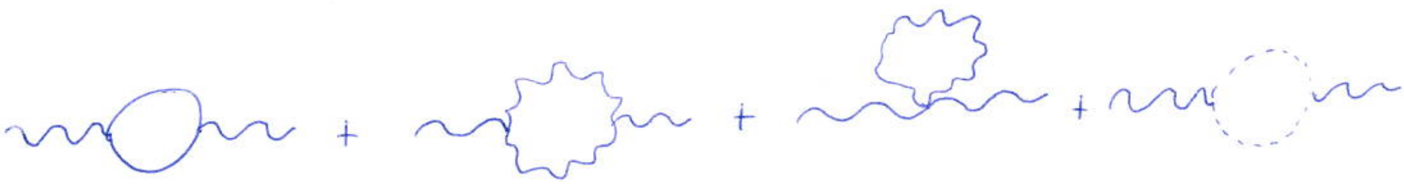
Gauge boson self-energy

$q^\mu \Big($  $\Big) = 0$: Ward-identity

$i(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi(q^2)$

Contribution until g^2

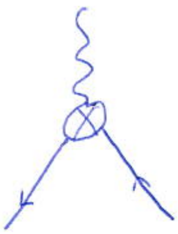


$$= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\delta^{ab}\left(-\frac{g^2}{(4\pi)^2}\,\frac{4}{3}\,n_f\,C(R)\,\Gamma(2-\tfrac{d}{2}) + \dots\right)$$

$$+\, i(g^{\mu\nu} q^2 - q^\mu q^\nu)\,\delta^{ab}\left(-\frac{g^2}{(4\pi)^2}\underbrace{\left(\frac{-13}{6}+\frac{5}{2}\right)}_{-\frac{5}{3}} C_2(G)\,\Gamma(2-d/2)\right) + \dots$$

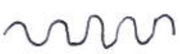
$= -i\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\delta^{ab}\,\delta_3$

$= i\not{p}\,\delta_2$



$= i g t^a \gamma^\mu \delta_1$

Gauge boson propagator:

$G^{(2)}(p) =$  $+$ (Lead order loop) $+$ 

$$\tilde{D}(k)^{ab}_{\mu\nu} = \frac{-i}{k^2 + i\epsilon}\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right]\delta_{ab}$$

(Lead Loop): $\tilde{D}(k)^{ac}_{\mu\alpha}\left[i\left(g^{\alpha\beta}k^2 - k^\alpha k^\beta\right)\delta^{cd} A \, \mathrm{Ln}\left(\frac{\Lambda^2}{-k^2}\right) + \cdots\right]\tilde{D}(k)^{db}_{\beta\nu}$

 : $\tilde{D}(k)^{ac}_{\mu\alpha}\left[-i\left(k^2 g^{\alpha\beta} - k^\alpha k^\beta\right)\delta^{cd}\right]\delta_3 \, \tilde{D}(k)^{db}_{\beta\nu}$

- $$\frac{k_\mu k_\alpha}{k^2}\left(k^2 g^{\alpha\beta} - k^\alpha k^\beta\right) = \frac{k_\mu}{k^2}\left(k^2 k^\beta - k^2 k^\beta\right) = 0$$

- $$\tilde{D}(k)^{ac}_{\mu\alpha}\left(k^2 g^{\alpha\beta} - k^\alpha k^\beta\right)\delta^{cd}\tilde{D}(k)^{db}_{\beta\nu} = \delta^{ac}\delta^{cd}\delta^{db}\left(-i\frac{g_{\mu\alpha}}{k^2}\right)$$
$$\times\left(k^2 g^{\alpha\beta} - k^\alpha k^\beta\right)\left(-i\frac{g_{\beta\nu}}{k^2}\right)$$
$$= -\frac{\delta^{ab}}{k^4}\left(k^2 g_{\mu\nu} - k_\mu k_\nu\right)$$

$$\frac{dG^{(2)}}{dM} = \left[\frac{i}{K^4}\left(K^2 g_{\mu\nu} - K_\mu K_\nu\right)\delta^{ab}\right]\frac{d}{dM}\delta^3$$

From Callan-Symanzik

$$M\frac{d}{dM}G^{(2)} + \beta(g)\frac{d}{dg}G^{(2)} + 2\gamma(g)\,G^{(2)} = 0$$

Comparing in order of g

$A = \mathcal{O}(g^2)$

$$\underbrace{\beta(g)}_{\mathcal{O}(g^n)}\underbrace{\frac{d}{dg}G^{(2)}}_{\mathcal{O}(g)} + 2\underbrace{\gamma(g)}_{\mathcal{O}(g^n)}\underbrace{G^{(2)}}_{\mathcal{O}(1)}$$

First order, ignore β

$$2\gamma_3(g)\,G^{(2)} = -M\frac{d}{dM}G^{(2)}$$

$$2\gamma_3(g)\left\{\frac{i}{K^2}\left[g_{\mu\nu} - \frac{K_\mu K_\nu}{K^2}\right]\delta_{ab}\right\} = M\left\{\frac{i}{K^4}\left(K^2 g_{\mu\nu} - K_\mu K_\nu\right)\delta^{ab}\right\}$$

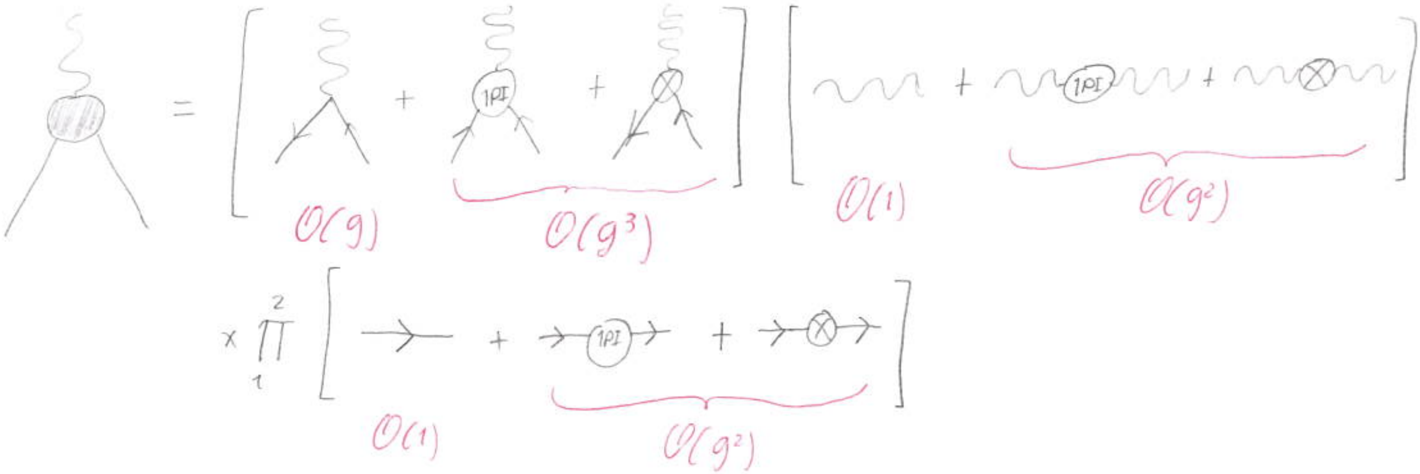
$$\times\frac{d}{dM}\delta_3$$

$$\boxed{\gamma_3(g) = \frac{1}{2}M\frac{d}{dM}\delta_3}$$

Same for fermion

$$\gamma_2(g) = \frac{1}{2} M \frac{d}{dM} \delta_2$$

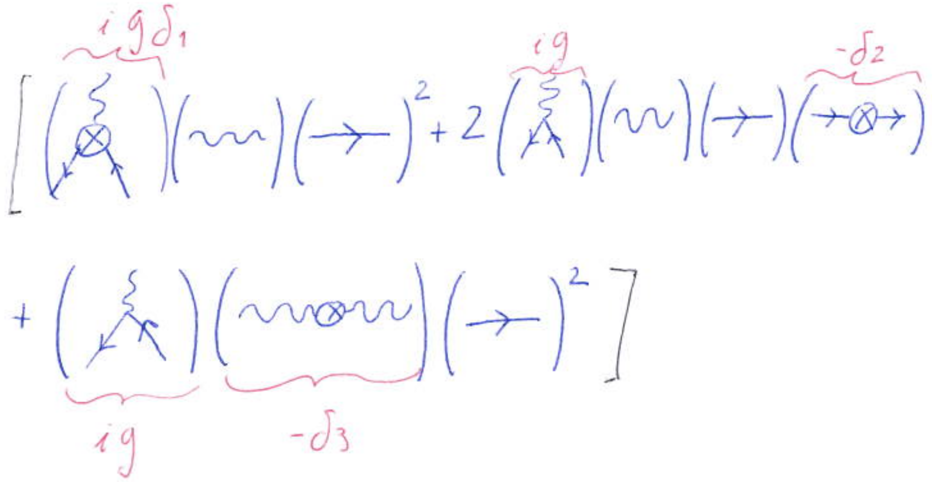
Green three-point function:



From Callan-Symanzik:

$$\left[M \frac{d}{dM} + \beta(g) \frac{d}{dg} + 2\gamma_2(g) + \gamma_3(g) \right] G^{(3)} = 0$$

$$M \frac{d}{dM} G^{(3)} = M \frac{d}{dM}$$



- $$M\frac{d}{dM}G^{(3)} = (\sim\!\sim)(\rightarrow)^2\, M\frac{d}{dM}\left(ig\,\delta_1 - i2g\,\delta_2 - ig\,\delta_3\right)$$

- $$\frac{d}{dg}G^{(3)} = i\,(\sim\!\sim)(\rightarrow)^2$$

L.O

$$G^{(3)} = ig\,(\sim\!\sim)(\rightarrow)^2$$

Replacing back into C-S.

$$(\sim\!\sim)(\rightarrow)^2\Big[M\frac{d}{dM}\left(ig\,\delta_1 - i2g\,\delta_2 - ig\,\delta_3\right) + i\beta(g)$$

$$+ igM\frac{d}{dM}\delta_2 + ig\frac{1}{2}M\frac{d}{dM}\delta_3\Big] = 0$$

$$\beta(g) = gM\frac{d}{dM}\left(-\delta_1 + 2\,\delta_2 + \delta_3 - \delta_2 - \frac{1}{2}\delta_3\right)$$

$$\beta(g) = gM\frac{d}{dM}\left[-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right]$$

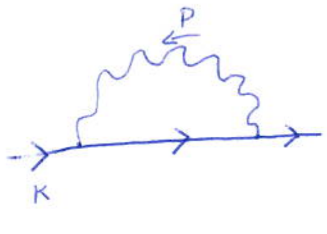
Determine counter terms

$$\delta_3 = -\frac{g^2}{(4\pi)^2}\left[\frac{4}{3} n_f C(R) + \left(-\frac{5}{3}\right) C_2(G)\right]$$

should cancel the divergence from Gauge boson self energy (order g^2)

$$\delta_3 = \frac{g^2}{(4\pi)^2}\left[\frac{5}{3} C_2(G) - \frac{4}{3} n_f C(R)\right] \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}}$$

Now δ_2:

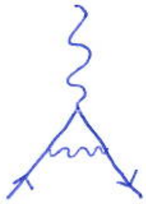


$$: \int \frac{d^4p}{(2\pi)^4} (ig)^2 \gamma^\mu t^a \frac{i(\not{p}+\not{k})}{(p+k)^2} \gamma_\mu t \frac{-i}{p^2}$$

$$= \underbrace{\frac{i g^2}{(4\pi)^2} \not{k}\, C_2(R)\, \Gamma(2-d/2)}_{\text{divergent part}} + \cdots$$

$$\delta_2 = -\frac{g^2}{(4\pi)^2} \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} C_2(R)$$

Now δ_1, come from vertex correction 1-loop.



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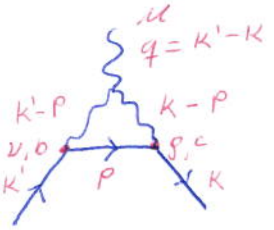

(a) (b)

$$(a): \int \frac{d^4p}{(2\pi)^4} g^3 t^b t^a t^b \frac{\gamma^\nu(\not{p} + \not{k}')\gamma^\nu(\not{p} + \not{k})\gamma_\nu}{(p+k')^2 \, (p+k)^2 \, p^2}$$

using, gauge group matrices

$$t^b t^a t^b = t^b t^b t^a + t^b [t^a, t^b]$$

$$= C_2(R)\, t^a + i\, t^b f^{abc} t^c$$

$$= C_2(R)\, t^a + i \frac{1}{2}\left[f^{abc} t^b t^c + f^{\overset{(-)}{abc}} t^a t^c\right]$$

$$= C_2(R) t^a + \frac{i}{2} f^{abc} [t^b, t^c]$$

$$= C_2(R) t^a + \frac{i}{2} f^{abc}\, i f^{bcd} t^d$$

$$= C_2(R)\, t^a - \frac{1}{2} f^{abc} f^{bcd} t^d$$

$$= \left[C_2(R) - \frac{1}{2} C_2(G)\right] t^a$$

then, the divergent part for large p

$$\sim g^3 \left[C_2(R) - \frac{1}{2} C_2(G) \right] t^a \int \frac{d^4p}{(2\pi)^4} \frac{\gamma^\nu \not{p} \gamma^\mu \not{p} \gamma_\nu}{(p^2)^3}$$

$$= \frac{i g^3}{(4\pi)^2} \left[C_2(R) - \frac{1}{2} C_2(G) \right] t^a \gamma^\mu \left[\Gamma(2 - d/2) + \cdots \right]$$

(b): $$\int \frac{d^4p}{(2\pi)^4} (ig\gamma_\nu t^b) \frac{i\not{p}}{p^2} (ig\gamma_\rho t^c) \frac{-i}{(k'-p)^2} \frac{-i}{(k-p)^2}$$

$$\times g f^{abc} \left[g^{\mu\nu}(q + k' - p)^\rho + g^{\nu\rho}(-k' - k + p)^\mu + g^{\rho\mu}(k - p - q)^\nu \right]$$

Note: $f^{abc} t^b t^c = \frac{1}{2} f^{abc} i f^{bcd} t^d = \frac{i}{2} C_2(G) t^a$

then

$$\underset{p \gg k', k, q}{\sim} \frac{g^3}{2} C_2(G) t^a \int \frac{d^4p}{(2\pi)^4} \gamma_\nu \not{p} \gamma_\rho \frac{g^{\mu\nu} p^\rho - 2 g^{\nu\rho} p^\mu + g^{\rho\mu} p^\nu}{(p^2)^3}$$

$$\sim \frac{i g^3}{(4\pi)^2} \frac{3}{2} C_2(G) t^a \gamma^\mu \left(\Gamma(2 - d/2) + \cdots \right)$$

the counter term should cancel this divergent part

$$\delta_1 = \frac{-g^2}{(4\pi)^2} \frac{\Gamma(2 - d/2)}{(M^2)^{2-d/2}} \left[C_2(R) + C_2(G) \right]$$

$$\beta(g) = g M \frac{d}{dM} \left[-\delta_1 + \delta_2 + \frac{1}{2} \delta_3 \right]$$

Note: $\bullet \; M \frac{d}{dM} \frac{1}{(M^2)^{2-d/2}} = (d-4) M^{d-4}$

$\bullet \; M^{d-4} \Gamma\left(2 - \frac{d}{2}\right) \sim -\frac{2}{d-4} + \cdots$

$$\beta(g) = -2g \left[\frac{g^2}{(4\pi)^2} \left(C_2(R) + C_2(G) \right) - \frac{g^2}{(4\pi)^2} C_2(R) \right.$$

$$\left. + \frac{1}{2} \frac{g^2}{(4\pi)^2} \left(\frac{5}{3} C_2(G) - \frac{4}{3} n_f C(R) \right) \right]$$

$$= -\frac{g^3}{(4\pi)^2} \left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(R) \right]$$

Negative for $\quad n_f \cdot C(R) < \frac{11}{4} C_2(G)$

$$\left.\begin{array}{l} SU(N): \; C_2(G) = N \\ \text{Fundamental Fermions} \quad C(N) = \frac{1}{2} \end{array}\right\} \quad n_f < \frac{11}{2} N$$