$$= \int D\bar\psi D\psi \exp\left\{ \int dt \left[-\bar\psi D_F^{-1} \psi \right] + i \int dt \left[\bar\psi\eta + \bar\eta\psi \right] \right\}$$

$$\therefore \boxed{Z[\eta,\bar\eta] = Z[0,0] \exp\left\{ - \int ds\, d\tau\, \bar\eta(s) D_F(s,\tau) \eta(\tau) \right\}}$$ (eq. 139.1)

$$\boxed{D_F(s,\tau) = \left[-i\left(i\partial_t - \omega \right) \right]^{-1} = i \int \frac{dE}{2\pi} \frac{e^{-iE(s-\tau)}}{E-\omega+i\epsilon} = \Theta(s-\tau)\, e^{-i\omega(s-\tau)}}$$ (eq. 139.2)

Note que: (1) Temos apenas um polo, em $E = \omega - i\epsilon$

(2) Isso significa que se fizemos a integral no hemisfério superior (*Im E > 0*) ela dá zero, e somos forçados a fazer isso se ($s < \tau$), portanto a integral é zero para ($s < \tau$). No outro hemisfério (obrigatório se $s > \tau$) pegamos o polo e obtemos o resultado não nulo acima.

$$-iE(s-\tau) \sim \mathrm{Im}(E)(s-\tau)$$

(3) Basta substituir a última expressão em 139.1 para obter 138.3 (incluindo o limite de integração, que impõe $s > \tau$)

Passando para o espaço Euclideano: $t = -it_E$

$$Z_E[\eta,\bar\eta] = \int D\bar\psi D\psi \operatorname{Exp}\left\{ \int dt_E\, \bar\psi_E \left(-\partial_{t_E} - \omega \right) \psi_E + \int dt_E\, \bar\psi_E \eta_E + \bar\eta_E \psi_E \right\}$$

suprimindo o "E"

$$= Z[0,0] \operatorname{Exp}\left\{ \int d\tau\, ds\, \bar\eta(s) D(s,\tau) \eta(\tau) \right\}$$

$$\boxed{-S_E = -\bar\psi \cdot D^{-1}\psi + \bar\psi\cdot\eta + \bar\eta\cdot\psi}$$ (eq. 139.3)

$$\boxed{D[s,\tau] = \left(\partial_t + \omega \right)^{-1} = i \int \frac{dE}{2\pi} \frac{e^{-iE(s-\tau)}}{E + i\omega}}$$ (eq. 139.4)

E rodando de volta para Minkowski com $E_E = (-i + \epsilon)E$ (para evitar o polo), voltamos ao propagador em 139.2

Agora basta aumentar o número de coordenadas espaciais para obter uma teoria de campo. A lagrangeana (em Mink.) é:

$$\mathcal{L}_F^{(M)} = \bar\psi \left(i\partial\!\!\!/ - m \right) \psi$$

Passando para o Euclideano:

$$\mathcal{L}_F^{(E)} = \bar{\psi}_E(\not{\partial} + m)\psi_E$$

Tomando o cuidado de manter a álgebra de Clifford funcionando: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$(\gamma^0)^2 = 1 \qquad (\gamma^i)^2 = -1$$

$$\boxed{\gamma_E^4 = \gamma^0} \qquad \hookrightarrow (\gamma_E^i)^2 = 1 \quad \Rightarrow \quad \gamma_E^i = -i\gamma^i$$

$$\psi_E(x_E) = \psi(-it_E, \vec{x})$$

$$\rightarrow \bar{\psi} = \psi^\dagger \gamma^0 = \psi^\dagger \gamma^4$$

E, nesta representação: $\gamma_\mu = \gamma^\mu = (\gamma^\mu)^\dagger$

A função de partição obtida é:

$$\boxed{Z_F^{(0)}[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ -\int d^4x\, \bar{\psi}(\not{\partial} + m)\psi + \int d^4x (\bar{\eta}\psi + \bar{\psi}\eta) \right\}} = \quad \text{(eq. 140.1)}$$

$$= Z(0,0)\, e^{\bar{\eta}(\not{\partial}+m)^{-1}\eta}$$

Atenção para o índice Espinorial:

$$\int d^4x\, \bar{\eta}\psi = \int d^4x\, \bar{\eta}_\alpha(x)\psi_\alpha(x) \qquad \alpha = 1,2,3,4$$

$$\bar{\eta}(\not{\partial}+m)^{-1}\eta = \int d^4x\, d^4y\, \bar{\eta}_\alpha(x) (\not{\partial}+m)^{-1}_{\alpha\beta}\, \eta_\beta(y)$$

$$\boxed{S_F(x,y) = (\not{\partial}_x + m)^{-1} = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{-\not{p} + im}} \quad \text{(eq. 140.2)}$$

$$(\not{\partial}_x + m) S_F(x,y) = \delta^4(x-y) \quad \because \quad \not{\partial}_x e^{ipx} = i\not{p}\, e^{ipx}$$

Usaremos com frequência a seguinte relação: $\dfrac{1}{-\not{p} + im} = \dfrac{-\not{p} - im}{p^2 + m^2} = \dfrac{-\not{p}_{\alpha\beta} - im\,1_{\alpha\beta}}{p^2+m^2}$

$$\not{p}\not{p} = \gamma^\mu p_\mu \gamma^\nu p_\nu = 2g^{\mu\nu}p_\mu p_\nu - \underbrace{\gamma^\nu\gamma^\mu p_\mu p_\nu}_{\gamma^\mu\gamma^\nu p_\nu p_\mu} \Rightarrow 2p^2 = 2\not{p}\not{p}$$

$$\boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}}$$

Voltando para Minkwoski, obtemos:

$$S_F^{(E)}(x,y) = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{-\not{p}+im} = i\int \frac{d^4p}{(2\pi)^4} \frac{-\not{p} - im}{p^2+m^2} e^{ip\cdot(x-y)}$$

$$p_E^0 x_E^0 = -i p^0 \, i x^0 = p^0 x_0$$

$$t_E = i t \qquad p_E^0 = (-i+\epsilon) p^0 \qquad p_E^2 + m^2 = -p^2 + m^2 - i\epsilon$$

$$\not{p}^E = \gamma_E^0 p_E^0 + \gamma_E^i p^i = \gamma^0(-i+\epsilon)p^0 - i\gamma^i p^i = -i\gamma^0 p_0 - i\vec{\gamma}\vec{p} + \epsilon\gamma^0 p_0$$

$$S_F^{(M)}(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{i[p^0(x^0-y^0)+p^i(x^i-y^i)]} \frac{+i\gamma^0 p_0 + i\vec{\gamma}\vec{p} - im}{-p^2+m^2-i\epsilon} =$$

$p_0 \to -p_0$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

Teorema de Wick para Campos Fermiônicos

Como um exemplo, consideremos uma teoria com um escalar ϕ (com fonte J) e um férmion ψ, interagindo por meio de um termo $S_I[\bar{\psi},\psi,\phi]$, neste caso poderíamos escrever:

$$\boxed{Z[\bar{\eta},\eta,J] = e^{-S_I\left[-\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta J}\right]} Z_F^{(0)}[\bar{\eta},\eta]\, Z_\phi^{(0)}[J]}$$ (eq. 141.1)

que é obtida segundo exatamente o mesmo procedimento usado na pag 110. A única diferença está no termo $-\frac{\delta}{\delta\eta}$ que tem este sinal pois o termo de fonte tem a forma: $\bar{\eta}\psi + \bar{\psi}\eta$

Logo: $$\psi\, e^{\int d^4x\, \bar{\eta}\psi + \bar{\psi}\eta} = -\frac{\delta}{\delta\eta} e^{\int d^4x\, \bar{\eta}\psi+\bar{\psi}\eta}$$

O lema de Coleman (eq. 115.2) também ganha um sinal pelo mesmo motivo:

$$\boxed{F\left(-\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}}\right) Z[\bar{\eta},\eta] = Z\left[-\frac{\delta}{\delta\psi},\frac{\delta}{\delta\bar{\psi}}\right]\left(F(\bar{\psi},\psi)\, e^{\bar{\psi}\eta+\bar{\eta}\psi}\right)\Big|_{\bar{\psi}=\psi=0}}$$ (eq. 141.2)

$$Z[\bar{\eta},\eta,J] = e^{-S_I\left(-\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}},\frac{\delta}{\delta J}\right)} e^{\bar{\eta}S_F\eta} Z_\phi^{(0)}[J] =$$

(eq. 141.3)

$$\boxed{\begin{aligned} &= e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}} e^{-S_I\left(\bar{\psi},\psi,\frac{\delta}{\delta J}\right)+\bar{\psi}\cdot\eta+\bar{\eta}\cdot\psi} Z_\phi^{(0)}[J]\Big|_{\psi=\bar{\psi}=J=0} = \\ &= e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}} e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}} \left\{e^{-S_I(\bar{\psi},\psi,\phi)+\bar{\psi}\cdot\eta+\bar{\eta}\cdot\psi+J\cdot\phi}\right\}_{\psi=\bar{\psi}=\phi=0}\end{aligned}}$$

Regras de Feynman para Férmions (Interação de Yukawa)

$$\mathcal{L}_y = g\,\bar{\Psi}\Psi\phi$$

$$Z[\bar{\eta},\eta,J] = e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}}\, e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\left\{ e^{-g\int d^4x\, \bar{\Psi}\Psi\phi + \bar{\Psi}\cdot\eta + \bar{\eta}\cdot\psi + J\cdot\phi}\right\}_{\psi=\bar{\psi}=\phi=0}$$

Comecemos com a função de dois pontos livre (onde os dois pontos são aplicações do campo fermiônico):

$G^{(N)}_{(n,m)}$ — N: ordem na expansão perturbativa

m: numero de pontos externos bosônicos

n: numero de pontos externos fermiônicos

$$G^{(0)}_{(2,0)} = \langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \frac{\delta}{\delta\bar{\eta}(x)_\alpha}\left(-\frac{\delta}{\delta\eta(y)_\beta}\right) Z_0(\bar{\eta},\eta,J)\Big|_{\eta=\bar{\eta}=J=0}$$

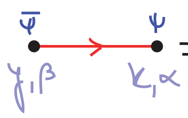
note que, trivialmente, temos $\langle 0|\psi\psi|0\rangle = \langle 0|\bar{\psi}\bar{\psi}|0\rangle = 0$

$$Z_0[\bar{\eta},\eta,J] = \underbrace{e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}}}\,\underbrace{e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}}_{\text{N CONTRIBUE}}\left\{e^{\bar{\psi}\cdot\eta+\bar{\eta}\cdot\psi+J\cdot\phi}\right\}_{\psi=\bar{\psi}=\phi=0} =$$

Só estou interessado na parte que contribui para $G^{(0)}_{(2,0)}$ ou seja com $\eta\cdot\bar{\eta}$

$$= -\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar{\psi}}\, e^{\bar{\psi}\eta+\bar{\eta}\cdot\psi} = \bar{\eta}\cdot S_F\cdot\eta\, e^{\bar{\psi}\eta+\bar{\eta}\cdot\psi}\Big|_{\bar{\psi}=\psi=0} = \bar{\eta}\cdot S_F\cdot\eta$$

$$G^{(0)}_{(2,0)} = \frac{\delta}{\delta\bar{\eta}(x)_\alpha}\left(-\frac{\delta}{\delta\eta(y)_\beta}\right)\bar{\eta}_\delta\cdot S_{F\,\delta\gamma}\cdot\eta_\gamma = \int d^4z_1\,d^4z_2\,\frac{\delta}{\delta\bar{\eta}(x)_\alpha}\bar{\eta}(z_1)_\delta\, S_F(z_1-z_2)_{\delta\gamma}\underbrace{\frac{\delta}{\delta\eta(y)_\beta}\eta(z_2)_\gamma}_{\delta^4(y-z_2)\,\delta_{\beta\gamma}} = S_F(x-y)_{\alpha\beta}$$



$$\boxed{= S_F(x-y)_{\alpha\beta} = i\int\frac{d^4p}{(2\pi)^4}\left(\frac{e^{ip(x-y)}}{-\not{p}+im}\right)_{\alpha\beta}}\qquad (\text{eq. } 142.1)$$

Note que agora o sinal do momento (ou a ordem de x e y) importa!

A regra para o vértice vem trivialmente da função de três pontos:

$$G^{(1)}_{(2,1)} = \frac{\delta}{\delta\bar{\eta}(x)_\alpha}\left(-\frac{\delta}{\delta\eta(y)_\beta}\right)\frac{\delta}{\delta J(y)}\, e^{-\frac{\delta}{\delta\psi}S_F\frac{\delta}{\delta\bar{\psi}}}\, e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\left\{e^{-g\int d^4w\,\bar{\Psi}\Psi\phi+\bar{\Psi}\cdot\eta+\bar{\eta}\cdot\psi+J\cdot\phi}\right\}_{\substack{\psi=\bar{\psi}=\phi=0\\ \eta=\bar{\eta}=J=0}} =$$

índice espinoriais subentendidos

$$= e^{-\frac{\delta}{\delta\psi} S_F \frac{\delta}{\delta\bar\psi}}\, e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}} \left\{ \phi(z)\,\psi_\alpha(x)\,\bar\psi_\beta(y)\, e^{\underbrace{-g\int d^4w\,\bar\psi\psi\phi}} \right\} =$$

só quero o termo O(g[1])

$$= e^{-\frac{\delta}{\delta\psi} S_F \frac{\delta}{\delta\bar\psi}}\, e^{\frac{1}{2}\frac{\delta}{\delta\phi}\cdot\Delta\cdot\frac{\delta}{\delta\phi}}\, (-g)\,\phi(z)\,\psi(x)\,\bar\psi(y)\int d^4w\, \bar\psi_\delta(w)\,\psi_\delta(w)\,\phi(w) =$$

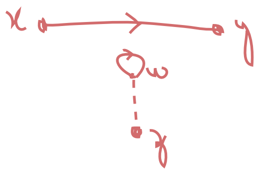
índice espinoriais subentendidos

$$= -g\int d^4w\;\Delta(z-w)\,\frac{1}{2}\left(-\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\right)\left(-\frac{\delta}{\delta\psi}\cdot S_F\,\frac{\delta}{\delta\bar\psi}\right)\psi(x)\,\bar\psi(y)\;\bar\psi(w)\,\psi(w) =$$

$$\left(-\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\right)\psi_\alpha(x)\,\bar\psi_\beta(y)\;\bar\psi_\delta(w)\,\psi_\delta(w) =$$

$$= \int d^4z_1\, d^4z_2 \left(-\frac{\delta}{\delta\psi_\xi(z_1)}\, S_F(z_1-z_2)_{\xi\chi}\right)\Big[-\psi_\alpha(x)\,\delta(z_2-y)\,\delta_{\chi\beta}\,\bar\psi_\delta(w)\,\psi_\delta(w) + \psi_\alpha(x)\,\bar\psi_\beta(y)\,\delta(w-z_2)\,\delta_{\chi\delta}\,\psi_\delta(w)\Big] =$$

Não quero o diagrama de bolha no vácuo:



$$= \int d^4z_1\, d^4z_2 \left(-S_F(z_1-z_2)_{\xi\chi}\right)\Big[-\psi_\alpha(x)\,\delta(z_2-y)\,\bar\psi_\delta(w)\,\delta(z_1-w)\,\delta_{\chi\beta}\,\delta_{\xi\delta} + \delta(x-z_1)\,\bar\psi_\beta(y)\,\delta(w-z_2)\,\psi_\delta(w)\,\delta_{\alpha\xi}\,\delta_{\chi\delta}\Big] + \text{BOLHAS}$$

$$= +S_F(w-y)_{\delta\beta}\,\psi_\alpha(x)\,\bar\psi_\delta(w) - S_F(x-w)_{\alpha\delta}\,\bar\psi_\beta(y)\,\psi_\delta(w) + \text{BOLHAS}$$

$$= -g\int d^4w\;\Delta(z-w)\,\frac{1}{2}\int d^4z_3\, d^4z_4 \left(-\frac{\delta}{\delta\psi(z_3)}\cdot S_F(z_3-z_4)\cdot\frac{\delta}{\delta\bar\psi(z_4)}\right)\Big[+S_F(w-y)\,\psi(x)\,\bar\psi(w) - S_F(x-w)\,\bar\psi(y)\,\psi(w)\Big] =$$

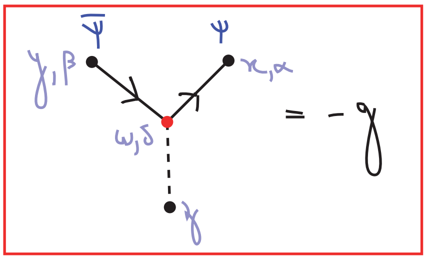
$$= -g\int d^4w\;\Delta(z-w)\,\frac{1}{2}\int d^4z_3\, d^4z_4 \left(-\frac{\delta}{\delta\psi(z_3)}\cdot S_F(z_3-z_4)\right)\Big[-S_F(w-y)\,\psi(x)\,\delta(w-z_4) - S_F(x-w)\,\delta(y-z_4)\,\psi(w)\Big] =$$

$$= -g\int d^4w\;\Delta(z-w)\,\frac{1}{2}\int d^4z_3\, d^4z_4 \left(-S_F(z_3-z_4)\right)\Big[-S_F(w-y)\,\delta(x-z_3)\,\delta(w-z_4) - S_F(x-w)\,\delta(y-z_4)\,\delta(w-z_3)\Big] =$$

$$= -g\int d^4w\;\Delta(z-w)\,\frac{1}{2}\Big[S_F(x-w)_{\alpha\delta}\,S_F(w-y)_{\delta\beta} + S_F(w-y)_{\delta\beta}\,S_F(x-w)_{\alpha\delta}\Big] =$$

$$G^{(1)}_{(2,1)} = -g\int d^4w\; \Delta(z-w)\, S_F(x-w)_{\alpha\delta}\, S_F(w-y)_{\delta\beta}$$

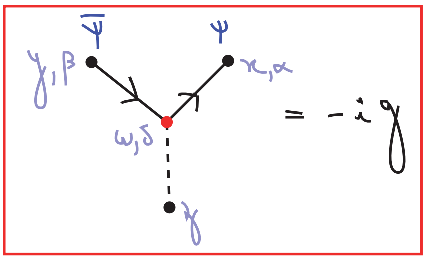
(eq. 144.1)



(Espaço das posições, Euclid.)

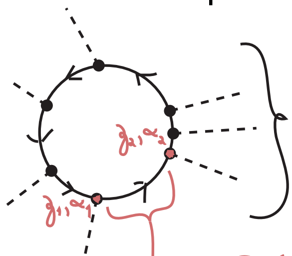
(eq. 144.2)

$\longleftrightarrow$



(Mink.)

A importância do ordamento do campo fermiônico cria uma importante diferença entre um loop fermiônico e um loop bosônico, pense no seguinte diagrama:



total de N linhas saindo do loop

$$S_F(z_2-z_1)_{\alpha_2\alpha_1} = -\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\left(\psi_{\alpha_2}(z_2)\bar\psi_{\alpha_1}(z_1)\right)$$

Temos vários termos deste tipo:

$$\left(-\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\right)\cdots\left(-\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar\psi}\right)\Big(\underbrace{(\bar\psi\psi)_1\cdots(\bar\psi\psi)_N}\Big) \longleftrightarrow (\bar\psi\psi)_i = \bar\psi(z_i)\,\psi(z_i)$$

note que estes vêm todos da interação, por isso a ordem $\bar\psi\,\psi$

Para obter a combinação cíclica (já que é um loop):

$$S_F(z_N-z_1)_{\alpha_N\alpha_1}\,S_F(z_1-z_2)_{\alpha_1\alpha_2}\cdots S_F(z_{N-1}-z_N)_{\alpha_{N-1}\alpha_N} = \mathrm{Tr}\left[S_F(z_1-z_2)\cdots S_F(z_{N-1}-z_N)\right]$$

(loops de férmions geram traços)

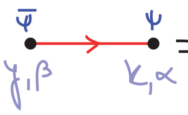
temos que trazer o último campo para a primeira posição e então aplicar as derivadas:

$$\left((\bar\psi\psi)_1\cdots(\bar\psi\,\psi)_N\right) = -\psi(z_N)(\bar\psi\psi)_1\cdots(\bar\psi\psi)_{N-1}\,\bar\psi_N(z_N)$$

passo por 2N-1 campos

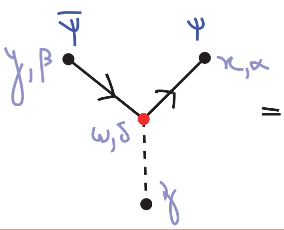
De onde vemos que, além de qualquer sinal que venha dos vértices $(-g)^N$, temos uma regra de Feynman nova, devemos multiplicar por sinal total negativo toda vez que aparecer um loop fermiônico. Você pode checar, por exemplo que o mesmo loop gerado em uma teoria $\lambda\phi^3$: não tem sinal algum além do que vem dos vértices.

Regras de Feymnan para interação de Yukawa:



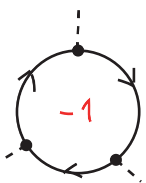
$$= S_F(x-y)_{\alpha\beta} = i\int\frac{d^4p}{(2\pi)^4}\left(\frac{e^{ip(x-y)}}{-\not p + im}\right)_{\alpha\beta}$$

(direção é importante)



$= -g$

$$= \int\frac{d^4p}{(2\pi)^4}\,\frac{e^{ip(x-y)}}{p^2+m^2}$$



(multiplico por -1^L, onde L = # loops fermiônicos)

(a contração dos índices espinoriais vai produzir um traço)

(Espaço das posições, Euclid.)

(eq. 144.3)