_\mu(k) = 0$ (from the gauge condition $\partial^\mu A_\mu = 0$ in momentum space). But note that the Lorenz gauge does not fix completely the gauge, there is a residual gauge symmetry, namely $\delta A_\mu = \partial_\mu \lambda$, with λ satisfying (since $\partial^\mu \delta A_\mu = 0$)

$$\partial^\mu \delta A_\mu = \Box \lambda = 0 \tag{16.4}$$

We can use this λ to fix also the gauge $A_0 = 0$, by transforming with $\partial_0 \lambda = -A_0$, since the KG equation for A_μ, (16.2), gives

$$\Box A_0 = 0 \Rightarrow \partial_0 \Box \lambda = 0 \tag{16.5}$$

so it is OK to use a λ restricted to $\Box \lambda = 0$ only. If $A_0 = 0$, the Lorenz gauge reduces to $\vec{\nabla} \cdot \vec{A} = 0$.

Note that this $A_0 = 0$ gauge is consistent only with $J^\mu = 0$ (no current, since otherwise $A_0 = \int J_0$), but we will consider such a case (no classical background). Therefore we consider the full gauge condition as

$$A_0 = 0; \quad \text{and}\vec{\nabla} \cdot \vec{A} = 0 \tag{16.6}$$

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known as *radiation, or Coulomb* gauge, and famliliar from electrodynamics (see e.g., Jackson's book). In this gauge there are only two propagating modes, the physical degrees of freedom (corresponding to the two polarizations of the electromagnetic field, transverse to the direction of propagation, or circular and anti-circular), since we removed A_0 and $\partial^\mu A_\mu$ from the theory.

The classical solution is

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} \sum_{\lambda=1,2} \vec{\epsilon}^{(\lambda)}(k)\left[e^{ik\cdot x}a^{(\lambda)}(k) + a^{(\lambda)\dagger}(k)e^{-ik\cdot x}\right] \qquad (16.7)$$

and with $k^2 = 0$ (from the KG equation) and $\vec{k}\cdot\vec{\epsilon}^{(\lambda)}(k) = 0$ (from the Lorenz gauge condition in momentum space). The normalization for the solutions was chosen, as in previous cases, so that under quantization, a and $a^\dagger$ act as creation and annihilation operators for the harmonic oscillator. We can also chose the *polarization vectors* $\vec{\epsilon}^{(\lambda)}(k)$ such that they are orthogonal,

$$\vec{\epsilon}^{(\lambda)}(\vec{k}) \cdot \vec{\epsilon}^{(\lambda')}(k) = \delta^{\lambda\lambda'} \qquad (16.8)$$

Quantization in physical gauge

We now turn to quantization of the system. As we saw last lecture, in Dirac formalism, we can drop A_0 and P^0 from the system entirely, which is now needed since we want to use the gauge $A_0 = 0$, and impose the remaining gauge condition $\vec{\nabla} \cdot \vec{A} = 0$ as an operatorial condition.

Consider the conjugate momenta

$$P^i(= \Pi^i) = F^{0i} = E^i \qquad (16.9)$$

We would be tempted to impose the canonical equal time quantization conditions

$$[A^i(\vec{x},t), E^j(\vec{x}',t)] = i\int \frac{d^3k}{(2\pi)^3}\delta^{ij}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = i\delta^{ij}\delta^3(\vec{x} - \vec{x}') \qquad (16.10)$$

but note that if we do so, applying $\nabla_{\vec{x},i}$ on the above relation does not give zero as it should (since we should have $\vec{\nabla} \cdot \vec{A} = 0$). So we generalize $\delta^{ij} \to \Delta^{ij}$ and from the condition $k_i\Delta^{ij} = 0$, we get

$$\begin{aligned}[A^i(\vec{x},t), E^j(\vec{x}',t)] &= i\int \frac{d^3k}{(2\pi)^3}\Delta^{ij}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = i\int \frac{d^3k}{(2\pi)^3}\left(\delta^{ij} - \frac{k^ik^j}{k^2}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\\ &= i\left(\delta^{ij} - \frac{\partial_i\partial_j}{\vec{\nabla}^2}\right)\delta^3(\vec{x} - \vec{x}')\end{aligned} \qquad (16.11)$$

and the rest of the commutators are zero,

$$[A^i(\vec{x},t), A^j(\vec{x}',t)] = [E^i(\vec{x},t), E^j(\vec{x}',t)] = 0 \qquad (16.12)$$

Replacing the mode decomposition of $\vec{A}$ in the above commutators, we obtain the usual harmonic oscillator creation/annihilation operator algebra

$$[a^{(\lambda)}(k), a^{(\lambda')\dagger}(k')] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\vec{k} - \vec{k}')$$

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$$[a^{(\lambda)}(k), a^{(\lambda')}(k')] = 0 = [a^{(\lambda)\dagger}(k), a^{(\lambda')\dagger}(k')] \tag{16.13}$$

We can then compute the energy and obtain the same sum over harmonic oscillator hamiltonians as in previous cases,

$$\begin{aligned} E &= H = \frac{1}{2}\int d^3x(\vec{E}^2 + \vec{B}^2) \\ &= \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\frac{k^0}{2}[a^{(\lambda)\dagger}(k)a^{(\lambda)}(k) + a^{(\lambda)}(k)a^{(\lambda)\dagger}(k)] \end{aligned} \tag{16.14}$$

The proof is similar to previous cases, so is left as an exercise. Again as in previous cases, we define normal ordering and find for the normal ordered Hamiltonian

$$: H := \sum_\lambda \int \frac{d^3k}{(2\pi)^3}k^0 a^{(\lambda)\dagger}(k)a^{(\lambda)}(k) \tag{16.15}$$

This quantization described here is good, but is cumbersome, since while the classical theory was Lorentz invariant, the quantization procedure isn't manifestly so, since we did it using the Lorentz-symmetry breaking radiation gauge condition (16.6). That means that in principle quantum corrections (in quantum field theory) could break Lorentz invariance, so we need to check at each step Lorentz invariance explicitly.

Therefore a better choice is to use a formalism that doesn't break manifest Lorentz invariance, like the

Lorenz gauge (covariant) quantization

If we only impose the KG equation, but no other condition, we still have all the 4 polarization modes, so we have the classical solution

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\sum_{\lambda=0}^{3}\epsilon_\mu^{(\lambda)}(k)\Big[e^{ik\cdot x}a^{(\lambda)}(k) + a^{(\lambda)\dagger}(k)e^{-ik\cdot x}\Big] \tag{16.16}$$

where again $k^2 = 0$ from the solution of the KG equation, but since the KG equation is obtained only in the Lorenz gauge, we still need to impose the Lorenz gauge in an operatorial fashion, as we will see later.

Let us fix the coordinate system such that the momentum solving the KG equation $k^2 = 0$ is in the 3rd direction, $k^\mu = (k, 0, 0, k)$, and define the 4 polarizations in the directions of the 4 axis of coordinates, i.e. $\epsilon_\mu^\lambda = \delta_\mu^\lambda$, or

$$\epsilon^{(\lambda)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ for } \lambda = 0, 1, 2, 3 \tag{16.17}$$

Then for $\lambda = 1, 2$ we obtain *transverse polarizations*, i.e. physical polarizations, transverse to the direction of the momentum,

$$k^\mu \epsilon_\mu^{(1,2)} = 0 \tag{16.18}$$

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and for the unphysical modes $\lambda = 0$ (*timelike polarization*) and $\lambda = 3$ (*longitudinal polarization*, i.e. parallel to the momentum),

$$k^\mu \epsilon_\mu^{(0)} = E_k = k^\mu \epsilon_\mu^{(3)}(k) \tag{16.19}$$

These modes are unphysical since they don't satisfy the gauge condition $\partial^\mu A_\mu = 0$, which would also be needed in order to obtain the KG equation for them. That means that these modes must somehow cancel out of the physical calculations, by imposing some operatorial gauge condition on physical states.

The natural guess would be to impose the Lorenz gauge condition on states, $\partial^\mu A_\mu |\psi> = 0$, which is what Fermi tried to do first. But in view of the mode expansion (16.16), this is not a good idea, since it contains also creation operators, so $\partial^\mu A_\mu$ would not be zero even on the vacuum $|0>$, since $a^\dagger |0> \neq 0$ (there are some details, but we can check that it is impossible). Instead, the good condition was found by Gupta and Bleuler, namely that *only the positive frequency part* of the Lorenz condition is imposed on states,

$$\partial^\mu A_\mu^{(+)}(x)|\psi> = 0 \tag{16.20}$$

(note the somewhat confusing notation: the plus means positive frequency, not dagger, in fact only the annihilation part must be zero on states), known as *Gupta-Bleuler* quantization condition.

Therefore the equal time commutation relations in the covariant gauge are

$$[A_\mu(\vec{x}, t), \Pi_\nu(\vec{x}', t)] = i g_{\mu\nu} \delta^{(3)}(\vec{x} - \vec{x}')$$
$$[A_\mu(\vec{x}, t), A_\nu(\vec{x}', t)] = [\Pi_\mu(\vec{x}, t), \Pi_\nu(\vec{x}', t)] = 0 \tag{16.21}$$

Note that since from the Lagrangean for electromagnetism we found $\Pi_0 = 0$, this contradicts the commutation relation above (and within this context we can't impose $\Pi_0 = 0$ on states, and even if we do, we would still obtain a contradiction with the commutation relation). One solution is to change the Lagrangean, and we will see later that we can in fact do exactly that.

Substituting the expansion (16.16) at the quantum level, we obtain *almost* the usual commutation relations for harmonic oscillator creation and annihilation operators, namely

$$[a^{(\lambda)}(k), a^{(\lambda')\dagger}(k')] = g^{\lambda\lambda'}(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')$$
$$[a^{(\lambda)}(k), a^{(\lambda')}(k')] = [a^{(\lambda)\dagger}(k), a^{(\lambda')\dagger}(k')] = 0 \tag{16.22}$$

For the $\lambda = 1, 2, 3$ modes that is OK, but for the $\lambda = 0$ mode not, since

$$[a^{(0)}(k), a^{(0)\dagger}(k')] = -(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \tag{16.23}$$

This means that these modes have negative norm, since

$$||a^\dagger|0>||^2 \equiv <0|aa^\dagger|0> = - <0|a^\dagger a|0> - <0|0> = -1 \tag{16.24}$$

so they are clearly unphysical. The Gupta-Bleuler condition means, substituing the form of A_μ and $k^\mu \epsilon_\mu^{(1,2)} = 0$,

$$(k^\mu \epsilon_\mu^{(0)}(k) a^{(0)}(k) + k^\mu \epsilon_\mu^{(3)} a^{(3)}(k))|\psi> = 0 \Rightarrow$$

$$[a^{(0)}(k) + a^{(3)}(k)]|\psi> = 0 \qquad (16.25)$$

Let's see how this looks on a state which is a linear combination of one one (0) and one (3) state:

$$
\begin{aligned}
& [a^{(0)}(k) + a^{(3)}(k)](a^{(0)\dagger}(k) + a^{(3)\dagger}(k))|0> \\
= \; & ([a^{(0)}(k), a^{(0)\dagger}(k)] + [a^{(3)}(k), a^{(3)\dagger}(k)])|0> = 0 \qquad (16.26)
\end{aligned}
$$

so it is indeed a physical state. In general, we see that we need to have the same number of (0) and (3) modes in the state (check).

So in general the contribution of $a^{(0)}$'s (timelike modes) and $a^{(3)}$'s (longitudinal modes) cancel each other; for instance, in loops, these modes will cancel each other, as we will see in QFT II.

Note that by imposing only the positive frequency part of the Lorenz gauge condition on $|\psi>$'s, we impose the full gauge condition on expectation values,

$$< \psi|\partial^\mu A_\mu|\psi> = < \psi|(\partial^\mu A_\mu^{(+)} + \partial^\mu A_\mu^{(-)})|\psi> = 0 \qquad (16.27)$$

since $(\partial^\mu A_\mu^{(+)}|\psi>)^\dagger = < \psi|\partial^\mu A_\mu^{(-)}$.

We have defined the above two methods of quantization, since they were historically first. But now, there are also other methods of quantization:

-in other gauges, like light cone gauge quantization.

-more powerful (modern) covariant quantization: BRST quantization. To describe it however, we would need to understand BRST symmetry, and this will be done in QFT II.

-path integral procedures.

- We can write a Hamiltonian path integral based on Dirac's formalism from last lecture. This path is taken for instance in Ramond's book, but we will not continue it here.

- We can use a procedure more easily generalized to the nonabelian case, namely Fadeev-Popov quantization. This is what we will describe here.

First, we come back to the observation that we need to modify the Lagrangean in order to avoid getting $\Pi_0 = 0$. It would be nice to find just the KG equation (what we have in the covariant gauge) from this Lagrangean, for instance. We try

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 \qquad (16.28)$$

and indeed, its equations of motion are

$$\Box A^\mu - (1 - \lambda)\partial^\mu(\partial_\lambda A^\lambda) = 0 \qquad (16.29)$$

and so for $\lambda = 1$ we obtain the KG equation $\Box A_\mu = 0$. $\lambda = 1$ is called "Feynman gauge", though it is a misnomer, since it is not really a gauge, just a choice. The Lagrangean above was just something that gives the right equation of motion from the Lorenz gauge, but we will see that it is what we will obtain in the Fadeev-Popov quantization procedure.

Fadeev-Popov path integral quantization

To properly define the path integral quantization, we must work in Euclidean space, as we saw. The Wick rotation is $x^0 = t \to -ix_4$. But now, since we have the vector A_μ that should behave like x_μ, we must also Wick rotate A_0 like $x_0 = -t$. Then we obtain

$$E_i^{(E)} = F_{4i} = \frac{\partial}{\partial x_4} A_i - \frac{\partial}{\partial x_i} A_4 = -iE_i^{(M)} \tag{16.30}$$

since $F_{0i} \to iF_{4i}$, like $x_0 \to ix_4$. Since the action is Wick rotated as $iS^{(M)} \to -S^{(E)}$, and we have $i \int dt(-F_{ij}^2) = -\int dx_4 F_{ij}^2$, we obtain for the Lagrangean in Euclidean space

$$\mathcal{L}_{em}^{(E)}(A) = +\frac{1}{4}F_{\mu\nu}^{(E)}F_{\mu\nu}^{(4)} = \frac{1}{2}((E_i^{(E)})^2 + (B_i^{(E)})^2) \tag{16.31}$$

and the action

$$S_{em}^{(E)} = \int d^4x \mathcal{L}_{em}^{(E)}(A) \tag{16.32}$$

Since we are in Euclidean space, we can do partial integrations without boundary terms (as we saw, the Euclidean theory is defined for periodic time, on a circle with an infinite radius, hence there are no boundary terms), so we can write

$$S_{em}[A] = \frac{1}{2}\int d^dx A_\mu(x)(-\partial^2\delta_{\mu\nu} + \partial_\mu\partial_\nu)A_\nu(x) \tag{16.33}$$

So we would be tempted to think that we can write as usual

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}A_\mu(x)\exp\left\{-\frac{1}{2}\int d^dx[A_\mu(-\partial^2\delta_{\mu\nu} + \partial_\mu\partial_\nu)A_\nu]\right\} \\
&" = " \quad Z[0]\exp\left\{\frac{1}{2}\int d^dx d^dy J_\mu(x)G_{\mu\nu}(x,y)J_\nu(y)\right\}
\end{aligned}
\tag{16.34}
$$

where $G_{\mu\nu}$ is the inverse of the operator $(-\partial^2\delta_{\mu\nu} + \partial_\mu\partial_\nu)$. However, that is not possible, since the operator has zero modes, $A_\mu = \partial_\mu\chi$, since

$$(-\partial^2\delta_{\mu\nu} + \partial_\mu\partial_\nu)\partial_\nu\chi = 0, \tag{16.35}$$

and an operator with zero modes (zero eigenvalues) cannot be inverted. This is a consequence of gauge invariance, since these zero modes are exactly the pure gauge modes, $\delta A_\mu = \partial_\mu\chi$. So in order to be able to invert the operator, we need to get rid of these zero modes, or gauge transformations, from the path integral. Since physical quantities are related to ratios of path integrals, it will suffice if we can *factorize* the integration over the gauge invariance,

$$Vol(G_{inv}) = \prod_{x\in\mathbb{R}^d}\int d\chi(x) \tag{16.36}$$

or the volume of the local $U(1)$ symmetry group,

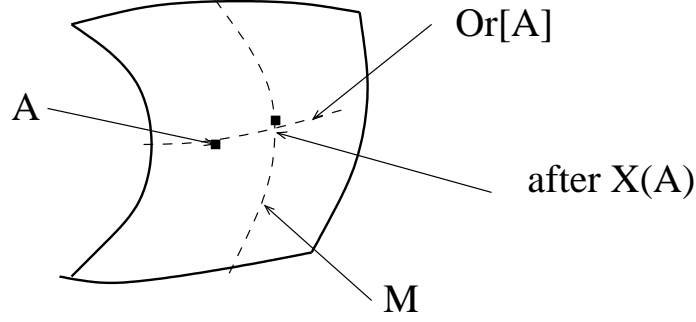$$G_{inv} = \prod_{x\in\mathbb{R}^d}U_x(1) \tag{16.37}$$

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Figure 37: The gauge fixed configuration is at the intersection of the orbit of A, the gauge transformations of a gauge field configuration A, and the space $\mathcal{M}$ of all possible gauge conditions.

To do this, we will consider the set of more general covariant gauge conditions

$$\partial_\mu A_\mu = c(x) \tag{16.38}$$

instead of the Euclidean Lorenz gauge $\partial_\mu A_\mu = 0$.

Consider an arbitrary gauge field A and the *orbit of A*, $Or(A)$, obtained from all possible gauge transformations of this A, see Fig.37. Then consider also the space all possible gauge conditions, $\mathcal{M}$. We should have that there is only one point at the intersection of $Or(A)$ and $\mathcal{M}$, i.e. that there is a unique gauge transformation of A, with $\chi^{(A)}$, that takes us into the gauge condition. We will suppose this is the case in the following (in fact, there is some issue with this assumption in the nonabelian case at least, it is not true for *large* nonabelian gauge transformations, there exists what are called *Gribov copies* which arise for large difference between A's, but we will ignore this here, since anyway we deal with the abelian case). Consider then the definition of χ^A,

$$\partial^2 \chi^{(A)}(x) = -\partial_\mu A_\mu(x) + c(x) \tag{16.39}$$

There is a unique solution in Euclidean space, considering that $c(x)$ falls off sufficiently fast at infinity, and similarly for A and $\chi^{(A)}$.

Define the gauge field A transformed by some gauge transformation χ as

$$^\chi A_\mu(x) \equiv A_\mu(x) + \partial_\mu \chi(x) \tag{16.40}$$

Then we have the following

Lemma

$$\int \prod_{x \in \mathbb{R}^d} d\chi(x) \prod_{y \in \mathbb{R}^d} \delta(-\partial_\mu(^\chi A_\mu(y)) + c(y)) = \frac{1}{\det(-\partial^2)} \tag{16.41}$$

Proof:

We have

$$-\partial_\mu(^\chi A_\mu) + c = -\partial^2 \chi - \partial_\mu A_\mu + c = -\partial^2 \chi + \partial^2 \chi^{(A)} \tag{16.42}$$

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so we shift the integration from $\int d\chi$ to $\chi \to \chi - \chi^{(A)}$, absorbing the second term above. Then we obtain

$$\int \prod_{x \in \mathbb{R}^d} d\chi(x) \prod_{y \in \mathbb{R}^d} \delta(-\partial_\mu(^X A_\mu(y)) + c(y)) = \int \prod_x d\chi(x) \prod_y \delta(-\partial^2 \chi(y)) \qquad (16.43)$$

But note that this is a continuum version of the discrete relation

$$\int \prod_{i=1}^n d\chi_i \prod_{j=1}^n \delta(\Delta_{ij}\chi_j) = \frac{1}{\det \Delta} \qquad (16.44)$$

for the operator

$$-\partial^2(x, y) \equiv -\partial_x^2 \delta^{(d)}(x - y) \qquad (16.45)$$

so we have proved the lemma. q.e.d.

We can then write the lemma with the determinant on the other side as

$$"1" = \int \prod_x d\chi(x) \det(-\partial^2) \prod_y \delta(-\partial_\mu(^X A_\mu(y)) + c(y)) \qquad (16.46)$$

and we can do a gaussian integration over the gauge conditions $c(x)$ of this result as

$$\begin{aligned} "1(\alpha)" &= \int \mathcal{D}c(x) e^{-\frac{1}{2\alpha}\int d^d x c^2(x)} "1" \\ &= \int \prod_x d\chi(x) \det(-\partial^2) e^{-\frac{1}{2\alpha}\int d^d x (\partial_\mu(^X A_\mu(x)))^2} \end{aligned} \qquad (16.47)$$

where in the second equality we used the lemma and used the delta function in it. Note that "$1(\alpha)$" means that it is a function of the arbitrary number α only, so it is not important, since this will cancel out in the ratio of path integrals which defines observables.

Then in the path integral for an obsevable $\mathcal{O}(A)$, we obtain

$$\int \mathcal{D}A e^{-S[A]} \mathcal{O}[A] "1(\alpha)" = \det(-\partial^2) \int \prod_x d\chi(x) \int \mathcal{D}A e^{-S[A]-\frac{1}{2\alpha}\int d^d x (\partial^X A)^2} \mathcal{O}(A) \qquad (16.48)$$

Now we change variables in the path integral from A to $A - \partial_\mu \chi$, such as to turn $\partial_\mu(^X A_\mu) \to \partial_\mu A_\mu$. Then the dependence on χ disappears from inside the integral, so the integration over the gauge modes finally factorizes,

$$\int \mathcal{D}A e^{-S[A]} \mathcal{O}[A] "1(\alpha)" = \left[\det(-\partial^2) \int \prod_x d\chi(x) \right] \int \mathcal{D}A e^{-S[A]-\frac{1}{2\alpha}\int d^d x (\partial A)^2} \mathcal{O}(A) \qquad (16.49)$$

That means that in correlator variables, defined as ratios of the path integral with insertion to the one without, to cancel out the bubble diagrams, the integration over the gauge modes cancels,

$$< \mathcal{O}(A...) > = \frac{\int \mathcal{D}(A...) \mathcal{O}(A...) e^{-S(A)} "1(\alpha)"}{\int \mathcal{D}(A...) e^{-S(A)} "1(\alpha)"}$$

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$$= \frac{\int \mathcal{D}(A...)\mathcal{O}(A...)e^{-S_{eff}(A,...)}}{\int \mathcal{D}(A...)e^{-S_{eff}(A...)}} \tag{16.50}$$

Here the effective Lagrangean is

$$\mathcal{L}_{eff} = \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\alpha}(\partial_\mu A_\mu) \tag{16.51}$$

where the extra term is called the *gauge fixing term*, as it is not gauge invariant, so the remaining path integral is without zero modes.

Photon propagator

We can again partially integrate and obtain for the effective action

$$S_{eff}(A) = \frac{1}{2}\int d^d x A_\mu \left(-\partial^2 \delta_{\mu\nu} + \left(1 - \frac{1}{\alpha}\right)\partial_\mu\partial_\nu\right)A_\nu \tag{16.52}$$

where the operator in in brackets, $(G^{(0)})^{-1}_{\mu\nu}$ is now invertible. We write in momentum space

$$S_{eff}(A) = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d} A_\mu(-k)(G^{(0)})^{-1}_{\mu\nu}(k)A_\nu(k) \tag{16.53}$$

Since

$$(G^{(0)})^{-1}_{\mu\nu}(k) = k^2 \delta_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right)k_\mu k_\nu, \tag{16.54}$$

we have

$$\left(k^2 \delta_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right)k_\mu k_\nu\right)G^{(0)}_{\nu\lambda} = \delta_{\mu\lambda} \tag{16.55}$$

meaning that finally the photon propagator is

$$G^{(0)}_{\mu\nu}(k) = \frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right) \tag{16.56}$$

Then in the case of $\alpha = 1$ ($\lambda = 1/\alpha = 1$), the "Feynman gauge" (a misnomer, as we said) we have the Klein-Gordon propagator

$$G^{(0)}_{\mu\nu}(k; \alpha = 1) = \frac{1}{k^2}\delta_{\mu\nu} \tag{16.57}$$

Important concepts to remember

- The physical gauge is $A_0 = 0$ and $\partial^\mu A_\mu = 0$, or $\vec{\nabla} \cdot \vec{A} = 0$, and here the equation of motion is KG.

- Quantization in the physical gauge is done for only the two physical transverse polarizations, for which we expand in usual harmonic oscillator creation and annihilation operators.

- Covariant gauge quantization is done keeping all the 4 polarization of the gauge field, but imposing $\partial^\mu A_\mu^{(+)} |\psi> = 0$ on physical states.

- Timelike modes have negative norm, but they cancel in calculations against longitudinal modes. In particular, the physical condition says the longitudinal and timelike modes should match inside physical states.

- Fadeev-Popov path integral quantization factorizes the gauge modes, and leaves an effective action which contains a gauge fixing term.

- The photon propagator in Feynman gauge is the KG propagator.

Further reading: See chapters 6.1,6.2,6.3.1 in [4], 7.3 in [1] and 7.1 in [3].

Exercises, Lecture 16

1) a) Prove that for quantization in physical gauge the quantum energy is

$$E = \frac{1}{2} \int d^3x (\vec{E}^2 + \vec{B}^2) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^3} \frac{k^0}{2} [a^{(\lambda)\dagger}(k)a^{(\lambda)}(k) + a^{(\lambda)}(k)a^{\dagger(\lambda)}(k)] \qquad (16.58)$$

b) Consider the state in covariant quantization

$$|\psi> = \left(a^{(1)\dagger}(k_1) - a^{(2)\dagger}(k_3)\right)\left(a^{(0)\dagger}(k_2) + a^{(3)\dagger}(k_4)\right)|0> \qquad (16.59)$$

Is it physical? Why?

2) Using the effective action S_{eff} (with gauge fixing term), for $k^\mu = (k,0,0,k)$, write down the equation of motion in momentum space, *separately* for the longitudinal, timelike and transverse modes, at arbitrary "gauge" α.

17 Lecture 17. Generating functional for connected Green's functions and the effective action (1PI diagrams)

Generating functional of connected Green's functions

As we saw, $Z[J]$ is the generating functional of the full (connected and disconnected) Green's functions.

But now we will prove that the generating functional of connected Green's functions is $-W[J]$, where

$$Z[J] = e^{-W[J]} \tag{17.1}$$

We define the n-point Green's functions in the presence of a nonzero source J as

$$G_n(x_1, ..., x_n)_J = \frac{\delta^n}{\delta J(x_1)...\delta J(x_n)} Z[J] \tag{17.2}$$

(before we had defined the Green's functions at nonzero J). We will denote them by a box with a J inside it, and n lines ending on points $x_1, ..., x_n$ coming out of it, as in Fig.38. A box without any lines from it is the 0-point function, i.e. $Z[J]$, see Fig.38.
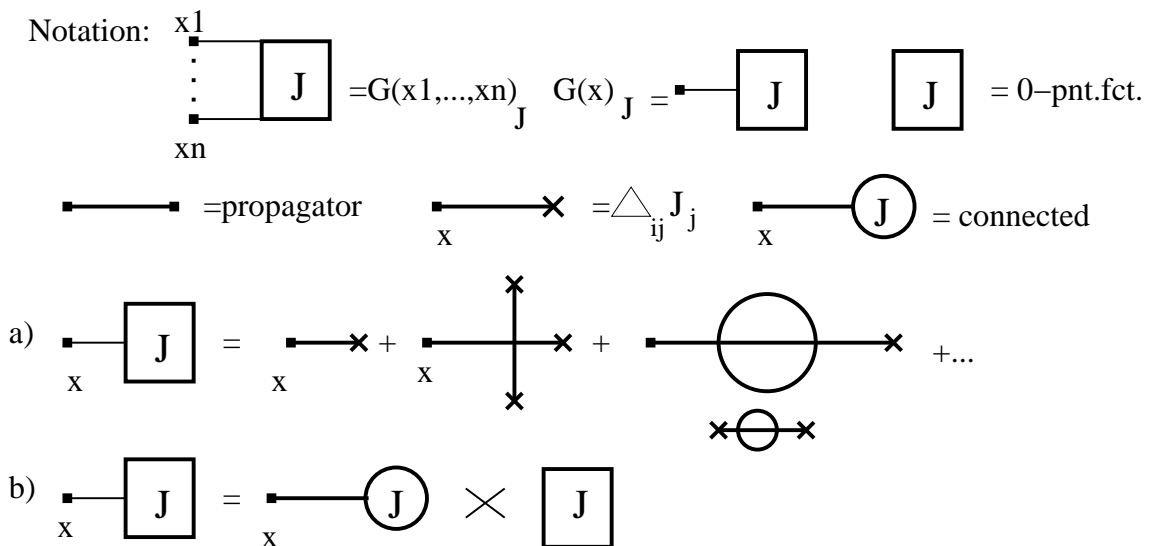

Figure 38: Notation for Green's functions, followed by the diagrammatic expansion for the 1-point function with source J (a), which can be seen diagrammatically to factorize into the connected part times the vacuum bubbles (b).

For the 1-point function in ϕ^4 theory, writing the perturbative expansion as we already did (before we put $J = 0$), we have

$$G(x)_J = \int d^d y \Delta(x - y) J(y) - g \int d^d z\, d^d y_1 d^d y_2 d^d y_3$$

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$$\Delta(x-z)\Delta(y_1-z)\Delta(y_2-z)\Delta(y_3-z)J(y_1)J(y_2)J(y_3) + ... \qquad (17.3)$$

and the Feynman diagrams that correspond to it are: line from x to a cross representing the source, line from x to a vertex from which 3 other lines end on crosses. There is also: line from x to a cross with a loop on it (setting sun), etc., see Fig.38 a for the diagrammatic form of the above equation.

A cross is $\int J(x)$, or $\sum_i J_i$ in the discretized version. A propagator is a line, and so a line between a point and a cross is $\Delta_{ij}J_j = \int d^d y \Delta(x-y)J(y)$, see Fig.38.

As we said, a box with J inside and a line ending on an external point is the 1-point function $G(x)_J$. We also draw the connected piece by replacing the box with a circle (Fig.38).

Then the full one-point function is the connected one-point function times the 0-point function (vacuum bubbles, or $Z[J]$), in exactly the same way as we showed it in the operator formalism, see Fig.38b. Note that this only works for the one-point functions, since for instance for the 2-point function we also have contributions like x_1 connected with a cross, and x_2 connected with another cross, as in Fig.39.
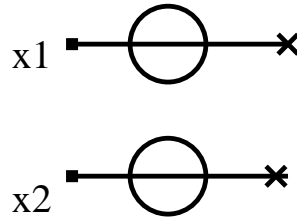

Figure 39: For the two-point function with a source, the factorization doesn't work anymore, since we can have disconnected pieces without vacuum bubbles, as exemplified here.

The above diagrammatic equation is written as

$$\frac{\delta Z[J]}{\delta J(x)} = -\frac{\delta W[J]}{\delta J(x)}Z[J] \qquad (17.4)$$

where here by definition $-W[J]$ is the generating functional of connected diagrams. The solution of this equation is

$$Z[J] = \mathcal{N}e^{-W[J]} \qquad (17.5)$$

as we said it should be. Here $W[J]$ is called free energy, since exactly like in thermodynamics, the partition function is the exponential of minus the free energy.

Effective action

Another functional of interest is the *effective action*, which we will define soon, and which we will find that is the generating functional of one-particle irreducible (1PI) Green's functions. One-particle irreducible means that the diagrams cannot be separated into two disconnected pieces by cutting a single propagator. An example of a one-particle reducible diagram would be two lines between two points, then one line to another, and then another
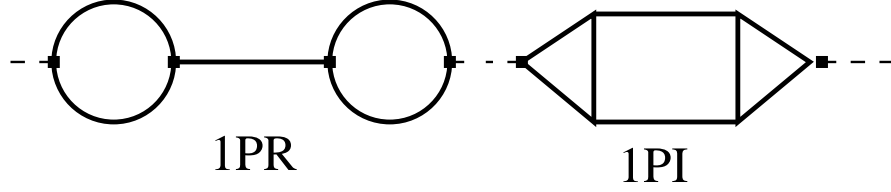
Figure 40: One particle reducible graph (left): the middle propagator can be cut, separating the graph into two pieces, versus one particle irreducible graph (right): one cannot separate the graph into 2 pieces by cutting a single propagator.

two to the last point, see Fig.40. To make it 1PI we would need to add another line for the points in the middle.

The effective action we will find is a Legendre transform of the free energy, exactly like we take a Legendre transform of the free energy in thermodynamics to obtain (Gibbs) potentials like $G = F - Q\Phi$.

We need to define first the object conjugate to the source J (just like the electric potential Φ is conjugate to the charge Q for G above). This is the *classical field* $\phi_{cl}[J]$ in the presence of an external current J. As the name suggests, it is the properly normalized VEV of the quantum field, which replaces the field of classical field theory at the quantum level,

$$\phi_{cl}[J] \equiv \frac{<0|\hat{\phi}(x)|0>_J}{<0|0>_J} = \frac{1}{Z[J]} \int \mathcal{D}\phi e^{-S[\phi]+J\cdot\phi}\phi(x) = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \frac{\delta(-W[J])}{\delta J(x)} = G_1^c(x; J)$$

(17.6)

that is, the connected one-point function in the presence of the external source.

e.g. free scalar field theory in the discretized version

Consider the action

$$S_0 - J \cdot \phi = \frac{1}{2} \int d^d x [(\partial_\mu \phi)^2 + m^2\phi^2] - J \cdot \phi = \frac{1}{2}\phi_i \Delta_{ij}^{-1}\phi_j - J_k\phi_k$$

(17.7)

The classical equation of motion for this is and its solution are

$$\Delta_{ij}^{-1}\phi_j - J_i = 0 \Rightarrow \phi_i = \Delta_{ij}J_j$$

(17.8)

which is the Coulomb law.

On the other hand, we saw that the free partition function is $Z_0 = e^{\frac{J_i\Delta_{ij}J_j}{2}}$, so that the free energy is

$$-W_0[J] = \frac{J_i\Delta_{ij}J_j}{2}$$

(17.9)

implying that in the free theory the classical field is

$$\phi_i^{cl(0)} = \frac{\delta}{\delta J_i}(-W_0[J]) = \Delta_{ij}J_j$$

(17.10)

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that is, the same as the solution of the classical equation of motion.

This is so, since free fields are *classical*, no interactions means there are no quantum fluctuations.

The classical field is represented by a propagator from the point i to a circle with a J inside it.

1PI Green's function

Some diagrams for the classical field in ϕ^4 theory are as follows: line from i to a cross; line from i to a vertex, then from the vertex 3 lines to crosses; the same just adding a setting sun on propagators; the vertex diagram with a circle surrounding the vertex and crossing all 4 propagators; etc., see Fig.41.
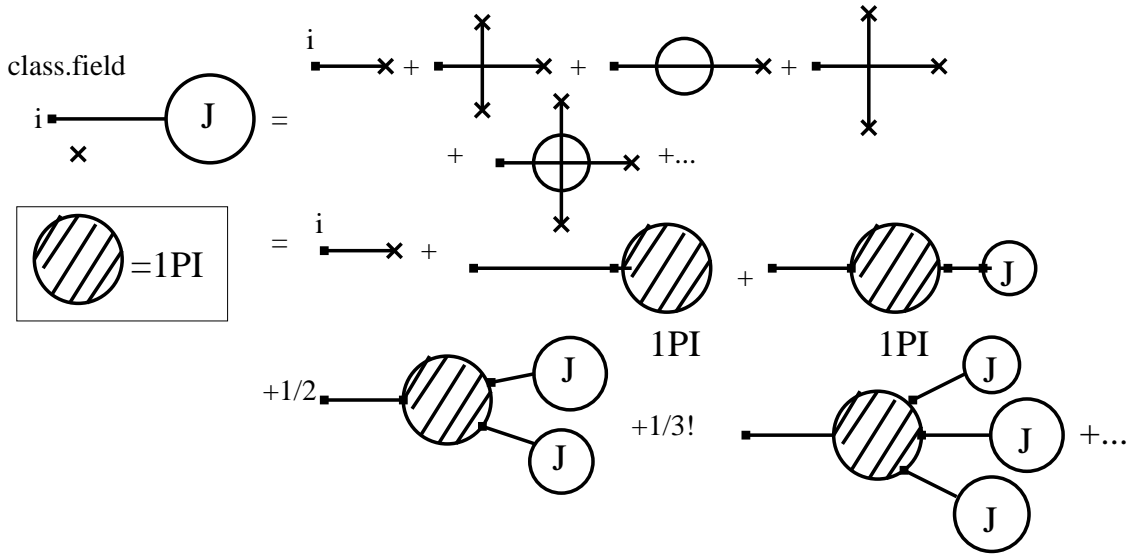

Figure 41: Diagrammatic expansion of the classical field. It can be reorganized into a self-consistent equation, in terms of the classical field and the 1PI n-point functions.

If we think about how to write this in general, and also what to write for a general theory instead of ϕ^4, denoting 1PI diagrams by a shaded blob, the relevant diagrams can be written in a self-consistent sort of way as: line from i to a cross; plus propagator from i to a 1PI blob with one external point; plus a propagator from i to a 1PI blob with two external points, and the external point connected with the same classical field, plus 1/2 times the same with 3-point 1PI with 2 external classical fields; etc, see Fig.41.

We then obtain the following self-consistency equation for the classical field

$$\phi_i^{cl} = \Delta_{ij}\Big[J_j - \Big(\Gamma_j + \Pi_{jk}\phi_k^{cl} + \frac{1}{2}\Gamma_{jkl}\phi_k^{cl}\phi_l^{cl} + ... + \frac{1}{(n-1)!}\Gamma_{ji_1...i_{n-1}}\phi_{i_1}^{cl}...\phi_{i_{n-1}}^{cl} + ...\Big)\Big]$$
(17.11)

Note that only the Π_{jk} was not called Γ_{jk}, the rest of the 1PI n-point functions were called $\Gamma_{i_1...i_n}$, we will see shortly why.

We define the generating functional of 1PI Green's functions

$$\hat{\Gamma}(\phi^{cl}) = \Gamma_i \phi_i^{cl} + \frac{1}{2}\Pi_{ij}\phi_i^{cl}\phi_j^{cl} + \frac{1}{3!}\Gamma_{ijk}\phi_i^{cl}\phi_j^{cl}\phi_k^{cl} + ... \tag{17.12}$$

such that the 1PI n-point functions are given by its multiple derivatives

$$\Gamma_{i_1...i_n} = \frac{\delta^n}{\delta\phi_{i_1}^{cl}...\delta\phi_{i_n}^{cl}}\hat{\Gamma}(\phi^{cl})|_{\phi^{cl}=0} \tag{17.13}$$

(only for two points we have Π_{ij}). Then the self-consistency equation (17.11) is written as

$$-\frac{\delta W}{\delta J_i} \equiv \phi_i^{cl} = \Delta_{ij}\left(J_j - \frac{\delta\hat{\Gamma}}{\delta\phi_j^{cl}}\right) \tag{17.14}$$

In turn, this is written as

$$\frac{\delta\hat{\Gamma}}{\delta\phi_i^{cl}} + \Delta_{ij}^{-1}\phi_j^{cl} = J_i \Rightarrow \frac{\delta}{\delta\phi_i^{cl}}\left[\hat{\Gamma}(\phi^{cl}) + \frac{1}{2}\phi_k^{cl}\Delta_{kj}^{-1}\phi_j^{cl}\right] = J_i \tag{17.15}$$

That means that we can define the *effective action*

$$\Gamma(\phi^{cl}) = \hat{\Gamma}(\phi^{cl}) + \frac{1}{2}\phi_k^{cl}\Delta_{kj}^{-1}\phi_j^{cl} \tag{17.16}$$

such that we have

$$\frac{\delta\Gamma(\phi^{cl})}{\delta\phi_i^{cl}} = J_i \tag{17.17}$$

just like the classical equation of motion is

$$\frac{\delta S}{\delta\phi_i} = J_i \tag{17.18}$$

In other words, ϕ^{cl} plays the role of the field ϕ in the classical field theory, and instead of the classical action S, we have the effective action Γ, which contains all the quantum corrections. For the effective action we define

$$\Gamma(\phi^{cl}) = \sum_{N\geq 1}\frac{1}{N!}\Gamma_{i_1...i_N}\phi_{i_1}^{cl}...\phi_{i_N}^{cl} \tag{17.19}$$

where the only difference from the generating functional of the 1PI diagrams is in the 2-point function, where

$$\Gamma_{ij} = \Pi_{ij} + \Delta_{ij}^{-1} \tag{17.20}$$

which does not have a diagrammatic interpretation (we have Δ^{-1}, not Δ), see Fig.42. Otherwise, the effective action contains only 1PI diagrams.

Theorem The effective action is the Legendre transform of the free energy, i.e.

$$e^{-\Gamma(\phi^{cl})+J_i\phi_i^{cl}} = \int \mathcal{D}\phi e^{-S[\phi]+J_i\phi_i} = e^{-W[J]} \tag{17.21}$$

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Figure 42: The effective action is not technically diagrammatic, because of the inverse free propagator.

or

$$\Gamma[\phi^{cl}] = W[J] + J_k \phi_k^{cl} \tag{17.22}$$

Proof: We take the derivative with respect to ϕ_i^{cl} of the needed relation above, and obtain

$$\frac{\delta\Gamma}{\delta\phi_i^{cl}} = J_i = \frac{\delta W}{\delta J_k}\frac{\delta J_k}{\delta\phi_i^{cl}} + \frac{\delta J_k}{\delta\phi_i^{cl}}\phi_k^{cl} + J_i = -\phi_k^{cl}\frac{\delta J_k}{\delta\phi_i^{cl}} + \frac{\delta J_k}{\delta\phi_i^{cl}}\phi_k^{cl} + J_i = J_i \tag{17.23}$$

i.e. we obtain an identity, as wanted. q.e.d.

The effective action contains all the information about the full quantum field theory, so in principle, if we were able to find the exact effective action, it would be equivalent to solving the quantum field theory.

The connected two-point function

The connected one-point function is

$$G_i^C[J] = -\frac{\delta W[J]}{\delta J_i} = \phi_i^{cl}[J] \tag{17.24}$$

and we obtain the connected two-point function by taking a derivative,

$$G_{ij}^C[J] = -\frac{\delta}{\delta J_i}\frac{\delta}{\delta J_j}W[J] = \frac{\delta\phi_i^{cl}}{\delta J_j} \tag{17.25}$$

Then we can substitute (17.11) and write

$$G_{ij}^C[J] = \Delta_{ij} - \Delta_{ik}\Pi_{kl}\frac{\delta\phi_l^{cl}}{\delta J_j} + ... \tag{17.26}$$

where the terms that were dropped vanish when $J = 0$, with the assumption that $\phi_i^{cl}[J = 0] = 0$, which is a reasonable requirement in quantum field theory (classically, at zero source we should have no field, and so quantum mechanically under the same condition we should have no field VEV, or rather, this should be removed by renormalization, of which we will learn next semester). Then at $J = 0$, the two point function obeys the equation

$$G_{ij}^C = \Delta_{ij} - \Delta_{ik}\Pi_{kl}G_{lj}^C \tag{17.27}$$

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which gives

$$(\delta_{ik} + \Delta_{il}\Pi_{lk})G^c_{kj} = \Delta_{ij} \Rightarrow$$
$$(\Delta^{-1}_{mk} + \Pi_{mk})G^C_{kj} = \delta_{mj} \tag{17.28}$$

which means that the connected two-point function is the inverse of the two-point term in the effective action,

$$\Gamma_{ik}G^C_{kj} = \delta_{ij} \tag{17.29}$$

or explicitly

$$G^C = (1 + \Delta\Pi)^{-1}\Delta = (1 - \Delta\Pi + \Delta\Pi\Delta\Pi - \Delta\Pi\Delta\Pi\Delta\Pi + ...)\Delta \Rightarrow$$
$$G^C_{ij} = \Delta_{ij} - \Delta_{il}\Pi_{lk}\Delta_{kj} + \Delta_{il}\Pi_{lk}\Delta_{km}\Pi_{mn}\Delta_{nj} - ... \tag{17.30}$$

So the connected two-point function is a propagator; minus a propagator, two-point 1PI, propagator again; plus a propagator, two-point 1PI, propagator, two-point 1PI, propagator; minus..., as in Fig.43.
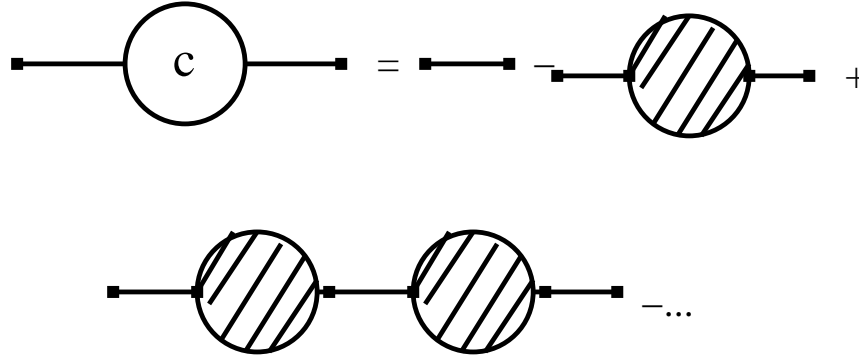

Figure 43: Diagrammatic equation relating the connected 2-point function and the 1PI 2-point function.

So apart from signs, it is obvious from diagrammatics.

Classical action as the generating functional of tree diagrams

In the classical $\hbar \to 0$ limit, the effective action turns into the classical action (all the quantum corrections vanish). In the same limit, all the 1PI diagrams disappear and we are left with only the vertices in the action (e.g. the 4-point vertex in the case of the ϕ^4 theory). Therefore from all the diagrams we are left with only the tree diagrams.

But note that having only tree diagrams is still *quantum mechanics*, since we still calculate amplitudes for quantum transition, probabilities, and can have interference, etc. The point however is that we don't have *quantum field theory* (*second* quantization), but we still have quantum mechanics. For the transition amplitudes we still have quantum mechanics rules, and we know for instance that some of the early successes of QED come from tree diagram calculations. The point is of course that when we talk about external particles, and

transitions for them, that is a quantum mechanics process: the classical field is a a collection of *many* particles, but if we have a single particle we use quantum mechanics.

As an example, let's consider the classical ϕ^4 theory,

$$S[\phi] = \frac{1}{2}\phi_i \Delta_{ij}^{-1} \phi_j + \frac{\lambda}{4!} \sum_i \phi_i^4 \qquad (17.31)$$

with equation of motion

$$\Delta_{ij}^{-1} \phi_j + \frac{\lambda}{3!} \phi_i^3 = J_i \qquad (17.32)$$

with solution

$$\phi_i = \Delta_{ij} J_j - \frac{\lambda}{3!} \Delta_{ij} \phi_j^3 \qquad (17.33)$$

This is a self-consistent solution (like for ϕ^{cl} at the quantum level), written as the classical field (line connected with a circle with J in the middle, at classical level)= line ending in cross $-\lambda/3! \times$ line ending in vertex connected with 3 classical fields.

It can be solved perturbatively, by replacing the classical field first with the line ending in cross, substituting on the right hand side of the equation, then the right hand side of the resulting equation is reintroduced instead of the classical field, etc. In this way, we obtain that the classical field (at classical level) equals the line ending in cross; $-\lambda/3! \times$ line ending in vertex, with 3 lines ending in cross; $+3(-\lambda/3!)^2 \times$ a tree with one external point and 5 crosses; etc. In this way we obtain all possible trees with one external point and many external crosses. This is also what we would get from ϕ^{cl} by keeping only the tree diagrams.

Important concepts to remember

- The generating functional of connected diagrams is $-W[J] = \ln Z[J]$, known as the free energy by analogy with thermodynamics.

- The classical field, defined as the normalized VEV of the quantum field, is the connected one-point function in the presence of a source J.

- The generating functional of 1PI diagrams is $\hat{\Gamma}(\phi^{cl})$, and the effective action Γ is the same, just in the two-point object we need to add Δ^{-1}, i.e. we have $\Gamma_{ij} = \Pi_{ij} + \Delta_{ij}^{-1}$.

- The effective action $\Gamma[\phi^{cl}]$ is the Legendre transform of the free energy $W[J]$.

- The connected two-point function at $J = 0$, G_{ij}^C, is the inverse of the two-point effective action Γ_{ij}.

- The classical action is the generating functional of tree diagrams. Tree diagrams still can be used for quantum mechanics, but we have no quantum field theory.

Further reading: See chapters 2.5, 4.1 in [4] and 3.1, 3.3 in [3].

Exercises, Lecture 17

1) Consider the hypothetical effective action in p space (here $\Delta^{-1} = p^2$, as usual for massless scalars)

$$\Gamma(\phi^{cl}) = \int \frac{d^4 p}{(2\pi)^4} \phi^{cl}(p)(p^2 + Gp^4)\phi^{cl}(-p)$$

$$+ \exp\left\{\lambda \int \left(\prod_{i=1}^{4} \frac{d^4 p_i}{(2\pi)^4}\right) \delta^4(p_1 + p_2 + p_3 + p_4)\phi^{cl}(p_1)\phi^{cl}(p_2)\phi^{cl}(p_3)\phi^{cl}(p_4)\right\} (17.34)$$

a) Calculate the 1PI 4-point Green's function $\Gamma_{p_1 p_2 p_3 p_4}$.
b) Calculate the 1PI 2-point function $\Pi_{p,-p}$ and the connected 2-point function.

2) Consider the theory

$$S[\phi] = \frac{1}{2}\phi_i \Delta_{ij}^{-1} \phi_j + \frac{\lambda}{3!} \sum_i \phi_i^3 \tag{17.35}$$

in the presence of sources J_i. Write down the perturbative diagrams for the classical equation of motion up to (including) order λ^3.

18 Lecture 18. Dyson-Schwinger equations and Ward identities

At the classical level, we have the classical equations of motion

$$\frac{\delta S[\phi]}{\delta \phi_i} - J_i = 0 \tag{18.1}$$

But we know from Ehrenfest's theorem for quantum mechanics that in the quantum theory these classical equations should hold on the quantum average, or VEV (note that we are talking about the quantum average of the equation for the field, not the classical equation for the quantum average of the field!). In the path integral formalism, these turn out to be the Dyson-Schwinger equations in a very easy (almost trivial) way. Historically, these equations were found in the operator formalism, and it was quite nontrivial to find them.

Consider the following simple identity

$$\int_{-\infty}^{+\infty} dx \frac{d}{dx} f(x) = 0 \tag{18.2}$$

if $f(\pm\infty) = 0$, and generalize it to the path integral case. In the path integral in Euclidean space, we have something similar, just that the boundary condition is even simpler: instead of fields going to zero at $t = \pm\infty$, we use periodic boundary conditions (Euclidean time is periodic, namely is a circle of radius $R \to \infty$), hence we have

$$0 = \int \mathcal{D}\phi \frac{\delta}{\delta \phi_i} e^{-S[\phi] + J \cdot \phi} \tag{18.3}$$

Writing explicitly, we have

$$\begin{aligned}
0 &= \int \mathcal{D}\phi \left[-\frac{\delta S[\phi]}{\delta \phi_i} + J_i \right] e^{-S[\phi] + J \cdot \phi} \\
&= \left[-\frac{\delta S}{\delta \phi_i} \Big|_{\phi = \frac{\delta}{\delta J_i}} + J_i \right] Z[J],
\end{aligned} \tag{18.4}$$

where in the second equality we used the usual fact that we have for example $\int \mathcal{D}\phi \phi_i e^{-S[\phi] + J \cdot \phi} = \int \mathcal{D}\phi (\delta/\delta J_i) e^{-S[\phi] + J \cdot \phi} = \delta/\delta J_i Z[J]$. This equation is the Dyson-Schwinger equation. Now it appears trivial, but it was originally derived from an analysis of Feynman diagrams, where it looks nontrivial. Now, we will derive the relation for Feynman diagrams from this.

To see it, we first specialize for a theory with quadratic kinetic term, which is a pretty general requirement. Therefore consider

$$S[\phi] = \sum_{ij} \frac{1}{2} \phi_i \Delta_{ij}^{-1} \phi_j + S_I[\phi] \tag{18.5}$$

Then we obtain (by differentiating and multiplying with Δ_{li})

$$\sum_i \Delta_{li} \frac{\delta S[\phi]}{\delta \phi_i} = \phi_l + \sum_i \Delta_{li} \frac{\delta S_I[\phi]}{\delta \phi_i} \tag{18.6}$$

and substituting it in the Dyson-Schwinger equation (18.4), we get

$$\left[-\frac{\delta}{\delta J_l} - \sum_i \Delta_{li} \frac{\delta S_I}{\delta \phi_i}\Big|_{\phi = \frac{\delta}{\delta J}} + \sum_i \Delta_{li} J_i \right] Z[J] = 0 \tag{18.7}$$

so that we get the Dyson-Schwinger equation for $Z[J]$,

$$\frac{\delta}{\delta J_l} Z[J] = \sum_i \Delta_{li} J_i Z[J] - \sum_i \Delta_{li} \frac{\delta S_I[\phi]}{\delta \phi_i}\Big|_{\phi = \frac{\delta}{\delta J}} Z[J] \tag{18.8}$$

From it, we can derive a Dyson-Schwinger equation for the full Green's functions (with a diagrammatic interpretation) if we choose a particular interaction term.

Let's consider a ϕ^3 plus ϕ^4 interaction

$$S_I[\phi] = \frac{g_3}{3!} \sum_i \phi_i^3 + \frac{g_4}{4!} \sum_i \phi_i^4 \tag{18.9}$$

Then the Dyson-Schwinger equation (18.8) becomes

$$\frac{\delta}{\delta J_l} Z[J] = \sum_i \Delta_{li} J_i Z[J] - \sum_i \Delta_{li} \left[\frac{g_3}{2!} \frac{\delta^2}{\delta J_i \delta J_i} + \frac{g_4}{3!} \frac{\delta^3}{\delta J_i \delta J_i \delta J_i} \right] Z[J] \tag{18.10}$$

We put $l = i_1$ and then take $(\delta/\delta J_{i_2})...(\delta/\delta J_{i_n})$ and finally put $J = 0$ to obtain a relation between (full) Green's functions,

$$G^{(n)}_{i_1...i_n} = \Delta_{i_1 i_2} G^{(n-2)}_{i_3...i_n} + \Delta_{i_1 i_3} G^{(n-2)}_{i_2 i_4...i_n} + ... + \Delta_{i_1 i_n} G^{(n-2)}_{i_2 i_3...i_{n-1}} - \sum_i \Delta_{i_1 i} \left[\frac{g_3}{2!} G^{(n+1)}_{i i i_2...i_n} + \frac{g_4}{3!} G^{(n+2)}_{i i i i_2...i_n} \right] \tag{18.11}$$

We can write a diagrammatic form of this equation. The full Green's function is represented by a box with n external points, $i_1, ..., i_n$. We need to choose one special point, i_1 above, and we can write it on the left of the box, and the other on the right of the box. There are $n - 1$ terms where we connect i_1 with one of the other by a propagator, disconnected from a box with the remaining $n - 2$ external points, minus $1/2! \times$ a term where the propagator from i_1 reaches first an i, where it splits into two before reaching the box, minus $1/3! \times$ a term where the propagator from i_1 reaches first an i where it splits into 3 before reaching the box, see Fig.44 for more details.

The whole perturbative expansion can be obtained by *iterating again* the Dyson-Schwinger equation.

Example We will see this in the example of the two-point function in the above theory. We first write the Dyson-Schwinger equation for it as above,

$$G^{(2)}_{i_1 i_2} = \Delta_{i_1 i_2} G^{(0)} - \frac{g_3}{2!} \sum_i \Delta_{i_1 i} G^{(3)}_{i i i_2} - \frac{g_4}{3!} \sum_i \Delta_{i_1 i} G^{(4)}_{i i i i_2} \tag{18.12}$$
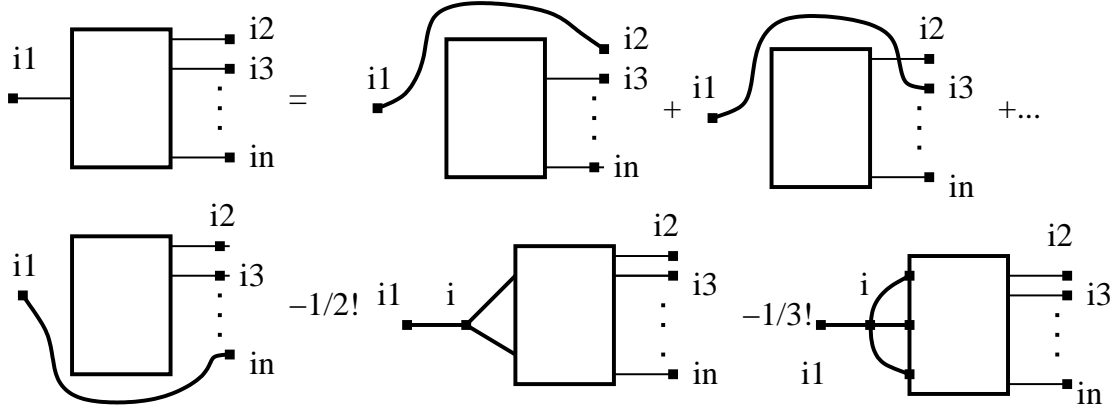
Figure 44: Dyson-Schwinger equation for the n-point function for the ϕ^3 plus ϕ^4 theory.

and then substitute in it the Dyson-Schwinger equations for $G^{(3)}_{iii_2}$ and $G^{(4)}_{iiii_2}$, with the first index on each considered as the special one (note that we cannot write a DS equation for $G^{(0)}$),

$$G^{(3)}_{iii_2} = \Delta_{ii}G^{(1)}_{i_2} + \Delta_{ii_2}G^{(1)}_{i} - \frac{g_3}{2!}\sum_j \Delta_{ij}G^{(4)}_{jjii_2} - \frac{g_4}{3!}\sum_j \Delta_{ij}G^{(5)}_{jjjii_2}$$

$$G^{(4)}_{iiii_2} = \Delta_{ii}G^{(2)}_{ii_2} + \Delta_{ii}G^{(2)}_{ii_2} + \Delta_{ii_2}G^{(2)}_{ii} - \frac{g_3}{2!}\sum_j \Delta_{ij}G^{(5)}_{jjiii_2} - \frac{g_4}{3!}\sum_j \Delta_{ij}G^{(6)}_{jjjiii_2} \quad (18.13)$$

These are represented in Fig.45.

We then obtain

$$G^{(2)}_{ij} = \Delta_{ij}G^{(0)} - \frac{g_3}{2!}\sum_i \Delta_{i_1i}\left[\Delta_{ii}G^{(1)}_{i_2} + \Delta_{ii_2}G^{(1)}_{i} - \frac{g_3}{2!}\sum_j \Delta_{ij}G^{(4)}_{jjii_2} - \frac{g_4}{3!}\sum_j \Delta_{ij}G^{(5)}_{jjjii_2}\right]$$

$$- \frac{g_4}{3!}\sum_i \Delta_{i_1i}\left[\Delta_{ii}G^{(2)}_{ii_2} + \Delta_{ii}G^{(2)}_{ii_2} + \Delta_{ii_2}G^{(2)}_{ii} - \frac{g_3}{2!}\sum_j \Delta_{ij}G^{(5)}_{jjiii_2} - \frac{g_4}{3!}\sum_j \Delta_{ij}G^{(6)}_{jjjiii_2}\right]$$

$$(18.14)$$

This is represented in Fig.46. By iterating it further, we can obtain the whole perturbative expansion.

Let's consider the expansion to order $(g_3)^0(g_4)^1$ of the above second iteration. Since for $G^{(0)}$, the nontrivial terms need to have at least two vertices, it only contributes to order 0+1, i.e. as $= 1 + g_4\Delta_{kk}\Delta_{kk}$. Also the first [] bracket does not contribute, since all the terms have at least one g_3 in them. In the second bracket, the last two terms are of orders $(g_3)^1(g_4)^1$ and $((g_4)^2$ respectively, so also don't contribute. Thus besides the trivial propagator term, we only have the first 3 terms in the second bracket contributing. But in it, the iterated two-point functions must be trivial to contribute to order $(g_3)^0(g_4)^1$ to $G^{(2)}_{ij}$, i.e. must be replaced by 1.
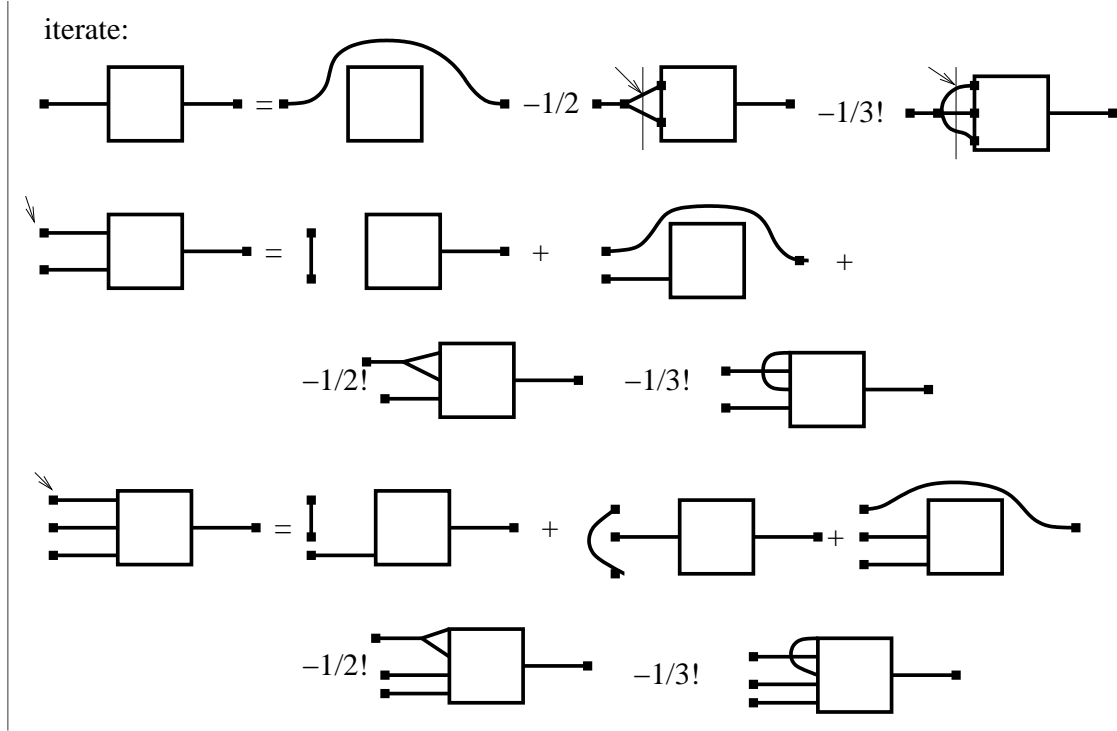
Figure 45: Ingredients for iterating the Dyson-Schwinger equation (step 1). Consider the first iteration for the 2-point function, and choose a specific leg on the rhs. The Dyson-Schwinger equations for the 3-point and 4-point functions with a special leg as chosen above are written below.

We then obtain that to order $(g_3)^0(g_4)^1$, the two-point function $G_{ij}^{(2)}$ is the propagator Δ_{ij} plus 3 times $1/3! \times$ the propagator self-correction, i.e.

$$G_{ij}^{(2)} = \Delta_{ij}(1 + g_4 \Delta_{kk} \Delta_{kk}) + \frac{3g_4}{3!} \sum_k \Delta_{ik} \Delta_{kk} \Delta_{kj} + ... \tag{18.15}$$

The connected part of this is written diagrammatically in Fig.47.

Note that in this way we have obtained the correct symmetry factor for the one-loop diagram, $S = 2$, since $1/S = 3/3!$. That means that the iterated Dyson-Schwinger equation is an algorithmic way to compute the symmetry factors in cases where they are very complicated and we are not sure of the algebra.

Note also that we wrote Dyson-Schwinger equations for the full Green's functions, but we can also write such equations for the connected and 1PI Green's functions, and these are also iterative and self-consistent. We will however not do it here.

Symmetries and Ward identities

Noether's theorem

We will first review Noether's theorem. Consider a global symmetry

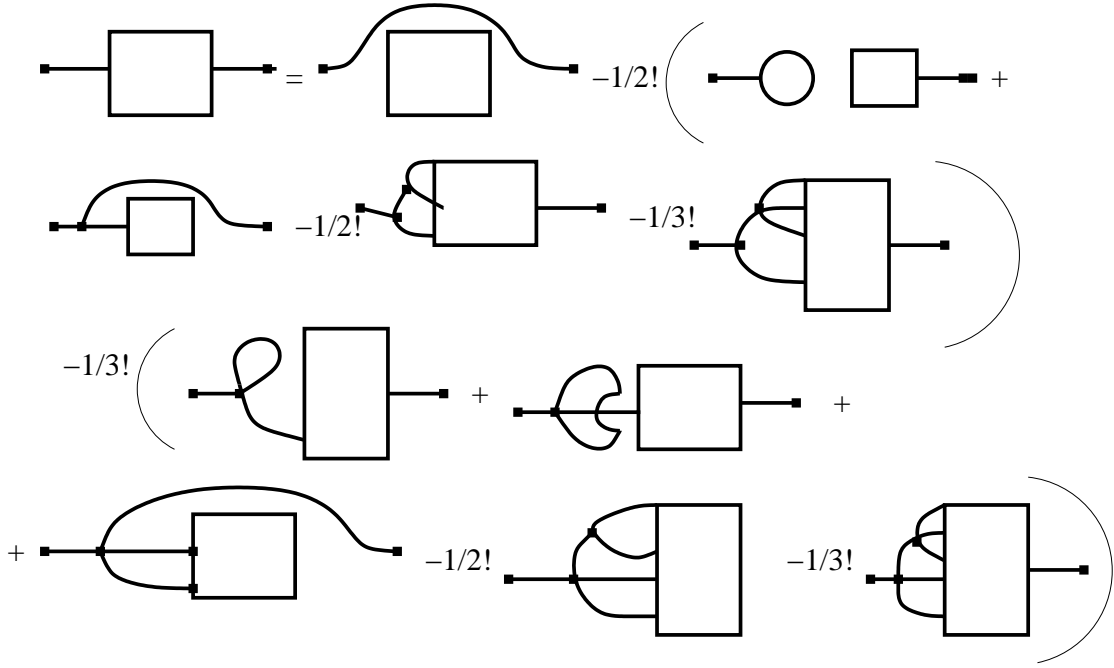$$\delta \phi^i = \epsilon^a (iT^a)_{ij} \phi^j \tag{18.16}$$

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Figure 46: The second iteration of the Dyson-Schwinger equation, using the ingredients from step 1.

where ϵ^a are constant symmetry parameters. We have

$$
\begin{aligned}
0 &= \delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi^i} \delta \phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i(x))} \partial_\mu \delta \phi^i \right] \\
&= \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i(x))} \right) \right) \delta \phi^i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i(x))} \delta \phi^i \right) \right]
\end{aligned}
\tag{18.17}
$$

If the equations of motion are satisfied, the first bracket is zero, and then substituting for $\delta \phi^i$ the transformation under the global symmetry, we obtain that classically (if the equations of motion are satisfied)

$$
(\delta \mathcal{L})_{\text{symm}}^{\text{class}} = i \sum_a \epsilon^a \partial_\mu \left[\sum_{ij} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i(x))} (T^a)_{ij} \phi^j(x) \right]
\tag{18.18}
$$

We then obtain that the current is

$$
j_\mu^a = \sum_{ij} \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^i(x))} (T^a)_{ij} \phi^j(x)
\tag{18.19}
$$

and is classically *conserved on-shell*, i.e.

$$
\partial^\mu j_\mu^a = 0
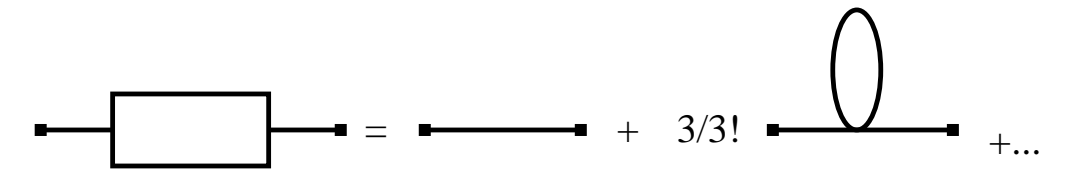\tag{18.20}
$$

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Figure 47: The result of iterating the Dyson-Schwinger equation for the 2-point function, at order $(g_3)^0(g_4)^1$.

which is the statement of the Noether theorem. Since $Q^a = \int d^3x j_0^a$, current conservation means that $dQ/dt = 0$, i.e. the charge is conserved in time.

Under a symmetry transformation, but off-shell (*if the equations of motion are not satisfied*), we still have $\delta S = 0$, which in turn means that

$$0 = \delta\mathcal{L} = i\epsilon^a(T^a)_{ij}\left[\left(\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))}\right)\right)\phi^j + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))}\phi^j\right)\right] \qquad (18.21)$$

We now promote the global transformation (18.16) to a local one, with parameter $\epsilon^a(x)$. Obviously then, it is not a symmetry anymore, so under it, δS is not zero anymore, but rather

$$\begin{aligned}
\delta S &= \int d^4x\, i\epsilon^a(x)(T^a)_{ij}\left[\left(\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))}\right)\right)\phi^j + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))}\phi^j\right)\right] \\
&+ i\int d^4x\sum_{a,i,j}(\partial_\mu\epsilon^a)(T^a)_{ij}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))}\phi^j(x)\right] \\
&= i\sum_a\int d^4x(\partial^\mu\epsilon^a(x))j_\mu^a(x) = -i\sum_a\int d^4x\epsilon^a(x)(\partial^\mu j_\mu^a(x)) \qquad (18.22)
\end{aligned}$$

where in the second equality we have used (18.21) and in the last step we have assumed that $\epsilon^a(x)$ "vanish at infinity" so that we can partially integrate without boundary terms. Therefore we have the variation under the local version of the global symmetry,

$$\delta S = i\sum_a\int d^4x(\partial^\mu\epsilon^a(x))j_\mu^a(x) = -i\sum_a\int d^4x\epsilon^a(x)(\partial^\mu j_\mu^a(x)) \qquad (18.23)$$

This formula is valid off-shell as we said, so we can use it inside the path integral. This gives a way to identify the current as the coefficient of $\partial_\mu\epsilon^a$ in the variation of the action under the local version of the global symmetry.

Ward identities

Since classically (on the classical equations of motion) we have $\partial^\mu j_\mu^a = 0$, by Ehrenfest's theorem we expect to have $\partial^\mu j_\mu^a = 0$ as a VEV (quantum average), and that will be the Ward identity.

However, it can happen that there are *quantum anomalies*, meaning that the classical symmetry is not respected by quantum corrections, and we then get that the classical equation $\partial^\mu j^a_\mu = 0$ is not satisfied as an average at the quantum level.

We consider the local version of the symmetry transformation,

$$\delta\phi^i(x) = i\sum_{a,j} \epsilon^a(x)(T^a)_{ij}\phi^j(x) \tag{18.24}$$

where $\phi' = \phi + \delta\phi$. Changing integration variables from ϕ to ϕ' (renaming the variable, really) does nothing, so

$$\int \mathcal{D}\phi' e^{-S[\phi']} = \int \mathcal{D}\phi e^{-S[\phi]} \tag{18.25}$$

However, now comes a crucial assumption: *if the jacobian from $\mathcal{D}\phi'$ to $\mathcal{D}\phi'$ is 1*, then $\mathcal{D}\phi' = \mathcal{D}\phi$, so

$$0 = \int \mathcal{D}\phi \left[e^{-S[\phi']} - e^{-S[\phi]} \right] = -\int \mathcal{D}\phi \delta S[\phi] e^{-S[\phi]} \tag{18.26}$$

This assumption however is not true in general. In the path integral formalism, quantum anomalies appear exactly as anomalous jacobians for the change of variables in the measure, and lead to breakdown of the symmetries at the quantum level. However, these will be described in the second semester of the course, so we will not discuss them further here.

Now we can use the previously derived form for δS under a local version of a global symmetry, eq. (18.23), obtaining

$$0 = \int d^4x i\epsilon^a(x) \int \mathcal{D}\phi e^{-S[\phi]} \partial^\mu j^a_\mu(x) \tag{18.27}$$

But since the parameters $\epsilon^a(x)$ are arbitrary, we can derive also that

$$\int \mathcal{D}\phi e^{-S[\phi]} \partial^\mu j^a_\mu(x) = 0 \tag{18.28}$$

which is indeed the quantum averaged version of the conservation law, namely the Ward indentity.

Note that here, j^a_μ is a function of ϕ also (the object integrated over), so it cannot be taken out of the path integral (this is an equation of the type $\int dx f(x) = 0$, from which nothing further can be deduced).

We can derive more general Ward identities by considering a general operator $A = A(\{\phi^i\})$, such that

$$\delta A(\{\phi^i\}) = \int d^4x \frac{\delta A(\{\phi^i\})}{\delta\epsilon^a(x)} \epsilon^a(x) \tag{18.29}$$

Then analogously we get

$$0 = \int \mathcal{D}\phi \delta \left[e^{-S[\phi]} A(\{\phi^i\}) \right] = i \int d^4x \epsilon^a(x) \int \mathcal{D}\phi e^{-S[\phi]} \left[\partial^\mu j^a_\mu(x) A - i\frac{\delta A(\{\phi^i\})}{\delta\epsilon^a(x)} \right] \tag{18.30}$$

so that we have the *more general Ward identities*

$$\int \mathcal{D}\phi\, e^{-S[\phi]}(\partial^\mu j^a_\mu(x))A = i\int \mathcal{D}\phi\, e^{-S[\phi]}\frac{\delta A(\{\phi^i\})}{\delta\epsilon^a(x)} \qquad (18.31)$$

There are also other forms of Ward identities we can write. For instance, choosing $A = e^{J\cdot\phi}$, or in other words adding a current J, giving

$$0 = \int \mathcal{D}\phi\, \frac{\delta}{\delta\epsilon^a(x)}e^{-S+J\cdot\phi} = i(T^a)_{ij}\int \mathcal{D}\phi\left[-\frac{\delta S}{\delta\phi^i(x)}\phi^j(x) + J_i\phi^j(x)\right]e^{-S+J\cdot\phi} \qquad (18.32)$$

Now using the usual trick of replacing ϕ^i inside the path integral with $\delta/\delta J$, we obtain the Ward identities

$$i(T^a)_{ij}\left[-\frac{\delta S}{\delta\phi^i(x)}\left\{\phi = \frac{\delta}{\delta J}\right\} + J_i(x)\right]\frac{\delta}{\delta J_j(x)}Z[J] = 0 \qquad (18.33)$$

Taking further derivatives with respect to J and then putting $J = 0$ we obtain Ward identities between the Green's functions, in the same way it happened with the Dyson-Schwinger equation. Also in this case we can write Ward identities for connected or 1PI Green's functions as well.

Ward identities play a very important role in the discussion of symmetries, since they constrain the form of the Green's functions.

For example, we will see later that in QED, the transversality condition $k^\mu A_\mu = 0$, coming from the Lorentz gauge condition, that fixes (part of) the (local) gauge invariance, gives a similar constraint on Green's functions. In particular, for the 1PI 2-point function $\Pi_{\mu\nu}$ we can write

$$k^\mu\Pi_{\mu\nu} = 0 \qquad (18.34)$$

which implies that the Lorentz structure of $\Pi_{\mu\nu}$ is fixed, namely

$$\Pi_{\mu\nu}(k) = (k^2\delta_{\mu\nu} - k_\mu k_\nu)\Pi(k^2) \qquad (18.35)$$

This constraint then is a local analogue of the Ward identities for global symmetries.

Important concepts to remember

- The Dyson-Schwinger equation is the quantum version of the classical equation of motion $\delta S/\delta\phi^i = J_i$, namely its quantum average (under the path integral).

- It can be written as an operator acting on $Z[J]$, and from it we can deduce a relation between Green's functions, for ϕ^k interactions relating $G^{(n)}$ with $G^{(n-2)}$ and $G^{(n+k-2)}$.

- By iterating the Dyson-Schwinger equation we can get the full perturbative expansion, with the correct symmetry factors. It therefore gives an algorithmic way to calculate the symmetry factors.

- The Noether conservation gives on-shell conservation of the current j_μ^a associated with a global symmetry.

- By making $\epsilon \to \epsilon(x)$, the variation of the action is $\delta S = i \int (\partial^\mu \epsilon^a) j_\mu^a$.

- Ward identities are quantum average versions of the classical (on-shell) current conservation equations $\partial^\mu j_\mu^a = 0$, perhaps with an operator A inserted, and its variation on the right hand side. However, there can be quantum anomalies which spoil it, manifesting themselves as anomalous Jacobians for the transformation of the path integral measue.

- Ward identities can also be written as an operator acting on $Z[J]$, and from it deriving relations between Green's functions.

Further reading: See chapters 4.2, 4.3 in [4], 3.1 in [6] and 9.6 in [2]

Exercises, Lecture 18

1) Consider the scalar theory with interaction term

$$S_I(\phi) = \frac{g_5}{5!} \sum_i \phi_i^5 \tag{18.36}$$

Write down the Dyson-Schwinger equation for the n-point function in this theory (equation and diagrammatic). Iterate it once more for the two-point function.

2) Consider the Euclidean action for N real scalars ϕ_i,

$$S = \int d^d x \left[\frac{1}{2} \sum_{i=1}^{N} (\partial_\mu \phi_j)^2 + \frac{\lambda}{4!} (\sum_i \phi_i^2)^2 \right], \tag{18.37}$$

invariant under rotations of the N scalars. Write down the explicit forms of the 2 types of Ward identities
   -for $A = \sum_i (\phi_i)^2$.
   -for the partition function $Z[J]$.

19 Lecture 19. Cross sections and the S-matrix

We now turn to experimentally measurable quantities.

We already said that the S-matrix is something like $S_{fi} = <f|S|i>$, with $S = U_I(+\infty, -\infty)$ (evolution operator in the interaction picture; this lecture we will use the Heisenberg picture however, next one we will use the interaction picture), and that the LSZ formula relates S-matrices to residues at all the $p_i^2 = -m_i^2$ external line poles of the momentum space Green's functions $\tilde{G}_n(p_1, ..., pn)$. Now we define better these facts and relate to experimental observables, in particular the cross section.

Cross sections

In almost all experiments, we scatter two objects, usually projectiles off a target ("laboratory reference frame"), or (like at the LHC) in a center of mass frame, collide two particles off each other.
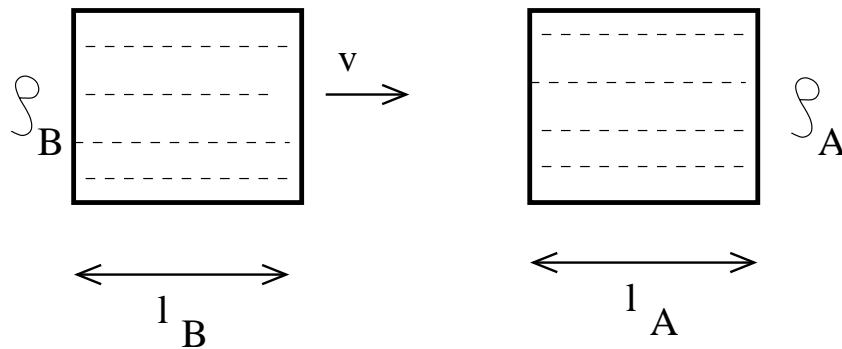

Figure 48: Scattering usually involves shooting a moving projectile at a fixed target.

If we have a box of particles target A, with density ρ_A and length l_A along the direction of impact, and a box of projectiles B, with density ρ_B and length l_B along the direction of impact, and with common cross sectional area (area of impact) A, as in Fig.48, the number of scattering events (when the projectiles are scattered) is easily seen to be proportional to all these, ρ_A, l_A, ρ_B, l_B and A. We can then define the *cross section σ* as

$$\sigma = \frac{\text{Nr. of scatt. events}}{\rho_A l_A \rho_B l_B A} \tag{19.1}$$

This is the definition used for instance by Peskin and Schroeder. But we have more intuitive ways to define it. Before that, let's notice that the dimension of σ is $[\sigma] = 1/[(L/L^3)(L/L^3)L^2] = L^2$, i.e. of area. The physical interpretation of the cross section is the effective (cross sectional) area of interaction per target particle. I.e., within such an area around one target, incident projectiles are scattered.

The cross section should be familiar from classical mechanics, where one can do the same. For instance, in the famous Rutherford experiment, proving that the atoms are not continuous media ("jellium model"), but rather a nucleus surrounded by empty space

and then orbiting electrons, one throws charged particles (e.g. electrons) against a target, and observes the scattered objects. The simple model which agrees with the experiment ("Rutherford scattering") is: one scatters the classical pointlike particle off a central potential $V(r) = Ze^2/r$, and calculates the classical cross section.

The cross section definition for scattering *on one single target* in that case is, more intuitively,

$$\sigma = \frac{\Delta N_{scatt}/\Delta t}{\phi_0} = \frac{\Delta N_{scatt}/\Delta t}{\Delta N_{in}/(\Delta t A)} = \frac{\Delta N_{scatt}}{n_B} \tag{19.2}$$

i.e. the ratio of scattered particles per time, over the incident flux, or particles per time and unit area. Here n_B is the number of incident particles per area. Also, the flux is

$$\phi_0 = \frac{\Delta N_{in}}{\Delta t A} = \frac{\rho_B(v\Delta t)A}{\Delta t A} = \rho_B v \tag{19.3}$$

since the incident volume in time Δt is $(v\Delta t)A$.

Now consider the case of N targets, like above, where $N = \rho_A l_A A$. Then to find the cross section (defined *per target*) we must divide by N, getting

$$\sigma = \frac{\Delta N_{scatt}/\Delta t}{\phi_0 N} = \frac{\Delta N_{scatt}}{(v\Delta t)\rho_B(\rho_A l_A A)} = \frac{\Delta N_{scatt}}{\rho_B l_B \rho_A l_A A} \tag{19.4}$$

as above. Note that in general, if both the target and projectiles are moving, v refers to the relative velocity, $v = v_{rel} = |\vec{v}_1 - \vec{v}_2|$.

The above *total* cross section is a useful quantity, but in experiments generally we measure the momenta (or directions, at least) of scattered particles as well, so a more useful quantity is the *differential cross section*, given momenta $\vec{p}_1, ..., \vec{p}_2$ (almost) well defined in the final state,

$$\frac{d\sigma}{d^3p_1...d^3p_n} \tag{19.5}$$

An important case, for instance for the classical Rutherford scattering above, is of $n = 2$, i.e. two final states. That means 6 momentum components, constrained by the 4-momentum conservation delta functions, leaving 2 independent variables, which can be taken to be two angles θ, ϕ. In the case of Rutherford scattering, these were the scattering angle θ relative to the incoming momentum, and the angle ϕ of rotation around the incoming momentum. Together, these two angles form a solid angle Ω. Therefore, in the case of $2 \to 2$ scattering, after using the 4-momentum conservation, one defines the differential cross section

$$\frac{d\sigma}{d\Omega} \tag{19.6}$$

Decay rate

There is one more process that is useful, besides the $2 \to n$ scattering.

Consider the decay of an unstable particle at rest into a final state made up of several particles. Then we define the decay rate

$$\Gamma = \frac{\#decays/time}{\#of\ part.} = \frac{dN}{Ndt} \tag{19.7}$$

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The lifetime τ of a particle with decay rates Γ_i into various channels is

$$\sum_i \Gamma_i = \frac{1}{\tau} \tag{19.8}$$

Consider an unstable atomic state in nonrelativistic quantum mechanics. In those cases, we know that the decay of the unstable state is found as a Breit-Wigner resonance in the scattering amplitude, i.e.

$$f(E) \propto \frac{1}{E - E_0 + i\Gamma/2} \tag{19.9}$$

which means that the cross section for decay, or probability, is proportional to $||^2$ of the above,

$$\sigma \propto \frac{1}{(E - E_0)^2 + \Gamma^2/4} \tag{19.10}$$

I.e., the resonance appears as a bell curve with width Γ, centered around E_0.

In relativistic quantum mechanics, or quantum field theory, the same happens. Initial particles combine to form an unstable (or metastable) particle, that then decays into others. In the amplitude, this is reflected in a relativistically invariant generalization of the Breit-Wigner formula, namely an amplitude proportional to

$$\frac{1}{p^2 + m^2 - im\Gamma} \simeq \frac{-1}{2E_p(p^0 - E_p + \frac{im}{E_p}\frac{\Gamma}{2})} \tag{19.11}$$

i.e., of the same form near the mass shell and for Γ small.

In-out states and the S-matrix

In quantum mechanics, we first need to define the states that scatter.

-We can consider states whose wavepackets are well isolated at $t = -\infty$, so we can consider them non-interacting then. But these (Heisenberg) states will overlap, and therefore interact at finite t. We call these Heisenberg states the in-states,

$$|\{\vec{p}_i\} >_{in} \tag{19.12}$$

-We can also consider states whose wavepackets are well isolated at $t = +\infty$, so we can consider them noninteracting there, but they will overlap at finite t. We call these Heisenberg states the out-states,

$$|\{\vec{p}_i\} >_{out} \tag{19.13}$$

Then the in-states, after the interaction, at $t = +\infty$, look very complicated, and reversely, the out states, tracked back to $-\infty$, look very complicated. But *all* the in-states, as well as *all* the out-states form complete sets,

$$\sum |\{\vec{p}_i\}; in >< \{\vec{p}_i\}; in| = \sum |\{\vec{p}_i\}; out >< \{\vec{p}_i\}; out| = \mathbf{1} \tag{19.14}$$

That means that we can expand one in the other, so the amplitude for an (isolated) out state, given an (isolated) in state is

$$_{out} < \vec{p}_1, \vec{p}_2, ... |\vec{k}_A, \vec{k}_B >_{in} \;\; = \;\; \lim_{T \to \infty} < \vec{p}_1, \vec{p}_2, ...(T)|\vec{k}_A, \vec{k}_B(-T) >$$

$$= \lim_{T\to\infty} < \vec{p}_1, \vec{p}_2, ... | e^{-iH(2T)} | \vec{k}_A, \vec{k}_B > \qquad (19.15)$$

where in the last form the states are defined at the same time, so we can define it as the time when Heisenberg=Schrödinger, thus think of them as Schrödinger states.

Then the *S-matrix* is

$$< \vec{p}_1, \vec{p}_2, ... | S | \vec{k}_A, \vec{k}_B > = {}_{out}< \vec{p}_1, \vec{p}_2, ... | \vec{k}_A, \vec{k}_B >_{in} \qquad (19.16)$$

which means that

$$S = e^{-iH(2T)} \qquad (19.17)$$

is the *Heisenberg picture evolution operator*.

Wavefunctions

The one-particle states are always isolated, so in that case

$$|\vec{p}_{in} > = |\vec{p}_{out} > = |\vec{p} > \ (= \sqrt{2E_p} a_p^\dagger |0 >) \qquad (19.18)$$

where in brackets we wrote the free theory result, and we can construct the one-particle states with wavefunctions

$$|\phi > = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(\vec{k}) |\vec{k} > \qquad (19.19)$$

With this normalization, the wavefunctions give the probabilities, since

$$\int \frac{d^3k}{(2\pi)^3} |\phi(\vec{k})|^2 = 1 \qquad (19.20)$$

and so $< \phi | \phi > = 1$. A wavefunction can be something like $e^{i\vec{k}\cdot\vec{x}}$, giving an $\vec{x}$ dependence.

In the case of 2-particle states, we can write the in-states as

$$|\phi_A \phi_B >_{in} = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(\vec{k}_A)\phi_B(\vec{k}_B)}{\sqrt{2E_A}\sqrt{2E_B}} e^{-i\vec{b}\cdot\vec{k}_B} |\vec{k}_A \vec{k}_B >_{in} \qquad (19.21)$$

Note that we could have absorbed the factor $e^{-i\vec{k}_B\cdot\vec{b}}$ in the B wavefunction, but we have written it explicitly since if the wavefunctions are centered around a momentum, like in a classical case, we have a separation between particles. In the Rutherford experiment, the imaginary line of the non-deflected incoming projectile passes at a $\vec{b}$ minimum distance close to the target. Note then that $\vec{b}$ is perpendicular to the collision direction, i.e. is transversal.

The out state of several momenta is defined as usual

$$_{out}< \phi_1, \phi_2...| = \prod_f \left(\frac{d^3p_f}{(2\pi)^3} \frac{\phi_f(p_f)}{\sqrt{2E_f}} \right) {}_{out}< p_1 p_2...| \qquad (19.22)$$

The S-matrix of states with wavefunctions is

$$S_{\beta\alpha} = < \beta_{out} | \alpha_{in} > \qquad (19.23)$$

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Because this matrix gives probabilities, and the S operator is, as we saw above, an evolution operator, it corresponds to a unitary operator, so

$$SS^\dagger = S^\dagger S = \mathbf{1} \tag{19.24}$$

But the operator S contains the case where particles go on without interacting, i.e. the identity $\mathbf{1}$. To take that out, we define

$$S = 1 + iT \tag{19.25}$$

Note that the i is conventional. Moreover, amplitudes always contain a momentum conservation delta function. Therefore, in order to define nonsingular and finite objects, we define the *invariant (or reduced) matrix element* $\mathcal{M}$ by

$$< \vec{p}_1, \vec{p}_2, ... | iT | \vec{k}_A, \vec{k}_B > = (2\pi)^4 \delta^4 \left(k_A + k_B - \sum p_f \right) i\mathcal{M}(k_A, k_B \to p_f) \tag{19.26}$$

Reduction formula (Lehmann, Symanzik, Zimmermann)

The LSZ formula relates S-matrices to Green's functions, as mentioned. We will not prove it here, since for that we need renormalization, and that will be done in the second semester. We will just state it.

We also introduce an operator $\mathcal{A}$ to make the formula a bit more general, but is not really needed. Define the momentum space Green's functions

$$\tilde{G}_{n+m}^{(A)}(p_i^\mu, k_j^\mu) = \int \prod_{i=1}^{n} \int d^4 x_i e^{-ip_i \cdot x_i} \prod_{j=1}^{m} \int d^4 y_j e^{ik_j \cdot y_j} < \Omega | T\{\phi(x_1)...\phi(x_n)\mathcal{A}(0)\phi(y_1)...\phi(y_m)\} | \Omega > \tag{19.27}$$

Then we have

$$_{out} < \{p_i\}n | \mathcal{A}(0) | \{k_j\}_m >_{in}$$
$$= \lim_{p_i^2 \to -m_i^2, k_j^2 \to -m_j^2} \frac{1}{(-i\sqrt{Z})^{m+n}} \prod_{i=1}^{n} (p_i^2 + m^2 - i\epsilon) \prod_{j=1}^{m} (k_j^2 + m^2 - i\epsilon) \tilde{G}_{n+m}^{(A)}(p_i^\mu, k_j^\mu) \tag{19.28}$$

For $\mathcal{A} = 1$, we obtain a formula for the S-matrix as the multiple residue at all the external line poles of the momentum space Green's functions, dividing by the extra factors of Z.

The full 2-point function behaves near the pole as

$$G_2(p) = \int d^4 x e^{-ip \cdot x} < \Omega | T\{\phi(x)\phi(0)\} | \Omega > \sim \frac{-iZ}{p^2 + m^2 - i\epsilon} \tag{19.29}$$

In other words, to find the S-matrix, we put the external lines on shell, and divide by the full propagators corresponding to all the external lines (but note that Z belongs to 2 external lines, hence the $\sqrt{Z}$). This implies a diagrammatic procedure called *amputation* that will be explained next lecture.

Note that the factor Z has a kind of physical interpretation, since we can define it as

$$Z \equiv |<\Omega|\phi(0)|\vec{p}>|^2 \tag{19.30}$$

In other words, it is the probability to create a state from the vacuum. Note that the factor $Z = 1 + \mathcal{O}(g^2)$, but the "correction" is an infinite loop correction (this is the oddity of renormalization, to be understood next semester). However, at tree level, $Z = 1$.

Cross sections from $\mathcal{M}$.

The probability to go from $|\phi_A\phi_B>$ to a state within the infinitesimal interval $d^3p_1...d^3p_n$ is

$$\mathcal{P}(AB \to 12...n) = \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}\right)|_{out} < p_1p_2...|\phi_A\phi_B >_{in}|^2 \tag{19.31}$$

since it must be proportional to the infinitesimal interval, and with the $||^2$ of the amplitude, and the rest is the correct normalization.

For one target, i.e. $N_A = 1$, and for n_B particles coming in per unit of transverse area, the number of scattered particles is

$$\Delta N = \int d^2 b n_B \mathcal{P}(\vec{b}) \tag{19.32}$$

If n_B is constant, we can take it out of the integral. Then the cross section is (as we saw in (19.2)

$$d\sigma = \frac{\Delta N}{n_B} = \int d^2 b \mathcal{P}(\vec{b}) \tag{19.33}$$

Replacing the form of $\mathcal{P}$ and on the states in the amplitudes there, we get

$$\begin{aligned}
d\sigma &= \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\int d^2b \prod_{i=A,B}\int \frac{d^3k_i}{(2\pi)^3}\frac{\phi_i(\vec{k}_i)}{\sqrt{2E_i}}\int \frac{d^3\bar{k}_i}{(2\pi)^3}\frac{\phi_i^*(\vec{\bar{k}}_i)}{\sqrt{2\bar{E}_i}} \\
&e^{i\vec{b}\cdot(\vec{\bar{k}}_B - \vec{k}_B)}\left(_{out} < \{p_f\}|\{k_i\} >_{in}\right)\left(_{out} < \{p_f\}|\{\bar{k}_i\} >\right)^*
\end{aligned} \tag{19.34}$$

To compute it, we use

$$\int d^2\vec{b} e^{i\vec{b}\cdot(\vec{\bar{k}}_B - \vec{k}_B)} = (2\pi)^2\delta^{(2)}(k_B^\perp - \bar{k}_B^\perp)$$

$$_{out} < \{p_f\}|\{k_i\} >_{in} = i\mathcal{M}(2\pi)^4\delta^{(4)}\left(\sum k_i - \sum p_f\right)$$

$$_{out} < \{p_f\}|\{\bar{k}_i\} >_{in}^* = -i\mathcal{M}^*(2\pi)^4\delta^{(4)}\left(\sum \bar{k}_i - \sum p_f\right) \tag{19.35}$$

We also use

$$\int d^3\bar{k}_A \int d^3\bar{k}_B \delta^{(4)}\left(\sum \bar{k}_i - \sum p_f\right)\delta^{(2)}(k_B^\perp - \bar{k}_B^\perp)$$

$$= (k_i^\perp = \bar{k}_i^\perp) \times \int d\bar{k}_A^z d\bar{k}_B^z \delta(\bar{k}_A^z + \bar{k}_B^z - \sum p_f^z)\delta\left(\bar{E}_A + \bar{E}_B - \sum E_f\right)$$

$$= \int d\bar{k}_A^z \, \delta \left(\sqrt{\bar{k}_A^2 + m_A^2} + \sqrt{\bar{k}_B^2 + m_B^2} - \sum E_f \right) \Bigg|_{\bar{k}_B^z = \sum p_f^z - \vec{k}_A^z}$$

$$= \frac{1}{\left| \frac{\bar{k}_A^z}{\bar{E}_A} - \frac{\bar{k}_B^z}{\bar{E}_B} \right|} = \frac{1}{|v_A - v_B|} \tag{19.36}$$

In the last line we have used that (since $E = \sqrt{k^2 + m^2}$ and there we have $\bar{k}_B^z = \sum p_f - \vec{k}_A^z$)

$$\frac{d\bar{E}_A}{d\bar{k}_A} = \frac{\bar{k}_A}{\bar{E}_A}; \quad \frac{d\bar{E}_B}{d\bar{k}_B} = \frac{\bar{k}_B}{\bar{E}_B} = -\frac{d\bar{E}_B}{d\bar{k}_A} \tag{19.37}$$

and the fact that $\int dx \delta(f(x) - f(x_0)) = 1/|f'(x_0)|$.

Putting everything together, we find

$$d\sigma = \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3}$$

$$\frac{|\mathcal{M}(k_A, k_B \to \{p_f\})|^2}{2E_A 2E_B |v_A - v_B|} |\phi_A(k_A)|^2 |\phi_B(k_B)|^2 (2\pi)^4 \delta^{(4)} \left(k_A + k_B - \sum p_f \right) \tag{19.38}$$

For states with wavefunctions centered sharply on a given momentum $\delta(\vec{k} - \vec{p})$, we obtain

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(k_A, k_B \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)} \left(k_A + k_B - \sum p_f \right) \tag{19.39}$$

In this formula, $|\mathcal{M}|^2$ is Lorentz invariant,

$$\int d\Pi_n = \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)} \left(k_A + k_B - \sum p_f \right) \tag{19.40}$$

is relativistic invariant as well, and is called *relativistically invariant n-body phase space*, however

$$\frac{1}{E_A E_B |v_A - v_B|} = \frac{1}{|E_B p_A^z - E_A p_B^z|} = \frac{1}{|\epsilon_{\mu x y \nu} p_A^\mu p_B^\nu|} \tag{19.41}$$

is therefore not relativistically invariant, meaning the cross section is not relativistically invariant either. However, if $\vec{k}_A \| \vec{k}_B$ (for instance in the center of mass frame, or the laboratory frame), this can be written as

$$\frac{1}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \tag{19.42}$$

So if we adopt this form in all reference frames, i.e. replace in the formula for the differential cross section

$$\frac{1}{E_A E_B |v_A - v_B|} \to \frac{1}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \tag{19.43}$$

then we obtain the *relativistically invariant cross section*. This is a theoretical concept, which is useful since we can write this cross section in simple way in terms of Mandelstam variables

s, t (to be defined later). But it is not measurable, so for comparison with experiment, we should remember to go back to the usual form (or just use center of mass frame or laboratory frame).

To understand the relativistically invariant n-body phase space better, consider the important case of $n = 2$, i.e. $2 \rightarrow 2$ scattering, and consider working in the center of mass frame, so $\vec{p}_{total} = 0$. Then we have $\vec{p}_1, \vec{p}_2$, but the delta function over 3-momenta imposes $\vec{p}_2 = -\vec{p}_1$, so we are left with

$$
\begin{aligned}
\int d\Pi_2 &= \int \frac{dp_1}{(2\pi)^3} \frac{p_1^2}{2E_1} \frac{d\Omega}{2E_2} 2\pi \delta(E_{CM} - E_1 - E_2) \\
&= \int d\Omega \frac{p_1^2}{16\pi^2 E_1 E_2} \left. \frac{1}{\frac{p_1}{E_1} + \frac{p_1}{E_2}} \right|_{p_1 = -p_2} = \int d\Omega \frac{|p_1|}{16\pi^2 E_{CM}}
\end{aligned}
\tag{19.44}
$$

Here $E_{CM} = E_1 + E_2$ and $E_1 = \sqrt{p_1^2 + m_1^2}, E_2 = \sqrt{p_1^2 + m_2^2}$.

Therefore finally we obtain that in the center of mass frame

$$
\left(\frac{d\sigma}{d\Omega} \right)_{CM} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\vec{p}_1|}{16\pi^2 E_{CM}} |\mathcal{M}(p_A, p_B \rightarrow p_1, p_2)|^2
\tag{19.45}
$$

In the case of identical masses for all the particles (A,B,1,2), we have

$$
\frac{E_{CM}}{2} = E_A = E_B; \quad |p_A| = |p_B| = |p_1| = |p_2|
\tag{19.46}
$$

and substituting above (together with $E_A E_B |v_A - v_B| = |E_A p_B - E_B p_A|$), we get

$$
\left(\frac{d\sigma}{d\Omega} \right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2}
\tag{19.47}
$$

Particle decay

We can now calculate particle decay very easily. Formally, all we have to do is keep only k_A's and drop the k_B's in the calculation above. Dropping the integrals $\int d^3 k_B / (2\pi)^3$ and $\int d^3 \bar{k}_B / (2\pi)^3$ implies that in the above we don't have $\int d\bar{k}_B^z$ anymore in (19.36), only

$$
\int d\bar{k}_A^z \delta \left(\bar{k}_A^z - \sum p_f^z \right) \delta \left(\bar{E}_A - \sum E_f \right) = \delta \left(\bar{E}_A - \sum E_f \right)
\tag{19.48}
$$

so in effect, we need to remove the factor $1/|v_A - v_B|$ from the calculation. We can then immediately write the result for the decay rate in the center of mass system,

$$
d\Gamma|_{CM} = \frac{1}{2m_A} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(m_A \rightarrow \{p_f\})|^2 (2\pi)^4 \delta^4 \left(p_A - \sum p_f \right)
\tag{19.49}
$$

We only have a problem of interpretation: what does this represent, since the state is unstable, so how do we define the asymptotic (noninteracting) state A? Nevertheless, the result is correct.

Important concepts to remember

- The cross section is the number of scattering events per time, divided by the incoming flux and the number of targets, and it measures the effective area of interaction around a single target.

- The decay rate is the number of decays per unit time, divided by the number of particles decaying. The sum of the decay rates gives the inverse of the lifetime.

- In a cross section, a resonance appears via a relativistic generalization of the Breit-Wigner formula, with amplitudes proportional to $1/(p^2 + m^2 - im\Gamma)$ for the resonance.

- In states are well separated states at $t = -\infty$, out states are well separated states at $+\infty$, and their overlap gives the S-matrix.

- The S operator is $1 + iT$, and extracting from the matrix elements of T the overall momentum conservation delta functions, we obtain the finite reduced matrix element $\mathcal{M}$.

- The LSZ formula says that the S-matrix is the residue at all the external poles of the momentum space Green's function, divided by the Z factors, or the Green's function divided by the full external propagators near the mass shell.

- The differential cross section equals $|\mathcal{M}|^2$, times the relativistically invariant n-body phase space, times $1/(E_A E_B |v_A - v_B|)$. If we replace the last factor with the relativistically invariant formula $1/\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$, we obtain the *relativistically invariant cross section*, a useful theoretical concept.

Further reading: See chapters 4.5 in [2], 2.5, 4.2, 4.3 in [1].

Exercises, Lecture 19

1) Consider the scattering of a massive field off a massless field ($m_1 = m_3 = m; m_2 = m_4 = 0$) in a theory where $i\mathcal{M} \simeq i\lambda =$ constant. (e.g. Fermi's 4-fermion theory for weak interactions). Calculate the *total, relativistically invariant* cross section $\sigma_{tot,rel}$ as a function of the center of mass energy.

2) Consider the theory

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi\phi \tag{19.50}$$

Then, if $M > 2m$, the first order amplitude for Φ to decay in 2 ϕ's is $|\mathcal{M}| = \mu$. Calculate the lifetime τ for Φ.

20 Lecture 20. S-matrix and Feynman diagrams

We now want to go from the Heisenberg picture used last lecture to the interaction picture, the same way as we did for Green's functions. Last lecture we wrote for the S-matrix

$$< \vec{p}_1 \vec{p}_2 ... |S| \vec{k}_A \vec{k}_B > = \lim_{T \to \infty} < \vec{p}_1 \vec{p}_2 ... |e^{-iH(2T)}| \vec{k}_A \vec{k}_B > \tag{20.1}$$

where the states are in the *full* theory (not free states), i.e. are eigenfunctions of H, and are defined at the same time. But as for the Green's functions, we want to replace them by *free* states, i.e. eigenfunctions of H_0. Before, we wrote

$$|\Omega > = \lim_{T \to \infty(1-i\epsilon)} (e^{-iE_0 T} < \Omega|0 >)^{-1} e^{-iHT}|0 > = \lim_{T \to \infty(1-i\epsilon)} (e^{-iE_0 T} < \Omega|0 >)^{-1} U_I(0, -T)|0 > \tag{20.2}$$

(by introducing a complete set of full states $|n> < n|$, with $|\Omega>$ having $H|\Omega> = E_0|\Omega>$ and the $T \to \infty(1-i\epsilon)$ guaranteeing only the lowest energy mode, i.e. the vacuum, remains). Now we want to write in a similar way

$$|\vec{k}_A \vec{k}_B > \propto \lim_{T \to \infty(1-i\epsilon)} e^{-iHT}|\vec{k}_A \vec{k}_B >_0 \tag{20.3}$$

where we wrote a proportionality sign, since proving the relation and getting the constant is now complicated: the external state is not a ground state anymore. Using this, we rewrite the right hand side of (20.1) as

$$\lim_{T \to \infty(1-i\epsilon)} {}_0 < \vec{p}_1 ... \vec{p}_n |e^{-iH(2T)}| \vec{p}_A \vec{p}_B >_0$$

$$\propto \lim_{T \to \infty(1-i\epsilon)} {}_0 < \vec{p}_1 ... \vec{p}_n |T \left\{ \exp \left[-i \int_{-T}^{T} dt H_I(t) \right] \right\} | \vec{p}_A \vec{p}_B >_0 \tag{20.4}$$

therefore

$$S_{fi}|_{int} = < f|U_I(+\infty, -\infty)|i > \tag{20.5}$$

The proportionality factors cancel out in the Green's functions case by using the Feynman and Wick theorems, and only the connected pieces remain. Here a similar story happens, though the proof is harder to do. We will explain later the result in terms of the LSZ formalism. The correct formula is

$$< \vec{p}_1 ... \vec{p}_n |iT| \vec{p}_A \vec{p}_B >$$
$$= (\sqrt{Z})^{n+2} \lim_{T \to \infty(1-i\epsilon)} \left({}_0 < \vec{p}_1 ... \vec{p}_n |T \left\{ \exp \left[-i \int_{-T}^{T} dt H_I(t) \right] \right\} | \vec{p}_A \vec{p}_B >_0 \right)_{\text{connected,amputated}} \tag{20.6}$$

We have iT instead of S, since the formula gives only the matrix elements where the initial and final states are different. To understand the meaning of "connected, amputated" we will do some examples. We first note that the factors of Z and equal to 1 at tree level, and we will understand the need to put them there later, from the LSZ formula.

We first consider the free term on the right hand side of (20.6). It is given by

$${}_0 < \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B >_0 \equiv \sqrt{2E_A 2E_B 2E_1 2E_2} < 0|a_1 a_2 a_A^\dagger a_B^\dagger|0 >$$

$$= 2E_A 2E_B (2\pi)^6 \left(\delta^{(3)}(\vec{p}_A - \vec{p}_1) \delta^{(3)}(\vec{p}_B - \vec{p}_2) + \delta^{(3)}(\vec{p}_A - \vec{p}_2) \delta^{(3)}(\vec{p}_B - \vec{p}_1) \right) \quad (20.7)$$

This has the diagrammatic interpretation of propagator lines connecting $(1A)(2B)$ and $(1B)(2A)$, see Fig.49, and corresponds to the $\mathbf{1}$ in $S = \mathbf{1} + iT$, therefore should be excluded from the right hand side of (20.6). It is excluded, as the two lines are not connected among each other.
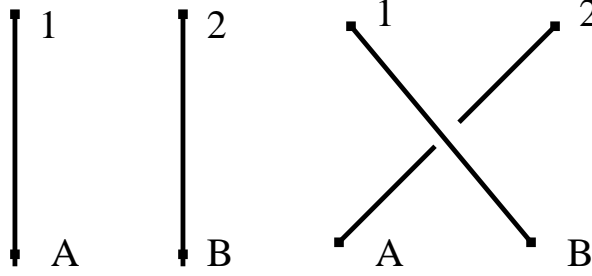

Figure 49: Free term for the scattering matrix: it is excluded (only interacting pieces considered).

We next consider the first order term in λ,

$$_0 < \vec{p}_1 \vec{p}_2 | T \left\{ -i\frac{\lambda}{4!} \int d^4 x \phi_I^4(x) \right\} | \vec{p}_A \vec{p}_B >_0 = {}_0 < \vec{p}_1 \vec{p}_2 | N \left\{ -i\frac{\lambda}{4!} \int d^4 x \phi_I^4(x) + \text{contractions} \right\}$$
$$(20.8)$$

where in the equality we used Wick's theorem.

But now we don't have $|0>$ anymore, but rather $|\vec{p}_A \vec{p}_B >_0$, so the "non-contracted" pieces give a nonzero result. Better said, there is another type of contraction. Consider the annihilation part of ϕ_I, ϕ_I^+, acting on an external state:

$$\phi_I^+(x)|\vec{p}>_0 = \left(\int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} a_{\vec{k}} e^{ik \cdot x} \right) \left(\sqrt{2E_p} a_{\vec{p}}^\dagger |0> \right)$$
$$= \int \frac{d^3 k}{(2\pi)^3} e^{ik \cdot x} \left((2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) \right) |0> = e^{ip \cdot x}|0> \quad (20.9)$$

Inside $< ..|N(\phi^k)|.. >=< ..|(\phi^-)^n (\phi^+)^{k-n}|.. >$, we have ϕ^+'s acting on a state on the right and ϕ^-'s acting on the left, so we define the contraction with an external state,

$$\overbracket{\phi_I(x)|\vec{p}} >= e^{ip \cdot x}|0> \quad (20.10)$$

and the conjugate

$$< \vec{p}| \overbracket{\phi_I(x)} =< 0|e^{-ip \cdot x} \quad (20.11)$$

We now analyze the first order term in the S matrix (20.8). It contains a term $\overbracket{\phi\phi}\phi\phi$, a term $\overbracket{\phi\phi\phi}\phi$, and a term $\overbracket{\phi\phi}\,\overbracket{\phi\phi}$.

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Consider the last term, with two $\phi\phi$ contractions. It is

$$-i\frac{\lambda}{4!}\int d^4x \overbracket{\phi\phi}\ \overbracket{\phi\phi}_0 <\vec{p}_1\vec{p}_2|\vec{p}_A\vec{p}_B>_0 \qquad (20.12)$$

It gives the figure eight vacuum bubble times the free term (zeroth order) above in Fig.49, so it is a disconnected piece where the initial state and final state are the same. Therefore again this is a piece that contributes to $\mathbf{1}$, not to iT, and again is disconnected, consistent with (20.6).

Next consider the second term, with a single contraction. After applying the normal ordering N operator, we have $\sim a^\dagger a^\dagger + 2a^\dagger a + aa$, but only the term with the same number of $a^\dagger$ and a contributes in the expectation value (e.g. $<p_1p_2|a^\dagger a^\dagger|p_Ap_B> \sim <0|aa(a^\dagger)^4|0> \sim [a,a^\dagger]^2 <0|a^\dagger a^\dagger|0>= 0$), i.e.

$$-i\frac{\lambda}{4!}\int d^4x 2\overbracket{\phi\phi}_0 <\vec{p}_1\vec{p}_2|\phi^-\phi^+|\vec{p}_A\vec{p}_B>_0 \qquad (20.13)$$

Doing the contractions of the ϕ's with the external states we obtain the same zeroth order term with a loop on one of the legs (thus a total of 4 terms), see Fig.50. Contractions with external states are as in Fig.51. Since the $\int d^4x$ interval gives a delta function, on the leg with a loop we also have the same momentum on both sides, so this is again a contribution to the identity $\mathbf{1}$, and not to iT, and is disconnected, consistent with (20.6).
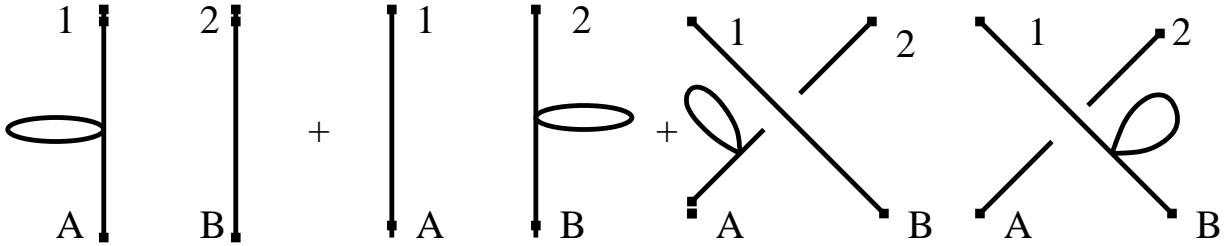

Figure 50: Other disconnected terms, this time with nontrivial contractions (loops), excluded from the scattering matrix.
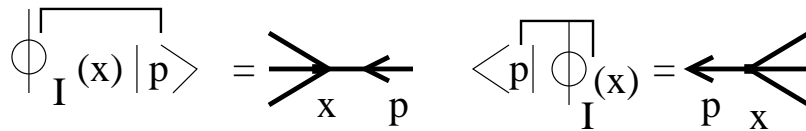

Figure 51: Feynman diagrammatic representation for contractions with external states.

Finally, the only nonzero term is the one where there are no contractions between the ϕ's, and only contractions with the external states, of the type

$$_0 < \vec{p}_1 \vec{p}_2 | \phi^- \phi^- \phi^+ \phi^+ (x) | \vec{p}_A \vec{p}_b >_0 \tag{20.14}$$

where all the ϕ's are at the same point, so this gives a term $e^{i(p_A + p_B - p_1 - p_2) \cdot x}$. But this comes from $N(\phi^4(x))$, with $\phi = \phi^+ + \phi^-$, so there are $4! = 24$ such terms as the above, since from the first ϕ, we can pick either ϕ^- or ϕ^+ to contract with either one of the 4 external states, $\vec{p}_1, \vec{p}_2, \vec{p}_A, \vec{p}_B$, the next ϕ can be contracted with either of the remaining (uncontracted) 3 external states, etc, giving $4 \cdot 3 \cdot 2 \cdot 1$ terms. The total result is then

$$4! \times \left(-i \frac{\lambda}{4!} \right) \int d^4 x e^{i(p_A + p_B - p_1 - p_2) \cdot x} = -i\lambda (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1 - p_2) \tag{20.15}$$

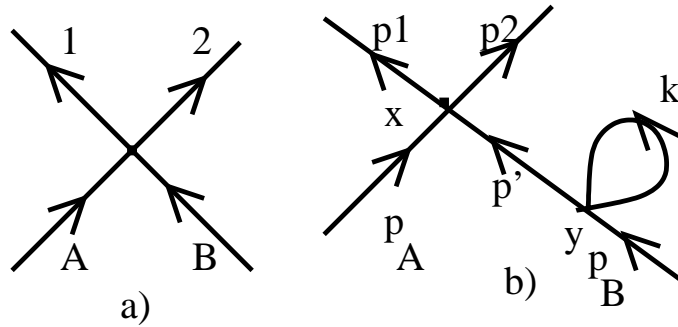which is just a vertex connecting the 4 external lines, as in Fig.52a.

Figure 52: a) The first order contribution to the S matrix. b) Even among the connected diagrams there are some that do not contribute to the S matrix, like this one. It is one which will be *amputated*.

Since iT is $(2\pi)^4 \delta(...) \times i\mathcal{M}$, we have $\mathcal{M} = -\lambda$. Replacing in the formula for the differential cross section in the center of mass frame from last lecture, we get

$$\left(\frac{d\sigma}{d\Omega} \right)_{CM} = \frac{\lambda^2}{64\pi^2 E_{CM}^2} \tag{20.16}$$

This is independent of the angles θ, ϕ, therefore it integrates trivially, by multiplying with $\Omega = 4\pi$, giving finally

$$\sigma_{tot} = \frac{\lambda^2}{32\pi E_{CM}^2} \tag{20.17}$$

Here we added a factor of $1/2$ since we have identical particles in the final state, so it is impossible to differentiate experimentally between 1 and 2.

So we need to consider only *fully connected* diagrams, where all the external lines are connected as well. But even among these diagrams there are some that need to be excluded.

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To see this, consider the second order diagram obtained as follows: a vertex diagram with a loop on one of the external states (p_B), with k in the loop and p' after it, as in Fig.52b. It gives

$$\frac{1}{2}\int\frac{d^4p'}{(2\pi)^4}\frac{-i}{p'^2+m^2}\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+m^2}(-i\lambda)(2\pi)^4\delta^{(4)}(p_A+p'-p_1-p_2)\times(-i\lambda)(2\pi)^4\delta^{(4)}(p_B-p')$$

(20.18)

where the delta functions coming from the integration over vertices. But doing the p' integration we get

$$\left.\frac{1}{p'^2+m^2}\right|_{p'=p_B}=\frac{1}{p_B^2+m^2}=\frac{1}{0}$$

(20.19)

so the result is badly defined. That means that this diagram must be excluded from the physical calculation, so we must find a procedure that excludes this kind of diagrams.

We now define *amputation* as follows. From the tip of an external leg, find the furthest point in the connected diagram where we can disconnect the external line from the rest by cutting just one propagator, and remove that whole piece (including the external line) from the diagrams. See examples of diagrams in Fig.53.
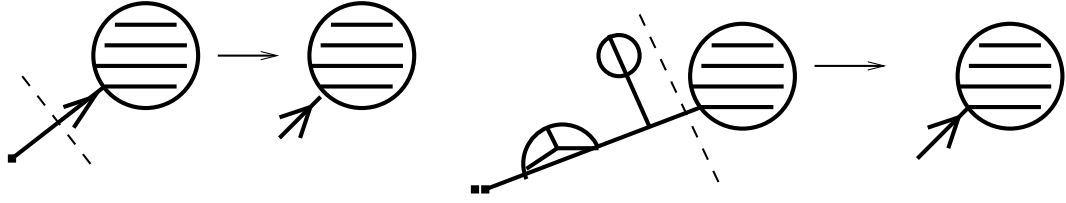

Figure 53: Examples for the diagrammatic representation of amputation, to obtain the Feynman diagram contributions to the S matrix.

Therefore we can say that

$$i\mathcal{M}(2\pi)^4\delta^{(4)}\left(p_A+p_B-\sum p_f\right)=\left(\sum\text{connected, amputated Feynman diag.}\right)\times(\sqrt{Z})^{n+2}$$

(20.20)

These pieces of diagrams that we are excluding are contributions to the two-point functions for the external lines (quantum corrections to the propagators). We will see the interpretation of this shortly.

But first we consider the **Feynman diagrams in x space.** See Fig.54, x space.
-Propagator $D_F(x-y)$.
-Vertex $-i\lambda\int d^4x$.
-External line $e^{ip\cdot x}$.
-Divide by the symmetry factor.
And the **Feynman diagrams in p space**. See Fig.54 p space.
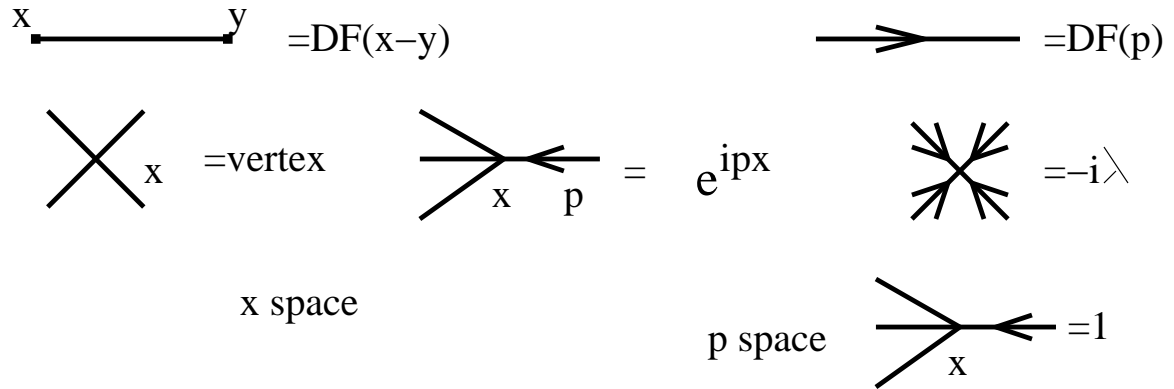-Propagator $D_F(p)=-i/(p^2+m^2-i\epsilon)$.
-Vertex $-i\lambda$.

Figure 54: Diagrammatic representation fot the relevant Feynman rules for S-matrices in a) x-space and b) p-space.

-External line = 1.
-Momentum conservation at vertices.
-$\int d^4p/(2\pi)^4$ for loops.
-Divide by the symmetry factor.

Given the LSZ formula (which we only wrote, we didn't derive), we can understand the above rule (20.20). Indeed, we can rewrite the LSZ formula as

$$S_{fi} - \delta_{fi} = <f|iT|i> = (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum p_f\right) i\mathcal{M}$$

$$= \lim_{p_i^2 + m^2 \to 0, k_j^2 + m^2 \to 0} (\sqrt{Z})^{m+n} \prod_{i=1}^{n}\left(\frac{p_i^2 + m^2 - i\epsilon}{-iZ}\right) \prod_{j=1}^{m}\left(\frac{k_j^2 + m^2 - i\epsilon}{-iZ}\right) \times \tilde{G}_{n+m}(p_i, k_j)$$

$$(20.21)$$

But in $\tilde{G}_{n+m}$ we have the connected Feynman diagrams, and the factors in the brackets are, near their mass shell, the inverse of the full propagators for the external lines,

$$G_2(p) \sim \frac{-iZ}{p^2 + m^2 - i\epsilon} \tag{20.22}$$

So amputation is just the removal (dividing by) these full on-shell 2-point functions for the external lines. Thus we obtain exactly the rule (20.20).

Important concepts to remember

- In the interaction picture, $S_{fi} \propto <f|U_I(+\infty, -\infty)|i>$.

- The matrix elements of iT (nontrivial S-matrices) are the sum over connected, amputated diagrams.

- Amputation means removing the parts that contribute to the (on-shell) 2-point functions of the external lines.

- The S matrix diagrammatic rules follow from the LSZ formula.

Further reading: See chapters 4.6 in [2], 4.1 and parts of 4.4 in [1].

Exercises, Lecture 20

1) Consider the second order term in the S matrix

$$_0 < \vec{p}_1\vec{p}_2|T\left\{\left(\frac{-i\lambda}{4!}\right)^2 \int d^4x\phi_I^4(x) \int d^4y\phi_I^4(y)\right\}|\vec{p}_A\vec{p}_B >_0 \qquad (20.23)$$

Using the Wick contractions and theorem, show the only Feynman diagrams which contribute to the S-matrix.

2) Consider the Feynman diagram in Fig. 55 in $\lambda\phi^4/4!$ theory.
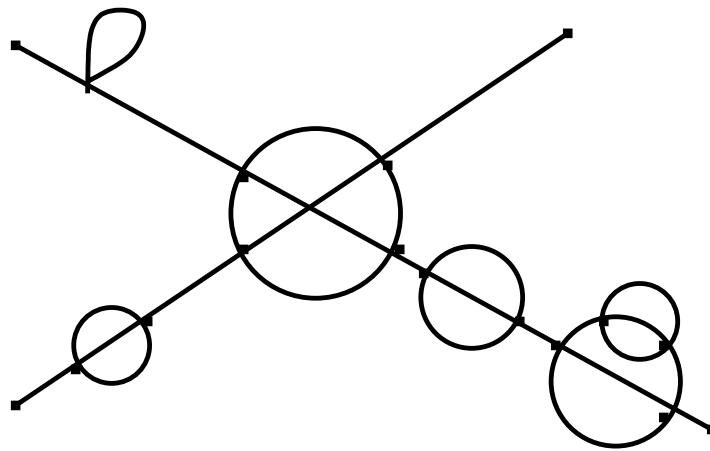

Figure 55: Example of Feynman diagram.

Write down the Feynman diagram corresponding to this in $_0 < \vec{p}_1\vec{p}_2|S|\vec{p}_3\vec{p}_4 >_0$ and write the integral expression for it using the Feynman rules.

21 Lecture 21. The optical theorem and the cutting rules

The optical theorem is a straightforward consequence of the *unitarity of the S-matrix*. In a good quantum theory, the operator defining the evolution of the system, in our case $S_{fi} =< f|U(+\infty, -\infty)|i >$, has to be unitary, in order to conserve probability (there are no "sources or sinks of probability"). Therefore, with $S = 1 + iT$, we have

$$S^\dagger S = \mathbf{1} = SS^\dagger \Rightarrow -i(T - T^\dagger) = T^\dagger T \tag{21.1}$$

which is the essence of the optical theorem. But the point is that in our definition of quantum field theory, and more importantly in our perturbative definition of S-matrices, there were many unknowns and approximations, so it is by no means obvious that the S matrix is unitary. We have to *prove it perturbatively (order by order in perturbation theory)*, and from the above, that is equivalent to proving the optical theorem.

To obtain the usual formulation of the optical theorem, we evaluate (21.1) between two 2-particle states, $|\vec{p}_1 \vec{p}_2 >$ and $|\vec{k}_1 \vec{k}_2 >$. Moreover, in between $T^\dagger T$, we introduce the identity expressed by considering the completeness relation for external states,

$$\mathbf{1} = \sum_n \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} |\{\vec{q}_i\} >< \{\vec{q}_i\}| \tag{21.2}$$

obtaining

$$< \vec{p}_1 \vec{p}_2 |T^\dagger T|\vec{k}_1 \vec{k}_2 >= \sum_n \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} < \vec{p}_1 \vec{p}_2 |T^\dagger|\{\vec{q}_i\} >< \{\vec{q}_i\}|T|\vec{k}_1 \vec{k}_2 > \tag{21.3}$$

But

$$< \{\vec{p}_f\}|T|\vec{k}_1 \vec{k}_2 >= \mathcal{M}(k_1, k_2 \to \{p_f\})(2\pi)^4 \delta^{(4)}\left(k_1 + k_2 - \sum p_f\right) \tag{21.4}$$

so by replacing in the above and taking out a common factor $(2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2)$, we obtain

$$-i[\mathcal{M}(k_1, k_2 \to p_1, p_2) - \mathcal{M}^*(p_1, p_2 \to k_1, k_2)]$$
$$= \sum_n \left(\prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \mathcal{M}^*(p_1 p_2 \to \{q_i\}) \mathcal{M}(k_1 k_2 \to \{q_i\})(2\pi)^4 \delta^{(4)}\left(k_1 + k_2 - \sum_i q_i\right) \tag{21.5}$$

or more schematically (here $d\Pi_f$ is the n-body relativistically invariant phase space defined in the next to last lecture)

$$-i[\mathcal{M}(a \to b) - \mathcal{M}^*(b \to a)] = \sum_f \int d\Pi_f \mathcal{M}^*(b \to f) \mathcal{M}(a \to f) \tag{21.6}$$

which is the general statement of the optical theorem on states.

An important particular case is when the initial and final states are the same ($a = b$), or *forward scattering*, $p_i = k_i$, in which case the right hand side of (21.5) contains the total cross section,

$$\text{Im}\mathcal{M}(k_1 k_2 \to k_1 k_2) = 2E_{CM}p_{CM}\sigma_{\text{tot}}(k_1 k_2 \to \text{anything}) \qquad (21.7)$$

or *the imaginary part of the forward scattering amplitude is proportional to the total cross section for the two particles to go to anything*, see Fig.56.
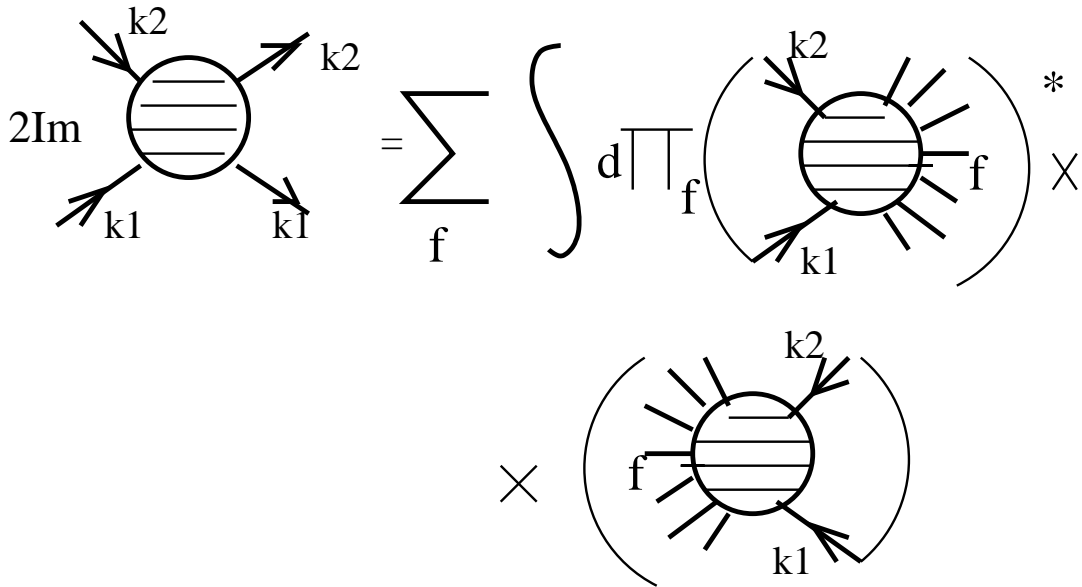

Figure 56: The optical theorem, diagrammatic representation.

But as we said, we supposed that the S-matrix is unitary, and *a priori* it could not be, we need to prove it in perturbative quantum field theory. It was proved for Feynman diagrams to all orders, first in $\lambda\phi^4$ theory by Cutkovsky, then in QED by Feynman and in gauge theories later. In particular, the proof for spontaneously broken gauge theories was done by 't Hooft and Veltman, which together with their proof of renormalizability of the same meant that spontaneously broken gauge theories are well-defined, and for which they got the Nobel prize. That proof was done using a more general formalism, of *the largest time equation*. Here we will just present the one-loop case in $\lambda\phi^4$.

Optical theorem at 1-loop in $\lambda\phi^4$ theory

The unitarity proof is done proving directly the optical theorem for Feynman diagrams.

Note that above the reduced amplitude $\mathcal{M}$ is defined formally, but we can define it purely by Feynman diagrams, and that is the definition we will use below, since we want to prove unitarity for Feynman diagrams. Then, $\mathcal{M}$ is defined at some center of mass energy, with $s = E_{CM}^2 \in \mathbb{R}_+$. But we analytically continue to an analytic function $\mathcal{M}(s)$ defined on the complex s plane.

Consider $\sqrt{s_0}$ the threshold energy for production of the lightest multiparticle state. Then, if $s < s_0$ and real, it means any possible intermediate state created from the initial

state cannot go on-shell, can only be virtual. In turn, that means that any propagators in the expression of $\mathcal{M}$ in terms of Feynman diagrams are always nonzero, leading to an $\mathcal{M}(s)$ real for $s < s_0$ real, i.e.

$$\mathcal{M}(s) = [\mathcal{M}(s^*)]^* \tag{21.8}$$

We then analytically continue to $s > s_0$ real, and since both sides of the above relation are analytical functions, we apply it for $s \pm i\epsilon$ with $s > s_0$ real, giving (since now the propagators can be on-shell, so we need to avoid the poles by moving s a bit away from the real axis)

$$\mathrm{Re}\mathcal{M}(s+i\epsilon) = \mathrm{Re}\mathcal{M}(s-i\epsilon); \quad \mathrm{Im}\mathcal{M}(s+i\epsilon) = -\mathrm{Im}\mathcal{M}(s-i\epsilon) \tag{21.9}$$

That means that for $s > s_0$ and real there is a branch cut in s (by moving accross it, we go between two Riemann sheets), with discontinuity

$$\mathrm{Disc}\mathcal{M}(s) = 2i\mathrm{Im}\mathcal{M}(s+i\epsilon) \tag{21.10}$$

It is much easier to compute the discontinuity only, than the full amplitude for complex s and taking the discontinuity.

The discontinuity of Feynman diagrams with loops, across cuts on loops (equal from the above to the Im parts of the same diagrams) will be related to $\int d\Pi |\bar{\mathcal{M}}|^2$, i.e. the optical theorem formula, see Fig.57. We will obtain the so-called "cutting rules", ironically found by Cutkovsky, so also called Cutkovsky rules.
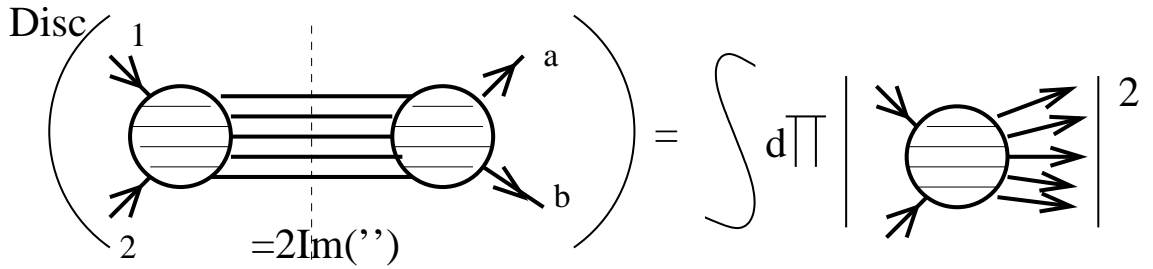

Figure 57: On the lhs of the optical theorem, represent the imaginary part of diagrams as a discontinuity (cut).

We consider the $2 \to 2$ one-loop diagram in $\lambda\phi^4$ where k_1, k_2 go to two intermediate lines, then to two final lines, called k_3, k_4. Define $k = k_1 + k_2$, and in the two intermediate lines define the momenta as $k/2 + q$ and $k/2 - q$, where q is the (integrated over) loop momentum, see Fig.58. Since this diagram has symmetry factor $S = 2$, we have for it (we call it $\delta\mathcal{M}$ since it could be part of a larger diagram)

$$i\delta\mathcal{M} = \frac{\lambda^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(k/2-q)^2+m^2-i\epsilon} \frac{1}{(k/2+q)^2+m^2-i\epsilon} \tag{21.11}$$

This amplitude has a discontinuity for $k^0 > 2m$ (in the physical region, since with center of mass energy k^0 we can create an on-shell 2-particle intermediate state of energy $2m$ only if $k^0 > 2m$), but we compute it for $k^0 < 2m$ and then analytically continue.
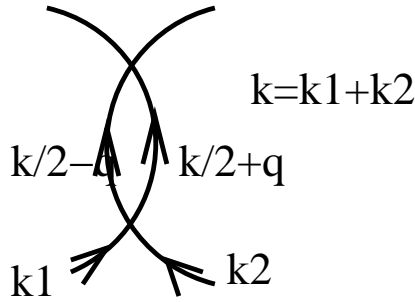
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Figure 58: The optical theorem is valid diagram by diagram. Consider this one-loop diagram, to derive the Cutkovsky rules.

In the center of mass system, $k = (k^0, \vec{0})$, where $s = (k^0)^2$, the poles of (21.11) are at $|k^0/2 \pm q| = \vec{q}^2 + m^2 \equiv E_q^2$, or at

$$
\begin{aligned}
q^0 &= \frac{k^0}{2} \pm (E_q - i\epsilon) \text{ and} \\
&= -\frac{k^0}{2} \pm (E_q - i\epsilon)
\end{aligned}
\tag{21.12}
$$
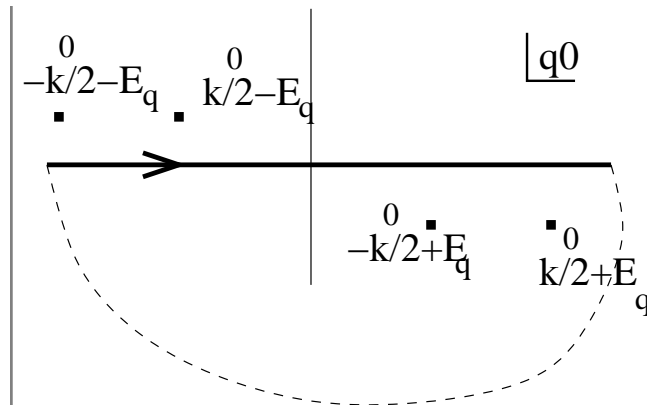
Figure 59: Closing the contour downwards we pick up the poles in the lower-half plane.

We now close the contour downwards, with a semi-circle in the lower-half plane, because of the prescription $T = \infty(1 - i\epsilon)$, therefore picking the poles in the lower-half planes, see Fig.59. We will see later that the pole at $+k^0/2 + E_q - i\epsilon$ does not contribute to the discontinuity, so only the pole at $-k^0/2 + E_q - i\epsilon$ does. But picking up the residue at the pole is equivalent to replacing

$$
\frac{1}{(k/2 + q)^2 + m^2 - i\epsilon} \simeq \frac{1}{q^0 + k^0/2 - E_q + i\epsilon} \frac{-1}{2E_q}
$$

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$$\rightarrow \quad 2\pi i \delta((k/2+q)^2 + m^2) = 2\pi i \frac{\delta(q^0 + k/2 - E_q)}{2E_q} \tag{21.13}$$

This gives

$$\begin{aligned}
i\delta\mathcal{M} &= -2\pi i \frac{\lambda^2}{2} \int \frac{d^3q}{(2\pi)^4} \frac{1}{2E_q} \frac{1}{(k^0 - E_q)^2 - E_q^2} \\
&= -2\pi i \frac{\lambda^2}{2} \frac{4\pi}{(2\pi)^4} \int_m^\infty dE_q E_q |q| \frac{1}{2E_q} \frac{1}{k^0(k_0 - 2E_q)}
\end{aligned} \tag{21.14}$$

In the first line we used the above delta function, and $(k/2-q)^2 + m^2 = -[(k^0 - E_q)^2 - E_q^2]$, and in the second $\int d^3q = \int d\Omega q^2 dq$, with $\int d\Omega = 4\pi$ and $\int q^2 dq = \int q E_q dE_q$, since $E_q^2 = q^2 + m^2$ (which also gives $E_q \geq m$). Note that if $k^0 < 2m$, the denominator in the last line has no pole on the integral path, as $k^0 < 2m < 2E_q$, so δM is real (has no discontinuity). [Also note that the pole in q^0 neglected, at $+k^0/1 + E_q - i\epsilon$ indeed does not contribute, since its residue has now no pole for the E_q integral, as we can easily check.] But if $k^0 > 2m$, we have a pole on the path of integration, so there is a discontinuity between $k^2 + i\epsilon$ and $k^2 - i\epsilon$. To find it, we can write

$$\frac{1}{k^0 - 2E_q \pm i\epsilon} = P\frac{1}{k^0 - 2E_q} \mp i\pi\delta(k^0 - 2E_q) \tag{21.15}$$

where P stands for principal part. Therefore for the discontinuity, equivalently, we can also replace the second propagator with the delta function

$$\begin{aligned}
\frac{1}{(k/2-q)^2 + m^2 - i\epsilon} &= \frac{1}{-(k^0 - E_q)^2 + E_q^2 - i\epsilon} = \frac{1}{k^0(k^0 - 2E_q)} \\
\rightarrow \quad 2\pi i \delta((k/2-q)^2 + m^2) &= 2\pi i \delta(k^0(k^0 - 2E_q)) = 2\pi i \frac{\delta(k^0 - 2E_q)}{k^0}
\end{aligned} \tag{21.16}$$

We now go back, to see what we did, and rewrite the original loop integration as the integration over momenta on the propagators, with a vertex delta function ($k/2 - q = p_1, k/2 + q = p_2$),

$$\int \frac{d^4q}{(2\pi)^4} = \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k) \tag{21.17}$$

In this formulation, the discontinuity is obtained by replacing the propagators with delta functions for them, i.e.

$$\frac{1}{p_i^2 + m_i^2 - i\epsilon} \rightarrow 2\pi i \delta(p_i^2 + m^2) \tag{21.18}$$

So by doing the integrals over the zero component of the momenta $\int d^4p_1 \int d^4p_2$, we put the momenta on-shell, and we finally get

$$\text{Disc}\mathcal{M}(k) = 2i\text{Im}\mathcal{M}(k) = \frac{i}{2} \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} |\tilde{\mathcal{M}}(k)|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k) \tag{21.19}$$

where the $1/2$ factor is interpreted as the symmetry factor for 2 identical bosons, $|\tilde{\mathcal{M}}(k)|^2 = \lambda^2$ corresponds in general to $\tilde{\mathcal{M}}(k_1, k_2 \rightarrow p_1, p_2)\tilde{\mathcal{M}}^*(k_3, k_4 \rightarrow p_1, p_2)$. On the left hand side

the discontinuity is in the one-loop amplitude $\mathcal{M}(k_1, k_2 \to k_3, k_4)$ (we wrote above for forward scattering $k_1 = k_3, k_2 = k_4$), where we cut the intermediate propagators, putting them on-shell, i.e. we replace the propagator with the delta function for the on-shell condition, as in the lhs of Fig.57. On the rhs we have the rhs of Fig.57.

Therefore we have verified the optical theorem at one-loop, and by the discussion at the beginning of the lecture, we have thus verified unitarity at one-loop in $\lambda\phi^4$. We can also write some one-loop diagrams in QED, cut them and write the corresponding optical theorem at one-loop, see 60. One can prove unitarity for them equivalently to the above.
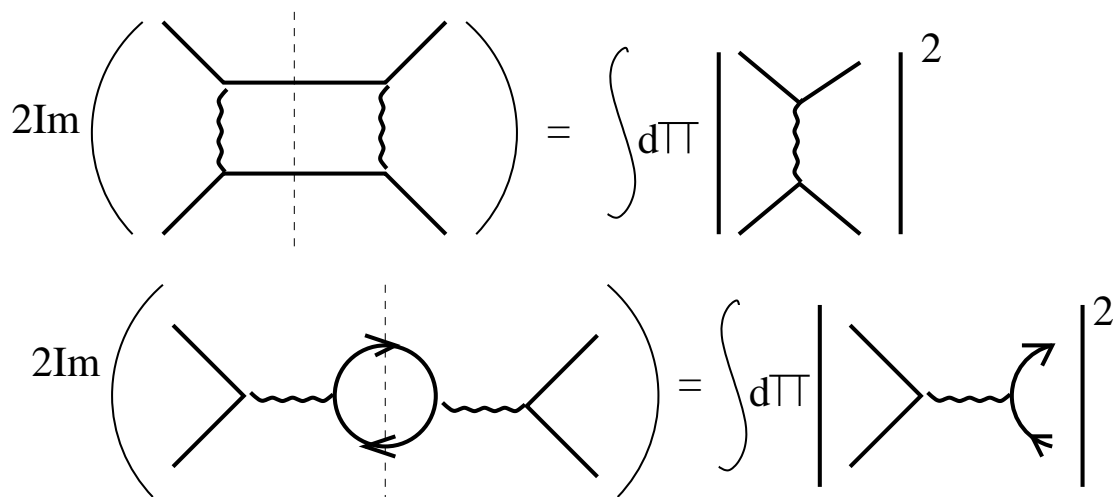
Figure 60: Examples of the optical theorem at one-loop in QED.

Finally, Cutkovski gave the **general cutting rules**, true in general (at all loops).

1) Cut the diagram in all possible ways such that cut propagators can be put *simultaneously* on-shell.

2) For each cut, replace

$$\frac{1}{p_i^2 + m_i^2 - i\epsilon} \to 2\pi i \delta(p^2 + m^2) \tag{21.20}$$

in the cut propagators, and then do the integral.

3) Sum over all possible cuts.

Then, he proved unitarity (equivalent to the optical theorem) in perturbation theory.

There exists a more general method, the method of the *largest time equation* (which is a generalization of the above method), that was used by 't Hooft and Veltman to prove the perturbative unitarity of spontaneously broken gauge theories, but we will not do it here.

Important concepts to remember

- The optical theorem is the statement $-i(T - T^\dagger) = T^\dagger T$ on states, which follows from unitarity of the S-matrix.

- The diagrammatic interpretation for it is that we cut over all possible intermediate states the amplitude for $2 \to 2$ scattering, and this equals the cut amplitude parts $\mathcal{M}(a \to f)\mathcal{M}^*(b \to f)$, integrated over the final phase space.

- It implies that the imaginary part of the forward scattering amplitude is proportional to the total cross section $\sigma_{\text{tot}}(k_1 k_2 \to \text{anything})$.

- In $\lambda\phi^4$ at one-loop, the discontinuity for $\mathcal{M}(k)$ is found by replacing the intermediate cut propagators with delta function of the mass shell condition, leading to the one-loop optical theorem (defined diagrammatically), which is equivalent to one-loop unitarity.

- The general cutting rules say that we should cut a diagram in all possible ways such that the cut propagators can be put simultaneously on-shell, then replace the cut progagators with their mass shell delta function, and sum over all possible cuts.

- Using this method, and its generalization, the largest time equation, perturbative unitarity was proven for $\lambda\phi^4$ theory, gauge theories, spontaneously broken gauge theories.

Further reading: See chapter 7.3 in [2].

Exercises, Lecture 21

1) ("**Black disk eikonal**") Consider $2 \rightarrow 2$ scattering of massless particles $k_1, k_2 \rightarrow k_3, k_4$, with all momenta defined as incoming, as in Fig.61, and $s = -(k_1 + k_2)^2$, $t = -\vec{q}^2 = -(k_1 + k_3)^2$, and the S-matrix

$$S = e^{i\delta(b,s)} \tag{21.21}$$

where $\delta(b, s)$ satisfies

$$\text{Re}[\delta(b, s)] = 0$$
$$\text{Im}[\delta(b, s)] = 0 \;\; \text{for} \;\; b > b_{max}(s); \quad \text{Im}[\delta(b, s)] = \infty \;\; \text{for} \;\; b < b_{max}(s) \tag{21.22}$$

and

$$\frac{1}{s}\mathcal{M}(s, t) = \int d^2 b\, e^{i\vec{q}\cdot\vec{b}} T(b, s) \tag{21.23}$$

where $\vec{b}$ is the impact parameter. Calculate $\sigma_{tot}(k_1, k_2 \rightarrow \text{anything})$.
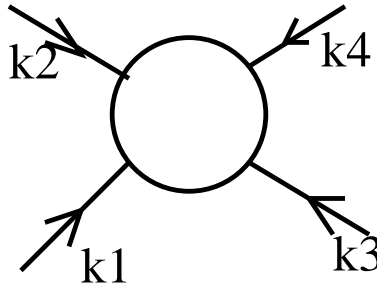
Figure 61: 2 to 2 scattering with incoming momenta.

2) Consider the two-loop Feynman diagram in $\lambda\phi^3$ theory (two incoming external lines merge into an intemediate one, which splits into a circle with a vertical line in it, then mirror symmetric: intermediate line splittng into the final two external states) in Fig.62.
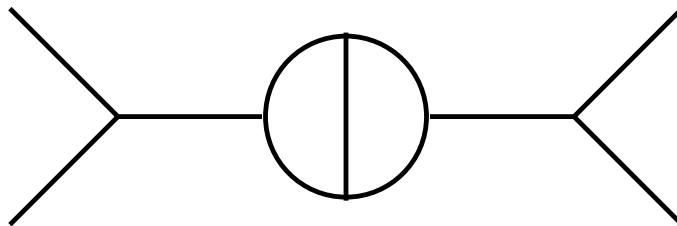
Figure 62: Two-loop diagram in ϕ^3 theory.

Write down the unitarity relation for it using the cutting rules. Which cuts depend on whether all lines have the same mass, and which ones don't?

22 QED: Definition and Feynman rules; Ward-Takahashi identities

We now turn to applications to the real world, in particular to QED (quantum electrodynamics). It is given by a gauge field (electromagnetic field) A_μ and a fermion (the electron field) ψ. For generality, we couple also to a complex scalar field ϕ.

The Lagrangean in *Minkowski space* is then

$$\mathcal{L}_{QED,M}(A,\psi,\phi) = -\frac{1}{4}F_{\mu\nu}^2 - \bar\psi(\slashed{D}+m)\psi - (D_\mu\phi)^*D^\mu\phi - V(\phi^*\phi) = \mathcal{L}(A)+\mathcal{L}(\psi,A)+\mathcal{L}(\phi,A) \tag{22.1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\slashed{D} = D_\mu\gamma^\mu$ and $D_\mu = \partial_\mu - ieA_\mu$, and $\bar\psi = \psi^\dagger i\gamma^0$, $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$.

The Lagrangean possesses a local $U(1)$ gauge invariance, under

$$\begin{aligned}\psi(x) &\to \psi'(x) = \psi(x)e^{ie\chi(x)}\\ \phi(x) &\to \phi'(x) = \phi(x)e^{ie\chi(x)}\\ A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x)\end{aligned} \tag{22.2}$$

We note that the Lagrangean at $A_\mu = 0$ has only global $U(1)$ invariance, and by the condition to make the invariance local (to "gauge it"), we need to introduce the gauge field $A_\mu(x)$ with the transformation law above.

In *Euclidean space*, since as usual $iS_M \to -S$, we get for the Euclidean fermion lagrangean

$$\mathcal{L}_E[\psi] = \bar\psi(\gamma^\mu\partial_\mu + m)\psi \tag{22.3}$$

where now $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$. We also saw that for the gauge field we have

$$\mathcal{L}_E[A] = +\frac{1}{4}F_{\mu\nu}^{(E)}F^{\mu\nu(E)} \tag{22.4}$$

Therefore the QED Lagrangean in Euclidean space is

$$\mathcal{L}_E(A,\psi,\phi) = +\frac{1}{4}F_{\mu\nu}^2 + \bar\psi(\slashed{D}+m)\psi + (D_\mu\phi)^*D^\mu\phi + V(\phi^*\phi) \tag{22.5}$$

Path integral

As we saw, defining the path integral by quantizing, we use a procedure like the Fadeev-Popov one, leading to a gauge fixing term, with

$$\mathcal{L}_{eff} = \mathcal{L}(A) + \mathcal{L}_{\text{gauge fix}} = \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 \tag{22.6}$$

The action with gauge fixing term can then be written as

$$S_{eff}[A] = \frac{1}{2}\int d^dx A_\mu\left(-\partial^2\delta_{\mu\nu} + \left(1-\frac{1}{\alpha}\right)\partial_\mu\partial_\nu\right)A_\nu = \frac{1}{2}\int d^dx A_\mu G_{\mu\nu}^{(0)-1}A_\nu \tag{22.7}$$

Then in momentum space the photon propagator is

$$G_{\mu\nu}^{(0)} = \frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right) \tag{22.8}$$

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From now on, we talk about pure QED, without ϕ.

The Euclidean space partition function is

$$Z[J_\mu, \xi, \bar{\xi}] = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S_{eff}[A,\psi,\bar{\psi},J,\xi,\bar{\xi}]} \tag{22.9}$$

where in the presence of sources

$$S_{eff} = \int d^dx[\mathcal{L}_{eff}(A, \psi, \bar{\psi}) - J_\mu A_\mu - \bar{\xi}\psi - \bar{\psi}\xi] \tag{22.10}$$

The VEV of an operator $\mathcal{O}(A, \bar{\psi}, \psi)$ is

$$< \mathcal{O}(A, \bar{\psi}, \psi) >= \frac{\int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{O}e^{-S_{eff}}}{\int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S_{eff}}} \tag{22.11}$$

The free energy F is defined by

$$Z[J, \xi, \bar{\xi}] = e^{-F[J,\xi,\bar{\xi}]} \tag{22.12}$$

The effective action is the Legendre transform of the free energy

$$\Gamma[A^{cl}, \bar{\psi}^{cl}, \psi^{cl}] = F[J, \xi, \bar{\xi}] + \int d^dx[J_\mu A_\mu^{cl} + \bar{\xi}\psi^{cl} + \bar{\psi}^{cl}\xi] \tag{22.13}$$

By taking derivatives with respect to $J_\mu, \xi, \bar{\xi}, A_\mu^{cl}, \psi^{cl}, \bar{\psi}^{cl}$ of the Legendre transform relation, we obtain

$$\frac{\delta F}{\delta J_\mu} = -A_\mu^{cl}; \quad \frac{\delta F}{\delta \bar{\xi}} = -\psi^{cl}; \quad \frac{\delta F}{\delta \xi} = \bar{\psi}^{cl}$$
$$\frac{\delta \Gamma}{\delta A_\mu^{cl}} = J_\mu; \quad \frac{\delta \Gamma}{\delta \psi^{cl}} = -\bar{\xi}; \frac{\delta \Gamma}{\delta \bar{\psi}^{cl}} = \xi \tag{22.14}$$

Feynman rules for Green's functions in Euclidean momentum space

It is easier to calculate Green's functions in Euclidean space, hence we first write the Feynman rules in Euclidean space (see Fig.63a):

- ψ propagator from α to β (arrow from α to β), with momentum p is

$$\left(\frac{1}{i\not{p} + m}\right)_{\alpha\beta} = \frac{(-i\not{p} + m)_{\alpha\beta}}{p^2 + m^2} \tag{22.15}$$

- photon propagator, from μ to ν is

$$\frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right) \tag{22.16}$$
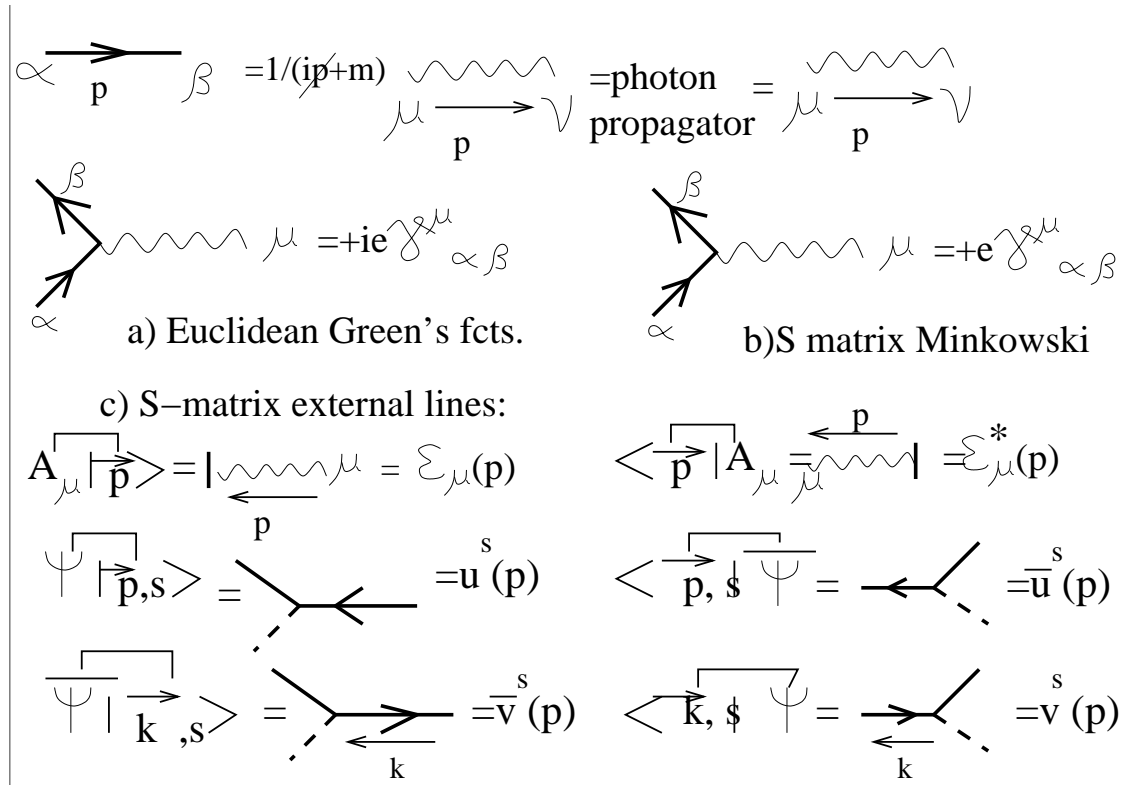
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Figure 63: Relevant Feynman rules for Green's functions and S-matrices.

- vertex $+ie(\gamma^\mu)_{\alpha\beta}$ for a photon with μ to come out of a line from α to β. This is so because the interaction term in the action is $S_{int} = -ie \int \bar{\psi}\gamma^\mu A_\mu \psi$, and we know that for an interaction $+g \int \prod_i \phi_i$, the vertex is $-g$.

- a minus sign for a fermion loop

Feynman rules for S-matrices in Minkowski space

S-matrices can only be defined in Minkowski space, since we need to have external states, satisfying $p^2 + m^2 = 0$, but that equation has no real solution in Euclidean space, only in Minkowski space. See Fig.63b,c.

- The fermion propagator can be found from the Euclidean space one by the replacement $i\not{p} + m \to i\not{p} + m$, $p^2 + m^2 \to p^2 + m^2 - i\epsilon$, and an extra $-i$ in front (from the x space continuation of t in the action), giving

$$-i\frac{-i\not{p} + m}{p^2 + m^2 - i\epsilon} = \frac{-(\not{p} + im)}{p^2 + m^2 - i\epsilon} \tag{22.17}$$

- photon propagator. The same continuation rules give

$$-\frac{i}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1 - \alpha)\frac{k_\mu k_\nu}{k^2 - i\epsilon}\right) \tag{22.18}$$

- vertex $+e(\gamma^\mu)_{\alpha\beta}$ (there is an i difference between the Euclidean and Minkowski vertices).

- a minus sign for fermion loops.

- external fermion lines: for the $\overline{\psi|\vec{p}, s} >$ contraction, $= u^s(p)$, for the $\overline{\bar{\psi}|\vec{p}}, s >$ contraction, $= \bar{v}^s(p)$, and for the bars, similar: $< \vec{p}, s\overline{|\bar{\psi}} = \bar{u}^s(p)$ and $< \vec{p}, s\overline{|\psi} = v^s(p)$.

- external photon lines, $\overline{A_\mu|\vec{p}} >= \epsilon_\mu(p)$ and $< \vec{p}\overline{|A_\mu} = \epsilon^*(p)$.

- We can also see that if we exchange the order of the external fermion line contractions, we get a minus sign, so we deduce that if we exchange identical external fermion lines in otherwise identical diagrams, we get a minus sign.

Ward-Takahashi identities

These are a type of Ward identities for QED, i.e. a local version of the global Ward identities we studied before. Consider an infinitesimal local gauge invariance transformation

$$\delta\psi(x) = ie\epsilon(x)\psi(x)$$
$$\delta A_\mu(x) = \partial_\mu\epsilon(x) \tag{22.19}$$

If we consider this as just a change (renaming) of integration variables in the path integral, the path integral should not change. On the other hand, *assuming that the regularization of the path integral respects gauge invariance*, we also expect that there is no nontrivial Jacobian, and $\delta(\mathcal{D}A\mathcal{D}\psi\mathcal{D}\phi) = 0$.

Note that this assumption is in principle nontrivial. There are classical symmetries that are broken by the quantum corrections. In the path integral formalism, the way this happens is that the regularization of the path integral measure (i.e. defining better $\mathcal{D}\phi$'s as $\prod_i d\phi(x_i)$'s) does not respect the symmetries (the only other ingredient of the path integral is the classical action, which is invariant). These are called *quantum anomalies*. But as we will see in QFT II, anomalies in gauge symmetries are actually bad, and make the theory sick, so should be absent from a well-defined regularization. Therefore we will assume that the regularization of the measure respects gauge invariance.

Since as we said, the change is just a redefinition of the path integral, we also have

$$\delta Z[J, \xi, \bar{\xi}] = 0 \tag{22.20}$$

Using the form of the partition function, and the fact that both the classical action and the measure are invariant, we only need to evaluate the change due to the gauge fixing and source terms. This gives

$$0 = \delta Z = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S_{eff}} \int d^dx \left[J^\mu\partial_\mu\epsilon + ie\epsilon(\bar{\xi}\psi - \bar{\psi}\xi) - \frac{1}{\alpha}(\partial_\mu A_\mu)\partial^2\epsilon \right] \tag{22.21}$$

where the last term comes from varying the gauge fixing term, and the others from varying the source terms.

Taking the derivative with respect to ϵ, and then expressing $A, \psi, \bar{\psi}$ as derivatives of S_{eff} (e.g., $A_\mu e^{-S_{eff}} = \delta/\delta J_\mu e^{-S_{eff}}$) , taking out of the path integral the remaining objects that are not integrated over, we are left with an operator that acts on Z, namely

$$0 = \frac{\delta Z}{\delta \epsilon(x)} = -\frac{1}{\alpha} \partial^2 \partial_\mu \frac{\delta Z}{\delta J_\mu} - (\partial_\mu J_\mu) Z + ie \left(\bar{\xi} \frac{\delta Z}{\delta \bar{\xi}} + \frac{\delta Z}{\delta \xi} \xi \right) \qquad (22.22)$$

Dividing by Z and using that $\ln Z = -F$, we get

$$0 = \frac{1}{\alpha} \partial^2 \partial_\mu \frac{\delta F}{\delta J_\mu} - \partial_\mu J_\mu - ie \left(\bar{\xi} \frac{\delta F}{\delta \bar{\xi}} + \frac{\delta F}{\delta \xi} \xi \right) \qquad (22.23)$$

Then replacing the equations (22.14) in the above, we finally get

$$-\frac{1}{\alpha} \partial^2 \partial_\mu A_\mu^{cl} - \partial_\mu \frac{\delta \Gamma}{\delta A_\mu^{cl}} - ie \left(\frac{\delta \Gamma}{\delta \psi^{cl}} \psi^{cl} + \bar{\psi}^{cl} \frac{\delta \Gamma}{\delta \bar{\psi}^{cl}} \right) = 0 \qquad (22.24)$$

These are called *generalized Ward-Takahashi identities*, or (for the 1PI functions, as we will shortly derive some examples) the *Lee-Zinn-Justin identities*.

Example 1. Photon propagator

By taking derivatives of the LZJ identities with respect to $A_\mu^{cl}(y)$ and putting $A^{cl}, \psi^{cl}, \bar{\psi}^{cl}$ to zero, i.e.

$$\frac{\delta}{\delta A_\nu^{cl}(y)} (LZJ)|_{A^{cl} = \psi^{cl} = \bar{\psi}^{cl} = 0} \qquad (22.25)$$

we get

$$-\frac{1}{\alpha} \partial_x^2 \partial_{x^\nu} \delta(x - y) - \partial_{x^\mu} \frac{\delta^2 \Gamma}{\delta A_\mu^{cl}(x) \delta A_\nu^{cl}(y)} \bigg|_{A^{cl} = \psi^{cl} = \bar{\psi}^{cl} = 0} = -\frac{1}{\alpha} \partial_x^2 \partial_{x^\nu} \delta(x - y) - \partial_{x^\mu} G_{\mu\nu}^{-1}(x, y) = 0 \qquad (22.26)$$

where $G_{\mu\nu}^{-1}$ is the inverse of the full connected propagator. Indeed, as we saw before, this equals the two-point effective action, i.e. the second derivative of the effective action. Taking the Fourier transform, we obtain

$$k^\mu \left[\frac{k^2}{\alpha} \delta_{\mu\nu} - G_{\mu\nu}^{-1}(k) \right] = 0 \qquad (22.27)$$

On the other hand, we have

$$G_{\mu\nu}^{-1}(k) = \Gamma_{\mu\nu}(k) = G_{\mu\nu}^{(0)-1} + \Pi_{\mu\nu} \qquad (22.28)$$

and since

$$G_{\mu\nu}^{(0)-1}(k) = k^2 \delta_{\mu\nu} - k_\mu k_\nu + \frac{1}{\alpha} k_\mu k_\nu \qquad (22.29)$$

we have

$$k^\mu G_{\mu\nu}^{(0)-1}(k) = \frac{k^2 k_\nu}{\alpha} \qquad (22.30)$$

Replacing it in (22.27), we get

$$k^\mu \Pi_{\mu\nu}(k) = 0 \tag{22.31}$$

That is, the 1PI corrections to the photon propagator are transverse (perpendicular to the momentum k^μ). That means that we can write them in terms of a single (scalar) function,

$$\Pi_{\mu\nu}(k) = (k^2 \delta_{\mu\nu} - k_\mu k_\nu)\Pi(k) \tag{22.32}$$

Example 2. n-photon vertex function for $n \geq 3$.
We now take $n-1$ derivatives with respect to A_μ^{cl}, i.e. put

$$\frac{\delta^{n-1}}{\delta A_{\mu_2}^{cl}(x_2)...\delta A_{\mu_n}^{cl}(x_n)}(LZJ)|_{A^{cl}=\psi^{cl}=\bar\psi^{cl}=0} \tag{22.33}$$

and find now

$$\frac{\partial}{\partial x_{\mu_1}} \frac{\delta^{(n)}\Gamma}{\delta A_{\mu_1}^{cl}(x_1)...\delta A_{\mu_n}^{cl}(x_n)}|_{A^{cl}=\psi^{cl}=\bar\psi^{cl}=0} \tag{22.34}$$

Going to momentum space we then find

$$k^{\mu_1} \Gamma_{\mu_1...\mu_n}^{(n)}(k^{(1)},...,k^{(n)}) = 0 \tag{22.35}$$

That is, the n-point 1PI photon vertex functions (from the effective action) are also transverse, like the 1PI 2-point function above.

Example 3. Original Ward-Takahashi identity
We take the derivatives with respect to ψ^{cl} and $\bar\psi^{cl}$, i.e.

$$\frac{\delta^2}{\delta \psi_\beta^{cl}(z)\delta \bar\psi_\alpha^{cl}(y)}(LZJ)|_{A^{cl}=\psi^{cl}=\bar\psi^{cl}=0} \tag{22.36}$$

to obtain

$$0 = \left[-\frac{\partial}{\partial x^\mu} \frac{\delta^3\Gamma}{\delta \psi_\beta^{cl}(z)\delta \bar\psi^{cl}(y)\delta A_\mu(x)} - ie\delta^{(d)}(x-y)\frac{\delta^2\Gamma}{\delta \psi_\beta^{cl}(z)\delta \bar\psi_\alpha^{cl}(x)} \right.$$
$$\left. + ie\delta^{(d)}(x-z)\frac{\delta^2\Gamma}{\delta \psi_\beta^{cl}(x)\delta \bar\psi_\alpha^{cl}(y)} \right]|_{A^{cl}=\psi^{cl}=\bar\psi^{cl}=0} \tag{22.37}$$

Replacing the derivatives with the corresponding n-point effective action terms, we get

$$-\frac{\partial}{\partial x^\mu}\Gamma_{\mu;\alpha\beta}(x;y,z) = -ie\delta^{(d)}(x-z)(S_F^{-1})_{\alpha\beta}(y-x) + ie\delta^{(d)}(x-y)(S_F^{-1})_{\alpha\beta}(x-z) \tag{22.38}$$

where S_F^{-1} is the inverse of the full connected fermion propagator, i.e. the 1PI 2-point effective action.

In momentum space, this relation becomes

$$p^\mu\Gamma_{\mu\alpha\beta}(p;q_2,q_1) = e(S_F^{-1}(q_2)_{\alpha\beta} - S_F^{-1}(q_1)_{\alpha\beta}) \tag{22.39}$$
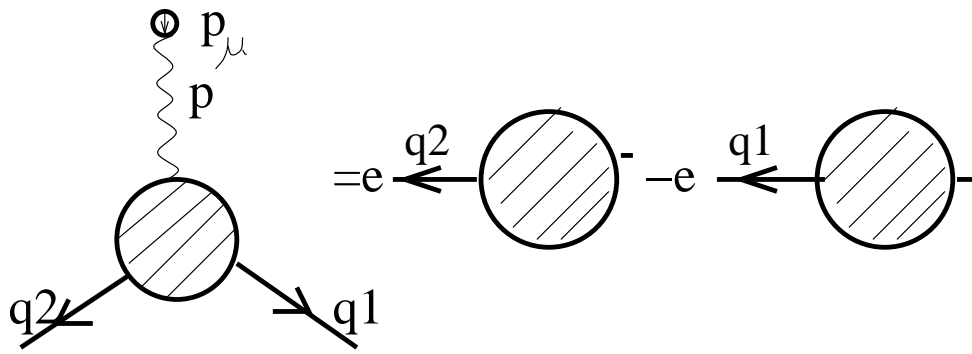
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Figure 64: Diagrammatic representation of the original Ward-Takahashi identity.

which is the original Ward-Takahashi identity: the photon-fermion-antifermion vertex contracted with the momentum gives the difference of the inverse fermion propagator (or 1PI 2-point function) for q_2 minus the one for q_1, where $q_{1,2}$ are the fermion momenta, as in Fig.64.

Important concepts to remember

- QED is a gauge field (electromagnetic field) coupled to a fermion (generalized to include a coupling to a complex scalar)

- The generalized Ward-Takahashi identities or Lee-Zin-Justin identities are identities between n-point effective actions, or 1PI n-point functions, resulting from local versions of Ward identities.

- The 1PI corrections to the photon propagator are transverse, $p^\mu \Pi_{\mu\nu}(k) = 0$.

- The 1PI n-photon vertex functions for $n \geq 3$ are also transverse, $k^{\mu_1} \Gamma_{\mu_1 \ldots \mu_n} = 0$.

- The original Ward-Takahashi identity relates the photon-fermion-antifermion vertex, contracted with the photon momentum, with the difference of the inverse photon propagator (2-point effective action) for the two fermions.

Further reading: See chapters 4.8 in [2], 6.3 in [4] and 8.1 in [1].

Exercises, Lecture 22

1) Write down the expression for the QED Feynman diagram in Fig.65 for the S-matrix $e^-(p_1)e^+(p_2) \to e^-(p_3)e^+(p_4)$.
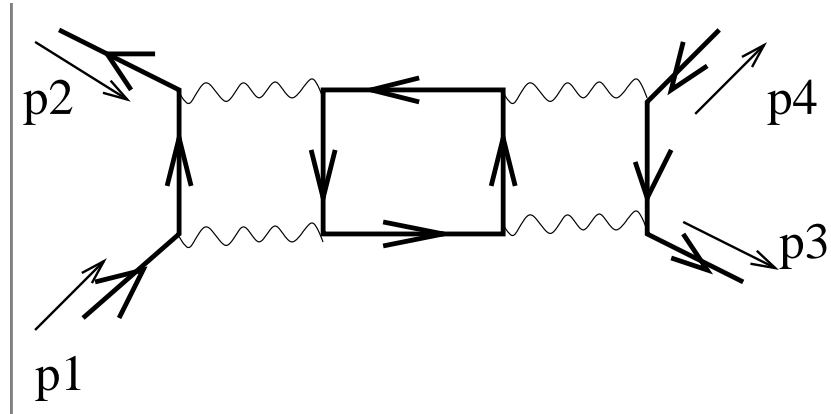

Figure 65: Feynman diagram for the S-matrix in QED.

2) Using the Lee-Zinn-Justin identities, derive a Ward identity for the vertex $\Gamma_{\mu_1\mu_2\alpha\beta}$.

23   Lecture 23. Nonrelativistic processes: Yukawa potential, Coulomb potential and Rutherford scattering

In this lecture we will rederive some formulas for classical, nonrelativistic processes using the quantum field theory formalism we have developed, to check that our formalism works, and understand how it is applied. In particular, we will calculate the Yukawa and Coulomb potentials from our formalism, and derive the differential cross section for Rutherford scattering.

Yukawa potential

Even though we are more interested in the QED example of the Coulomb potential, we will start with the simplest example, the Yukawa potential, as a warm-up.

In Yukawa theory, the interaction of two fermions, $\psi(p) + \psi(k) \to \psi(p') + \psi(k')$, happens via the exchange of a scalar particle ϕ, of nonzero mass m_ϕ. We want to see that this gives indeed the well-known Yukawa potential.
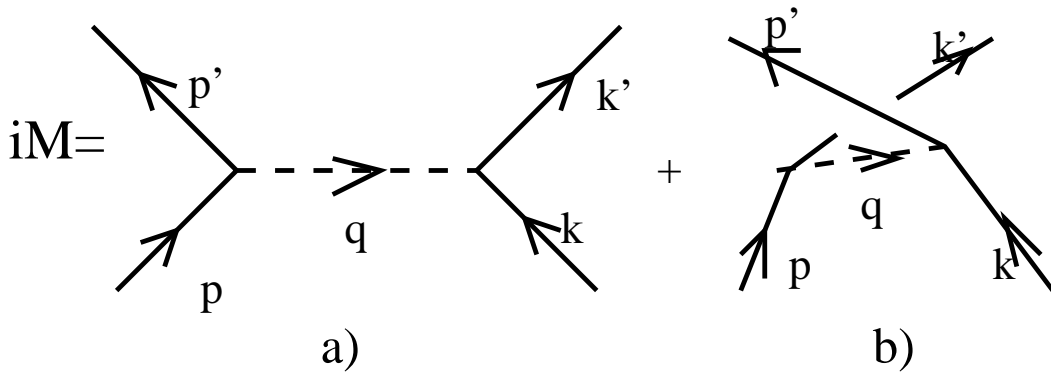
Figure 66: The two Feynman diagrams for the potential in the case of Yukawa interaction. For distinguishable particles, only the diagram a) contributes.

There are two diagrams that in principle contribute to $i\mathcal{M}$. One where the scalar with momentum $q = p - p'$ is exchanged between the two fermions, see Fig.66a, and one where the final fermions are interchanged (the fermion with momentum p' comes out of the one with k, and the fermion with momentum k' comes out of the one with p), see Fig.66b. But we will work with *distinguishable fermions*, i.e. we will assume that there is some property that we can measure that will distinguish between them (for instance, we could consider one to be an e^-, and another to be a μ^-). In this case, the second diagram will not contribute. Since this is what happens classically, this is a good assumption in order to get a classical potential.

In the nonrelativistic limit, we have

$$p \simeq (m, \vec{p}); \quad p' \simeq (m, \vec{p}'); \quad k = (m, \vec{k}); \quad k' = (m, \vec{k}')$$

$$(p - p')^2 \simeq (\vec{p} - \vec{p'})^2$$

$$u^s(p) \simeq \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \qquad (23.1)$$

where $\xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $s = 1, 2$. The last property means that

$$\bar{u}^{s'}(p')u^s(p) = u^{\dagger s'}(i\gamma^0)u^s = m \begin{pmatrix} \xi^{\dagger s'} & \xi^{\dagger s'} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} = 2m\xi^{\dagger s'}\xi^s = 2m\delta^{ss'} \qquad (23.2)$$

Then, writing both terms in $i\mathcal{M}$ for completeness, we have

$$\begin{aligned} i\mathcal{M} &= (-ig)\bar{u}(p')u(p)\frac{-i}{(p-p')^2 + m_\phi^2}(-ig)\bar{u}(k')u(k) \\ &\quad -(-ig)\bar{u}(p')u(k)\frac{-i}{(p-k')^2 + m_\phi^2}(-ig)\bar{u}(k')u(p) \end{aligned} \qquad (23.3)$$

but as we said, we will drop the second term since we consider distinguishable fermions. Some comments are in order here. We have considered a Yukawa coupling g, which means a vertex factor $-ig$ in Minkowski space. The fermions are contracted along each of the fermion lines separately. The overall sign for the fermions is conventional, only the relative sign between diagrams is important, and that can be found from the simple rule that when we interchange two fermions between otherwise identical diagrams (like interchanging $u(p)$ with $u(k)$ in the two terms above), we add a minus sign.

But we can choose a convention. We can define

$$|\vec{p}, \vec{k}> \sim a_{\vec{p}}^\dagger a_{\vec{k}}^\dagger|0>; \quad < \vec{p'}, \vec{k'}| = (|\vec{p'}, \vec{k'}>)^\dagger \sim < 0|a_{\vec{k'}}a_{\vec{p'}} \qquad (23.4)$$

so that in the matrix element

$$< \vec{p'}, \vec{k'}|(\bar{\psi}\psi)_x(\bar{\psi}\psi)_y|\vec{p}, \vec{k}> \sim < 0|a_{\vec{k'}}a_{\vec{p'}}(\bar{\psi}\psi)_x(\bar{\psi}\psi)_y a_{\vec{p}}^\dagger a_{\vec{k}}^\dagger|0 > \qquad (23.5)$$

the contraction of $\vec{k'}$ with $\bar{\psi}_x$, $\vec{p'}$ with $\bar{\psi}_y$, $\vec{p}$ with ψ_y and $\vec{k}$ with ψ_x, corresponding to the first diagram (the one we keep), gives a plus sign (we can count how many jumps we need for the fermions to be able to contract them, and it is an even number). Then we can easily see that the second diagram, where we now contract $\vec{k'}$ with $\bar{\psi}_y$ and $\vec{p'}$ with $\bar{\psi}_x$ instead, has a minus sign. As we said, the simple Feynman rule above accounts for this relative sign.

We then finally have for the contribution of the first diagram (for distinguishable fermions)

$$i\mathcal{M} \simeq +\frac{ig^2}{|\vec{p} - \vec{p'}|^2 + m_\phi^2}(2m\delta^{ss'})(2m\delta^{rr'}) \qquad (23.6)$$

But we then can compare with the Born approximation to the scattering amplitude in nonrelativistic quantum mechanics, written in terms of the potential in momentum space,

$$< \vec{p'}|iT|\vec{p}> = -iV(\vec{q})2\pi\delta(E_{p'} - E_p) \qquad (23.7)$$

where $\vec{q} = \vec{p}' - \vec{p}$. When comparing, we should remember a few facts: the $2m$ factors are from the relativistic normalization we used for the fermions, whereas the Born approximation above uses nonrelativistic normalization, so we should drop them in the comparison. Then $iT = 2\pi\delta(E_f - E_i)i\mathcal{M}$, so we drop the $2\pi\delta(E_{p'} - E_p)$ in the comparison; and also the formula is at $s = s', r = r'$. We finally get

$$V(\vec{q}) = \frac{-g^2}{|\vec{q}|^2 + m_\phi^2} \qquad (23.8)$$

We should also comment on the logic: it is impossible to directly measure the potential $V(\vec{q})$ in the scattering, since this is a classical concept, so we can only measure things that depend on it. But for expedience's sake, we used the comparison with nonrelativistic quantum mechanics, where we have a quantum mechanics calculation, yet in a classical potential, in order to directly extract the result for $V(\vec{q})$.

The x-space potential is (here $|\vec{x}| \equiv r$)

$$\begin{aligned}
V(\vec{x}) &= \int \frac{d^3q}{(2\pi)^3} \frac{-g^2}{|\vec{q}|^2 + m_\phi^2} e^{i\vec{q}\cdot\vec{x}} \\
&= -\frac{g^2}{(2\pi)^3} 2\pi \int_0^\infty q^2 dq \frac{e^{iqr} - e^{-iqr}}{iqr} \frac{1}{q^2 + m_\phi^2} \\
&= -\frac{g^2}{4\pi^2 ir} \int_{-\infty}^{+\infty} dq \frac{qe^{iqr}}{q^2 + m_\phi^2} \qquad (23.9)
\end{aligned}$$

Here in the second equality we have used $\int_0^\pi \sin\theta d\theta e^{iqr\cos\theta} = \int_0^1 d(\cos\theta)e^{(iqr)\cos\theta} = (e^{iqr} - e^{-iqr})/(iqr)$ and in the last equality we have rewritten the second term as an integral from $\int_{-\infty}^0$. The integral has complex poles at $q = \pm im_\phi$, so we can close the contour over the real axis with an infinite semicircle in the upper-half plane, since then e^{iqr} decays exponentially at infinity. We then pick up the residue of the pole $q = im_\phi$, giving finally

$$V(r) = \frac{-g^2}{4\pi r} e^{-m_\phi r} \qquad (23.10)$$

the well-known attractive Yukawa potential.

Coulomb potential

We can now move to the case of nonrelativistic scattering in QED, as in Fig.67. The difference is now that we have a vector exchanged between the two fermions, with a (massless) vector propagator, in the Feynman gauge just $g_{\mu\nu}$ times the KG propagator, and a vertex factor $+e\gamma^\mu$ in Minkowski space.

We then get

$$i\mathcal{M} = (+e)\bar{u}(p')\gamma^\mu u(p) \frac{-ig_{\mu\nu}}{(p - p')^2} (+e)\bar{u}(k')\gamma^\nu u(k) \qquad (23.11)$$

But in the nonrelativistic limit we can check that the only nonzero components are the zero ones (as expected), so we only need to calculate

$$\bar{u}(p')\gamma^0 u(p) = u^\dagger(p')i(\gamma^0)^2 u(p) \simeq -im \begin{pmatrix} \xi^{\dagger s'} & \xi^{\dagger s'} \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} = -2im\delta^{ss'} \qquad (23.12)$$
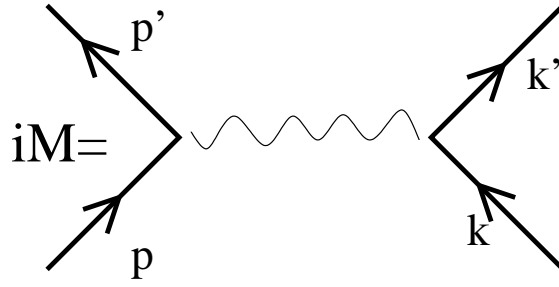
Figure 67: Feynman diagram for the Coulomb interaction between distinguishable fermions.

(since $(\gamma^0)^2 = -1$, from $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$), finally obtaining

$$i\mathcal{M} \simeq \frac{ie^2 g_{00}}{|\vec{p} - \vec{p'}|^2}(2m\delta^{ss'})(2m\delta^{rr'}) = -\frac{ie^2}{|\vec{p'} - \vec{p}|^2}(2m\delta^{ss'})(2m\delta^{rr'}) \qquad (23.13)$$

Thus there is a sign change with respect to the Yukawa potential, meaning that the Coulomb potential is repulsive. Also, the mass is now zero. To go to the configuration space, it is easier to take the zero mass limit of the Yukawa potential, getting

$$V(r) = \frac{+e^2}{4\pi r} = \frac{\alpha}{r} \qquad (23.14)$$

Note that $\alpha = e^2/(4\pi) \simeq 1/137$ is the electromagnetic coupling, which is very weak.

Particle-antiparticle scattering
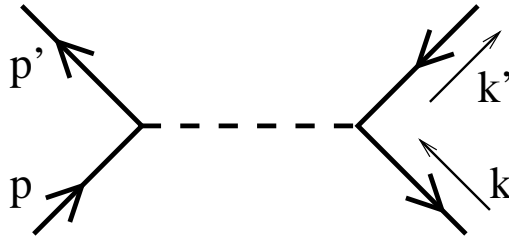

Figure 68: Feynman diagram for the Yukawa potential interaction between particle and antiparticle.

Yukawa

We now consider scattering a fermion off an antifermion in Yukawa theory. Since now the second fermion line is antifermionic, with momentum k in and momentum k' out, as in Fig.68, we have a few changes from the calculation before. First, we replace the u's with v's

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for the second line, i.e.

$$\bar{u}^{s'}(k')u^s(k) \to \bar{v}^s(k)v^{s'}(k') \simeq m \begin{pmatrix} \xi^{\dagger s} & -\xi^{\dagger s} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \xi^{s'} \\ -\xi^{s'} \end{pmatrix} = -2m\delta^{ss'} \tag{23.15}$$

which gives a minus sign with respect to the fermion-fermion calculation. But also changing the fermion into antifermion amounts to exchanging the fermion lines with momenta k and k', as we can check. Another way to say this is that if we still do the contraction like in the fermion-fermion case, i.e. such that the order is: k' first, then k, we obtain $v(k')\bar{v}(k)$, and in order to obtain the Lorentz invariant $\bar{v}(k)v(k')$ we need to change the order of the fermions.

All in all, we obtain two minus signs, i.e. a plus. Thus the potential between fermion and antifermion is the same as between two fermions: scalar particle exchange gives an universally attractive potential.

Coulomb potential

Considering now fermion-antifermion scattering in QED, we again have the same minus sign for exchanging the fermions, and exchanging the u's with v's in the second fermion line gives now

$$\bar{v}(k)\gamma^0 v(k') = v^\dagger(k)i(\gamma^0)^2 v(k) = -im \begin{pmatrix} \xi^{\dagger s} & -\xi^{\dagger s} \end{pmatrix} \begin{pmatrix} \xi^s \\ -\xi^s \end{pmatrix} = -2im\delta^{ss'} \tag{23.16}$$

instead of

$$\bar{u}(k)\gamma^0 u(k') = -2im\delta^{ss'} \tag{23.17}$$

so no change here. Thus overall, we have a minus sign with respect to fermion-fermion scattering, i.e. the fermion-antifermion potential is now *attractive*. Indeed, e^+e^- attract, whereas e^-e^- repel.

In conclusion, for the exchange of a vector particle (electromagnetic field in this case), like charges repel, and opposite charges attract.

We note that the repulsive nature of the fermion-fermion potential was due to the presence of g_{00} in (23.13).

We can now guess also what happens for an exchange of a tensor, i.e. spin 2, particle (the graviton), without doing the full calculation. Since a tensor has two indices, we will have propagation between a point with indices $\mu\nu$ and one with indices $\rho\sigma$, which in some appropriate gauge (the equivalent of the Feynman gauge) should be proportional to $g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}$, and in the nonrelativistic limit only the 00 components should contribute, giving $\propto (g_{00})^2 = +1$.

Therefore we can guess that gravity is attractive, and moreover universally attractive, like we know from experiments to be the case.

Rutherford scattering

We now calculate an example of nonrelativistic scattering cross section, namely the case of Rutherford scattering, of a charged particle ("electron") off a fixed electromagnetic field ("field of a nucleus").

We should remember that in the classic experiment of Rutherford that determined that the atoms are made up of a positively charged nucleus, surrounded by electrons (as opposed to

a "raisin pudding", a ball of constant density of positive charged filled with electrons), where charged particles were scattered on a fixed target made of a metal foil, Rutherford made a classical calculation for the scattering cross section, which was confirmed by experiment, thus proving the assumption.

We will consider the scattering of an electron off the constant electromagnetic field of a fixed nucleus. We will thus treat the electromagnetic field classically, and just the fermion quantum mechanically.

The interaction Hamiltonian is

$$H_I = \int d^3x e\bar{\psi}i\gamma^\mu\psi A_\mu \qquad (23.18)$$

Then the S-matrix contribution, coming from $\sim e^{-i\int H_I dt}$, the first order contribution is

$$
\begin{aligned}
< p'|iT|p > \ &= \ < p'|T\left\{-i\int d^4x\bar{\psi}ie\gamma^\mu\psi A_\mu\right\}|p > \\
&= \ -ie\bar{u}(p')i\gamma^\mu u(p)\int d^4xe^{ip\cdot x}e^{-ip'\cdot x}A_\mu(x) \\
&= \ +e\bar{u}(p')\gamma^\mu u(p)A_\mu(p-p')
\end{aligned} \qquad (23.19)
$$

But if we consider that $A_\mu(x)$ is time-independent, meaning that

$$A_\mu(p-p') = A_\mu(\vec{p}-\vec{p}')2\pi\delta(E_f-E_i) \qquad (23.20)$$

and since the $\mathcal{M}$ matrix is defined by

$$< p'|iT|p >= i\mathcal{M}2\pi\delta(E_f-E_i) \qquad (23.21)$$

we see that in the Feynman rules for $i\mathcal{M}$ we can add a rule for the interaction with a classical field, namely add $+e\gamma^\mu A_\mu(\vec{q})$, with $\vec{q}=\vec{p}-\vec{p}'$, with a diagram: off the fermion line starts a wavy photon line that ends on a circled cross, as in Fig.69.
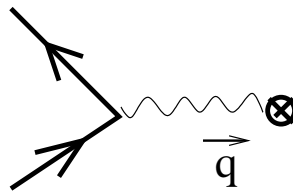

Figure 69: Feynman diagram representation for the interaction of a fermion with a classical potential.

We now calculate the cross section for scattering of the electron off the classical field centered at a point situated at an impact parameter $\vec{b}$ from the line of the incoming electron (or rather, the impact parameter is the distance of the incoming electron from the nucleus),

as in Fig.70. This is in principle a $2 \to 2$ scattering, even though we treat the nucleus and its electromagnetic field classically. As such, we must consider a situation similar to the one in $2 \to 2$ scattering, by writing incoming wavefunctions with impact parameter $\vec{b}$, as

$$\phi_i = \int \frac{d^3 k_i}{(2\pi)^3} \frac{\phi(k_i)}{\sqrt{2E_i}} e^{i\vec{k}_i \cdot \vec{b}} \tag{23.22}$$
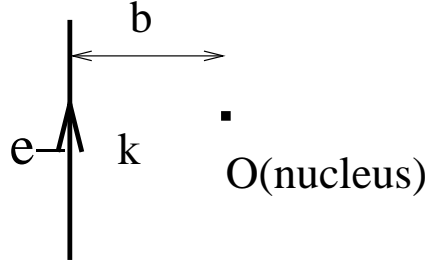

Figure 70: The impact parameter for scattering of an electron off a nucleus (fixed classical object).

As usual, the probability for $i \to f$ is

$$d\mathcal{P}(i \to f) = \frac{d^3 p_f}{(2\pi)^3 2E_f} |_{out} < p_f | \phi_A >_{in} |^2 \tag{23.23}$$

and the cross section element is

$$
\begin{aligned}
d\sigma &= \int d^2\vec{b} \, d\mathcal{P}(\vec{b}) \\
&= \int d^2 b \frac{d^3 p_f}{(2\pi)^3 2E_f} \int \frac{d^3 k_i \phi(k_i)}{(2\pi)^3 \sqrt{2E_i}} \int \frac{d^3 \bar{k}_i \phi(\bar{k}_i)^*}{(2\pi)^3 \sqrt{2E_i}} e^{i\vec{b} \cdot (\vec{k}_i - \vec{\bar{k}}_i)} \, {}_{out} < p_f | k_i >_{in} ({}_{out} < p_f | k_i >_{in})^*
\end{aligned}
\tag{23.24}
$$

We then use the relations

$$
\begin{aligned}
{}_{out} < p_f | k_i >_{in} &= i\mathcal{M}(i \to f) 2\pi \delta(E_f - E_i) \\
({}_{out} < p_f | k_i >_{in})^* &= -i\mathcal{M}^*(i \to f) 2\pi \delta(E_f - \bar{E}_i) \\
\int d^2 b \, e^{i\vec{b} \cdot (\vec{k}_i - \vec{\bar{k}}_i)} &= (2\pi)^2 \delta^2(k_i^\perp - \bar{k}_i^\perp) \\
\delta(E_f - E_i) \delta(E_f - \bar{E}_i) &= \delta(E_f - E_i) \frac{\delta(\bar{k}_i^z - k_i^z)}{|\frac{\bar{k}_i^z}{E_i}|} \\
\int \frac{d^3 p_f}{(2\pi)^3 2E_f} 2\pi \delta(E_f - E_i) &= d\Omega \int \frac{dp_f p_f^2}{(2\pi)^3 2E_f} 2\pi \delta(E_f - E_i) = \frac{d\Omega}{8\pi^2} \int \frac{dp_f p_f^2}{E_f} \frac{E_f}{p_f} \delta(p_f - p_i) \\
&= \frac{d\Omega}{8\pi^2} \int dp_f p_f \delta(p_f - p_i)
\end{aligned}
\tag{23.25}
$$

where we have used $\delta(\bar{E}_i - E_i) = \delta(\bar{k}_i^z - k_i^z)/|\frac{\bar{k}_i^z}{E_i}|$ and $\delta(E_f - E_i) = \delta(p_f - p_i)/|\frac{p_f}{E_f}|$. Using the wavefunctions peaked on the value p_i, i.e. $|\phi(k_i)|^2 = (2\pi)^3\delta^3(k_i - p_i)$, and since $\bar{k}_i^z = v_i E_i$, putting everything together, we finally have

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\frac{1}{v_i E_i}\int dp_f p_f \delta(p_f - p_i)|\mathcal{M}(p_i \to p_f)|^2 \tag{23.26}$$

where $E_i \simeq m$.

We now apply to the case of the electric field of a nucleus of charge $+Ze$, i.e. take

$$A_0 = \frac{+Ze}{4\pi r} \tag{23.27}$$

with Fourier transform

$$A_0(q) = \frac{Ze}{|\vec{q}|^2} \tag{23.28}$$

inside the matrix element

$$i\mathcal{M} = e\bar{u}(p_f)\gamma^\mu u(p_i)A_\mu(\vec{p}_f - \vec{p}_i) \tag{23.29}$$

In the nonrelativistic limit, as we saw (and summing over s', final state helicities),

$$|\bar{u}(p_f)\gamma^\mu u(p_i)|^2 \simeq \sum_{s'}|2m(\xi^{\dagger s'}\xi^s)|^2\delta^{\mu 0} = 4m^2\delta^{\mu 0} \tag{23.30}$$

giving

$$|\mathcal{M}|^2 = e^2 4m^2|A_0(\vec{p}_f - \vec{p}_i)|^2 = e^2 4m^2\frac{Z^2 e^2}{|\vec{p}_f - \vec{p}_i|^4} \tag{23.31}$$

Since $E_f = E_i$ (energy conservation), we have $|\vec{p}_f| = |\vec{p}_i|$, and if they have an angle θ between them, $|\vec{p}_f - \vec{p}_i| = 2p_i \sin\theta/2 \simeq 2mv_i \sin\theta/2$, see Fig.71.
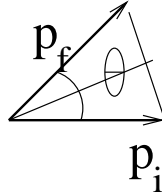

Figure 71: The scattering angle.

Putting everything together, we find

$$\begin{aligned}\frac{d\sigma}{d\Omega} &\simeq \frac{1}{8\pi^2}\frac{mv_i}{v_i 2m}\frac{e^2 4m^2 Z^2 e^2}{16m^4 v_i^4 \sin^4\theta/2} \\ &= \frac{Z^2\alpha^2}{4m^2 v_i^4 \sin^4\theta/2}\end{aligned} \tag{23.32}$$

which is the Rutherford formula, originally found by Rutherford with a classical calculation.

Important concepts to remember

- For distinguishable fermions, fermion-fermion scattering has only one diagram, the exchange of the force particle, which can be scalar for Yukawa interaction, vector for Coulomb interaction, tensor for gravitational interaction (etc.)

- In the nonrelativistic limit, the scalar exchange diagram gives rise to the Yukawa potential. Since the potential is not directly measurable, we calculate it by matching with the nonrelativistic quantum mechanical calculation of the Born approximation for scattering in a given potential. The Yukawa potential is attractive.

- In the Coulomb case, due to the $g_{00} = -1$ in the vector propagator, we have a repulsive process for like charges (fermion-fermion scattering).

- For fermion-antifermion scattering, we find the opposite sign in the potential for Coulomb, and the same sign in the potential for Yukawa.

- To derive the classical Rutherford formula for nonrelativistic scattering of an electron (or charged particle in general) off a fixed nucleus, we treat the electromagnetic field generated by the nucleus as a time-independent classical field, interacting with the quantum field of the incoming electron.

Further reading: See chapters 4.7, 4.8 and exercises in [2].

Exercises, Lecture 23

1) Consider a Yukawa interaction with 2 scalars, ϕ_1, ϕ_2 with masses m_1, m_2. Using the quantum mechanical argument in the text, calculate the potential $V(\vec{x})$. What happens for $m_1 \ll m_2$? How is the above result for $V(\vec{x})$ consistent with the linear principle of quantum mechanics (wave functions add, not probabilities)?

2) Integrate $d\sigma/d\Omega$ for Rutherford scattering to find σ_{tot} (comment on the result). The $d\sigma/d\Omega$ formula is the same as for classical scattering in a Coulomb potential (that's how Rutherford computed it). Yet we used a quantum field theory calculation. Argue how would the various steps of the QFT calculation turn into steps of a classical calculation.

24 Lecture 24. $e^+e^- \to l\bar{l}$ unpolarized cross section

In the previous lecture we re-derived some formulas for classical, nonrelativistic processes using the formalism of quantum field theory, in order to gain some confidence in the formalism. This lecture we turn to the first new calculation, for the $e^+e^- \to l\bar{l}$ scattering at first order (tree level). This will be a relativistic calculation, and also it is quantum field theory, i.e. it has no classical or (nonrelativistic) quantum mechanics counterpart, since in this process an electron and a positron annihilate, to create a par of lepton-antilepton, and this is a purely quantum field theoretic process.

The leptons are e^-, μ^-, τ^- and their antiparticles, together with the corresponding neutrinos (ν_e, ν_μ, ν_τ and their antiparticles), but the neutrinos have no charge, so within QED (interaction only via photons, coupling to charged particles) there is no process creating neutrinos. The case when $l = e^-$, of *Bhabha scattering*, is special, and there we have more diagrams, but for $l = \mu^-$ or τ^- we have only one diagram: $e^-(p)e^+(p')$ annihilate into a photon, who then creates $l(k)\bar{l}(k')$. For definiteness we will assume that we have a μ^-. Note that we need sufficient energy to create the lepton pair, so only for $E_{CM} > 2m_l$ is the process possible. Therefore, for $E_{CM} = 2E_{e,CM} < 2m_\mu$, we can create only e^+e^-, for $2m_\mu < E_{CM} < 2m_\tau$ we can create e^+e^- or $\mu^+\mu^-$, and for $E_{CM} > 2m_\tau$ we can create all, $e^+e^-, \mu^+\mu^-, \tau^+\tau^-$.
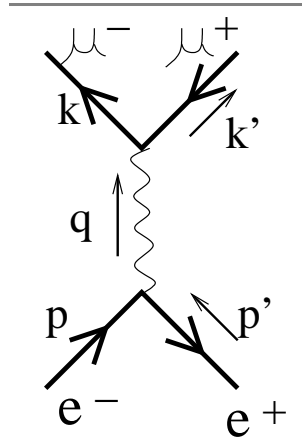
Figure 72: Feynman diagram for $e^+e^- \to \mu^+\mu^-$ scattering at tree level.

Using the Feynman rules for Fig.72, we can write the amplitude (in Feynman gauge)

$$
\begin{aligned}
i\mathcal{M} &= \bar{v}^{s'}(p')(+e\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(+e\gamma^\nu)v^{r'}(k') \\
&= -i\frac{e^2}{q^2}\left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^s(k)\gamma_\mu v^{s'}(k')\right)
\end{aligned}
\tag{24.1}
$$

Note that the vector propagator is $-ig_{\mu\nu}/(q^2 - i\epsilon)$, but in this tree level process q is not integrated, but rather $q = p + p'$, so $q^2 \neq 0$, therefore we can drop the $i\epsilon$.

For the large brackets, since they are numbers, their complex conjugates (equal to the adjoint for a number) give

$$(\bar{v}i\gamma^\mu u)^* = u^\dagger (i\gamma^\mu)^\dagger (i\gamma^0)^\dagger v = u^\dagger i\gamma^0 i\gamma^\mu v = \bar{u}i\gamma^\mu v \tag{24.2}$$

where we have used $(i\gamma^0)^\dagger = i\gamma^0$ and $(\gamma^\mu)^\dagger i\gamma^0 = -i\gamma^0(\gamma^\mu)$. Therefore without the i, we have $(\bar{v}\gamma^\mu u)^* = -\bar{u}\gamma^\mu v$.

Then we can calculate $|\mathcal{M}|^2$ as

$$|\mathcal{M}|^2 = (-)^2 \frac{e^4}{q^4} \left[\left(\bar{v}^{s'}(p')\gamma^\mu u^s(p) \right) \left(\bar{u}^s(p)\gamma^\nu v^{s'}(p') \right) \right] \left[\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k') \right) \left(\bar{v}^{r'}(k')\gamma_\nu u^r(k) \right) \right] \tag{24.3}$$

But in many experiments, we don't keep track of spins (helicities), usually because it is difficult to do so. Therefore, we must *average over the initial spins* (since we don't know the values of the initial spins), and *sum over the final spins* (since we measure the final particles no matter what their spin is). We call the resulting probabilities the *unpolarized scattering cross section*.

It is then obtained from

$$\left(\frac{1}{2} \sum_s \right) \left(\frac{1}{2} \sum_{s'} \right) \sum_r \sum_{r'} |\mathcal{M}(s, s' \to r, r')|^2 = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \tag{24.4}$$

But in order to do the sums over spins, we use the results derived in lecture 14, that

$$\sum_s u^{s\alpha}(p)\bar{u}^s_\beta(p) = -i\not{p}^\alpha{}_\beta + m\delta^\alpha_\beta; \quad \sum_s v^{s\alpha}(p)\bar{v}^s_\beta(p) = -i\not{p}^\alpha{}_\beta - m\delta^\alpha_\beta \tag{24.5}$$

That means that

$$\sum_{ss'} \bar{v}^{s'}_\alpha(p')(\gamma^\mu)^\alpha{}_\beta u^{s\beta}(p)\bar{u}^s_\delta(p)(\gamma^\nu)^\delta{}_\epsilon v^{s'\epsilon}(p') = (-i\not{p}' - m_e)^\epsilon{}_\alpha (\gamma^\mu)^\alpha{}_\beta (-i\not{p} + m_e)^\beta{}_\delta (\gamma^\nu)^\delta{}_\epsilon$$
$$= \text{Tr}[(-i\not{p}' - m_e)\gamma^\mu(-i\not{p} + m_e)\gamma^\nu] \tag{24.6}$$

Then similarly we obtain

$$\sum_{rr'} \left[\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k') \right) \left(\bar{v}^{r'}(k')\gamma_\nu u^r(k) \right) \right] = \text{Tr}[(-i\not{k} + m_\mu)\gamma_\mu(-i\not{k}' - m_\mu)\gamma_\nu] \tag{24.7}$$

so that finally

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4q^2} \text{Tr}[(-i\not{p}' - m_e)\gamma^\mu(-\not{p} + m_e)\gamma^\nu] \, \text{Tr}[(-i\not{k} + m_\mu)\gamma_\mu(-i\not{k}' - m_\mu)\gamma_\nu] \tag{24.8}$$

To calculate this further, we need to use gamma matrix identities, therefore we stop here and now write all possible gamma matrix identities which we will need here and later on.

Gamma matrix identities

First, we remember our conventions for gamma matrices, with the representation

$$\gamma^0 = -i \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}; \quad \gamma^i = -i \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \tag{24.9}$$

We then see that at least in this representation, we have

$$\mathrm{Tr}[\gamma^\mu] = \mathrm{Tr}[\gamma^5] = 0 \tag{24.10}$$

But this result is more general, since, using the fact that in general $(\gamma^5)^2 = 1$, and $\{\gamma^\mu, \gamma^5\} = 0$, we have

$$\mathrm{Tr}[\gamma^\mu] = \mathrm{Tr}[(\gamma^5)^2\gamma^\mu] = -\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^5] = -\mathrm{Tr}[(\gamma^5)^2\gamma^\mu] = -\mathrm{Tr}[\gamma^\mu] \tag{24.11}$$

and therefore $\mathrm{Tr}[\gamma^\mu] = 0$. Here we have used the anticommutation of γ^5 with γ^μ, and then the cyclicity of the trace, to put back the γ^5 from the end of the trace, to the beginning.

Using the same cyclicity of the trace, we have

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = \mathrm{Tr}\left[\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\right] = g^{\mu\nu}\,\mathrm{Tr}[\mathbf{1}] = 4g^{\mu\nu} \tag{24.12}$$

We can also calculate that the trace of an odd number of gammas gives zero, by the same argument as for the single gamma above:

$$\mathrm{Tr}[\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = \mathrm{Tr}[(\gamma^5)^2\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = (-1)^{2n+1}\,\mathrm{Tr}[\gamma^5\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}\gamma^5]$$
$$= -\mathrm{Tr}[(\gamma^5)^2\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = -\mathrm{Tr}[\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] \tag{24.13}$$

and therefore

$$\mathrm{Tr}[\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}] = 0 \tag{24.14}$$

In general, the method for calculating the traces of products of gamma matrices is to expand in the complete basis of 4×4 matrices,

$$\mathcal{O}_\mathcal{I} = \{\mathbf{1}, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \gamma^{\mu\nu}\} \tag{24.15}$$

since for $\mathcal{O}_I \neq \mathbf{1}$, we can check that $\mathrm{Tr}[\mathcal{O}_I] = 0$. Here and always, a gamma matrix with more than one index means that the indices are totally antisymmetrized, i.e. $\gamma^{\mu\nu} = 1/2[\gamma^\mu, \gamma^\nu]$, $\gamma^{\mu\nu\rho} = 1/6[\gamma^\mu\gamma^\nu\gamma^\rho - 5 \text{ terms}]$, etc. Indeed, we have

$$\mathrm{Tr}[\gamma^{\mu\nu}] = \frac{1}{2}\mathrm{Tr}[[\gamma^\mu, \gamma^\nu]] = 4g^{(\mu\nu)} = 0 \tag{24.16}$$

(the antisymmetric part of $g^{\mu\nu}$ is zero), and also by gamma matrix anticommutation,

$$\mathrm{Tr}[\gamma^\mu\gamma^5] = -\mathrm{Tr}[\gamma^5\gamma^\mu] \tag{24.17}$$

therefore

$$\mathrm{Tr}[\gamma^\mu\gamma^5] = 0 \tag{24.18}$$

We also have the relations

$$\gamma^{\mu\nu}\gamma^5 = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_{\rho\sigma}$$
$$\gamma^{\mu\nu\rho} = -i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5$$
$$\gamma^{\mu\nu\rho}\gamma_5 = -i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma$$
$$\gamma^{\mu\nu\rho\sigma} = i\epsilon^{\mu\nu\rho\sigma}\gamma_5 \qquad (24.19)$$

To prove this, it suffices it to show them for some particular indices. Consider the first relation for $\mu = 0, \nu = 1$. Note that for matrices gamma with several indices, the indices must be different (the totally antisymmetric part is nonzero only if the indices are different). But for different indices, the product of gamma matrices is already antisymmetrized, due to the anticommutation property of the gamma matrices, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$, if $\mu \neq \nu$. Therefore we have e.g. $\gamma^{01} = \gamma^0\gamma^1$, etc. On the left hand side (lhs) of the first relation, we have

$$\gamma^0\gamma^1\gamma^5 = \gamma^0\gamma^1(-i)\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^2\gamma^3, \qquad (24.20)$$

(where we have used $\gamma^0\gamma^1 = -\gamma^1\gamma^0$ and $(\gamma^0)^2 = -1, (\gamma^1)^2$, which follow from $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$) and on the right hand side (rhs), we have $-i/2 \times 2\epsilon^{0123}\gamma_{23} = -i\gamma_2\gamma^3 = -i\gamma^2\gamma^3$, therefore the same result. (the factor of 2 is because we can have sum over (23) or (32)).

The second relation is proved choosing, e.g. $\mu = 0, \nu = 1, \rho = 2$, in which case on the lhs we have $\gamma^0\gamma^1\gamma^2$, and on the rhs we have

$$-i\epsilon^{0123}\gamma_3(-i)\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = +\gamma^0\gamma^1\gamma^2(\gamma^3)^2 = +\gamma^0\gamma^1\gamma^2 \qquad (24.21)$$

The third relation follows from multiplication with γ^5 and using $(\gamma^5)^2 = 1$. The fourth relation can be proven by choosing the only nonzero case, $\mu = 0, \nu = 1, \rho = 2, \sigma = 3$ (of course, we can also have permutations of these), in which case the lhs is $\gamma^0\gamma^1\gamma^2\gamma^3$ and the rhs is

$$i\epsilon^{0123}(-i)\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^0\gamma^1\gamma^2\gamma^3 \qquad (24.22)$$

To calculate the trace of a product of 4 gamma matrices, we decompose in the basis $\mathcal{O}_I$, which has Lorentz indices, times a corresponding constant Lorentz structure that gives a total of 4 Lorentz indices. But in 4 dimensions the only possibilities are $g^{\mu\nu}, \epsilon^{\mu\nu\rho\sigma}$, and products of them. That means that there are no constant Lorentz tensors with odd number of indices, and correspondingly in the product of 4 gamma matrices we cannot have γ^μ and $\gamma^\mu\gamma_5$ in the $\mathcal{O}_I$ decomposition, since it would need to be multiplied by a 3-index constant Lorentz structure. We can only have $\mathbf{1}$, γ_5 or $\gamma^{\mu\nu}$.

For $\mathbf{1}$ and γ_5 we can multiply with two possible Lorentz structures, $\epsilon^{\mu\nu\rho\sigma}$, and $g^{\mu\nu}g^{\rho\sigma}$ and permutations. But we already know that $\gamma^{\mu\nu\rho\sigma}$, which should certainly appear in the product of gammas, can be written as $i\epsilon^{\mu\nu\rho\sigma}\gamma_5$, and therefore the $g^{\mu\nu}g^{\rho\sigma}$'s should multiply the $\mathbf{1}$ term. Then finally we can write the general formula

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \; & c_1\epsilon^{\mu\nu\rho\sigma}\gamma_5 + c_2 g^{\mu\nu}\gamma^{\rho\sigma} + c_3 g^{\mu\rho}\gamma^{\nu\sigma} + c_4 g^{\mu\sigma}\gamma^{\nu\rho} + c_5 g^{\nu\rho}\gamma^{\mu\sigma} \\ & + c_6 g^{\nu\sigma}\gamma^{\mu\rho} + c_7 g^{\rho\sigma}\gamma^{\mu\nu} + c_8 g^{\mu\nu}g^{\rho\sigma} + c_9 g^{\mu\rho}g^{\nu\sigma} + c_{10}g^{\mu\sigma}g^{\nu\rho} \qquad (24.23)\end{aligned}$$

Since $\mathrm{Tr}[\gamma_5] = \mathrm{Tr}[\gamma_{\mu\nu}] = 0$, we have

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = (c_8 g^{\mu\nu}g^{\rho\sigma} + c_9 g^{\mu\rho}g^{\nu\sigma} + c_{10}g^{\mu\sigma}g^{\nu\rho})\,\mathrm{Tr}[\mathbf{1}] = 4(c_8 g^{\mu\nu}g^{\rho\sigma} + c_9 g^{\mu\rho}g^{\nu\sigma} + c_{10}g^{\mu\sigma}g^{\nu\rho}) \tag{24.24}$$

To determine c_8, c_9, c_{10}, we consider indices for which only one structure gives a nonzero result. We can choose $\mu = \nu = 1, \rho = \sigma = 2$. Then c_1 does not contribute, and also $c_2 - c_7$, since they have symmetric×antisymmetric indices, unlike here, whereas c_9, c_{10} do not contribute due to the fact that $\rho = \sigma \neq \mu = \nu$. Then the lhs is $\gamma^1\gamma^1\gamma^2\gamma^2 = +1$, and the rhs is $+c_8$, meaning $c_8 = 1$. If we choose $\mu = \rho = 1, \nu = \sigma = 2$ instead, in the same way, we isolate c_9. Then the lhs is $\gamma^1\gamma^2\gamma^1\gamma^2 = -1$, and the rhs is $+c_9$, meaning $c_9 = -1$. If we choose $\mu = \sigma = 1, \nu = \rho = 2$, we isolate c_{10}. Then the lhs is $\gamma^1\gamma^2\gamma^2\gamma^1 = +1$, and the rhs is $+c_{10}$, meaning $c_{10} = +1$. Therefore

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{24.25}$$

This relation can be derived in another way as well, by commuting one of the gamma matrices past all the other ones, and then using cyclicity of the trace at the end:

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= \mathrm{Tr}[(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma^\rho\gamma^\sigma] = \\ \dots &= \mathrm{Tr}[2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho] - \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] \\ &= \mathrm{Tr}[2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho] - \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]\end{aligned} \tag{24.26}$$

which implies the same relation as above. We have calculated it via decomposition in $\mathcal{O}_I$ since the method generalizes easily to other cases. In particular, now we can direcly use it to calculate $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5]$, by multiplying (24.23) with γ_5. Then again $c_2 - c_7$ do not contribute, since $\gamma_{\mu\nu}\gamma_5 \propto \gamma_{\rho\sigma}$ as we proved above, so it has zero trace. But now $c_8 - c_{10}$ are multiplied by γ_5, so also have zero trace, and the only nonzero result in the trace comes from c_1, since it multiplies $(\gamma^5)^2 = 1$. Then

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = 4c_1\epsilon^{\mu\nu\rho\sigma} \tag{24.27}$$

To calculate c_1, we isolate indices that only contribute to c_1, namely $\mu = 0, \nu = 1, \rho = 2, \sigma = 3$. Then the lhs of (24.23) is $\gamma^0\gamma^1\gamma^2\gamma^3$, and the rhs is $c_1\epsilon^{0123}(-i)\gamma^0\gamma^1\gamma^2\gamma^3 = -ic_1\gamma^0\gamma^1\gamma^2\gamma^3$, and therefore $c_1 = i$, giving finally

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = 4i\epsilon^{\mu\nu\rho\sigma} \tag{24.28}$$

We will also use contractions of ϵ tensors, so we write the results for them here:

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} &= 4!\epsilon^{0123}\epsilon_{0123} = -24 \\ \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} &= 3!\delta^\mu_\nu\epsilon^{0123}\epsilon_{0123} = -6\delta^\mu_\nu \\ \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} &= 2!(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)\epsilon^{0123}\epsilon_{0123} = -2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)\end{aligned} \tag{24.29}$$

In the first relation we have 4! terms, for the permutations of 0123, in the second we have $\mu \neq \alpha\beta\gamma$ and also $\nu \neq \alpha\beta\gamma$, which means that $\mu = \nu$, and for given $\mu = \nu$, $\alpha\beta\gamma$ run over the remaining 3 values, with 3! permutations. For the third relation, we have $\mu\nu \neq \alpha\beta$, and also

$\rho\sigma \neq \alpha\beta$, meaning $\mu\nu = \rho\sigma$, or $\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho$, and for given $\mu\nu$, $\alpha\beta$ run over 2 values, giving 2! terms.

We also have

$$\text{Tr}[\gamma^\mu \gamma^\nu ...] = \text{Tr}[C\gamma^\mu C^{-1} C\gamma^\nu C^{-1}...C^{-1}] = (-1)^n \text{Tr}[(\gamma^\mu)^T (\gamma^\nu)^T ...]$$
$$= (-1)^n [\text{Tr}[(...\gamma^\nu \gamma^\mu)^T] = (-1)^n \text{Tr}[...\gamma^\nu \gamma^\mu] \tag{24.30}$$

but since in any case for odd n the trace is zero, it means that the trace of many gammas is equal to the trace of the gammas in the opposite order

$$\text{Tr}[\gamma^\mu \gamma^\nu ...] = \text{Tr}[...\gamma^\nu \gamma^\mu] \tag{24.31}$$

Finally, we write the sums over $\gamma^\mu (...) \gamma_\mu$ with some other gammas in the middle. First, since $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, it means that

$$\gamma^\mu \gamma_\mu = \delta^\mu_\mu = 4 \tag{24.32}$$

and further, using this, we get

$$\gamma^\mu \gamma^\nu \gamma_\mu = -\gamma^\nu \gamma^\mu \gamma_\mu + 2g^{\mu\nu} \gamma_\mu = -2\gamma^\nu \tag{24.33}$$

In turn, using this, we can calculate the trace with two gammas in the middle,

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = -\gamma^\nu \gamma^\mu \gamma^\rho \gamma_\mu + 2g^{\mu\nu} \gamma^\rho \gamma_\mu = 2\gamma^\nu \gamma^\rho + 2\gamma^\rho \gamma^\nu = 4g^{\nu\rho} \tag{24.34}$$

And finally, using this, we can calculate the trace with three gammas in the middle,

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -\gamma^\nu \gamma^\mu \gamma^\rho \gamma^\sigma \gamma_\mu + 2g^{\mu\nu} \gamma^\rho \gamma^\sigma \gamma_\mu = -4\gamma^\nu g^{\rho\sigma} + 2(-\gamma^\sigma \gamma^\rho + 2g^{\rho\sigma}) = -2\gamma^\sigma \gamma^\rho \gamma^\nu \tag{24.35}$$

Cross section for unpolarized scattering

We can now go back to the calculation of the cross section. Since the trace of three gamma matrices gives zero, in the two traces in (24.8), only the traces of two and four gamma matrices contribute.

We then have for the first trace

$$\text{Tr}[(-i\slashed{p}' - m_e)\gamma^\mu (-i\slashed{p} + m_e)\gamma^\nu] = -4p'_\rho p_\sigma(g^{\rho\mu} g^{\nu\sigma} - g^{\rho\sigma} g^{\mu\nu} + g^{\nu\rho} g^{\mu\sigma}) - m_e^2 4g^{\mu\nu}$$
$$= 4[-p'^\mu p^\nu - p'^\nu p^\mu - g^{\mu\nu}(-p \cdot p' + m_e^2)] \tag{24.36}$$

and similarly for the second trace

$$\text{Tr}[(-i\slashed{k} + m_\mu)\gamma_\mu (-i\slashed{k}' - m_\mu)\gamma_\nu] = -4k_\rho k'_\sigma(g^{\mu\rho} g^{\nu\sigma} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\nu} g^{\mu\sigma}) - 4m_\mu^2 g_{\mu\nu}$$
$$= 4[-k_\mu k'_\nu - k_\nu k'_\mu - g_{\mu\nu}(-k \cdot k' + m_\mu^2)] \tag{24.37}$$

But since $m_e/m_\mu \simeq 1/200$, we neglect m_e and only keep m_μ.

Then we obtain for the unpolarized $|\mathcal{M}|^2$,

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{4e^4}{q^4} [p'^\mu p^\nu + p'^\nu p^\mu - p \cdot p' g^{\mu\nu}][k_\mu k'_\nu + k_\nu k'_\mu + g_{\mu\nu}(-k \cdot k' + m_\mu^2)]$$

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$$= \frac{4e^4}{q^4}[2(p \cdot k)(p' \cdot k') + 2(p \cdot k')(p' \cdot k) + 2p \cdot p'(-k \cdot k' + m_\mu^2)$$
$$+2k \cdot k'(-p \cdot p') + 4(-p \cdot p')(-k \cdot k' + m_\mu^2)]$$
$$= \frac{8e^4}{q^4}[(p \cdot k)(p' \cdot k') + (p \cdot k')(p' \cdot k) - m_\mu^2 p \cdot p'] \qquad (24.38)$$

CM frame cross section

Since muons are created, we need to have $2E_e > 2m_\mu \sim 400 m_e$, therefore the electrons are ultrarelativistic, and can be considered as massless. Therefore for the electron and positron we have $p = (E, E\hat{z})$ and $p' = (E, -E\hat{z})$. Since the $\mu^+ \mu^-$ have the same mass, $E_e = E_\mu$, but they are not necessarily ultrarelativistic, so $k = (E, \vec{k})$ and $k' = (E, -\vec{k})$, and $|\vec{k}| = \sqrt{E^2 - m_\mu^2}$, see Fig.73. We also define the angle θ between the electrons and the muons, i.e. $\vec{k} \cdot \hat{z} = |\vec{k}| \cos\theta$.
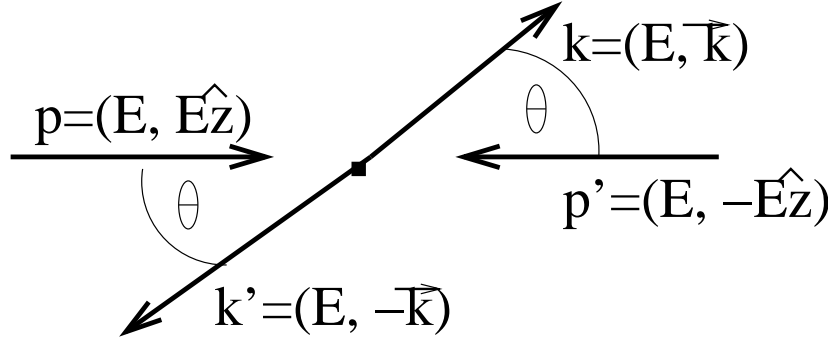
Figure 73: The kinematics of center of mass scattering for $e^+ e^- \to \mu^+ \mu^-$.

Then also
$$q^2 = (p + p')^2 = -4E^2 = 2p \cdot p' \Rightarrow p \cdot p' = -2E^2 \qquad (24.39)$$

We also obtain
$$p \cdot k = p' \cdot k' = -E^2 + E|\vec{k}| \cos\theta$$
$$p \cdot k' = p' \cdot k = -E^2 - E|\vec{k}| \cos\theta \qquad (24.40)$$

Then substituting these formulas in $|\mathcal{M}|^2$, we get

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{16E^4}[E^2(E - |k| \cos\theta)^2 + E^2(E + |k| \cos\theta)^2 + 2E^2 m_\mu^2]$$

$$= \frac{e^4}{E^2}[E^2 + k^2 \cos^2\theta + m_\mu^2]$$

$$= e^4 \left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right] \qquad (24.41)$$

The differential cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|p_1|}{(2\pi)^2 4E_{CM}} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \qquad (24.42)$$

But $2E_A = 2E_B = 2E = E_{CM}$, and $\vec{v}_A = \vec{p}_A/E_A = \hat{z}$, and $\vec{v}_B = \vec{p}_B/E_B = -\hat{z}$, so $|v_A - v_B| = 2$. Then

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|k|}{2E_{CM}^2 16\pi^2} \frac{1}{2E} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{\alpha^2}{4E_{CM}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$
$$(24.43)$$

In the ultrarelativistic limit $E \gg m_\mu$, it becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM,ultrarel.} = \frac{\alpha^2}{4E_{CM}^2}(1 + \cos^2\theta) \qquad (24.44)$$

The total cross section is found by integrating over the solid angle, $d\Omega = 2\pi\sin\theta d\theta = -2\pi d(\cos\theta)$, and using $\int_{-1}^{1} d(\cos\theta) = 2$ and $\int_{-1}^{1} d(\cos\theta)\cos^2\theta = 2/3$, we get

$$\begin{aligned}
\sigma_{tot} &= 2\pi \int_{-1}^{1} d(\cos\theta) \frac{d\sigma}{d\Omega}(\cos\theta) \\
&= 2\pi \frac{\alpha^2}{4E_{CM}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[\frac{8}{3} + \frac{4}{3}\frac{m_\mu^2}{E^2}\right] \\
&= \frac{4\pi}{3} \frac{\alpha^2}{E_{CM}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[1 + \frac{m_\mu^2}{2E^2}\right] \qquad (24.45)
\end{aligned}$$

In the ultrarelativistic limit, $E \gg m_\mu$, it becomes

$$\sigma_{tot} \to \frac{4\pi}{3} \frac{\alpha^2}{E_{CM}^2} \qquad (24.46)$$

Important concepts to remember

- The $e^+e^- \to l\bar{l}$ is quantum field theoretic, and for $l = \mu, \tau$, we have only one Feynman diagram.

- The complex conjugation changes $(\bar{u}\gamma v)$ into $\bar{v}\gamma u$, meaning that the sum over spins in $|\mathcal{M}|^2$ generates sums over spins $\sum_s v_s(p)\bar{v}^s(p)$, allowing us to convert all the u's and v's into traces of gamma matrices.

- In calculations with gamma matrices, we use the Clifford algebra, anticommutations, Lorentz invariant structures, the complet set $\mathcal{O}_I$, etc.

- The total unpolarized $e^+e^- \to l\bar{l}$ cross section is finite, and given in the ultrarelativistic limit and in the CM frame by $4\pi/3\alpha^2/E_{CM}^2$.

Further reading: See chapters 5.1 of [2] and 8.1 of [1].

Exercises, Lecture 24

1) Consider Bhabha scattering, $e^+e^- \to e^+e^-$. Write down the Feynman diagrams, and find the expression for $|\mathcal{M}|^2$ in terms of $u^s(p)$'s and $v^s(p)$'s.

2) Consider the gamma matrices in 3 Euclidean dimensions, γ^i, satisfying the Clifford algebra $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. Calculate $\text{Tr}[\gamma^i\gamma^j], \text{Tr}[\gamma^i\gamma^j\gamma^k]$ and $\text{Tr}[\gamma^i\gamma^j\gamma^k\gamma^l]$. You can use a specific representation for γ^i if you want (though it's not needed).

25 Lecture 25. $e^+e^- \to l\bar{l}$ polarized cross section; crossing symmetry

In this lecture we show how to calculate the same process as last lecture, but for given spin, i.e. the polarized cross section. Then we will explore an important symmetry of QFT processes called crossing symmetry and how it appears in the Mandelstam variables (which we will introduce).

For simplicity we will work in the ultrarelativistic case $m_e, m_\mu \to 0$.

We have seen that in the Weyl representation for gamma matrices, we have

$$\gamma_5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \tag{25.1}$$

That means that $P_L = (1 + \gamma_5)/2$ projects a fermion ψ into its upper two components, called ψ_L, and $P_R = (1 - \gamma_5)/2$ projects the fermion unto its lower two components, called ψ_R, i.e. $P_L \psi = \psi_L$, $P_R \psi = \psi_R$. P_L, P_R are *projectors*, i.e. $P_L^2 = P_L$, $P_R^2 = P_R$ and $P_L P_R = P_R P_L = 0$, i.e. $P_R \psi_L = 0$ and $P_L \psi_R = 0$.

Here ψ_L and ψ_R have given *helicities*, i.e. eigenvalues of the projection of the spin onto the direction of the momentum, i.e. $h = \vec{S} \cdot \vec{p}/|\vec{p}| = \pm 1/2$. Therefore measuring a given spin, or more precisely a given helicity, means considering only ψ_L or ψ_R fermions.

Note that if $\psi_L = P_L \psi = (1 + \gamma_5)/2 \psi$, then (since $\gamma_5^\dagger = \gamma_5$ and γ_5 anticommutes with the other gammas)

$$\overline{\psi_L} = \overline{\frac{1 + \gamma_5}{2} \psi} = \left(\frac{1 + \gamma_5}{2} \psi \right)^\dagger i \gamma^0 = \psi^\dagger \frac{1 + \gamma_5}{2} i \gamma^0 = \psi^\dagger i \gamma^0 \frac{1 - \gamma_5}{2} = \bar{\psi} \frac{1 - \gamma_5}{2} = (\bar{\psi})_R \tag{25.2}$$

so for instance $\overline{u_L} = (\bar{u})_R$.

Consider a Lorentz invariant object like the one we calculated before, $\bar{v}(p')\gamma^\mu u(p)$. We want to calculate $\bar{v}(p')^{s'}\gamma^\mu u^s(p)$ with s, s' corresponding to u_R and $(\bar{v})_L = \overline{v_R}$. Since $P_R u_R = u_R$, we can introduce for free a P_R in front of u_R. Then, also on the bar side we have a nonzero element, since

$$\bar{v}(p')\gamma^\mu \frac{1 - \gamma_5}{2} u(p) = \bar{v}(p') \frac{1 + \gamma_5}{2} \gamma^\mu u(p) = \overline{\frac{1 - \gamma_5}{2} v(p')} \gamma^\mu u(p) = \bar{v}(p')\gamma^\mu u(p) \tag{25.3}$$

But since $P_R u_L = 0$, and also $(\bar{\psi})_R P_L = 0$, we can for free add the terms with the other spins (u_L and/or $(\bar{v})_R$), since they give zero when we have P_R inside, and therefore calculate the sum over spins for the e^+e^- factor in $|\mathcal{M}^2|$ (note that now it is a sum over spins, not an average, because now *we do know the initial spin*, we don't average over it, we just choose to add contributions of spins which give zero in the sum)

$$\sum_{spins} |\bar{v}(p')\gamma^\mu \left(\frac{1 - \gamma_5}{2} \right) u(p)|^2 = \sum_{spins} \bar{v}(p')\gamma^\mu \left(\frac{1 - \gamma_5}{2} \right) u(p)\bar{u}(p)\gamma^\nu \left(\frac{1 - \gamma_5}{2} \right) v(p') \tag{25.4}$$

Using the sums over spins, $\sum_s u^s(p)\bar{u}^s(p) = -i\not{p} + m \simeq -i\not{p}$ and $\sum_s v^s(p)\bar{v}^s(p) = -i\not{p} - m \simeq -i\not{p}$, we obtain

$$- \text{Tr}\left[\not{p}'\gamma^\mu \left(\frac{1-\gamma_5}{2}\right)\not{p}\gamma^\nu \left(\frac{1-\gamma_5}{2}\right)\right] = -\text{Tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu \frac{1-\gamma_5}{2}\right] \tag{25.5}$$

where we have commuted the first P_R past two gammas, thus giving P_R again, and then used $P_R^2 = P_R$. Now we have two terms, one is the same as in the unpolarized cross section, with 4 gammas inside the trace, and a new one that has an extra γ_5 (the trace with 4 gammas and a γ_5 was calculated last lecture also). Substituting the result of the traces, we get

$$-\frac{4}{2}[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}p\cdot p'] + \frac{1}{2}4ip'_\rho p_\sigma \epsilon^{\rho\mu\sigma\nu} = -2[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}p\cdot p' - i\epsilon^{\rho\mu\sigma\nu}p'_\rho p_\sigma] \tag{25.6}$$

Similarly, we can calculate the sum over spins in the $\mu^+\mu^-$ factor in $|\mathcal{M}|^2$, which gives

$$\sum_{spins} |\bar{u}(k)\gamma_\mu \left(\frac{1-\gamma_5}{2}\right)v(k')|^2 = -\text{Tr}\left[\not{k}\gamma_\mu \left(\frac{1-\gamma_5}{2}\right)\not{k}'\gamma_\nu \left(\frac{1-\gamma_5}{2}\right)\right]$$
$$= -2\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k' - ik^\alpha k'^\beta \epsilon_{\alpha\mu\beta\nu}\right] \tag{25.7}$$

Then we get for $|\mathcal{M}|^2$ (where besides the two factors calculated above we have the factor e^4/q^4)

$$\begin{aligned}
|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 &= \frac{4e^4}{q^4}[2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) - \epsilon^{\rho\mu\sigma\nu}\epsilon_{\alpha\mu\beta\nu}p'_\rho p_\sigma k^\alpha k'^\beta] \\
&= \frac{4e^4}{q^4}[2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) + 2((p\cdot k')(p\cdot k') - (p\cdot k)(p'\cdot k'))] \\
&= \frac{16e^4}{q^4}(p\cdot k')(p'\cdot k) \tag{25.8}
\end{aligned}$$

where we used $\epsilon^{\rho\mu\sigma\nu}\epsilon_{\alpha\mu\beta\nu} = -2(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha)$. Using also (from last lecture) $q^2 = -4E^2$ and $p\cdot k' = p'\cdot k = -E(E + k\cdot\cos\theta)$, and in the ultrarelativistic limit $E = k$, we obtain

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 = e^4(1 + \cos\theta)^2 \tag{25.9}$$

Then, since in the center of mass frame, in the ultrarelativistic limit we have (as we saw before)

$$\frac{d\sigma}{d\Omega}|_{CM,m_i\to 0} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2} \tag{25.10}$$

we finally obtain

$$\frac{d\sigma}{d\Omega}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|_{CM,m_i\to 0} = \frac{\alpha^2}{4E_{CM}^2}(1 + \cos\theta)^2 \tag{25.11}$$

We can similartly calculate the cross sections for other polarizations. For the process $e_R^- e_L^+ \to \mu_L^- \mu_R^+$, for instance, only one of the two factors has $\gamma_5 \to -\gamma_5$, meaning that there

is a minus sign in front of the $\epsilon^{\rho\mu\sigma\nu}$ in the first term, therefore changing the relative sign in the next to last line in (25.8). The details are left as an exercise, but it follows that we have

$$\frac{d\sigma}{d\Omega}(e_R^- e_L^+ \to \mu_L^- \mu_R^+)|_{CM,m_i\to 0} = \frac{\alpha^2}{4E_{CM}^2}(1 - \cos\theta)^2 \qquad (25.12)$$

It is then clear that the other two nonzero processes give

$$\frac{d\sigma}{d\Omega}(e_L^- e_R^+ \to \mu_R^- \mu_L^+)|_{CM,m_i\to 0} = \frac{\alpha^2}{4E_{CM}^2}(1 - \cos\theta)^2$$

$$\frac{d\sigma}{d\Omega}(e_L^- e_R^+ \to \mu_L^- \mu_R^+)|_{CM,m_i\to 0} = \frac{\alpha^2}{4E_{CM}^2}(1 + \cos\theta)^2 \qquad (25.13)$$

All the other helicity combinations give zero by the fact that $P_L P_R = P_R P_L = 0$.

There is another way of calculating the polarized cross section that we will just sketch, but not follow through. Namely, we can use explicit forms for the spinors:

$$u(p) = \begin{pmatrix} \sqrt{-p\cdot\sigma}\,\xi \\ \sqrt{-p\cdot\bar\sigma}\,\xi \end{pmatrix} = \begin{pmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\xi \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\xi \end{pmatrix} \overset{E\to\infty}{\to} \sqrt{2E}\begin{pmatrix} \frac{1}{2}(1+\hat p\cdot\vec\sigma)\xi \\ \frac{1}{2}(1-\hat p\cdot\vec\sigma)\xi \end{pmatrix} \qquad (25.14)$$

and similarly

$$v(p) = \begin{pmatrix} \sqrt{-p\cdot\sigma}\,\xi \\ -\sqrt{-p\cdot\bar\sigma}\,\xi \end{pmatrix} \overset{E\to\infty}{\to} \sqrt{2E}\begin{pmatrix} \frac{1}{2}(1+\hat p\cdot\vec\sigma)\xi \\ -\frac{1}{2}(1-\hat p\cdot\vec\sigma)\xi \end{pmatrix} \qquad (25.15)$$

One can then use the explicit form of the gamma matrices in the Weyl representation together with the above formulae to compute the $|\mathcal{M}|^2$'s. We will not do it here, but the details can be found for instance in Peskin and Schroeder.

Crossing symmetry

Feynman diagrams contain more information than the specific processes we analyze, since by simply rotating the diagrams (considering time running in a different direction) we obtain a new process, related to the old one in a simple way, by a transformation called *crossing*. We will find that we can transform the momenta as well in order to obtain the same value for the amplitude $\mathcal{M}$, obtaining *crossing symmetry*. Of course, if we apply the transformation of momenta on the functional form of a given amplitude, in general we will not obtain a symmetry. We will discuss this more later.

In the case of the $e^+ e^- \to \mu^+ \mu^-$ process, the "crossed diagram" ("crossed process"), obtained by a 90^0 rotation, or by time running horizontally instead of vertically in the diagram, is $e^-(p_1)\mu^-(p_2) \to e^-(p_1')\mu^-(p_2')$, as in Fig.74, giving

$$i\mathcal{M} = \bar u(p_1')(+e\gamma^\mu)u(p_1)\frac{-ig_{\mu\nu}}{q^2}\bar u(p_2')(+e\gamma^\nu)u(p_2) = -i\frac{e^2}{q^2}\bar u(p_1')\gamma^\mu u(p_1)\bar u(p_2')\gamma_\mu u(p_2) \quad (25.16)$$

Then the unpolarized $|\mathcal{M}|^2$ gives

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\text{Tr}[(-i\not p_1 + m_e)\gamma^\nu(-i\not p_1' + m_e)\gamma^\mu]\,\text{Tr}[(-i\not p_2' + m_\mu)\gamma_\mu(-i\not p_2 + m_\mu)\gamma_\nu] \quad (25.17)$$
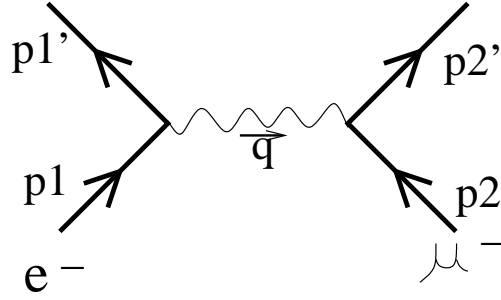
Figure 74: Feynman diagram for $e^- \mu^- \to e^- \mu^-$ scattering at tree level.

Here $q = p_1 - p_1'$. This is the same result as for $e^+ e^- \to \mu^+ \mu^-$, except for renaming the momenta, and some signs. More precisely, when we change antiparticles to particles, we change $v^s(p)$'s to $u^s(p)$'s, and since $\sum_s u^s(p)\bar{u}^s(p) = -i\slashed{p}+m$, but $\sum_s v^s(p)\bar{v}^s(p) = -i\slashed{p}-m$, and since we do this for two terms, changing $-i\slashed{p}+m \to -i\slashed{p}-m = -(i\slashed{p}+m)$. Thus we finally see that the rules for replacing the original diagram with the crossed diagram are

$$ p \to p_1; \quad p' \to -p_1'; \quad k \to p_2'; \quad k' \to -p_2 \tag{25.18} $$

Thus we obtain

$$ \frac{1}{4}\sum_{spins} |\mathcal{M}(e^- \mu^- \to e^- \mu^-)|^2 = \frac{8e^4}{q^4}[(p_1 \cdot p_2')(p_1' \cdot p_2) + (p_1 \cdot p_2)(p_1' \cdot p_2') + m_\mu^2 p_1 \cdot p_1'] \tag{25.19} $$
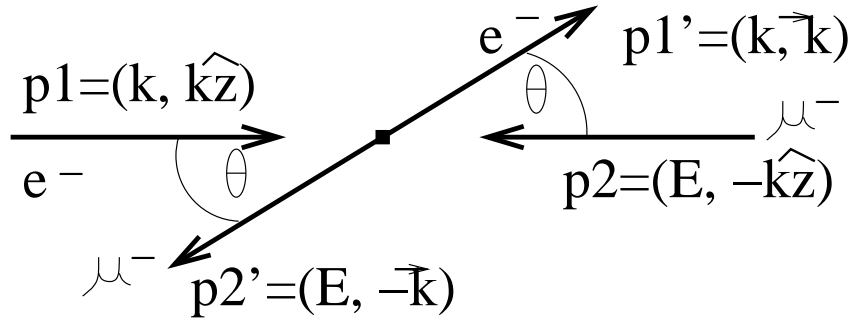
Figure 75: Kinematics of $e^- \mu^- \to e^- \mu^-$ scattering in the center of mass frame.

In this process, in the center of mass frame, we have for the initial electron $p_1 = (k, k\hat{z})$, and for the initial muon we have opposite initial momentum (but higher energy), $p_2 = (E, -k\hat{z})$, with $E^2 = k^2 + m_\mu^2$. For the final electron we have $p_1' = (k, \vec{k})$ and for the final muon $p_2' = (E, -\vec{k})$, see Fig.75. The angle between the initial and final directions being θ, we have $\vec{k} \cdot \hat{z} = k \cos\theta$. We also have the center of mass energy $E_{CM} = E + k$ and for the

various invariants

$$
\begin{aligned}
p_1 \cdot p_2 &= p_1' \cdot p_2' = -k(E + k) \\
p_1' \cdot p_2 &= p_1 \cdot p_2' = -k(E + k \cos \theta) \\
p_1 \cdot p_1' &= -k^2(1 - \cos \theta) \\
q^2 &= -2p_1 \cdot p_1' = 2k^2(1 - \cos \theta)
\end{aligned}
\tag{25.20}
$$

Then the sum over spins in the center of mass frame is

$$
\frac{1}{4} \sum_{spins} |\mathcal{M}|^2 = \frac{2e^4}{k^2(1 - \cos \theta)^2}[(E + k)^2 + (E + k \cos \theta)^2 - m_\mu^2(1 - \cos \theta)]
\tag{25.21}
$$

and the differential cross section is

$$
\frac{d\sigma}{d\Omega}_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{CM}^2} = \frac{\alpha^2}{2k^2(1 - \cos \theta)^2(E + k)^2}[(E + k)^2 + (E + k \cos \theta)^2 - m_\mu^2(1 - \cos \theta)]
\tag{25.22}
$$

In the ultrarelativistic limit, $E \gg m_\mu$, we have

$$
\frac{d\sigma}{d\Omega}_{CM} \to \frac{\alpha^2}{2E_{CM}^2(1 - \cos \theta)^2}[4 + (1 + \cos \theta)^2] \overset{\theta \to 0}{\propto} \frac{1}{\theta^4}
\tag{25.23}
$$

The fact that $d\sigma/d\Omega \to 1/\theta^4$ as $\theta \to 0$ means that there is a strong divergence at $\theta = 0$ (since $\int d\theta|_0 \sim \int d\theta/\theta^4 \to \infty$ in the massless limit $m \to 0$; remember that the exchanged photon is massless, as is the electron in the ultrarelativistic limit). This is indicative of something that will be dealt with in the second semester, infrared (IR) divergences for scattering of massless particles.

To complete this portion, we now can define the general crossed diagram by generalizing the case $e^+ e^- \to \mu^+ \mu^-$ into $e^- \mu^- \to e^- \mu^-$, and the corresponding transformation that leaves the amplitude invariant as

$$
\mathcal{M}(\phi(p) + ... \to ...) = \mathcal{M}(... \to ... + \bar{\phi}(k))
\tag{25.24}
$$

Here ϕ is a particle, and $\bar{\phi}$ is the corresponding antiparticle, i.e. we change particle to antiparticle, exchanging in and out, and the momenta as $k = -p$, see Fig.76. This equality defines crossing symmetry. Note that since $k = -p$, we can't have $k^0 > 0$ and $p^0 > 0$ simultaneously, which means that when a particle is real, the crossed particle is virtual, and viceversa. Thus really, crossing symmetry also corresponds to an analytical continuation in energy.

Mandelstam variables

Crossing symmetry is easier to describe in Mandelstam variables, so we will define them better now.

For a $2 \to 2$ scattering process, with incoming momenta p, p' and outgoing momenta k, k', as in Fig.77, we define the Mandelstam variables s, t, u by

$$
s = -(p + p')^2 = -(k + k')^2 = E_{CM}^2
$$

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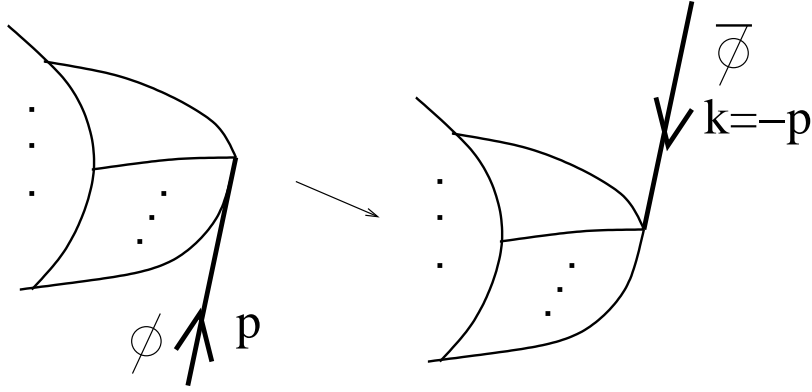
Figure 76: Crossing symmetry (includes a charge conjugation).
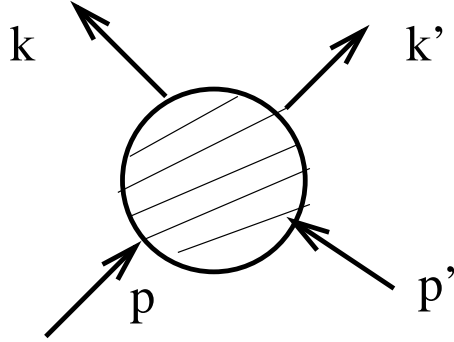
Figure 77: Mandelstam variables.

$$
\begin{aligned}
t &= -(k-p)^2 = -(k'-p')^2 \\
u &= -(k'-p)^2 = -(k-p')^2
\end{aligned}
\tag{25.25}
$$

Note that $s = E_{CM}^2$. Also, in the more interesting case that the particles with momenta p and k are the same one (elastic scattering), $E_k = E_p$, meaning that $t = -(\vec{k}-\vec{p})^2$ is minus the invariant *momentum transfer squared* $(\vec{k}-\vec{p} = -(\vec{k'}-\vec{p'})$ is the momentum transferred between the scattering particles).

Now consider again the process $e^+e^- \to \mu^+\mu^-$. Then

$$
\begin{aligned}
s &= -(p+p')^2 = -q^2 \\
t &= -(k-p)^2 = 2p \cdot k = 2p' \cdot k' \\
u &= -(p-k')^2 = 2p \cdot k' = 2p' \cdot k
\end{aligned}
\tag{25.26}
$$

and the sum over spins is

$$
\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \frac{8e^4}{s^2}\left[\left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right]
\tag{25.27}
$$

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Figure 78: 90^0 rotation followed by mirror image of a), giving crossed diagram b).

Consider the 90^0 rotation of $e^-(p)e^+(p') \to \mu^+(k)\mu^-(k')$ (followed by a mirror image exchanging left with right that doesn't change physics), as in Fig.78. By the above crossing rules, it gives for $e^-\mu^- \to e^-\mu^-$, $e^-(p)\mu^-(-k) \to e^-(-p)\mu^-(k')$, the Mandelstam variables for the process are

$$
\begin{aligned}
s &= -(p-k)^2 = \bar{t} \\
t &= -(p-(-p'))^2 = \bar{s} \\
u &= -(p-k')^2 = \bar{u}
\end{aligned}
\tag{25.28}
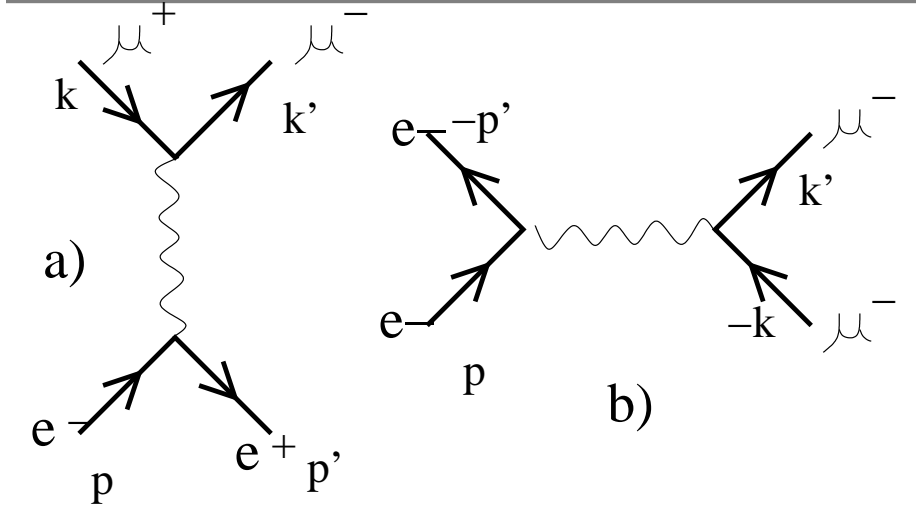$$

where the quantities with bar are defined using the original (uncrossed) diagram. Therefore if we have a functional form $\mathcal{M}(s,t)$ for the original amplitude, the form of the amplitude for the crossed process is obtained by exchanging s with t and keeping u unchanged. This in general will not give the same result. In particular, in our case we obtain

$$
\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \frac{8e^4}{t^2}\left[\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right]
\tag{25.29}
$$

which is not the same as for the original process. If nevertheless we obtain the same formula, we say the amplitude is *crossing symmetric*.

The Mandelstam variables are not independent. In fact, a simple counting argument tells us there are only 2 independent variables. There are 4 momenta, and for each 3 independent components (since $E^2 = \vec{p}^2 + m^2$), for a total of 12 components. But there are 4 momentum conservation conditions, leaving 8 components. But there are Lorentz transformations $\Lambda^\mu{}_\nu$ that we can make on the reference frame, corresponding to an antisymmetric 4×4 matrix ($\mu, \nu = 1, ..., 4$), or equivalently 3 rotations (around the 3 axis of coordinates) and 3 Lorentz boosts (speed of the frame in 3 directions), for a total of 6 frame-dependent components, leaving only 2 Lorentz invariants. In fact, we can write a simple relation between s, t, u

$$
s + t + u = -(p+p')^2 - (k-p)^2 - (k'-p)^2
$$

$$
\begin{aligned}
&= -3p^2 - p'^2 - k^2 - k'^2 - 2p \cdot p' + 2p \cdot k + 2p \cdot k' \\
&= 2p \cdot (k + k' - p') - 3p^2 - p'^2 - k^2 - k'^2 = -p^2 - p'^2 - k^2 - k'^2 \\
&= \sum_{i=1}^{4} m_i^2
\end{aligned}
\tag{25.30}
$$
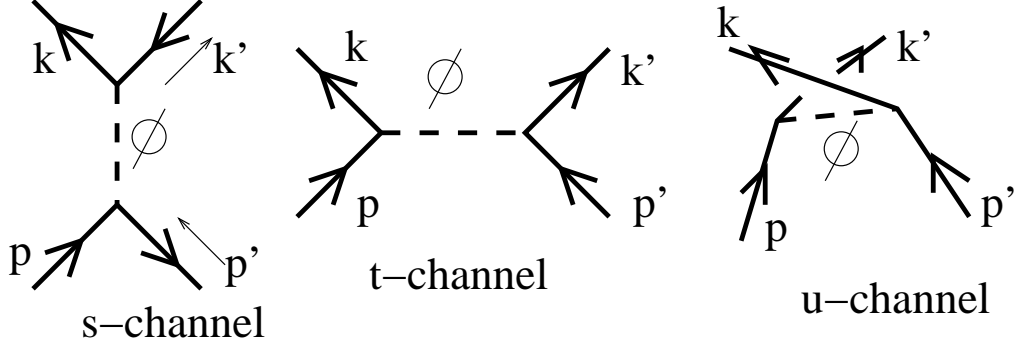
Figure 79: In 2 to 2 scattering, we can have the s-channel, the t-channel and the u-channel, depending on whether the physical pole of the S-matrix is in s, t or u, corresponding to effective Feynman diagrams shown here.

In the case of $2 \to 2$ scattering and a single virtual particle being exchanged between the two, we talk of "channels". Consider a particle-antiparticle pair of momenta p, p' annihilating, and creating a particle ϕ which then creates a different particle-antiparticle pair. Since then the momentum on ϕ is $p + p'$, the diagram has a propagator for ϕ with a $1/[(p+p')^2 + m_\phi^2] = -1/[s - m_\phi^2]$ factor, meaning

$$
\mathcal{M} \propto \frac{1}{s - m_\phi^2}
\tag{25.31}
$$

i.e. it has a pole in s. We thus refer to this as the "s-channel", see Fig.79 for details. Consider now the crossed diagram with the particle with momentum p going into the particle with momentum k, and the ϕ particle being exchanged with the other particle. In this case, the propagator for ϕ has a factor of $1/[(p-k)^2 + m_\phi^2] = -1/[t - m_\phi^2]$, so now

$$
\mathcal{M} \propto \frac{1}{t - m_\phi^2}
\tag{25.32}
$$

i.e. it has a pole in t. We refer to this as the "t-channel". Finally, consider a different crossing, where the particle of momentum p goes into the particle with momentum k' instead, and ϕ exchanged with the other particle. Then the ϕ propagator gives a factor of $1/[(p-k')^2 + m_\phi^2] = -1/[u - m_\phi^2]$, and therefore

$$
\mathcal{M} \propto \frac{1}{u - m_\phi^2}
\tag{25.33}
$$

i.e. it has a pole in u. We call this the "u-channel".

We now define a relativistically invariant differential cross section for elastic scattering (the same particles are in the initial and final states). We have defined $d\sigma/d\Omega$ before, even a relativistically invariant formula that becomes the usual $d\sigma/d\Omega$ in the center of mass frame. But now we can define a formula only in terms of the independent relativistic invariants s and t. We can write a $d\sigma/dt$. Consider that $t = -(p - p')^2$ and since $p = (E, \vec{p}_{CM})$ and $k = (E, \vec{k}_{CM})$ (if p and k correspond to the same particle, it has the same initial and final energy),

$$t = -(\vec{p}_{CM}^2 - \vec{k}_{CM})^2 = -(p_{CM}^2 + k_{CM}^2 - 2k_{CM}p_{CM}\cos\theta) \tag{25.34}$$

That means that

$$\frac{dt}{d\cos\theta} = 2|k_{CM}||p_{CM}| \tag{25.35}$$

Consider then that the solid angle element defined by a fixed θ would be $d\phi d(\cos\theta)$, and integrating over the angle ϕ at fixed θ between the initial and final momenta, we have $2\pi d(\cos\theta)$, therefore

$$\frac{dt}{d\Omega} = \frac{dt}{2\pi d(\cos\theta)} = \frac{1}{\pi}|k_{CM}||p_{CM}| \tag{25.36}$$

We already defined the relativistically invariant formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{rel.inv.} = \frac{1}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{\vec{k}_{CM}}{64\pi^2 E_{CM}}|\mathcal{M}|^2 \tag{25.37}$$

where $p_1 \to p$, $p_2 \to p'$, and since

$$s = -(p_1 + p_2)^2 = -p_1^2 - p_2^2 - 2p_1 \cdot p_2 = m_1^2 + m_2^2 - 2p_1 \cdot p_2$$
$$\Rightarrow (p_1 \cdot p_2)^2 - m_1^2 m_2^2 = \frac{1}{4}(s - m_1^2 - m_2^2) - m_1^2 m_2^2 \tag{25.38}$$

On the other hand, in the center of mass frame,

$$\begin{aligned}(p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= (E_1 E_2 + p_{CM})^2 - m_1^2 - m_2^2 = 2p_{CM}^4 + p_{CM}^2(m_1^2 + m_2^2) + 2E_1 E_2 p_{CM}^2 \\ &= p_{CM}^2(2p_{CM} + 2E_1 E_2 + m_1^2 + m_2^2) = p_{CM}^2 s \tag{25.39}\end{aligned}$$

where we have used $s = -(p_1 + p_2)^2 = m_1^2 + m_2^2 - 2p_1 \cdot p_2 = m_1^2 + m_2^2 + 2E_1 E_2 + 2\vec{p}_{CM}^2$.

Then we have

$$\begin{aligned}\frac{d\sigma}{dt} &= \frac{\pi}{|k_{CM}||p_{CM}|}\left(\frac{d\sigma}{d\Omega}\right)_{rel.inv.} = \frac{1}{64\pi|p_{CM}|\sqrt{s}\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}|\mathcal{M}|^2 \\ &= \frac{1}{64\pi[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]}|\mathcal{M}|^2 \tag{25.40}\end{aligned}$$

We can write for the total cross section

$$\sigma_{tot} = \int dt \frac{d\sigma}{dt}(s,t) = \sigma_{tot}(s) \tag{25.41}$$

Note that t is the momentum transfer, and as such it can in principle be anything, though of course we expect that for very large t (in which case at least one of the outgoing 3-momenta, and therefore also its energy, will be very large) the amplitude for this process would be very small or zero.

In the case of equal masses, $m_1 = m_2 = m$, (25.38) gives $s(s - 4m^2)/4$ and therefore finally

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s(s - 4m^2)}|\mathcal{M}(s,t)|^2 \tag{25.42}$$

Important concepts to remember

- In order to calculate polarized cross section, i.e. for given helicities, we can introduce projectors onto helicities P_L, P_R for free, since $P_L\psi_L = \psi_L$, $P_R\psi_R = \psi_R$, and then sum over spins since the other spins have zero contributions, because of $P_L\psi_R = P_R\psi_L = 0$.

- The crossed diagrams are diagrams were we rotate the direction of time, and reinterpret the diagram by changing particles into antiparticles.

- By changing the particle momenta into minus antiparticle momenta when crossing, we obtain the same value for the amplitude, hence crossing symmetry means $\mathcal{M}(\phi(p) + ... \to ...) = \mathcal{M}(... \to ...\bar{\phi}(k))$, with $k = -p$.

- If we keep the functional form of the amplitude fixed, crossing symmetry corresponds in Mandelstam variables to exchanging s with t and keeping u fixed. If we obtain the same amplitude after the exchange, we say that the amplitude is crossing symmetric.

- In the $2 \to 2$ scattering, there are only two independent Lorentz invariants, s and t. We have $s + t + u = \sum_{i=1}^{4} m_i^2$.

- In the diagrams with only an intermediate particle ϕ, we talk of s-channel, t-channel and u-channel for the diagrams which contain poles in s, t and u respectively, due to the ϕ propagator having s,t,u as $-p^2$.

- In $2 \to 2$ scattering, we can define the relativistic invariant differential cross section $d\sigma/dt(s,t)$, which integrates to $\sigma_{tot}(s)$.

Further reading: See chapters 5.2, 5.4 in [2] and 8.1 in [1].

Exercises, Lecture 25

1) Check that

$$\frac{d\sigma}{d\Omega}(e_L^- e_R^+ \to \mu_R^- \mu_L^+) = \frac{\alpha^2}{4E_{CM}^2}(1 - \cos\theta)^2 \tag{25.43}$$

as in the text (check all the details).

2) Consider the amplitude

$$\mathcal{M}(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \tag{25.44}$$

(Veneziano), where $\alpha(x) = a + bx$ $(a, b > 0)$. Show that it is crossing symmetric. Write it as an (infinite) sum of s-channel contributions (processes), and equivalently as a sum of t-channel contributions (processes).

26 Lecture 26. (Unpolarized) Compton scattering

In this lecture we will study the most (calculationally) complicated process so far, the last example of applications of the quantum field theory formalism to QED, namely Compton scattering, $e^-\gamma \to e^-\gamma$. After that, in the last 3 lectures, we will preview the treatment of divergences in the next semester, by showing what are the divergences and how we regularize. Also, in this lecture, we will have a new ingredient compared with the previous lectures, the sum over photon polarizations. It is also the first example we calculate when we have a sum of diagrams. The scattering we will calculate is the unpolarized one, when we can't measure the spin of the particles.

We have an electron of momentum p scattering with a photon of momentum k, resulting in a final electron of momentum p' and a final photon of momentum k'. There are 2 possible Feynman diagrams, with the photon being absorbed by the electron line before the final photon is emitted from the electron line (or we can write it as the intermediate electron line being vertical in the diagrams), or first the final photon is emitted, and then the incoming photon is absorbed (or we can write it as the intermediate electron line being horizontal in diagram), see Fig.80. In the first diagram the intermediate electron line has momentum $p + k$, in the second one it has momentum $p - k'$.
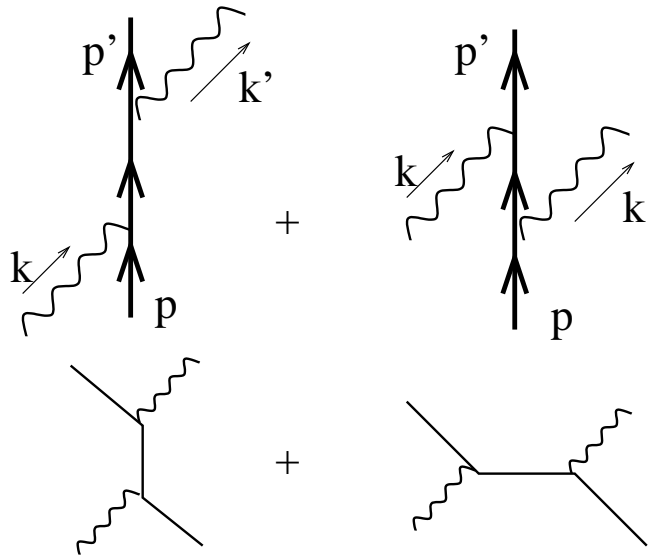
Figure 80: Feynman diagrams for Compton scattering, shown in two ways: above as scatterings off an electron line, and below as s-channel and t-channel diagrams.

Using the Feynman rules, we can write the expression for the amplitude as (note that the two fermion lines are identical, so there is no relative minus sign between the diagrams)

$$i\mathcal{M} = \bar{u}(p')(+e\gamma^\mu)\epsilon_\mu^*(k')(-)\frac{\not{p}+\not{k}+im}{(p+k)^2+m^2}(+e\gamma^\nu)\epsilon_\nu(k)u(p)$$

$$+\bar{u}(p')(+e\gamma^\nu)\epsilon_\nu(k)(-)\frac{\not{p}-\not{k}'+im}{(p-k')^2+m^2}(+e\gamma^\mu)\epsilon_\mu^*(k')u(p)$$

$$= -e^2\epsilon_\mu^*(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+im)\gamma^\nu}{(p+k)^2+m^2}+\frac{\gamma^\nu(\not{p}-\not{k}'+im)\gamma^\mu}{(p-k')^2+m^2}\right]u(p) \quad (26.1)$$

But note that $(p+k)^2+m^2 = p^2+k^2+2p\cdot k+m^2 = 2p\cdot k$ (since $k^2=0$ for the external photon, and $p^2+m^2=0$ for the external electron), and similarly $(p-k')^2+m^2 = p^2+k'^2-2p\cdot k'+m^2 = -2p\cdot k'$.

We now also use the fact that $u(p)$ and $v(p)$ satisfy the Dirac equation, i.e. $(\not{p}-im)u(p) = 0$ and $(\not{p}+im)v(p)=0$. Then we have

$$(\not{p}+im)\gamma^\nu u(p) = (-\gamma^\nu\not{p}+2g^{\mu\nu}p_\mu+im\gamma^\nu)u(p) = 2p^\nu u(p)-\gamma^\nu(\not{p}-im)u(p) = 2p^\nu u(p) \quad (26.2)$$

where we have used $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu+2g^{\mu\nu}$ and the Dirac equation above. Substituting this formula in the square bracket in $\mathcal{M}$, we find

$$i\mathcal{M} = -e^2\epsilon_\mu^*(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu+2\gamma^\mu p^\nu}{2p\cdot k}+\frac{-\gamma^\nu\not{k}'\gamma^\mu+2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p) \quad (26.3)$$

As we saw last lecture,

$$(\bar{u}\gamma^\mu\tilde{u})^* = -\bar{\tilde{u}}\gamma^\mu u \quad (26.4)$$

due to the fact that for a number, complex conjugation is the same as adjoint, and then we can write the transposed matrices in the opposite order, and use $(\gamma^\mu)^\dagger i\gamma^0 = -i\gamma^0\gamma^\mu$ (where $\bar{u} = u^\dagger i\gamma^0$). Here $u,\tilde{u}$ can be either u or v. Then we can generalize this to

$$(\bar{u}\gamma^{\mu_1}...\gamma^{\mu_n}\tilde{u})^* = (-)^n\bar{\tilde{u}}\gamma^{\mu_n}...\gamma^{\mu_1}u \quad (26.5)$$

We then find

$$-i\mathcal{M}^* = +e^2\epsilon_\rho(k')\epsilon_\sigma^*(k)\bar{u}(p)\left[\frac{\gamma^\sigma\not{k}\gamma^\rho+2\gamma^\rho p^\sigma}{2p\cdot k}+\frac{-\gamma^\mu\not{k}'\gamma^\sigma+2\gamma^\sigma p^\rho}{-2p\cdot k'}\right]u(p') \quad (26.6)$$

Photon polarization sums

We note that if we sum over polarizations, in $|\mathcal{M}|^2$ we have $\sum_{pol.}\epsilon_\mu^*(k)\epsilon_\nu(k)$. We then have the theorem that in $|\mathcal{M}|^2$, such sums over polarizations can be done with the replacement

$$\sum_{pol.}\epsilon_\mu^*(k)\epsilon_\nu(k) \to g_{\mu\nu} \quad (26.7)$$

More precisely, if

$$i\mathcal{M} = \epsilon_\mu^*(k)\mathcal{M}^\mu(k) \quad (26.8)$$

then

$$\sum_{pol.}\mathcal{M}\mathcal{M}^* = \sum_{pol.}\epsilon_\mu^*(k)\epsilon_\nu(k)\mathcal{M}^\mu(k)\mathcal{M}_\nu^*(k) = g_{\mu\nu}\mathcal{M}^\mu(k)\mathcal{M}^{*\nu}(k) \quad (26.9)$$

Indeed, Ward identities imply that $k_\mu\mathcal{M}^\mu(k) = 0$. Indeed, we have seen from the Ward-Takahashi identities, that we have the same for the 1PI functions, $k^\mu\Pi_{\mu\nu}(k) = 0$, $k^{\mu_1}\Gamma_{\mu_1...\mu_n} = $

0. The S-matrix corresponds to a connected, amputated n-point function, contracted with external lines for the states, e.g. for the photons with $\epsilon_\mu(k)$. As it is reasonable to assume that the connected and amputated n-point functions have the same property, we should have $k_\mu \mathcal{M}^\mu = 0$. Another approximate way to see this is that $\mathcal{M}^\mu$ is the Fourier transform of an electromagnetic current in between some initial and final state,

$$\mathcal{M}^\mu(k) = \int d^4x e^{ik\cdot x} < f|j^\mu(x)|i > \tag{26.10}$$

where $j^\mu = \bar{\psi}\gamma^\mu\psi$. The reason is that external photons are created by the interaction term $e\int d^4x j^\mu A_\mu$. We can check this fact for our amplitude, where each half of a diagram (initial, and final halfs) have this property.

Then, consider an on-shell photon momentum ($k^2 = 0$), for instance $k^\mu = (k, 0, 0, k)$ (moving in the 3 direction). The two physical polarizations are then transverse to the momentum, and of unit norm, i.e. $\epsilon^\mu_{(1)} = (0, 1, 0, 0)$ and $\epsilon^\mu_{(2)} = (0, 0, 1, 0)$. Then $k_\mu \mathcal{M}^\mu(k) = 0$ is written as

$$-k\mathcal{M}^0(k) + k\mathcal{M}^3(k) = 0 \Rightarrow \mathcal{M}^3(k) = \mathcal{M}^0(k) \tag{26.11}$$

Thus finally the sum over photon polarizations gives

$$\sum_{pol.} \epsilon^*_\mu(k)\epsilon_\nu(k)\mathcal{M}^\mu(k)\mathcal{M}^{*\nu}(k) = |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 = -|\mathcal{M}^0|^2 + |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2$$

$$= g_{\mu\nu}\mathcal{M}^\mu(k)\mathcal{M}^{*\nu}(k) \tag{26.12}$$

q.e.d.

We now go back to the Compton scattering calculation, and find for the unpolarized $|\mathcal{M}|^2$, (doing also the sum over electron spins in the usual manner)

$$\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = -\frac{e^4}{4}g_{\mu\rho}g_{\nu\sigma}\text{Tr}\left\{(-i\slashed{p}' + m)\left[\frac{\gamma^\mu\slashed{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu\slashed{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]\right.$$

$$\left.(-i\slashed{p} + m)\left[\frac{\gamma^\sigma\slashed{k}\gamma^\rho + 2\gamma^\rho p^\sigma}{2p\cdot k} + \frac{-\gamma^\rho\slashed{k}'\gamma^\sigma + 2\gamma^\sigma p^\rho}{-2p\cdot k'}\right]\right\}$$

$$\equiv -\frac{e^4}{4}\left[\frac{I}{(2p\cdot k)^2} + \frac{II}{(2p\cdot k)(2p\cdot k')} + \frac{III}{(2p\cdot k')(2p\cdot k)} + \frac{IV}{(2p\cdot k')^2}\right] \tag{26.13}$$

But we note that $II = III$, and that $I(k = -k') = IV$, so we only need to calculate I and II.

We have

$$I = \text{Tr}\left[(-i\slashed{p}' + m)(\gamma^\mu\slashed{k}\gamma^\nu + 2\gamma^\mu p^\nu)(-i\slashed{p} + m)(\gamma_\nu\slashed{k}\gamma_\mu + 2\gamma_\mu p_\nu)\right] \tag{26.14}$$

and we can split it into 8 nonzero traces. Indeed, there are $2^4 = 16$ terms, but as we already saw, only the trace of an even number of gammas is nonzero, so half of the terms are zero. We calculate each of the 8 traces separately.

First, the longest trace is

$$I_1 = (-i)^2\text{Tr}[\slashed{p}'\gamma^\mu\slashed{k}\gamma^\nu\slashed{p}\gamma_\nu\slashed{k}\gamma_\mu] = \text{Tr}[\gamma^\mu\slashed{p}'\gamma_\mu\slashed{k}(-2\slashed{p})\slashed{k}] = -4\,\text{Tr}[\slashed{p}'\slashed{k}\slashed{p}\slashed{k}]$$

$$= -4\operatorname{Tr}[\not{p}'\not{k}(-\not{k}\not{p} + 2p \cdot k)] = -8p \cdot k\operatorname{Tr}[\not{p}'\not{k}] = -32(p \cdot k)(p' \cdot k) \qquad (26.15)$$

Here we have used the fact that $\gamma^\mu\gamma_\nu\gamma_\mu = -2\gamma_\nu$, thus $\gamma^\nu\not{p}\gamma^\mu = -2\not{p}$, that the trace is cyclically symmetric, that $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2g^{\mu\nu}$, thus $\not{p}\not{k} = -\not{k}\not{p} + 2p \cdot k$, and $\not{k}\not{k} = k^2 = 0$ (the photon is massless), and finally $\operatorname{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$, thus $\operatorname{Tr}[\not{p}'\not{k}] = 4p' \cdot k$.

Similarly,

$$
\begin{aligned}
I_2 &= (-i)^2\operatorname{Tr}[\not{p}'\gamma^\mu\not{k}2p^\nu\gamma_\nu\not{p}\gamma_\mu] = -2\operatorname{Tr}[\gamma_\mu\not{p}'\gamma^\mu\not{k}(-m^2)] = -4m^2\operatorname{Tr}[\not{p}'\not{k}] = -16m^2p' \cdot k \\
I_3 &= (-i)^2\operatorname{Tr}[\not{p}'\gamma^\mu2p^\nu\gamma_\nu\not{p}\not{k}\gamma_\mu] = -2\operatorname{Tr}[\gamma_\mu\not{p}'\gamma^\mu(-m^2)\not{k}] = -4m^2\operatorname{Tr}[\not{p}'\not{k}] = -16m^2p' \cdot k \\
I_4 &= (-i)^2(2p_\nu 2p^\nu)\operatorname{Tr}[\not{p}'\gamma^\mu\not{p}\gamma_\mu] = +4m^2\operatorname{Tr}[\not{p}(-2\not{p})] = -32m^2p \cdot p'
\end{aligned}
\qquad (26.16)
$$

Here we have used also $p^2 = -m^2$ for the external electron, besides the others, already used. These are all the terms with the $(-i\not{p}')$ and $(-i\not{p})$ factors. There are also terms with $(-i\not{p}')m$ and with $m(-i\not{p})$, but these are zero, since they have an odd number of gammas in the trace. Therefore we only have the 4 terms with m^2 left, which give

$$
\begin{aligned}
I_5 &= m^2\operatorname{Tr}[\gamma^\mu\not{k}\gamma^\nu\gamma_\nu\not{k}\gamma_\mu] = 4m^2\operatorname{Tr}[\gamma^\mu\gamma_\mu\not{k}\not{k}] = 0 \\
I_6 &= 2m^2\operatorname{Tr}[\gamma^\mu\not{k}p^\nu\gamma_\nu\gamma_\mu] = 8m^2\operatorname{Tr}[\not{k}\not{p}] = 32m^2k \cdot p \\
I_7 &= 2m^2\operatorname{Tr}[\gamma^\mu\not{p}\not{k}\gamma_\mu] = 8m^2\operatorname{Tr}[\not{p}\not{k}] = 32m^2p \cdot k \\
I_8 &= 4m^2p^\nu p_\nu\operatorname{Tr}[\gamma^\mu\gamma_\mu] = -64m^4
\end{aligned}
\qquad (26.17)
$$

Then for the term I we have

$$
\begin{aligned}
I &= \sum_{i=1}^{8}I_i = -32(p \cdot k)(p' \cdot k) - 32m^2p' \cdot k - 32m^2p \cdot p' + 64m^2p \cdot k - 64m^4 \\
&= -32[(p \cdot k)(p' \cdot k) + m^2p' \cdot k + m^2p \cdot p' - 2m^2p \cdot k + 2m^4]
\end{aligned}
\qquad (26.18)
$$

Similarly, we find (the details are left as an exercise, similarly to the above I term)

$$
\begin{aligned}
II = III &= -16[-2k \cdot k'p \cdot p' + 2k \cdot pp' \cdot p + m^2p' \cdot k - 2k' \cdot pp \cdot p' - m^2p' \cdot k' + m^2p \cdot p' \\
&\quad - m^2k \cdot k' + 2m^2p \cdot k - 2m^2p \cdot k' - m^4]
\end{aligned}
\qquad (26.19)
$$

We now translate into Mandelstam variables. We have

$$
\begin{aligned}
s &= -(p+k)^2 = -p^2 - k^2 - 2p \cdot k = m^2 - 2p \cdot k \\
&= -(p'+k')^2 = -p'^2 - k'^2 - 2p' \cdot k' = m^2 - 2p' \cdot k' \\
t &= -(p-p')^2 = -p^2 - p'^2 + 2p \cdot p' = 2m^2 + 2p \cdot p' \\
&= -(k-k')^2 = -k^2 - k'^2 + 2k \cdot k' = 2k \cdot k' \\
u &= -(k'-p)^2 = -k'^2 - p^2 + 2k' \cdot p = m^2 + 2k' \cdot p \\
&= -(k-p')^2 = -k^2 - p'^2 + 2k \cdot p' = m^2 + 2k \cdot p'
\end{aligned}
\qquad (26.20)
$$

which allow us to write all the inner products of external momenta in terms of s, t, u. We have also to remember that

$$s + t + u = \sum_{i=1}^{4}m_i^2 = 2m^2 \qquad (26.21)$$

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As well, to calculate IV we need to replace $k \leftrightarrow -k'$, which we easily see to imply exchanging $s \leftrightarrow u$.

We then find

$$
\begin{aligned}
I &= -32 \left[\frac{m^2 - s}{2} \frac{u - m^2}{2} + m^2 \frac{u - m^2}{2} + m^2 \frac{t - 2m^2}{2} - 2m^2 \frac{m^2 - s}{2} + 2m^4 \right] \\
&= -32 \left[-\frac{su}{4} + m^2 \left(\frac{3}{4}u + \frac{t}{2} + \frac{s}{4} \right) - \frac{3}{4}m^4 \right] \\
&= -32 \left[-\frac{su}{4} + \frac{m^2}{4}(3s + u) + \frac{m^4}{4} \right] \\
&= -16 \left[-\frac{1}{2}(s - m^2)(u - m^2) + m^2(s - m^2) + 2m^4 \right]
\end{aligned}
\tag{26.22}
$$

where in the next to last line we used $(s + t + u)/2 = m^2$.

Then we immediately find the expression for IV, by just switching $u \leftrightarrow s$,

$$
IV = -16 \left[-\frac{1}{2}(s - m^2)(u - m^2) + m^2(u - m^2) + 2m^4 \right]
\tag{26.23}
$$

Similarly, after a similar algebra left as an exercise, we find

$$
II = III = 16 \left[\frac{m^2}{2}(s - m^2) + \frac{m^2}{2}(u - m^2) + 2m^4 \right]
\tag{26.24}
$$

Finally then, we have

$$
\begin{aligned}
\frac{1}{4} \sum_{spins} |\mathcal{M}|^2 &= \frac{e^4}{4} \left\{ \frac{16}{(s - m^2)^2} \left[-\frac{1}{2}(s - m^2)(u - m^2) + m^2(s - m^2) + 2m^4 \right] \right. \\
&\quad + \frac{16}{(u - m^2)^2} \left[-\frac{1}{2}(s - m^2)(u - m^2) + m^2(u - m^2) + 2m^4 \right] \\
&\quad \left. + \frac{2 \cdot 16}{(s - m^2)(u - m^2)} \left[\frac{m^2}{2}(s - m^2) + \frac{m^2}{2}(u - m^2) + 2m^4 \right] \right\} \\
&= 4e^4 \left\{ -\frac{s - m^2}{2(u - m^2)} - \frac{u - m^2}{2(s - m^2)} \right. \\
&\quad \left. + \frac{2m^2}{u - m^2} + \frac{2m^2}{s - m^2} + 2m^4 \left(\frac{1}{u - m^2} + \frac{1}{s - m^2} \right)^2 \right\}
\end{aligned}
\tag{26.25}
$$

The relativistically invariant differential cross section is

$$
\frac{d\sigma}{dt} = \frac{1}{64\pi[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]} \frac{1}{4} \sum_{spins} |\mathcal{M}|^2
\tag{26.26}
$$

But in our case, with $p_1 = p, p_2 = k$, we have

$$
(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = \left(\frac{s - m^2}{2} \right)^2 - 0 = \frac{(s - m^2)^2}{4}
\tag{26.27}
$$

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giving

$$
\begin{aligned}
\frac{d\sigma}{dt} &= \frac{e^4}{4\pi(s-m^2)^2}\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 \\
&= \frac{4\pi\alpha^2}{(s-m^2)^2}\left\{-\frac{s-m^2}{2(u-m^2)}-\frac{u-m^2}{2(s-m^2)}\right. \\
&\quad \left.+\frac{2m^2}{u-m^2}+\frac{2m^2}{s-m^2}+2m^4\left(\frac{1}{u-m^2}+\frac{1}{s-m^2}\right)^2\right\}
\end{aligned}
\tag{26.28}
$$

As we see, this formula is not that simple.

We can also analyze this formula in the lab frame, obtaining the *spin-averaged Klein-Nishina formula*

$$
\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right]
\tag{26.29}
$$

where ω and ω' are the angular frequencies of the incoming and outgoing photons in the lab frame (where the electron target is fixed), related to each other by

$$
\frac{\omega'}{\omega} = \frac{1}{1+\frac{\omega}{m}(1-\cos\theta)}
\tag{26.30}
$$

We will not prove the Klein-Nishina formula here.

Important concepts to remember

- For sums over photon polarizations, if $i\mathcal{M} = \epsilon^*_\mu(k)\mathcal{M}^\mu(k)$, we replace $\sum_{pol} \epsilon^*_\mu(k)\epsilon_\nu(k)$ by $g_{\mu\nu}$ in $|\mathcal{M}|^2$.

- When we have fermion propagators in a Feynman diagrams, we can use the fact that the $u(p)$ and $v(p)$'s satisfy the Dirac equation, $(\not{p} - im)u(p) = 0$ and $(\not{p} + im)v(p) = 0$.

Further reading: See chapters 5.5 in [2] and 8.2 in [1].

Exercises, Lecture 26

1) Show that

$$\text{Tr}\left[(-i\not{p}' + m)(\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu)(-i\not{p} + m)(\gamma^\mu \not{k}'\gamma^\nu - 2\gamma^\nu p^\mu)\right]$$
$$= 16\left[\frac{m^2}{2}(s - m^2) + \frac{m^2}{2}(u - m^2) + 2m^4\right] \tag{26.31}$$

2) Using the formulas in the text, calculate the total relativistically invariant cross section, $\sigma_{tot}(s)$.

27 Lecture 27. One-loop determinants, vacuum energy and zeta function regularization

We now start discussing infinities that appear in quantum field theory, with the simplest example, that we already saw a bit before, namely the infinite zero point energy.

For a harmonic oscillator, the vacuum has a "zero-point energy" (vacuum energy) $E_0 = \hbar\omega/2$, and we have seen that analogously, for a free bosonic field, for instance for a real scalar field, we have an sum over all the harmonic oscillator modes of the field, i.e. (in $D-1$ space dimensions)

$$E_0 = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\hbar\omega_k}{2} (2\pi)^{D-1}\delta(0) \to E_0 = \sum_{\vec{k}} \frac{\hbar\omega_{\vec{k}}}{2} = \infty \qquad (27.1)$$

where $\int d^{D-1}k/(2\pi)^{D-1} \to 1/V \sum_{\vec{k}}$ and $\delta(\vec{p}-\vec{k}) \to V\delta_{\vec{p}\vec{k}}$.

In the free theory, we have removed this infinity using *normal ordering*. But the issue remains, is this something physical, that has to do with quantum field theory, or just a pure artifact of the formalism? The answer is that it does have in fact a physical significance. Not the infinite result of course, but we can measure *changes* in it due to changes in geometry. This is known as the *Casimir effect*, and has already been experimentally observed, meaning the existence of the zero-point energy has been experimentally confirmed.
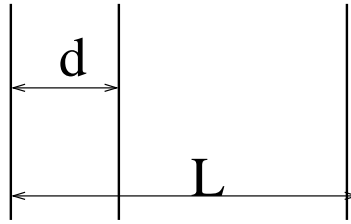

Figure 81: Parallel plates at a distance d, inside a box of length L.

Since the result is not only formally infinite (an infinite sum), but also continuous for a system of infinite size, we will consider a space that has a finite size (length) L, see Fig.81 and we will take $L \to \infty$ at the end of the calculation. Also, in order to consider the geometry dependence, consider two parallel perfectly conducting plates situated at a distance d. Strictly speaking, this would be for the physical case of fluctuation of the electromagnetic field, the only long range field other than gravity. The conducting plates would impose constant potential on them, i.e. $A_0 = V$ =const., thus more generally, Dirichlet boundary conditions for the fluctuating field A_μ. In our simple case then, consider a scalar field with Dirichlet boundary conditions at the plates, such that the eigenmodes between the plates have

$$k_n = \frac{\pi n}{d} \qquad (27.2)$$

One of the plates can be considered to be a boundary of the system without loss of generality,

so the system is separated into d and $L - d$. (We can consider the more general case where there is space on both sides, but it will not change the final result).

If the y and z directions are infinite, then the energy per unit area (area of x, y directions) of fluctuations *between the plates* is given by

$$\frac{E}{A} = \frac{\hbar}{2} \sum_n \int \frac{dk_y dk_z}{(2\pi)^2} \sqrt{\left(\frac{\pi n}{d}\right)^2 + k_y^2 + k_z^2} \tag{27.3}$$

and we can do this calculation, related to the real case (experimentally measured), just that it will be more involved. It is simpler to work directly with a system in 1+1 dimensions, meaning there is no k_y, k_z.

The total vacuum energy of the system is the energy of fluctuations between the plates (boundary conditions) at distance d and the energy outside the plates. If the space is considered to be periodic, the boundary at $x = L$ is the same (after the equivalence) as the first plate at $x = 0$, and the other boundary of the system is at $x = d$ (the second plate), just that now the space in between the plates is the "outside" one, of length $L - d$. Therefore the energy is

$$E = f(d) + f(L - d) \tag{27.4}$$

where

$$f(d) = \frac{\hbar\pi}{2d} \sum_{n=1}^{\infty} n \tag{27.5}$$

As we already noted, $f(d)$ is infinite, but let us consider a regularization. This is so, since at $n \to \infty$, modes have infinite energy, so they clearly need to be suppressed. If for nothing else, when the energy reaches the Planck energy, we expect that the spacetime will start getting curved, so this naive notion of perturbation modes in flat space cannot be a good approximation anymore. In any case, it is physically clear that the contribution of modes of infinite energy should be suppressed. We consider the regulator $e^{-a\omega_n}$, i.e. the regulated function is

$$
\begin{aligned}
\frac{\tilde{f}(d)}{\hbar} &= \frac{\pi}{2d} \sum_{n \geq 1} n e^{-a\omega_n} = \frac{\pi}{2d} \sum_{n \geq 1} n e^{-a\frac{n\pi}{d}} = -\frac{1}{2}\frac{\partial}{\partial a} \sum_{n \geq 1} \left(e^{-a\frac{\pi}{d}}\right)^n \\
&= -\frac{1}{2}\frac{\partial}{\partial a} \frac{1}{1 - e^{-a\frac{\pi}{d}}} = \frac{\pi}{2d} \frac{e^{-a\frac{\pi}{d}}}{(1 - e^{-a\frac{\pi}{d}})^2}
\end{aligned}
\tag{27.6}
$$

Now taking $a \to 0$, we obtain

$$\frac{\tilde{f}(d)}{\hbar} \simeq \frac{d}{2\pi a^2} - \frac{\pi}{24d} + \ldots \tag{27.7}$$

Adding now the contributions of the two pieces (inside and outside the plates) to the Casimir effect, we obtain

$$E_0 = f(d) + f(L - d) = \frac{L}{2\pi a^2} - \frac{\pi}{24d} + \mathcal{O}\left(\frac{1}{L^2}; a^2\right) \tag{27.8}$$

and now the infinite term is *constant* (d-independent), but there is a finite d-dependent term.

In other words, now by varying the geometry, i.e. by varying d, we obtain a force on the plates,

$$F = \frac{\partial E}{\partial d} = +\frac{\pi}{24d^2} + ... \tag{27.9}$$

We now consider a generalization of this case to an interacting quantum field theory for a scalar ϕ with potential $U(\phi)$, with minimum at $\phi = \phi_0$ and expand around it. We keep in the expansion of the action and the associated potential energy only the constant and quadratic terms (since the linear term is zero by the equations of motion). The potential energy of the system is then

$$V[\phi] \simeq V[\phi_0] + \int d^3\vec{x}\frac{1}{2}\left\{\delta\phi(\vec{x},t)\left[-\vec{\nabla}^2 + \left(\frac{d^2U}{d\phi^2}\right)\bigg|_{\phi=\phi_0}\right]\delta\phi(\vec{x},t) + ...\right\} \tag{27.10}$$

We then need to find the eigenvalues (and eigenfunctions) of the operator in square brackets, namely

$$\left(-\vec{\nabla}^2 + \frac{d^2U}{d\phi^2}\bigg|_{\phi=\phi_0}\right)\eta_i(\vec{x}) = \omega_i^2\eta_i(\vec{x}) \tag{27.11}$$

The eigenfunctions are orthonormal

$$\int d^3x\eta_i^*(\vec{x})\eta_j(\vec{x}) = \delta_{ij} \tag{27.12}$$

We can then expand the fluctuation in the eigenfunctions

$$\delta\phi(\vec{x},t) = \sum_i c_i(t)\eta_i(\vec{x}) \tag{27.13}$$

Thus we obtain the total energy

$$\begin{aligned} E &= \int d^4x\frac{1}{2}(\dot{\phi}(\vec{x},t))^2 + \int dt V[\phi] \\ &\simeq \int dt \sum_i \left[\frac{1}{2}\dot{c}_i(t)^2 + \frac{1}{2}\omega_i^2 c_i(t)^2\right] \end{aligned} \tag{27.14}$$

which is a sum of harmonic oscillators, thus the energy of the vacuum of the system is

$$E_0 = V[\phi_0] + \frac{\hbar}{2}\sum_i \omega_i + ... \tag{27.15}$$

where we ignored corrections, which can be identified as higher loop corrections (whereas the calculated term is only one-loop, as we will see shortly). Note that this formula has many subtleties. In particular, for nonzero interactions, there will be renormalization of the parameters of the interaction (as we will show in the next semester). But at a formal level, the above formula is correct.

As an example, let's see what happens for the free theory, with $U = 0$. Then $\eta_i = e^{i\vec{k}_i \cdot \vec{x}}$, so

$$(-\vec{\nabla}^2)\eta_i = \vec{k}_i^2 \eta_i \equiv \omega_i^2 \eta_i \qquad (27.16)$$

therefore

$$E_0 = \frac{\hbar}{2} \sum_i \sqrt{\vec{k}_i^2} \qquad (27.17)$$

Moreover, for $U = m^2\phi^2/2$ (just a mass term),

$$(-\vec{\nabla}^2 + m^2)\eta_i = (\vec{k}_i^2 + m^2)\eta_i \equiv \omega_i^2 \eta_i \qquad (27.18)$$

therefore

$$E_0 = \frac{\hbar}{2} \sum_i \sqrt{\vec{k}_i^2 + m^2} \qquad (27.19)$$

Let us now understand better the regularization that we used and its generality (or uniqueness of the nontrivial piece). The quantity we want is $\sum_{n \geq 1} n$. It is of course infinite, but in mathematics there is a well-defined finite value associated with it by analytical continuation.

Indeed, one can define the *Riemann zeta function*,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \qquad (27.20)$$

For real s, it is convergent for $s > 1$. However, one can define it in the complex plane $\mathbb{C}$ (Riemann). In this case, away from the real axis, the function $\zeta(s)$ is well-defined close to -1. One can then uniquely define $\zeta(-1)$ by analytical continuation as $\zeta(s \to -1)$.* We then have

$$\zeta(-1) = -\frac{1}{12} \qquad (27.21)$$

Then we can write directly for the Casimir effect with two plates,

$$E_0 \simeq \left(\frac{\hbar}{2d} + \frac{\hbar}{2(L-d)} \right) \sum_{n \geq 1} n = -\frac{\hbar\pi}{24d} \qquad (27.22)$$

the same as we obtained before, but with the infinite constant already removed. Therefore the zeta function procedure is a good regularization.

Zeta function regularization

We now generalize the above regularization procedure using the zeta function to a general operator.

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*We can write an integral representation for the zeta function, by writing an integral representation for each term in the sum, as $\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n \geq 1} e^{-nt} = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}e^{-t}}{1-e^{-t}}$, which form is shown to be well-defined (analytic) over the whole complex plane, with the only pole at $s = +1$. For $s = -2m$, with m integer, it has in fact (trivial) poles. The Riemann hypothesis, one of the most important unsolved (unproven) problems in mathematics, is that all the nontrivial zeroes of the Riemann zeta function (other than the trivial ones above) lie on the line $Re(s) = +1/2$.

Consider an operator with positive, real and discrete eigenvalues $a_1, ..., a_n$ and corresponding eigenfunctions f_n,

$$A f_n(x) = a_n f_n(x) \tag{27.23}$$

Then we define

$$\zeta_A(s) = \sum_{n \geq 1} \frac{1}{a_n^s} \tag{27.24}$$

Then

$$\frac{d}{ds} \zeta_A(s)|_{s=0} = -\sum_n \ln a_n e^{-s \ln a_n}|_{s \to 0} = -\ln \prod_n a_n \tag{27.25}$$

So finally,

$$e^{\mathrm{Tr} \ln A} = \det A = \prod_n a_n = e^{-\zeta_A'(0)} \tag{27.26}$$

As we will shortly see, the object that we want to calculate is $\det A = e^{\mathrm{Tr} \ln A}$, so we can calculate it if we know $\zeta_A(s)$.

Heat kernel regularization

Sometimes we can't calculate directly $\det A$ or $\zeta_A(s)$, but it may be easier to calculate another object, called the associated "heat kernel", defined as

$$G(x, y, \tau) \equiv \sum_n e^{-a_n \tau} f_n(x) f_n^*(y) \tag{27.27}$$

It satisfies the generalized heat equation

$$A_x G(x, y, \tau) = -\frac{\partial}{\partial \tau} G(x, y, \tau) \tag{27.28}$$

as we can easily check. We can also easily check that, since the $f_n(x)$ are orthornormal, the heat kernel satisfies the boundary condition

$$G(x, y; \tau = 0) = \delta(x - y) \tag{27.29}$$

Sometimes it is easier to solve the generalized heat equation with the above boundary condition, and then we can find $\zeta_A(s)$ from it as

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \int d^4 x \, G(x, x, \tau) \tag{27.30}$$

since $\int d^4 x |f_n(x)|^2 = 1$.

If we find $\zeta_A(s)$, then we can calculate $\det A$, as we saw.

Saddle point evaluation

We now see how to calculate the sum over frequencies from the the path integral formulation. For that, we need to make a quadratic approximation around a minimum (sometimes it is only a local minimum, not a global minimum) of the action.

This is based on the idea of saddle point evaluation of an integral. If we write the integrand as an exponential of minus something that has a minimum,

$$I \equiv \int dx\, e^{-a(x)} \tag{27.31}$$

with

$$a(x) \simeq a(x_0) + \frac{1}{2}(x - x_0)^2 a''(x_0) \tag{27.32}$$

with $a''(x_0) > 0$, then we can approximate the integral as a gaussian around the minimum, i.e. as

$$I \simeq e^{-a(x_0)} \int dx\, e^{-\frac{1}{2}(x-x_0)^2 a''(x_0)} = e^{-a(x_0)} \sqrt{\frac{2\pi}{a''(x_0)}} \tag{27.33}$$

Path integral formulation

Consider the case of field theory in Euclidean space, with partition function

$$Z[J] = N \int \mathcal{D}\phi\, e^{-S[\phi]+J\cdot\phi} \equiv N \int \mathcal{D}\phi\, e^{-S_E[\phi,J]} \tag{27.34}$$

We can again make a quadratic approximation around a minimum,

$$S_E[\phi, J] = S_E[\phi_0, J] + \frac{1}{2}\left\langle \frac{\delta^2 S_E}{\delta\phi_1 \delta\phi_2}(\phi - \phi_0)_1(\phi - \phi_0)_2 \right\rangle_{1,2} + \dots \tag{27.35}$$

where $\phi_1 = (\phi - \phi_0)_1$ and $\phi_2 = (\phi - \phi_0)_2$, with 1,2 corresponding to different variables of integration (x, y), and $<>_{1,2}$ to integrating over them.

Then the saddle point evaluation of the partition function gives

$$\begin{aligned}
Z[J] &\simeq N' e^{-S_E[\phi_0]+\phi_0\cdot J} \int \mathcal{D}\phi \exp\left\{ -\frac{1}{2}\left\langle \phi_1 \frac{\delta^2 S_E}{\delta\phi_1 \delta\phi_2}\phi_2 \right\rangle_{1,2} \right\} \\
&= N'' e^{-S_E[\phi_0]+\phi_0\cdot J} \det\left[(-\partial_\mu \partial^\mu + U''(\phi_0))\delta_{1,2} \right]^{-1/2}
\end{aligned} \tag{27.36}$$

Since $\det A = e^{\operatorname{Tr}\ln A}$, we have

$$-W[J] = \ln Z[J] = -S[\phi_0] + \phi_0 \cdot J - \frac{1}{2}\operatorname{Tr}\ln\left[(-\partial_\mu \partial^\mu + U''(\phi_0))\delta_{1,2} \right] \tag{27.37}$$

and as we saw, we can calculate

$$-\frac{1}{2}\operatorname{Tr}\ln[A] = \frac{1}{2}\frac{d\zeta_A}{ds}\bigg|_{s=0} \tag{27.38}$$

To make contact with what we had before, we continue to Minkowski space, where the eigenvalue-eigenfunction problem is

$$\left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 - U''(\phi_0) \right)\phi_k(\vec{x}, t) = a_k \phi_k(\vec{x}, t) \tag{27.39}$$

We separate variables writing
$$\phi_k(\vec{x}, t) = \eta_r(\vec{x}) f_n(t) \tag{27.40}$$

(i.e. $k = (rn)$), with
$$(-\vec{\nabla}^2 + U''(\phi_0))\eta_r(\vec{x}) = \omega_r^2 \eta_r \tag{27.41}$$

Then we have
$$\left(-\frac{\partial^2}{\partial t^2} - \omega_r^2\right)\sin\frac{n\pi t}{T} = \left(\frac{n^2\pi^2}{T^2} - \omega_r^2\right)\sin\frac{n\pi t}{T} \tag{27.42}$$

So that finally
$$\det\left[\partial_\mu\partial^\mu - U''(\phi_0)\right]^{-1/2} = \prod_{n\geq 1}\left(\frac{n^2\pi^2}{T^2} - \omega_r^2\right)^{-1/2} \tag{27.43}$$

One can calculate the partition function then, and obtain
$$Z = \sum_n e^{-\frac{iE_n T}{\hbar}} \tag{27.44}$$

and as before
$$E_0 = \frac{\hbar}{2}\sum_r \omega_r \tag{27.45}$$

The first observation is that $E_0 \propto \hbar$, and it is obtained by quantum fluctuations around the classical minimum of the action, so it is the first quantum correction, i.e. one-loop. It is one-loop since in the quadratic (free) approximation we used, calculating $Z_0[J]$, the only Feynman diagrams there are are vacuum bubbles, i.e. circles, which form one loop. Moreover, we have seen that the result is given by determinants of the operator acting on fluctuations. Hence one refers to these quantum corrections as *one-loop determinants*.

Fermions

In the case of fermions, as we know, gaussian integration gives the determinant in the numerator instead of the denominator, i.e.
$$\int d^n x e^{x^T A x} = 2^{n/2}\sqrt{\det A} = N e^{+\frac{1}{2}\mathrm{Tr}\ln A} \tag{27.46}$$

Moreover, in the operator formalism we saw that the energy is
$$H = \sum_i \sum_{n_i} \frac{\hbar\omega_i}{2}(b_i^\dagger b_i - b_i b_i^\dagger) = \sum_i \sum_{n_i} \hbar\omega\left(N_i - \frac{1}{2}\right) \tag{27.47}$$

so in both ways we obtain
$$E_0 = -\frac{\hbar}{2}\sum_i \omega_i \tag{27.48}$$

which are opposite in sign to the bosonic corrections. In supersymmetric theories, the bosonic and fermionic zero-point energies cancel each other.

Important concepts to remember

- We can measure changes in the infinite zero point energy of a field due to geometry, but not the infinite constant piece. This is known as the *Casimir effect.*

- We obtain the relevant finite piece by using a regularization, either by introducing a $e^{-a\pi n/d}$ factor with $a \to 0$, or better, by using the finite number associated to $\sum_n n$ by the zeta function, $\zeta(-1) = -1/12$.

- We generalize to an arbitrary field theory by calculating the eigenvalue-eigenfunction problem for the potential energy operator, and summing over $\hbar\omega_i/2$.

- In the path integral formulation, the saddle point evaluation gives a determinant of fluctuations around the minimum of the action. It leads to the same form of the zero-point energy. This corresponds to one-loop corrections, namely vacuum bubbles. Higher loop corrections appear from higher order terms in the action (neglected here).

- One can define zeta function regularization by generalizing the harmonic oscillator case via $\zeta_A(s) = \sum_{n\geq 1} 1/(a_n)^s$.

- The determinant of an operator is then $\det A = e^{-\zeta'_A(0)}$.

- We can define the heat kernel associated with an operator, that satisfies the generalized heat equation, with a boundary condition $G(x, y; \tau = 0) = \delta(x - y)$, which sometimes is easier to solve. Then we can find $\zeta_A(s)$ from the heat kernel.

- Fermions give $(\det A)^{+1/2}$ instead of $(\det A)^{-1/2}$, and a negative sign in front of the zero point energy, so they can in principle cancel against the bosonic corrections.

Further reading: See chapters 3.4, 3.5 in [3].

Exercises, Lecture 27

1) Consider two scalar fields with potential $(\lambda_1, \lambda_2 > 0)$

$$U(\phi, \chi) = \lambda_1(m_1\phi - \chi^2)^2 + \lambda_2(m_2\chi - \phi^2)^2 \tag{27.49}$$

Write down the infinite sum for the zero-point energy in 1+1 dimensions *for this model* (no need to calculate the sum) around the vacuum at $\phi = \chi = 0$.

2) Write down an expression for the zeta function regularization for the above sum, and for the heat kernel.

28 Lecture 28. One-loop divergences for scalars; power counting

In this lecture we will analyze possible divergences in loop integrals, in particular we will look at one-loop, and how to determine if a theory contains divergences (using power counting). In the final lecture we will understand how to regularize these divergences in order to calculate something finite.
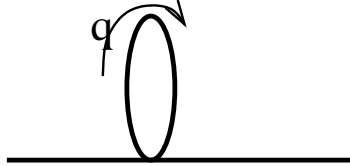

Figure 82: One-loop divergence in ϕ^4 theory for the 2-point function.

Up to now, we have only calculated explicitly tree processes, which are finite, and we have ignored the fact that loop integrals can be divergent. For example, in $\lambda\phi^4$ theory in Euclidean space, consider the unique one-loop $\mathcal{O}(\lambda)$ diagram, a loop connected to the free line (propagator) at a point, see Fig.82. It is

$$-\lambda \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \tag{28.1}$$

Since the integral is

$$\simeq -\lambda \frac{\Omega_{D-1}}{(2\pi)^D} \int q^{D-1} dq \frac{1}{q^2 + m^2} \sim \int dq \, q^{D-3} \tag{28.2}$$

it is divergent in $D \geq 2$ and convergent only for $D < 2$. In particular, in $D = 4$ it is quadratically divergent

$$\sim \int^\Lambda q dq \sim \Lambda^2 \tag{28.3}$$

We call this kind of divergence "ultraviolet", or UV divergence, from the fact that it is at large energies (4-momenta), or large frequencies.

Note also that we had one more type of divergence for loop integrals that was easily dealt with, the fact that when integrating over loop momenta in Minkowski space, the propagators can go on-shell, leading to a pole, which needed to be regulated. But the $i\epsilon$ prescription dealt with that. Otherwise, we can work in Euclidean space and then analytically continue to Minkowski space at the end of the calculation. This issue did not appear until now, because we only calculated tree level processes, when the propagators have fixed momenta, and are not on-shell.

Let us now consider a one-loop diagram for the $2k$-point function in $\lambda\phi^{k+2}$ theory with k momenta in, then two propagators forming a loop, and then k lines out, as in Fig.83. The
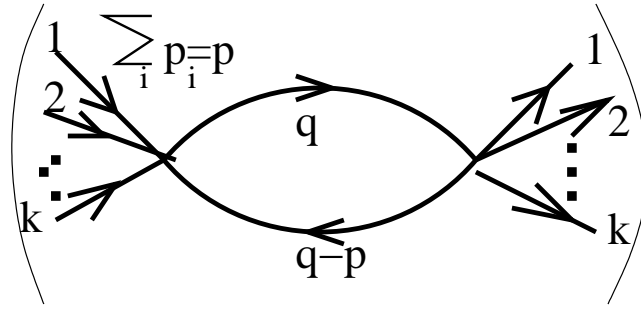
Figure 83: One-loop diagram in ϕ^{k+2} theory, for the 2k-point function.

incoming momenta are called $p_1, ..., p_k$ and sum to $p = \sum_{i=1}^{k} p_i$. Then the two propagators have momenta q (loop variable) and $q - p$, giving

$$\frac{\lambda^2}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)((q - p)^2 + m^2)} \tag{28.4}$$

Again, we can see that at large q, it behaves as

$$\int \frac{q^{D-1} dq}{q^4} \tag{28.5}$$

so is convergent only for $D < 4$. In particular, in $D = 4$ it is (log) divergent, and this again is an UV divergence. From this example we can see that various diagrams are divergent in various dimensions.

But this diagrams has also another type of divergence, namely at low q ($q \to 0$). This divergence appears only *if we have $m^2 = 0$ AND $p^2 = 0$*. Thus only if we have massless particles, and all the particles that are incoming on the same vertex sum up to something on-shell (in general, the sum of on-shell momenta is not on-shell). Then the integral is

$$\sim \int d\Omega \int \frac{q^3 dq}{q^2(q^2 - 2q \cdot p)} \tag{28.6}$$

and in the integral over angles, there will be a point where the unit vector on q, $\hat{q}$, satisfies $\hat{q} \cdot \hat{p} = 0$ with respect to the (constant) unit vector on p, $\hat{p}$. Then we obtain

$$\int \frac{dq}{q} \tag{28.7}$$

i.e., log divergent. We call this kind of divergences "infrared" or IR divergences, since they occur at low energies (low 4-momenta), i.e. low frequencies.

Thus we have two kinds of potential divergences, UV and IR divergences. The UV divergences are an artifact of perturbation theory, i.e. of the fact that we were forced to introduce asymptotic states as states of the free theory, and calculate using Feynman

diagrams. As such, they can be removed by redefining the parameters of the theory (like masses, couplings, etc.), a process known as *renormalization*, which will be studied next semester. A nonperturbative definition is not in general available, in particular for scattering processes it isn't. But for things like masses and couplings of bound states (like the proton mass in QCD, for instance), one can define the theory nonperturbatively, for instance on the lattice, and then we always obtain finite results. The infinities of perturbation theory manifest themselves only in something called the renormalization group, which will also be studied next semester.

By contrast, the IR divergences are genuine divergences from the point of view of the Feynman diagram (can't be reabsorbed by redefining the parameters). But they arise because the Feynman diagram we are interested in, in the case of a theory with massless external states, and with external states that are on-shell at the vertex, are not quantities that can be experimentally measured. Indeed, for a massless external state ($m = 0$), of energy E, experimentally we cannot distinguish between the process with this external state, or with it and another emitted "soft and/or collinear particle", namely one of $m = 0$ and $E \simeq 0$ and/or parallel to the first. If we include the tree level process for that second process, and sum it together with the first (loop level), we obtain a finite differential cross section (which can be experimentally measured), for a given cut-off in energy and/or angle of resolution between two particles.

Thus the physical processes are always finite, in spite of the infinities in the Feynman diagram.

Analytical continuation

A question which one could have already asked is: Is Wick rotation of the final result the same with Wick rotation of the integral to Minkowski space, followed by evaluation?

Let us look at the simplest one-loop diagram in Euclidean space (in $\lambda\phi^4$, already discussed above)

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \tag{28.8}$$

In Minkowski space it becomes

$$-i \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2 - i\epsilon} = +i \int \frac{d^{D-1} q}{(2\pi)^{D-1}} \int \frac{dq_0}{2\pi} \frac{1}{q_0^2 - \vec{q}^2 - m^2 + i\epsilon} \tag{28.9}$$

where now $q^2 = -q_0^2 + \vec{q}^2$.

Then the poles are at $\tilde{q}_0 - i\epsilon$ and $-\tilde{q}_0 + i\epsilon$, where $\tilde{q}_0 = \sqrt{\vec{q}^2 + m^2}$. The Minkowski space integration contour is along the real axis in the q_0 plane, in the increasing direction, called C_R. On the other hand, the Euclidean space integration contour C_I is along the imaginary axis, in the increasing direction, see Fig.84. As there are no poles in betweeen C_R and C_I (in the quadrants I and III of the complex plane, the poles are in the quadrants II and IV), the integral along C_I is equal to the integral along C_R (since we can close the contour at infinity, with no poles inside). Therefore, along C_I, $q_0 = iq_D$, with q_D real and increasing, and therefore $dq_0 = idq_D$, so

$$\int_{C_R} dq_0(...) = \int_{C_I} dq_0(...) = \int \frac{d^{D-1} q}{(2\pi)^D} (-i) i \int \frac{dq_D}{\vec{q}^2 + (q_D)^2 + m^2 - i\epsilon} \tag{28.10}$$
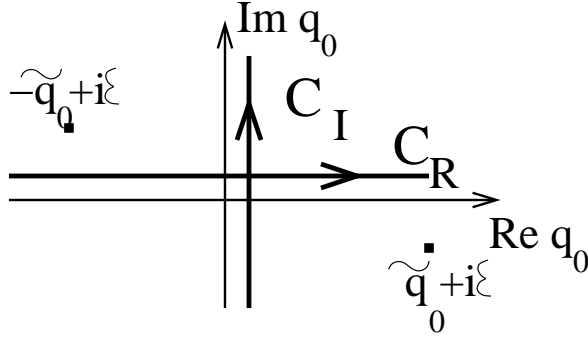
Figure 84: Wick rotation of the integration contour.

which gives the same result as the Euclidean space integral, after we drop the (now unnecessary) $-i\epsilon$.

However, in general it is not true that we can easily analytically continue. Instead, we must *define the Euclidean space integral and Wick rotate the final result*, since in general this will be seemingly different than the continuation of the Minkowski space integral (rather, it means that the Wick rotation of the integrals is subtle). But the quantum field theory perturbation in Euclidean space is well-defined, unlike the Minkowski space one, as we already saw, so is a good starting point.

Let's see an example of this situation. Consider the second integral we analyzed, now in Minkowski space

$$-\frac{\lambda^2}{2}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^2+m^2-i\epsilon}\frac{1}{(q-p)^2+m^2-i\epsilon} \tag{28.11}$$

We deal with the two different propagators in the loop integral using the Feynman trick. We will study it in more detail next class, but this time we will just use the result for two propagators. The Feynman trick for this case is the observation that

$$\frac{1}{AB}=\int_0^1 dx[xA+(1-x)B]^{-2} \tag{28.12}$$

which allows one to turn the two propagators with different momenta into a single propagator squared. Indeed, now we can write

$$-\frac{\lambda^2}{2}\int\frac{d^Dq}{(2\pi)^D}\int_0^1 dx[x(q-p)^2+(1-x)q^2+(x+1-x)(m^2-i\epsilon)]^{-2} \tag{28.13}$$

The square bracket equals $q^2+xp^2-2xq\cdot p+m^2-i\epsilon$. Changing variables to $q'^\mu=q^\mu-xp^\mu$ allows us to get rid of the term linear in q. We can change the integration variable to q', since the Jacobian for the transformation is 1, and then the square bracket becomes $q'^2+x(1-x)p^2+m^2-i\epsilon$. Finally, the integral is

$$\frac{(-i\lambda)^2}{2}\int\frac{d^Dq}{(2\pi)^D}\int_0^1 dx[q'^2+x(1-x)p^2+m^2-i\epsilon]^{-2} \tag{28.14}$$

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which has poles at

$$\tilde{q}_0^2 = \vec{q}^2 + m^2 - i\epsilon + x(1-x)p^2 \tag{28.15}$$

If $p^2 > 0$, this is the same as the in the previous example, we just redefine m^2: the poles are outside quadrants I and III, so we can make the Wick rotation of the integral without problem. However, if $p^2 < 0$ and sufficiently large in absolute value, we can have $q_0^2 < 0$, so the poles are now in quadrants I and III, and we cannot simply rotate the contour C_R to the contour C_I, since we encounter poles along the way. So in this case, the Wick rotation is more subtle: apparently, the Minkowski space integral gives a different result from the Euclidean space result, Wick rotated. However, the latter is better defined, so we can use it.

Power counting

We now want to understand how we can figure out if a diagram, and more generally a theory, contains UV divergences. We do this by *power counting*. We consider here scalar $\lambda_n \phi^n$ theories.

Consider first just a (Euclidean space) diagram, with L loops and E external lines, and I internal lines and V vertices. The loop integral will be

$$I_D(p_1, ..., p_E; m) = \int \prod_{\alpha=1}^{L} \frac{d^d q_\alpha}{(2\pi)^d} \prod_{j=1}^{I} \frac{1}{q_j^2 + m^2} \tag{28.16}$$

where $q_j = q_j(p_i, q_\alpha)$ are the momenta of the internal lines (which have propagators $1/(q_j^2 + m^2)$). More precisely, they are linear combinations of the loop momenta and external momenta,

$$q_j = \sum_{\alpha=1}^{L} c_{j\alpha} q_\alpha + \sum_{i=1}^{E} c_{ji} p_i \tag{28.17}$$

Like we already mentioned in Lecture 11, $L = I - V + 1$, since there are I momentum variables, constrained by V delta functions (one at each vertex), but one of the delta functions is the overall (external) momentum conservation.

If we scale the momenta and masses by the same multiplicative factor t, we can also change the integration variables (loop momenta q_α) by the same factor t, getting $\prod_{\alpha=1}^{L} d^d q \to t^{LD} \prod_{\alpha=1}^{L} d^d q$, as well as $q_j \to t q_j$, and $q^2 + m^2 \to t^2(q^2 + m^2)$, giving finally

$$I_D(t p_i; t m) = t^{\omega(D)} I_D(p_i; m) \tag{28.18}$$

where

$$\omega(D) = dL - 2I \tag{28.19}$$

is called the *superficial degree of divergence* of the diagram D, since it is the overall dimension for the scaling above.

Theorem This gives rise to the following theorem: $\omega(D) < 0$ is *necessary* for the convergence of I_D. (Note: but is not sufficient!)

Proof: We have

$$\prod_{i=1}^{I} (q_i^2 + m^2) \leq (\sum_{i=1}^{I} q_i^2 + m^2)^I \tag{28.20}$$

Then for large enough q_α, there is a constant C such that

$$\sum_{i=1}^{I}(q_i^2 + m^2) = \sum_{i=1}^{I}\left[\left(\sum_{\alpha=1}^{L} c_{i\alpha}q_\alpha + \sum_{j=1}^{E} c_{ij}p_j\right)^2 + m^2\right] \leq C\sum_{\alpha=1}^{L} q_\alpha^2 \qquad (28.21)$$

as we can easily see. Then we have

$$I_D > \frac{1}{C^I}\int_{\sum q_\alpha^2 > \Lambda^2}\prod_{\alpha=1}^{L}\frac{d^dq}{(2\pi)^d}\frac{1}{(\sum_{\alpha=1}^{L} q_\alpha^2)^I} > \int_{r > \Lambda}\frac{r^{DL-1}dr}{r^{2I}} \qquad (28.22)$$

where we used the fact that $\sum_{\alpha=1}^{L} q_\alpha^2 \equiv \sum_{M=1}^{dL} q_M^2$ is a sum of dL terms, which we can consider as a dL-dimensional space, and the condition $\sum_\alpha q_\alpha^2 > \Lambda^2$, stated before as q_α being large enough, now becomes the fact that the modulus of the dL-dimensional q_M is bounded from below. We finally see that if $\omega(D) = dL - 2I > 0$, I_D is divergent. The opposite statement is that if I_D is convergent, then $\omega(D) < 0$, i.e. $\omega(D) < 0$ is a necessary condition for convergence. q.e.d.

As we said, the condition is necessary, but not sufficient. Indeed, we can have subdiagrams that are superficially divergent ($\omega(D_s) \geq 0$), therefore divergent, then the full diagram is also divergent, in spite of having $\omega(D) < 0$.
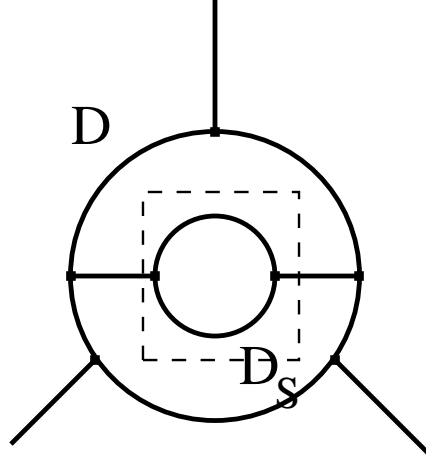

Figure 85: Power counting example: diagram is power counting convergent, but subdiagram is actually divergent.

We can take an example in $\lambda\phi^3$ theory in $D = 4$ the one in Fig.85: a circle with 3 external lines connected with it, with a subdiagram D_s connected to the inside of the circle: a propagator line that has a loop made out of two propagators in the middle. The diagram has $I_D = 9, V_D = 7$, therefore $L_D = I_D - V_D + 1 = 9 - 7 + 1 = 3$, and then $\omega(D) = dL_D - 2I_D = 4 \cdot 3 - 2 \cdot 9 = -6 < 0$. However, the subdiagram has $I_{D_s} = 2, V_{D_s} = 2$, therefore

$L_{D_s} = 2 - 2 + 1 = 1$, and then $\omega(D_s) = 4 \cdot 1 - 2 \cdot 2 = 0$, therefore we have a logarithmically divergent subdiagram, and therefore the full diagram is also logarithmically divergent.

We can then guess that we have the following

Theorem (which we will not prove here) $\omega(D_s) < 0 \forall D_s$ 1PI subdiagrams of $D \Leftrightarrow I_D(p_1, ..., p_E)$ is an absolutely convergent integral.

We note that the $\Rightarrow$ implication should obviously be true, and moreover is valid for any field theory. But the $\Leftarrow$ implication is true only for scalar theories. If there are spin 1/2 and spin 1 fields, then $\omega(D) < 0$ is not even necessary, since there can be cancellations between different spins, giving a zero result for a superficially divergent diagram (hence the name *superficial* degree of divergence, it is not necessarily the *actual* degree of divergence).

We can now write a formula from which we derive a condition on the type of divergencies we can have.

We note that each internal line connects to two vertices, and each external line connects only one vertex. In a theory with $\sum_n \lambda_n \phi^n/n!$ interactions, we can have a number $n = n_v$ of legs at each vertex v, meaning we have

$$2I + E = \sum_{v=1}^{V} n_v \qquad (28.23)$$

Then the superficial degree of divergence is

$$\omega(D) \equiv dL - 2I = (d-2)I - dV + d = d - \frac{d-2}{2}E + \sum_{v=1}^{V} \left(\frac{d-2}{2} n_v - d \right) \qquad (28.24)$$

where in the second equality we used $L = I - V + 1$ and in the last equality we have used $2I + E = \sum_v n_v$.

Since the kinetic term for a scalar is $- \int d^d x (\partial_\mu \phi)^2/2$, and it has to be dimensionless, we need the dimension of ϕ to be $[\phi] = (d-2)/2$. Then since the interaction term is $- \int d^d x \lambda_n \phi^n/n!$, we have

$$[\lambda_{n_v}] = d - n_v[\phi] = d - \frac{d-2}{2} n_v \qquad (28.25)$$

meaning that finally

$$\omega(D) = d - \frac{d-2}{2}E - \sum_{v=1}^{V} [\lambda_v] \qquad (28.26)$$

Thus we find that if $[\lambda_v] \geq 0$, there are only a finite number (and very small) of divergent diagrams. Indeed, first, we note that by increasing the number of external lines, we get to $\omega(D) < 0$. For instance, consider the limiting case of $[\lambda_v] = 0$, and $d = 4$. Then $\omega(D) = 4 - E$, and only $E = 0, 1, 2, 3, 4$ give divergent results, irrespective of V. Since $E = 0, 1$ are not physical ($E = 0$ is a vacuum bubble, and $E = 1$ should be zero in a good theory), we have only $E = 2, 3, 4$, corresponding to 3 physical parameters (physical parameters are defined by the number of external lines, which define physical objects like n-point functions). For $[\lambda_v] > 0$, any vertices lower $\omega(D)$, so we could have even a smaller number of E's for divergent n-point functions, since we need at least a vertex for a loop

diagram. In higher dimensions, we will have a slightly higher number of divergent n-point functions, but otherwise the same idea applies.

Such theories, where there are a finite number of n-point functions that have divergent diagrams, is called *renormalizable*, since we can absorb the infinities in the redefinition of the (finite number of) parameters of the theory.

By contrast, if $[\lambda_v] > 0$, we can make divergent diagrams for any n-point function (any E) just by increasing V. Therefore there are an infinite number of divergent n-point functions that would need redefinition, so we can't make this by redefining the parameters of the theory. Such a theory is called *nonrenormalizable*. Note that a nonrenormalizable theory can be so only in perturbation theory, there exist examples of theories that are perturbatively nonrenormalizable, but the nonperturbative theory is well-defined. Also note that we can work with nonrenormalizable theories in perturbation theory, just by introducing new parameters at each loop order. Therefore we can compute quantum corrections, though the degree of complexity of the theory quickly increases with the loop order.

Important concepts to remember

- Loop diagrams can contain UV divergence (at high momenta), divergent in diagram-dependent dimensions, and IR divergences, which appear only for massless theories and for on-shell total external momenta at vertices.

- UV divergences can be absorbed in a redefinition of the parameters of the theory (renormalization), and IR divergences can be cancelled by adding the tree diagrams for emission of low momentum ($E \simeq 0$) particles, perhaps parallel to the original external particle.

- Wick rotation of the result of the Euclidean integrals can in general not be the same as Wick rotation of the Euclidean integral, since there can be poles in between the Minkowskian and Euclidean contours for the loop energy integration. We can work in Euclidean space and continue the final result, since the Euclidean theory is better defined.

- Power counting gives the superficial degree of divergence of a diagram as $\omega(D) = dL - 2I$.

- In a scalar theory, $\omega(D) < 0$ is necessary for convergence of the integral I_D, but in general not sufficient.

- In a scalar theory, $\omega(D_s) < 0$ for any 1PI subdiagram D_s of a diagram $D \Leftrightarrow I_D$ is absolutely convergent.

- Theories with couplings satisfying $[\lambda_v] \geq 0$ are renormalizable, i.e. one can absorb the infinities in redefinitions of the parameters of the theory, while theories with $[\lambda_v] < 0$ are nonrenormalizable, since we can't (there are an infinite number of different infinities to be absorbed).

Further reading: See chapters 5.1,5.2 in [4], 9.1 in [1] and 4.2 in [3].

Exercises, Lecture 28

1) Consider the one-loop diagram for arbitrary masses of the various lines (see class). Check whether there are any divergences.
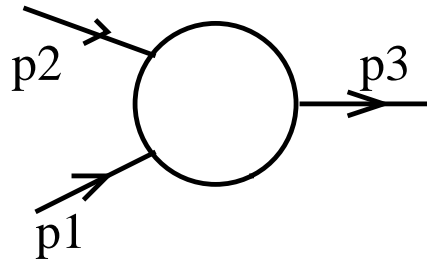

Figure 86: One loop Feynman diagram. Check for divergences.

2) Check whether there are any UV divergences in the $D = 3$ diagram in $\lambda\phi^4$ theory in Fig.87.
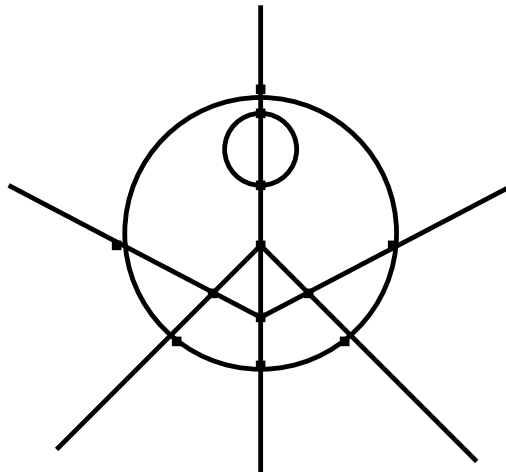

Figure 87: Check for UV divergences in this Feynman diagram.

29 Lecture 29.   Regularization, definitions: cut-off, Pauli-Villars, dimensional regularization

In this lecture we will preview methods of regularization, that will be used in the second semester for renormalization. We already saw a few methods of *regularization*, i.e. making finite the integrals.

The simplest is *cut-off regularization*, which means just putting upper and lower bounds on the integral over the modulus of the momenta, i.e. a $|p|_{max} = \Lambda$ for the UV divergence, and an $|p|_{min} = \epsilon$ for the IR divergence. It has to be over the modulus only (the integral over angles is not divergent), and then the procedure works best in Euclidean space (since then we don't need to worry about the fact that $-(p_0)^2 + \vec{p}^2 = \Lambda^2$ has a continuum of solutions for Minkowski space). Note that having a $|p|_{max} = \Lambda$ is more or less the same as considering a lattice of size Λ^{-1} in Euclidean space, which breaks Euclidean ("Lorentz") invariance (since translational and rotational invariance are thus broken). For this reason, very seldom we consider the cut-off regularization.

There are many regularizations possible, and in general we want to consider a regularization that preserves all symmetries that play an important role at the quantum level. If there are several that preserve the symmetries we want, all of them can in principle be used (we could even consider cut-off regularization, just that then we would have a hard time showing that our results are consistent with the symmetries we want to preserve).
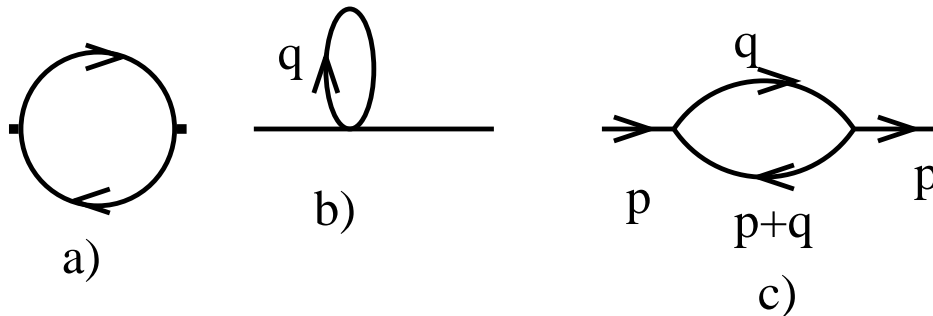

Figure 88: Examples of one-loop divergent diagrams.

Let's see the effect of cut-off regularization on the simplest diagram we can write, a loop for a massless field, with no external momentum, but two external points, i.e. two equal loop propagators inside (see Fig.88a):

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2)^2} = \frac{\Omega_3}{(2\pi)^4} \int_\epsilon^\Lambda \frac{p^3 dp}{p^4} = \frac{1}{8\pi^2} \ln \frac{\Lambda}{\epsilon} \tag{29.1}$$

As we said, we see that this has both UV and IR divergences.

We also already saw that sometimes, we can turn integrals into infinite sums, like in the case of the zero point energy, where a first regularization, putting the system in a box of

size L, allowed us to get a sum instead of an integral: $\sum_n \hbar\omega_n/2$. The sum however was still divergent. Then there were various ways of regularizing the sum. We can deduce that if the result depends on the method of regularization, there are two possibilities:

- perhaps the result is unphysical (can't be measured). This is what happened to the *full* zero-point energy. Certainly the infinite constant (which in $e^{-a\omega_n}$ regularization was something depending only on L) was not measurable.

- perhaps some of the methods of regularization are unphysical, so we have to choose the physical one.

An example of a physical calculation is the *difference* of two infinite sums. We saw the example of the Casimir effect, i.e. the difference between two geometries (two values of d). In general,

$$\sum_n \frac{\hbar\omega_n^{(1)}}{2} - \sum_n \frac{\hbar\omega_n^{(2)}}{2} \tag{29.2}$$

Another physical case of this type arises when we consider quantum corrections to masses of solitons. Then we can consider the difference between quantum fluctuations ($\sum_n \hbar\omega_n/2$) in the presence of the soliton, and in the vacuum (without the soliton), and this would give the physical calculation of the quantum correction to the mass of the soliton.

Note that it would seem that we could write

$$\sum_n \frac{\hbar\omega_n^{(1)} - \hbar\omega_n^{(2)}}{2} \tag{29.3}$$

and calculate this, but this amounts to a choice of regularization, called mode number (n) regularization. Indeed, we have now $\infty - \infty$, so unlike the case of finite integrals, now giving the $\sum_n$ operator as a common factor is only possible if we choose the *same number N as the upper bound* in both sums (if one is N, and the other $N + a$ say, with $a \sim \mathcal{O}(1)$, then obviously we obtain a different result for the difference of two sums).

This may seem natural, but there is more than one other way to calculate: for instance, we can turn the sums into integrals in the usual way, and then take the *same upper limit in the integral (i.e., in energy)*, obtaining *energy/momentum cut-off regularization*. The difference of the two integrals gives a result differing from mode number cut-off by a finite piece.

For $\sum_n \hbar\omega_n/2$, we saw other regularizations as well: zeta function regularization, heat kernel regularization, and $\sum_n \omega_n \to \sum_n \omega_n e^{-a\omega_n}$.

Returning to the loop integrals appearing in Feynman diagrams, we have other ways to regulate. One of the oldest ways used is *Pauli-Villars regularization*, and its generalizations. These fall under the category of modifications to the propagator that cut off the high momentum modes in a smoother way than the hard cut-off $|p|_{max} = \Lambda$. The generalized case corresponds to making the substitution

$$\frac{1}{q^2 + m^2} \to \frac{1}{q^2 + m^2} - \sum_{i=1}^{N} \frac{c_i(\Lambda; m^2)}{q^2 + \Lambda_i^2} \tag{29.4}$$

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and we can adjust the c_i such that at large momentum q, the redefined propagator behaves as

$$\sim \frac{\Lambda^{2N}}{q^{2N+2}} \tag{29.5}$$

In other words, if in a loop integral, we need to have the propagator at large momentum behave as $1/q^{2N+2}$ in order for the integral to become finite, we choose a redefinition with the desired N, and with c_i's chosen for the required high momentum behaviour.

In particular, the original Pauli-Villars regularization is

$$\frac{1}{q^2+m^2} \to \frac{1}{q^2+m^2} - \frac{1}{q^2+\Lambda^2} = \frac{\Lambda^2-m^2}{(q^2+m^2)(q^2+\Lambda^2)} \sim \frac{\Lambda^2}{q^4} \tag{29.6}$$

We can easily see that it cannot be obtained from a normal modification of the action, because of the minus sign, however it corresponds to subtracting the contribution of a very heavy particle. Indeed, physically it is clear that heavy particle cannot modify anything physical (for instance, a Planck mass particle cannot influence Standard Model phyics). But it is equally obvious that subtracting its contribution will cancel heavy momentum modes in the loop integral, cancelling the unphysical infinities of the loop.

However, there is a simple modification that has the same result as the above Pauli-Villars subtraction at high momentum, and has a simple physical interpretation as the effect of a higher derivative term in the action. Specifically, consider the replacement of the propagator

$$\frac{1}{q^2+m^2} \to \frac{1}{q^2+m^2+q^4/\Lambda^2} \tag{29.7}$$

The usual propagator comes from

$$\int d^4x \frac{1}{2}(\partial_\mu \phi)^2 = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2}\phi(p)p^2\phi(-p) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{2}\phi(p)\Delta^{-1}(p)\phi(-p) \tag{29.8}$$

so the above is obtained by adding to the action a higher derivative term:

$$\int d^4x \frac{1}{2}\left[(\partial_\mu \phi)^2 + \frac{(\partial^2\phi)^2}{\Lambda^2}\right] = \int \frac{d^4p}{(2\pi)^4}\phi(p)\left[p^2 + \frac{(p^2)^2}{\Lambda^2}\right]\phi(-p) \tag{29.9}$$

Now consider a non-Pauli-Villars, but similar modification of the the loop integral, that is strictly speaking not a modification of the propagator, but of its square. Consider the same simplest loop integral, with two equal propagators, i.e.

$$I = \int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^2} \to \int \frac{d^4p}{(2\pi)^4}\left[\left(\frac{1}{p^2+m^2}\right)^2 - \left(\frac{1}{p^2+\Lambda^2}\right)^2\right] = I(m^2) - I(\Lambda^2) \tag{29.10}$$

The new object in the square brackets is

$$\frac{2p^2(\Lambda^2-m^2)+\Lambda^4-m^4}{(p^2+m^2)^2(p^2+\Lambda^2)^2} \sim \frac{2\Lambda^2}{p^6} \tag{29.11}$$

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so is now UV convergent. Since the object is UV convergent, we can use any method to calculate it. In particular, we can take a derivative $\partial/\partial m^2$ of it, and since $I(\Lambda^2)$ doesn't contribute, we get for the integral

$$\frac{\partial}{\partial m^2}[I(m^2) - I(\Lambda^2)] = \int \frac{d^4p}{(2\pi)^4} \frac{-2}{(p^2 + m^2)^3} = -2\frac{\Omega_3}{(2\pi)^4} \int_0^\infty \frac{p^3 dp}{(p^2 + m^2)^3} \qquad (29.12)$$

where $\Omega_3 = 2\pi^2$ is the volume of the 3 dimensional unit sphere. Considering $p^2 + m^2 = x$, so $p^3 dp = (x - m^2)dx/2$, we get

$$\frac{\partial}{\partial m^2} I(m^2, \Lambda^2) = -\frac{1}{8\pi^2} \int_{m^2}^\infty \frac{(x - m^2)dx}{x^3} = -\frac{1}{8\pi^2} \frac{1}{2m^2} \qquad (29.13)$$

Integrating this, we obtain $I(m^2)$, then

$$I(m^2, \Lambda^2) = I(m^2) - I(\Lambda^2) = \frac{1}{16\pi^2} \ln \frac{\Lambda^2}{m^2} \qquad (29.14)$$

This object is UV divergent, as $\Lambda \to \infty$, and also divergent as $m \to 0$ (IR divergent).

However, note that in the way we calculated, we really introduced another type of regularization. It was implicit, since we first found a finite result by subtracting the contribution with $m \to \Lambda$, and then calculated this finite result using what was a simple trick.

However, if we keep the original integral and do the same derivative on it, after the derivative we obtain a finite result,

$$\frac{\partial}{\partial m^2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^2} = -2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^3} = -\frac{1}{16\pi^2 m^2} = \text{finite.} \qquad (29.15)$$

and now the integral (and its result) is UV convergent, despite the integral before the derivative being UV divergent. Hence the derivative with respect to the parameter m^2 is indeed a regularization. Both the initial and final results are however still IR divergent as $m^2 \to 0$. Now integrating, we obtain

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^2} = -\frac{1}{16\pi^2} \ln \frac{m^2}{\epsilon} \qquad (29.16)$$

which is still IR divergent as $\epsilon \to 0$.

However, all the regularizations we analyzed until now don't respect a very important invariance, namely gauge invariance. Therefore, 't Hooft and Veltman introduced a new regularization to deal with the spontaneously broken gauge theories, namely *dimensional regularization*, rather late (early seventies), since it is a rather strange concept.

Dimensional regularization

Dimensional regularization means that we analytically continue in D, the dimension of spacetime. This seems like a strange thing to do, given that the dimension of spacetime is an integer, so it is not clear what can physically mean a real dimension, but we nevertheless choose $d = 4 \pm \epsilon$. The sign has some significance as well, but here we will just consider $d = 4 + \epsilon$.

We already saw a way to continue a result defined for integers to a result for real numbers, in the case of the Riemann zeta function: defined initially for integers, but extended for complex numbers by writing an integral formula for it. A more relevant example for us is the case of the Euler gamma function, which is an extension of the factorial, $n!$, defined for integers, to the complex plane. Again this is done by writing an integral formula,

$$\Gamma(z) = \int_0^\infty d\alpha \, \alpha^{z-1} e^{-\alpha} \tag{29.17}$$

Indeed, one easily shows that $\Gamma(n) = (n-1)!$, for $n \in \mathbb{N}_*$, but the integral formula can be extended to the complex plane, defining the Euler gamma function. The gamma function satisfies

$$z\Gamma(z) = \Gamma(z+1), \tag{29.18}$$

an extension of the factorial property. But that means that we can find the behaviour at $z = \epsilon \to 0$, which is a simple pole, since

$$\epsilon\Gamma(\epsilon) = \Gamma(1+\epsilon) \simeq \Gamma(1) = 1 \Rightarrow \Gamma(\epsilon) \simeq \frac{1}{\epsilon} \tag{29.19}$$

We can repeat this process,

$$(-1+\epsilon)\Gamma(-1+\epsilon) = \Gamma(\epsilon) \Rightarrow \Gamma(-1+\epsilon) \simeq -\frac{1}{\epsilon}$$
$$(-2+\epsilon)\Gamma(-2+\epsilon) = \Gamma(-1+\epsilon) \Rightarrow \Gamma(-2+\epsilon) \simeq \frac{1}{2\epsilon} \tag{29.20}$$

etc. We see then that the gamma function has simple poles at all $\Gamma(-n)$, with $n \in \mathbb{N}$. In fact, these poles are exactly the one we obtain in dimensional regularization, as we now show.

Consider first the simplest case, the tadpole diagram, with a single loop, of momentum q, connected at a point to a propagator line, as in Fig.88b:

$$I = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \tag{29.21}$$

We now write

$$\frac{1}{q^2 + m^2} = \int_0^\infty d\alpha \, e^{-\alpha(q^2+m^2)} \tag{29.22}$$

and then

$$\begin{aligned}
I &= \int_0^\infty d\alpha \, e^{-\alpha m^2} \int \frac{d^D q}{(2\pi)^D} e^{-\alpha q^2} = \int_0^\infty d\alpha \, e^{-\alpha m^2} \frac{\Omega_{D-1}}{(2\pi)^D} \int_0^\infty dq \, q^{D-1} e^{-\alpha q^2} \\
&= \int_0^\infty d\alpha \, e^{-\alpha m^2} \frac{\Omega_{D-1}}{(2\pi)^D} \frac{1}{2\alpha^{D/2}} \int_0^\infty dx \, x^{\frac{D}{2}-1} e^{-x} \tag{29.23}
\end{aligned}$$

and we use the fact that $\int_0^\infty dx \, x^{D/2-1} e^{-x} = \Gamma(D/2)$, and that the volume of the D-dimensional sphere is

$$\Omega_D = \frac{2\pi^{\frac{D+1}{2}}}{\Gamma\left(\frac{D+1}{2}\right)}, \tag{29.24}$$

which we can easily test on a few examples, $\Omega_1 = 2\pi/\Gamma(1) = 2\pi$, $\Omega_2 = 2\pi^{3/2}/\Gamma(3/2) = 2\pi^{3/2}/(\sqrt{\pi}/2) = 4\pi$, $\Omega_3 = 2\pi^2/\Gamma(2) = 2\pi^2$. Then we have $\Omega_{D-1}\Gamma(D/2) = 2\pi^{D/2}$, so

$$I = \int_0^\infty d\alpha \, e^{-\alpha m^2}(4\pi\alpha)^{-\frac{D}{2}} = \frac{(m^2)^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(1 - \frac{D}{2}\right) \qquad (29.25)$$

Taking derivatives $(\partial/\partial m^2)^{n-1}$ on both sides (both the definition of I and the result), we obtain in general

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)^n} = \frac{\Gamma(n - d/2)}{(4\pi)^n \Gamma(n)}\left(\frac{m^2}{4\pi}\right)^{\frac{D}{2}-n} \qquad (29.26)$$

We see that in $D = 4$, this formula has a pole at $n = 1, 2$, as expected from the integral form. In these cases, the divergent part is contained in the gamma function, namely $\Gamma(-1)$ and $\Gamma(0)$.

We now move to a more complicated integral, which we will solve with Feynman parametrization, cited last lecture. Specifically, we consider the diagram for a one-loop correction to the propagator in ϕ^3 theory, with momentum p on the propagator and q and $p + q$ in the loop, as in Fig.88c, i.e.

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \frac{1}{(q + p)^2 + m^2} \qquad (29.27)$$

We now prove the Feynman parametrization in this case of two propagators. We do the trick used in the first integral (tadpole) twice, obtaining

$$\frac{1}{\Delta_1 \Delta_2} = \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 e^{-(\alpha_1 \Delta_1 + \alpha_2 \Delta_2)} \qquad (29.28)$$

We then change variables in the integral as $\alpha_1 = t(1 - \alpha)$, $\alpha_2 = t\alpha$, with Jacobian $\left|\begin{pmatrix} 1-\alpha & \alpha \\ -t & t \end{pmatrix}\right| = t$, so

$$\frac{1}{\Delta_1 \Delta_2} = \int_0^1 d\alpha \int_0^\infty dt \, t e^{-t[(1-\alpha)\Delta_1 + \alpha\Delta_2]} = \int_0^1 d\alpha \frac{1}{[(1-\alpha)\Delta_1 + \alpha\Delta_2]^2} \qquad (29.29)$$

We finally want to write the square bracket as a new propagator, so we redefine $q^\mu = \tilde{q}^\mu - \alpha p^\mu$, obtaining

$$(1 - \alpha)\Delta_1 + \alpha\Delta_2 = (1 - \alpha)q^2 + \alpha(q + p)^2 + m^2 = \tilde{q}^2 + m^2 + \alpha(1 - \alpha)p^2 \qquad (29.30)$$

Finally, we obtain for the integral

$$I = \int_0^1 d\alpha \int \frac{d^D \tilde{q}}{(2\pi)^D} \frac{1}{[\tilde{q}^2 + (\alpha(1 - \alpha)p^2 + m^2)]^2} = \frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}} \int_0^1 d\alpha [\alpha(1 - \alpha)p^2 + m^2]^{\frac{D}{2}-1} \qquad (29.31)$$

and again we obtained the divergence as just an overall factor coming from the simple pole of the gamma function at $D = 4$.

The Feynman parametrization can in fact be generalized, to write instead of a product of propagators, a single "propagator" raised at the corresponding power, with integrals over parameters α left to do. Then we can use the formula (29.26) to calculate the momentum integral, and we are left with the integration over parameters. So the dimensional regularization procedure described above is general, and we see that the divergence always appears as the simple pole of the gamma function at $D = 4$.

So we saw that the loop integrals of the above type are OK to dimensionally continue, but is it OK to dimensionally continue the Lagrangeans?

For scalars, it is OK, but we have to be careful. The Lagrangean is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda_n}{n!}\phi^n \tag{29.32}$$

and since the action $S = \int d^D x$ must be dimensionless, the dimension of the scalar is $[\phi] = (D - 2)/2$ and thus the dimension of the coupling is $[\lambda_n] = D - n(D - 2)/2$. For instance, for $D = 4$ and $n = 4$, we have $[\lambda_4] = 0$, but for $D = 4 + \epsilon$, we have $[\lambda_4] = -\epsilon$. That means that outside $D = 4$, we must redefine the coupling with a factor μ^ϵ, where μ is some scale that appears dynamically. This process is called dynamical transmutation, but we will not say more about it here.

For higher spins however, we must be more careful. The number of components of a field depends on dimension, which is a subtle issue. We must then use dimensional continuation of various gamma matrix formulae

$$g^{\mu\nu}g_{\mu\nu} = D$$
$$\gamma_\mu \slashed{p} \gamma^\mu = (2 - D)\slashed{p} \tag{29.33}$$

etc. On the other hand, the gamma matrices still satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. But the dimension of the (spinor) representation of the Clifford algebra depends on dimension in an unusual way, $n = 2^{[D/2]}$, which means it is 2 dimensional in $D = 2, 3$ and 4 dimensional in $D = 4, 5$. That means that we cannot continue dimensionally n to $D = 4 + \epsilon$. Instead, we must still consider the gamma matrices as 4×4 even in $D = 4 + \epsilon$, and thus we still have

$$\text{Tr}[\gamma_\mu \gamma_\nu] = 4g_{\mu\nu} \tag{29.34}$$

This is not a problem, however there is another fact that is still a problem. The definition of γ_5 is

$$\gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{29.35}$$

and that cannot be easily dimensionally continued. Since chiral fermions, i.e. fermions that are eigenvalues of the chiral projectors $P_{L,R} = (1 \pm \gamma_5)/2$, appear in the Standard Model, we would need to be able to continue dimensionally chiral fermions. But that is very difficult to do.

Therefore we can say that there are no perfect regularization procedures, always there is something that does not work easily, but for particular cases we might prefer one or another.

Next semester we will see that the divergences that we have regularized in this lecture can be absorbed in a redefinition of the parameters of the theory, leaving only a finite piece giving quantum corrections. But for now, this is the only thing we will say.

Important concepts to remember

- We must regularize the infinities appearing in loop integrals, and the infinite sums.

- Cut-off regularization, imposing upper and lower limits on $|p|$ in Euclidean space, regulates integrals, but is not very used because it is related to breaking of Euclidean ("Lorentz") invariance, as well as breaking of gauge invariance.

- Often the difference of infinite sums is a physical observable, and then the result is regularization-dependent. In particular, we can have mode-number cut-off (giving the sum operator as a common factor), or energy cut-off (giving a resulting energy integral as a common factor). We must choose one that is more physical.

- The choice of regularization scheme for integrals is dictated by what symmetries we want to preserve. If several respect the wanted symmetries, they are equally good.

- (Generalized) Pauli-Villars regularization removes the contribution of high energy modes from the propagator, by subtracting the propagator a very massive particle from it. A related version for it is obtained from a term in the action which is higher derivative.

- By taking derivatives with respect to a parameter (e.g. m^2), we obtain derivative regularization, which also reduces the degree of divergence of integrals.

- Dimensional regularization respects gauge invariance, and it corresponds to analytically continuing the dimension, as $D = 4 + \epsilon$. It is based on the fact that we can continue $n!$ away from the integers to the Euler gamma function.

- In dimensional regularization, the divergences are the simple poles of the gamma function at $\Gamma(-n)$, and appear as a multiplicative $1/\epsilon$.

- For scalars, dimensional regularization of the action is OK, if we remember that couplings have extra mass dimensions away from $D = 4$. For higher spins, we must regulate the number of components, including things like $g^{\mu\nu}g_{\mu\nu}$ and gamma matrix identities.

- The dimension of gamma matrices away from $D = 4$ is still 4, so traces of gamma matrices still give a factor of 4, and the γ_5 cannot be continued away from $D = 4$, which means analytical continuation in dimension that involve chiral fermions are very hard.

' **Further reading:** See chapters 5.3 in [4] 4.3 in [3] and 9.3 in [1].

Exercises, Lecture 29

1) Calculate

$$\int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m_1^2} \frac{1}{p^2 + m_2^2} \qquad (29.36)$$

in Pauli-Villars regularization.

2) Calculate

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m_1^2)^2} \frac{1}{(q + p)^2 + m_2^2} \qquad (29.37)$$

in dimensional regularization (it is divergent in $D = 6$).

References

[1] George Sterman, "An introduction to Quantum Field Theory" [GS]

[2] M.E. Peskin and D.V. Schroeder, "An introduction to Quantum Field Theory" [PS]

[3] Pierre Ramond, "Field theory: A modern primer" [PR]

[4] Jan Ambjorn and Jens Lyng Petersen, "Quantum field theory", Niels Bohr Institute lecture notes, 1994. [NBI]

[5] P.A.M.Dirac, "Lectures in Quantum Mechanics"

[6] Thomas Banks, "Modern Quantum Field Theory".