Introduction to path integral methods in string theory

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Contents

I. INTRODUCTION

The fundamentals of path integral methods in string theory are reviewed with emphasis in the development of the Faddeev Popov ghost that emerge from overcounting of configurations and the conditions that the string theory requires for maintaining Weyl invariance. Since the lecture is presented for a course that does not assume previous knowledge in string theory, I include an introduction to both bosonic and superstring theory (in the RNS formalism).

This text begins with an introduction to bosonic string theory, we study the symmetries of the Polyakov action, we work out the canonical quantization of the theory to calculate its central charge. Later the path integral formulation is introduced with the appearance of Faddeev-Popov ghosts, we consider their canonical quantization for computing the central charge. The total central charge generates an anomaly in the Weyl invariance of the theory. The vanishing of such term has powerful implications that yield conditions related with the possible physical states and the dimensions of the theory. We also introduce the BRST quantization and construction of the spectrum of the theory by means of representation theory. Once the fundamental are presented we do essentially the same steps with superstring theory, excluding the part of BRST quantization; since the procedure is very similar to the bosonic case, it will be a brief discussion emphasizing mostly with fermionic coordinates.

This review does not include several important topics including background fields (such as the graviton and dilaton) in the path integral formulation of bosonic and superstring theory (which is extensively discussed in [1] and [2]), or the calculation of scattering amplitudes with path integrals (again can be studied in [1] and [2]) and related topics of global topological properties of string theory. Also we do not consider more than the RNS formalism

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for superstring, since it's uncommon of finding in the literature (it's particularly difficult for the GS formalism [2], although it arises naturally in the pure spinor formalism [3]). We will adopt natural units with $\hbar = c = 1$, and use the convention roman indices (such as μ, ν) for space-time variables, and latin indices (such as a, b) for world sheet variables. The Minkowski signature is chosen to be $(-, +, \ldots, +)$ in D space-time dimensions.

II. FUNCTIONAL METHODS IN BOSONIC STRING THEORY

A. Polyakov action

Let us recall that the motion of a relativistic particle of mass m in a curved D-dimensional space-time can be formulated as a variational problem. Since the classical motion of a point particle is along geodesics, the action should be proportional to the invariant length of the particle's trajectory:

$$S_0 = -\alpha \int ds \tag{1}$$

where the line element is given by $ds^2 = -g_{\mu\nu}(X)dX^\mu dX^\nu$, and α is a constant which leads to the correct non-relativistic limit only if it's equal to the particle's mass m, that is $\alpha = m$. Here $g_{\mu\nu}(X)$, with μ, $\nu = 0, \ldots, D-1$, describes the background geometry, which is chosen to have Minkowski signature. The particle's trajectory $X_\mu(\tau)$, also called the world line of the particle, is parametrized by a real parameter τ, but the action is independent of the parametrization, which can be easily checked. The resulting action

$$S = -m \int \sqrt{-g_{\mu\nu}(X)dX^\mu dX^\nu}$$

contains a square root, so that it is difficult to quantize. Furthermore, this action obviously cannot be used to describe a massless particle. These problems can be circumvented by introducing an action equivalent to the previous one at the classical level in the sense that it leads to the same classical equations.

The action (1) can be generalized to the case a p-brane sweeping out a (p + 1)-dimensional world volume in D-dimensional space-time, constraint to $p < D$, by:

$$S_p = -T_p \int d\mu_p. \tag{2}$$

where T_p is the p-brane tension and $d\mu_p$ is the $(p+1)$-dimensional volume element given by:

$$d\mu_p^2 = g_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu d^{p+1}\sigma \tag{3}$$

where $a, b = 0, \ldots, p$. The brane world volumes can be parametrized by the coordinates $\sigma_0 = \tau$, which is time-like, and σ_i, which are p space-like coordinates. We shall study the case of strings (1-branes) in a D dimensional space-time. The string sweeps out a two-dimensional surface as it moves through space-time, which is called the world sheet. The points on the world sheet are parametrized by the two coordinates $\sigma^0 = \tau$, which is time-like, and $\sigma^1 = \sigma$, which is space-like. If the variable σ is periodic, it describes a closed string. If it covers a finite interval, the string is open. The space-time embedding of the string world sheet is described by functions $X_\mu(\sigma, \tau)$. The particular action of a string moving in flat space-time is the Nambu-Goto action

$$S = T \int d^2\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2}$$

The integral appearing in this action describes the area of the world sheet. As a result, the classical string motion minimizes the world-sheet area, just as classical particle motion makes the length of the world line extremal by moving along a geodesic. The quantization of the Nambu-Goto action is complicated because of the square root, so we may instead write an action that reproduces the same classical equations of motion as the Nambu-Goto action, which is the string sigma model action, or also called the Polyakov action. This action is expressed in terms of an auxiliary world- sheet metric $h_{ab}(\sigma, \tau)$ (whereas $g_{\mu\nu}$ denotes a space-time metric), and it's given by:

$$S[h, X] = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu. \tag{4}$$

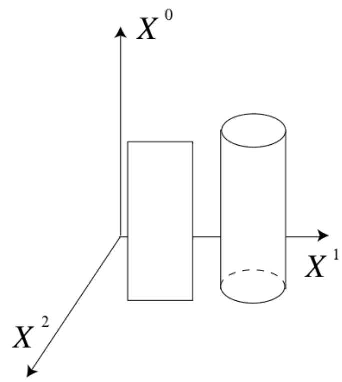

FIG. 1: The world sheet for the free propagation of an open string is a rectangular surface, while the free propagation of a closed string sweeps out a cylinder.

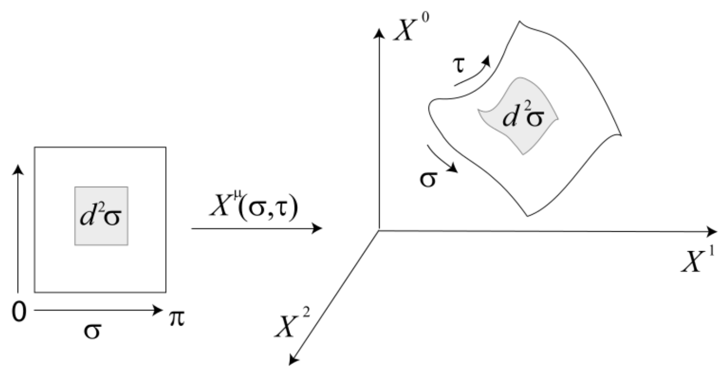

FIG. 2: The embedding of the string world sheet in space-time

where $h = \det(h_{ab})$ and $h^{ab} = (h^{-1})_{ab}$. It's important to calculate the world-sheet energy momentum tensor to make canonical quantization later on. Since there is no kinetic term for h_{ab} in the action, the energy momentum will be zero by the Euler-Lagrange equations, that is:

$$T_{ab} \equiv -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S[X,h]}{\delta h^{ab}} = \partial_a X \cdot \partial_b X - \frac{1}{2}h_{ab}h^{cd}\partial_c X \cdot \partial_d X = 0. \tag{5}$$

B. Symmetries

- **Poincaré transformations**. These are global symmetries under which the world-sheet fields transform as

$$\delta X_\mu = a_{\mu\nu}X^\nu + b_\mu \quad \text{and} \quad \delta h_{ab} = 0.$$

  Here the constants $a_{\mu\nu}$ (with $a_{\mu\nu} = -a_{\nu\mu}$) describe infinitesimal Lorentz transformations and b_μ describe space-time translations.

- **Reparametrizations**. The string world sheet is parametrized by two coordinates τ and σ, but a change in the parametrization does not change the action. Indeed, the transformations

$$\sigma^a \to f^a(\sigma) = \sigma'^a, \quad \text{and} \quad h_{ab}(\sigma) = \frac{\partial f^c}{\partial \sigma^a}\frac{\partial f^d}{\partial \sigma^b}h_{cd}.$$

  leave the action invariant. These local symmetries are also called diffeomorphisms. This implies that the transformations and their inverses are infinitely differentiable.

- **Weyl transformations**. The action is invariant under the rescaling:

$$h_{ab} \rightarrow e^{\phi(\sigma,\tau)} h_{ab} \quad \text{and} \quad \delta X_\mu = 0,$$

because $\sqrt{-h} \rightarrow e^{\phi}\sqrt{-h}$ and $h_{ab} \rightarrow e^{-\phi} h^{ab}$ give cancelling factors. This local symmetry is the reason that the energy–momentum tensor is traceless.

Poincaré transformations are global symmetries, whereas reparametrizations and Weyl transformations are local symmetries, which can be used to choose a gauge.

C. Equations of motion and boundary conditions

It is very convenient to introduce world-sheet light-cone coordinates for solving the equations of motion. Let's define $\sigma^{\pm} = \tau \pm \sigma$. In these coordinates the derivatives and the two dimensional Lorentz metric takes the form:

$$\partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \quad \text{and} \quad \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

In light-cone coordinates the wave equation for the embedding is:

$$\partial_a \partial^a X^\mu = \left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}\right) X^\mu = \partial_+ \partial_- X^\mu = 0. \tag{7}$$

The general solution involves right $X_R(\sigma^+)$ and left $X_L(\sigma^+)$ movers, with the constraint $(\partial_- X_R)^2 = (\partial_+ X_L)^2 = 0$. In this coordinates, the vanishing of the energy momentum tensor becomes:

$$T_{++} = \partial_+ X^\mu \partial_+ X^\mu = 0, \qquad T_{--} = \partial_- X^\mu \partial_- X^\mu = 0, \tag{8}$$

while $T_{+-} = T_{-+} = 0$ automatically. For extremizing the action we require the following boundary terms to vanish:

$$-T \int d\tau \, \frac{\mathrm{d}X_\mu}{\mathrm{d}\sigma}\Bigg|_{\sigma=0}^{\sigma=\pi} = 0, \tag{9}$$

where we have chosen $\sigma \in [0,\, \pi]$ (we could have chosen any length ℓ for the interval instead of π, it's just a matter of convenience that does not affect the quantization of the string modes). The only boundary conditions that are consistent with (9) are:

- **Closed string**: The embeddings are periodic $X^\mu(\sigma,\, \tau) = X^\mu(\sigma + \pi,\, \tau)$. The most general solution of (7) with this boundary condition is:

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\sigma_-) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in\sigma_-} \tag{10}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\sigma_+) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma_+} \tag{11}$$

where x^μ is a center-of-mass position, p^μ is the total string momentum in the string center of mass, and l_s is called the string length scale, related with the string tension and the open-string Regge slope parameter α' by

$$T = \frac{1}{4\pi\alpha'}, \qquad \alpha' = \frac{1}{2}l_s^2.$$

- **Open string with Neumann boundary conditions**: The component of the momentum normal to the boundary of the world sheet vanishes, that is: $\frac{\partial X_\mu}{\partial\sigma} = 0$ at $\sigma = 0,\, \pi$. The general solution:

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0} \frac{\alpha_m^\mu}{m} e^{-im\tau} \cos m\sigma.$$

- **Open string with Dirichlet boundary conditions**: The positions of the two string ends are fixed, so $\delta X^\mu = 0$, $X^\mu|_{\sigma=0} = X_0^\mu$ and $X^\mu|_{\sigma=\pi} = X_\pi^\mu$, where $\mu = 1, \ldots, D-p-1$. Neumann boundary conditions break Poincaré invariance, so they were not consider until it was realized that they are of fundamental importance for the existence of Dp-branes (hypersurfaces where the string can end, so the total system conserves Poincaré invariance). The general solution:

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0} \frac{\alpha_m^\mu}{m} e^{-im\tau} \sin m\sigma.$$

  This solution can be obtained from a T-duality applied to an open string with Neumann boundary condition, see for example section 6.1 of [4].

In any of these cases the canonical momentum conjugate to X^μ is given by:

$$P^\mu(\sigma,\, \tau) = \frac{\delta S[X,h]}{\delta \partial_\tau X_\mu} = T\frac{\partial X^\mu}{\partial \tau}. \tag{12}$$

With this definition of the canonical momentum, we may calculate the relevant Poisson brackets $\{P^\mu(\sigma,\,\tau),\, X^\nu(\sigma,\,\tau)\}_{P.B.} = \delta^{\mu\nu}\delta(\sigma-\sigma')$ in terms of the Fourier coefficients $\alpha_n^\mu,\, \alpha_n^\nu$. The world-sheet theory can be quantized by replacing the Poisson brackets by commutators in the case of bosons, $\{\ldots\}_{P.B.} \to i[\cdots]$, leading to:

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde\alpha_m^\mu, \tilde\alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad [\alpha_m^\mu, \tilde\alpha_n^\nu] = 0. \tag{13}$$

where $\alpha_m,\ m>0$ destroys particles, and $\alpha_m,\ m<0$ creates particles. One may insert the closed-string mode expansions for X_L and X_R into the energy momentum tensor of Eq. (8) to obtain:

$$T_{--} = 2l_s^2 \sum_m L_m e^{-2im\sigma_-}, \qquad T_{++} = 2l_s^2 \sum_m \tilde L_m e^{-2im\sigma_+} \tag{14}$$

where the Fourier coefficients are Virasoro generators

$$L_m = \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n, \qquad \tilde L_m = \frac{1}{2}\sum_m \tilde\alpha_{m-n}\cdot\tilde\alpha_n. \tag{15}$$

In quantum theory these operators are defined to be normal-ordered, that is

$$L_m = \frac{1}{2}\sum_n :\alpha_{m-n}\cdot\alpha_n:$$

such that in the lowering operators always appear to the right of the raising operators, i.e.

$$:\alpha_m\alpha_n: := \begin{cases} \alpha_m\alpha_n, & \text{if } m\leq n \\ \alpha_n\alpha_m, & \text{if } n<m. \end{cases}. \tag{16}$$

This prescription is motivated by the fact that the Hamiltonian (which follows from Eq. (4))

$$H = \frac{T}{2}\int_0^\pi \left[\left(\frac{\partial X}{\partial\tau}\right)^2 + \left(\frac{\partial X}{\partial\sigma}\right)^2\right] d\sigma \tag{17}$$

for an open string is given by:

$$H = \sum_n \alpha_{-n}\cdot\alpha_n \tag{18}$$

so that acting on vacuum it should be anhilated instead of raised which is the case when $n<0$ in the previous expression, therefore to correctly describe the quantum problem we require the normal ordered expression

$$H = \sum_n :\alpha_{-n}\cdot\alpha_n: .$$

Using the commutators for the modes α_n^μ, one can show that in the quantum theory the Virasoro generators for the embeding X satisfy the relation:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}. \tag{19}$$

where $c = d$ is the number of space-time dimensions. In the classical theory we would had found that $[L_m, L_n]_{P.B.} = i(m-n)L_{m+n}$. The term proportional to c is a quantum effect, the term accompanying c is called the central extension, and c is called the central charge.

Also notice that from the normal ordering ambiguity when imposing the constraint that the zero mode of the energy–momentum tensor should vanish, the only requirement in the case of the open string is that there exists some constant a such that

$$(L_0 - a)\,|\phi\rangle = 0$$

Here $|\phi\rangle$ is any physical on-shell state in the theory, and the constant a will be determined later.

D. Polyakov path integral

From the action for bosonic string theory, we may attempt to quantize the theory starting from a generating functional and implement the Faddev-Popov techniques for interpreting the gauge theory. The integral runs over all metrics and over all embeddings $X^\mu(\sigma,\, \tau)$ of the world-sheet in Minkowski spacetime:

$$Z = \int \mathcal{D}h(\sigma,\, \tau)\, \mathcal{D}X(\sigma,\, \tau)\, e^{iS[h,\, X]} \tag{20}$$

where S is the Polyakov action, and $\int \mathcal{D}h(\sigma, \tau)$ denotes an integral over the three independent components of the metric h_{00}, h_{10} and h_{11} (because it's a symmetric tensor), or equivalently in the light cone coordinates h_{++}, h_{--}, h_{-+}. We could had done a Wick rotation and work with an Euclidean signature in h_{ab}, it make no difference.

There are 2 popular paths that we could follow to develop the functional integral. In Ref. [2], the functional integral over $\mathcal{D}h$ is performed on the independent components of the metric h_{++} and h_{--} and the world-sheet metric is gauge fixed to $h_{ab} = e^\phi \eta_{ab}$. In Ref. [1] the functional integration over $\mathcal{D}h$ is performed on the diffeomorphism and Weyl parameters, and the metric is gauge fixed to a generic form $\hat{h}_{ab}$. We shall follow this later treatment since it's more formal and the interpretation of Faddeev–Popov becomes easier.

Let's start from the fact that the action is invariant under a general diffeomorphism $\sigma^a \to \sigma^a - \epsilon^a(\sigma, \tau)$ combined with a local Weyl rescalings $h_{ab} \to e^{2\Lambda(\sigma,\, \tau)} h_{ab}$, so that:

$$X^\mu \to X^\mu + \epsilon^a \partial_a X^\mu, \tag{21}$$

$$h_{ab} \to h_{ab} + (P\cdot\epsilon)_{ab} + \left(2\Lambda + \frac{1}{2}D\cdot\epsilon\right)h_{ab} \tag{22}$$

where $(P\cdot\epsilon)_{ab} = D_a\epsilon_b + D_b\epsilon_a - h_{ab}(D\cdot\epsilon)$, and $D_a\sigma_b = \partial_a\sigma_b - \Gamma^c_{ab}\sigma_c$ is the covariant derivative for a curved world sheet, which involves Christoffel symbols Γ^c_{ab}. The operator P maps vectors to a symmetric traceless 2-tensors:

$$(P\cdot\epsilon)_{ab} = P^c_{ab}\epsilon_c, \qquad \text{with } P^c_{ab} = \delta^c_{(b}D_{a)} - h_{ab}D^c$$

where $f_{(a}g_{b)} = f_a g_b + f_b g_a$. Also notice that the effect of $(P\cdot\epsilon)_{ab}$ on h_{ab} can be undone by Weyl rescaling. The corresponding ϵ_a are called conformal Killing vectors.

If for fixed $\hat{h}_{ab}$ the parameters $\zeta = (\epsilon^a,\, \Lambda)$ run over all diffeomorphisms and Weyl rescalings then $\hat{h}^\zeta = \hat{h} + \delta h$ runs over all metrics. Given some functional of the metric, $F[h]$, we can rewrite the path integral:

$$\int \mathcal{D}h\; F[h] = \int \mathcal{D}(P\cdot\epsilon)\mathcal{D}\tilde{\Lambda} F[h^g]\; \det\frac{\partial P\cdot\epsilon,\, \tilde{\Lambda}}{\partial\epsilon,\, \Lambda}. \tag{23}$$

where we introduced the Jacobian determinant for the integral over the gauge parameters ζ, which can be trivially evaluated:

$$\det\frac{\partial P\cdot\epsilon,\, \tilde{\Lambda}}{\partial\epsilon,\, \Lambda} = \begin{vmatrix} P & 0 \\ * & 1 \end{vmatrix} = \det P$$

where the $*$ indicates that the value doesn't matter since it's multiplied by 0 in the evaluation of the determinant. Therefore, Eq. (20) becomes:

$$Z = \int \mathcal{D}\zeta\, \mathcal{D}X \;\det P \exp\left[iS[X,\, \hat{h}^{\zeta}]\right] \tag{24}$$

Note that $S[X, \hat{h}^{\zeta}] = S[X^{\zeta^{-1}}, \hat{h}]$ because the action invariant under combined diffeomorphisms and Weyl transformations. **If and only if** the functional measure is also invariant, then we can perform such a gauge transformation to obtain:

$$Z = \int \mathcal{D}\zeta \mathcal{D}X^{\zeta^{-1}}\, e^{iS\left[X^{\zeta^{-1}},\, \hat{h}\right]} \det P \tag{25}$$

and we can relabel $X^{\zeta^{-1}} \to X$. Then $\mathcal{D}\zeta$ factors out and yields an overall volume of the group of diffeomorphisms and Weyl rescalings, which was our goal. Omitting this overall factor we arrive at

$$Z = \int \mathcal{D}X\; e^{iS[X,\, \hat{h}]} \det P \tag{26}$$

Notice however that the measure $\mathcal{D}\zeta\mathcal{D}X$ is in general invariant only under diffeomorphisms, not under Weyl rescalings. Later we will find that criticality (i.e. $a = 1$, $d = 26$) is equivalent to the absence of this total Weyl anomaly in the quantum measure.

Also we assumed that every metric h can be written as $h = \hat{h}^{\zeta}$ for precisely one ζ. However, the conformal Killing transformations are residual gauge symmetries not fixed in the previous treatment. These extra parametrisations must not be included in the path integral in order to avoid overcounting, so they must be fix when computing scattering amplitudes. Also if the string worldhseets have complicated topology the metric contains extra parameters, called moduli, not accounted for by local gauge transformations ζ. We must therefore sum over these moduli separately.

We may use the Faddeev-Popov procedure to write the determinant of P^{c}_{ab} in terms of fermionic ghost c and antighosts b:

$$\det P = \int \mathcal{D}b_{(ab)} \mathcal{D}c^{d} \exp\left(\frac{1}{4\pi}\int d^2\sigma \sqrt{-\hat{h}} b^{ab} P^{d}_{ab} c_d\right) \tag{27}$$

Here $b_{(ab)}(\sigma^a)$ transforms as a symmetric traceless tensor on the worldsheet and $c^d(\sigma^a)$ as a vector, and the factor $\frac{1}{4\pi}$ in the exponential is merely conventional. *These ghosts are fermionic objects with integer spin.* After integration by parts in the ghost action:

$$Z = \int \mathcal{D}X \mathcal{D}b \mathcal{D}c\, e^{i(S_X + S_g)} \tag{28}$$

where:

$$S_X = -\frac{1}{2\alpha'}\int d^2\sigma \sqrt{-\hat{h}}\hat{h}^{ab}\partial_a X \cdot \partial_b X, \qquad S_g = -\frac{i}{2\pi}\int d^2\sigma \sqrt{-\hat{h}}\hat{h}^{ab} c^d D_a b_{(ab)}.$$

- The equation of motion for c^a is given by $P \cdot c = 0$. Therefore the normalisable solutions for c^a are in one-to-one correspondence with the conformal Killing vectors, which are the generators of the residual symmetry.
- The equation of motion for $b_{(ab)}$ is $D_a b^{ab} = 0$.

Let's use the Minkowski metric as the reference metric: $\hat{h}_{ab} = \eta_{ab}$. The matter and ghost action in flat lightcone coordinates read

$$S_X + S_g[b, c] = \frac{1}{\pi}\int d^2\sigma \left[\frac{1}{\alpha'}\partial_+ X \cdot \partial_- X + i(c^+\partial_- b_{++} + c^-\partial_+ b_{--})\right].$$

From the traceless of the symmetric tensor b_{ab} we see that $b_{+-} = b_{-+} = 0$. The equations of motion for the ghosts become:

$$\partial_+ b_{--} = \partial_- b_{++} = 0, \qquad \partial_+ c^- = \partial_- c^+ = 0. \tag{29}$$

provided the boundary terms obey

$$\int d\tau \, (c^+ \delta b_{++} - c^- \delta b_{--})\Big|_{\sigma=0}^{\sigma=\pi} = 0$$

that we pick in the process of varying the action. As before we can classify the boundary conditions and solutions by:

- **Closed string**: $b(\sigma+\pi) = b(\sigma)$ and $c(\sigma+\pi) = c(\sigma)$. The most general solution:

$$b_{++} = 4\sum_n \tilde{b}_n e^{-2in\sigma_+}, \qquad b_{--} = 4\sum_n b_n e^{-2in\sigma_-}, \tag{30}$$

$$c^+ = \frac{1}{2}\sum_n \tilde{c}_n e^{-2in\sigma_+}, \qquad c^- = \frac{1}{2}\sum_n c_n e^{-2in\sigma_-}. \tag{31}$$

  The normalization is chosen such as to lead to nice expressions for the anti-commutator in the quantum theory.

- **Open string**: Boundary terms must vanish at $\sigma = 0$ and $\sigma = \pi$ separately. One way of doing that:

$$c^+(\sigma^+)\Big|_{\sigma=0,\,\pi} = c^-(\sigma^-)\Big|_{\sigma=0,\,\pi} \quad \text{and} \quad b_{++}(\sigma^+)\Big|_{\sigma=0,\,\pi} = b_{--}(\sigma^-)\Big|_{\sigma=0,\,\pi} \tag{32}$$

  The most general solution:

$$c^\pm = \sum_n c_n e^{-in\sigma^\pm}, \qquad b^{\pm\pm} = \sum_n b_n e^{-in\sigma^\pm} \tag{33}$$

The conjugate momentum of the anti-ghost field $b_{\pm\pm}$ follows from the action as

$$\Pi_{b_{\pm\pm}} = \frac{\delta S_g[b,\, c]}{\delta \partial_\tau b_{\pm\pm}} = \frac{i}{2\pi} c^\pm \tag{34}$$

with canonical Poisson-bracket relation

$$\big\{ b_{\pm\pm}(\tau,\sigma),\, \Pi_{b_{\pm\pm}}(\tau,\sigma') \big\}_{P.B.} = \delta(\sigma - \sigma').$$

To quantise this system we must take into account the fermionic nature of the b and c-fields. As is well-known from quantisation of fermions in QFT, the correct procedure is to replace the Poisson-bracket by $-i$ times the anti-commutator. Thus,

$$\big\{ b_{++}(\tau,\sigma),\, c^+(\tau,\sigma') \big\} = 2\pi\delta(\sigma-\sigma'), \qquad \big\{ b_{--}(\tau,\sigma),\, c^-(\tau,\sigma') \big\} = 2\pi\delta(\sigma-\sigma') \tag{35}$$

This corresponds to the anti-commutator relations for the modes:

$$\{c_m,\, b_n\} = \delta_{m+n,0}, \quad \{c_m,\, c_n\} = \{b_m,\, b_n\} = 0 \tag{36}$$

with the same relations obeyed in addition by $\tilde{c}'_n$, $\tilde{b}_n$ for the closed string.

The ghost-energy momentum tensor $T^{(g)}$ follows from the full non-gauge fixed action as

$$T^{(g)}_{ab} = \frac{4\pi}{\sqrt{-h}} \frac{\delta S_g[b,c,h]}{\delta h^{ab}}. \tag{37}$$

In the lightcone gauge its non-vanishing components are:

$$T^{(g)}_{++} = -i\big(2b_{++}\partial_+ c^+ + (\partial_+ b_{++})c^+\big) = \frac{1}{2}\sum_n \tilde{L}^{(g)}_n e^{-2in\sigma}, \tag{38}$$

$$T^{(g)}_{--} = -i\big(2b_{--}\partial_- c^- + \partial_- b_{--}\big)c^-\big) = \frac{1}{2}\sum_n L^{(g)}_n e^{-2in\sigma} \tag{39}$$

with corresponding Virasoro generators, for the closed string:

$$L^{(g)}_n = -\frac{1}{4\pi}\int_0^\pi d\sigma e^{-2in\sigma} T_{--}, \quad \tilde{L}^{(g)}_n = -\frac{1}{4\pi}\int_0^\pi d\sigma e^{2in\sigma} T_{++}$$

and similar formulae for the open string. In terms of modes we can check that:

$$L_m^{(g)} = \sum_n (m-n) b_{m+n} c_{-n} \tag{40}$$

which is valid classically. At the quantum level the Virasoro operators are defined as the normal ordered analogue of this classical expression. Normal ordering requires lower-level modes to the left. Due to the anti-commuting nature of the modes, we pick up a minus sign in this process if we have to change the order of the modes, i.e.

$$: b_m b_n := \begin{cases} b_m b_n, & \text{if } m \leq n \\ -b_n b_m, & \text{if } n < m. \end{cases} \tag{41}$$

and similarly for $: c_m c_n :$ as well as for $: b_m c_n :$. Then,

$$L_m^{(g)} = \sum_n (m-n) : b_{m+n} c_{-n} : . \tag{42}$$

This yields the Ghost Virasoro-algebra (check it as an exercise):

$$\left[L_m^{(g)}, L_n^{(g)}\right] = (m-n) L_{m+n}^{(g)} + \frac{1}{6}(m - 13m^3)\delta_{m+n,0}.$$

Notice that the central charge would be -26. The generators $L_m^{(g)}$ are bilinears in the fermionic ghost modes and thus bosonic. This is why they indeed satisfy commutator (as opposed to anti-commutator) relations. The algebra of the conformal transformations of the full action $S = S_X + S_g$ is generated by the combined Virasoro generators

$$L_m^{tot} = L_m^{(X)} + L_m^{(g)} - a^{tot}\delta_{m,0}, \tag{43}$$

where we conventionally include a total normal ordering constant a^{tot} into the definition of L_m^{tot}, which is the sum of the normal ordering constant for the X and for the ghost fields,

$$a^{tot} = a^{(X)} + a^{(g)}.$$

with a^{tot} as the total normal ordering constant in the definition of L_m^{tot},

$$a^{tot} = a^{(X)} + a^{(g)} = a.$$

$a^{(X)}$ corresponds to $d-2$ physical transverse of the X-oscillations together with a contribution from the X^0 and X^{d-1} unphysical components because we are in a covariant gauge. We could compute $a^{(g)}$ and $a^{(g)}$ in terms of the number of space-time dimensions [5]. However we can directly derive the total contribution which would involve only the physical degrees of freedom, since the ghost system cancels the contribution from the unphysical non-transverse polarisations a feature that also appears in the BRST quantisation.

It's easy to verify that the combined Virasoro generators satisfy the commutation relations

$$\left[L_m^{tot}, L_n^{tot}\right] = (m-n)L_{m+n}^{tot} + \delta_{m+n,0}\left(\frac{c^{tot}}{12}(m^3 - m) + 2m(a-1)\right) \tag{44}$$

where $c^{tot} = d - 26$ is called a central term. This central term generates an anomaly in the Weyl invariance of the full action $S_X + S_g$, because the Virasoro algebra for the generators is modified. The only way to eliminate the Weyl anomaly is when

$$d = 26, \qquad a = 1.$$

This tells us that the X-theory must cancel the conformal anomaly of the ghost system, so that the anomaly of the full quantum theory is absent; which is called criticality. What is actually fixed is not the number of spacetime dimensions, but the central extension of the embedding $c^{(X)} = d$.

E. BRST quantization

The full action $S_X + S_g$ after gauge fixing $h_{ab} = \eta_{ab}$ enjoys a global, fermionic, residual symmetry. Let ϵ be a constant Grassmann parameter. Then this symmetry is generated by the transformations:

$$\delta_\epsilon X^\mu = \epsilon(c^+\partial_+ + c^-\partial_-)X^\mu, \tag{45}$$

$$\delta_\epsilon c^\pm = \epsilon(c^+\partial_+ + c^-\partial_-)c^\pm, \tag{46}$$

$$\delta_\epsilon b_{\pm\pm} = i\epsilon(T^{(x)}_{\pm\pm} + T^{(g)}_{\pm\pm}). \tag{47}$$

Note that the transformations of X^μ are just the conformal Killing transformations with fermonic parameter $\epsilon c^\pm$. This symmetry is named BRST symmetry (after Becchi, Rouet, Stora, Tyutin). Via Noether's theorem one can define a BRST charge operator Q_B as the conserved charge associated with a suitable BRST current.

As always, this charge will then generate the underlying symmetry. Explicit application of the Noether procedure confirms that the BRST charge is fermonic as expected. It generates the BRST symmetry in the sense that

$$\delta_\epsilon X^\mu = \epsilon[Q_B,\, X^\mu], \quad \delta_\epsilon c^\pm = \epsilon\{Q_B,\, c^\pm\}, \quad \delta_\epsilon b_{\pm\pm} = \epsilon\{Q_B,\, b_{\pm\pm}\}. \tag{48}$$

One can show explicitly that (for open strings):

$$Q_B = \sum_m : (L^{(X)}_{-m} + L^{(g)}_{-m} - a\delta_{m,0})c_m : \tag{49}$$

does the job (and analogously for the left- and right-moving charges in the closed string). In particular $Q_B^\dagger = Q_B$. An important property of the BRST symmetry is that it is nilpotent:

$$\delta_\epsilon \delta_{\epsilon'} \Phi = 0 \quad \text{for } \Phi \in \{X^\mu,\, b,\, c\},$$

which means that the charge is nilpotent $Q_B^2 = 0$. In the quantum theory, the evaluation of $Q_B^2 = \frac{1}{2}\{Q_B,\, Q_B\}$ is complicated by normal ordering subtleties. By an explicit computation it can be found that:

$$Q_B^2 = \frac{1}{2}\{Q_B,\, Q_B\} = \frac{1}{2}\sum_{m,n}(\left[L^{tot}_m, L^{tot}_n\right] + (m-n)L^{tot}_{m+n})c_{-m}c_{-n}, \tag{50}$$

which vanishes if and only if the full Virasoro algebra is non-anomalous, the same case for the critical string with ($d = 26,\ a = 1$). This means that *consistency of the BRST symmetry is equivalent to absence of the total Weyl anomaly.*

The nice property of BRST symmetry is that it gives the correct physical state condition. A physical state must be gauge invariant. Given the relation between the gauge transformations and the BRST symmetry it is therefore reasonable to expect that a physical state must be invariant under a BRST transformation. A necessary condition for a state to be physical is that:

$$Q_B \left|\text{phys}\right\rangle = 0.$$

Indeed since Q_B acts on X as the (residual) symmetry this implements in particular the constraints resulting from gauge fixing (here the Virasoro constraints). This, however, is not enough. Namely there exists a large set of trivially physical states given by

$$\left|\chi\right\rangle = Q_B \left|\Psi\right\rangle, \qquad \text{for } \left|\Psi\right\rangle \text{ arbitrary} \tag{51}$$

because of the nilpotence of Q_B. These states are null, i.e. they are orthogonal to all physical states including themselves, $\langle \text{phys}|\chi\rangle = \langle\text{phys}| Q_B \left|\Psi\right\rangle = \langle\text{phys}| Q_B^\dagger \left|\Psi\right\rangle = 0.$ and $\langle\chi|\chi\rangle = \langle\Psi| Q_B^2 \left|\Psi\right\rangle = 0$. States in the kernel of Q_B , $\left|\chi\right\rangle$ such that $Q_B\left|\chi\right\rangle = 0$, are called Q-closed, while those in the image of Q_B, $\left|\chi\right\rangle = Q_B\left|\Psi\right\rangle$, are called Q-exact.

To define a positive norm Hilbert space we need to divide the set of Q-closed states by the set of Q-exact states, which is given by

$$\mathcal{H}_{BRST} = \frac{\mathcal{H}_{\text{closed}}}{\mathcal{H}_{\text{exact}}} = \text{cohomology of } Q_B.$$

States differing by elements of $\mathcal{H}_{\text{exact}}$ are in the same equivalence class:

$$|\Psi\rangle \equiv |\Psi\rangle + Q_B\,|\chi\rangle \tag{52}$$

The concept of a cohomology as the kernel over the image is defined in mathematics for every nilpotent operator. The probably most famous example is the exterior derivative d that maps a p-form to a $p+1$-form. In this context the p-th cohomology group is defined as $\mathcal{H}^p = \frac{\text{closed p-forms}}{\text{exact p-forms}}$.

F. Representation theory

o make all of this explicit we need to define a vacuum $|0\rangle = \left|0^{(X)}\right\rangle \otimes \left|0^{(g)}\right\rangle$ for the full theory defined by $S^{tot} = S^{(X)} + S^{(g)}$, act with creation operators associated with X, b and c on each factor and then implement the physical state condition. As before c_n, b_n, for $n < 0$ act a s creators, while for $n > 0$ act as annihilators. This is consistent with the normal ordering prescription ("creators to the left") and with the form of the zero-level Virasoro generator

$$L_0^{(g)} = \sum_{n=1}^{\infty}(nb_{-n}c_n + nc_{-n}b_n).$$

Since the ghost Hamiltonian $H^{(g)} \propto L_0^{(g)}$ we are reassured that b_{-n}, c_{-n} take the role of creators. There is an important difference compared to the X-sector, though: Since the zero modes b_0, c_0 do not appear in $L_0^{(g)}$ they must be treated separately. From the anti-commutation relations we deduce that the zero-modes c_0, b_0 form an algebra defined by

$$c_0^2 = b_0^2 = 0, \qquad \{c_0,\, b_0\} = 1 \tag{53}$$

A state in the ghost sector must furnish a representation of this algebra. The smallest representation contains two states $|\uparrow\rangle$, $|\downarrow\rangle$ such that

$$c_0\,|\downarrow\rangle = |\uparrow\rangle\,, \quad c_0\,|\uparrow\rangle = 0, \quad b_0\,|\uparrow\rangle = |\downarrow\rangle\,, \quad b_0\,|\downarrow\rangle = 0. \tag{54}$$

We could make two inequivalent choices for the full vacuum:

- $$\left|0^{\text{tot}}\right\rangle = \left|0^{(X)}\right\rangle \otimes |\uparrow\rangle \equiv |0,\uparrow\rangle$$

  The vacuum is annihilated by c_0 and by α_n , c_n, b_n with $n > 0$.

- $$\left|0^{\text{tot}}\right\rangle = \left|0^{(X)}\right\rangle \otimes |\downarrow\rangle \equiv |0,\downarrow\rangle$$

  The vacuum is annihilated by b_0 and by α_n , c_n, b_n with $n > 0$.

To gain some intuition which is the correct one we evaluate the BRST condition on the special subset $\left|\Psi^{(X)}\right\rangle \otimes |\uparrow\rangle$ and, respectively, $\left|\Psi^{(X)}\right\rangle \otimes |\downarrow\rangle$ contained in the spectrum that results from the two choices of vacua.

Consider first case 2.). The physical state condition implies $Q_B\,|\chi\rangle = 0$. For $|\chi\rangle = \left|\Psi^{(X)}\right\rangle \otimes |\downarrow\rangle$ this gives

$$0 = Q_B\,|\chi\rangle = \sum_m :(L_{-m}^{(X)} + \frac{1}{2}L_{-m}^{(g)} - a\delta_{m,0})c_m :|\chi\rangle\,. \tag{55}$$

Using $c_m\,|\chi\rangle = b_m\,|\chi\rangle = 0$ for $m > 0$ and setting $a = 1$ this becomes

$$0 = Q_B\,|\chi\rangle = \left[(L_0^{(X)} - 1)c_0 + \sum_{m>0} c_{-m} L_m^{(X)}\right] |\chi\rangle\,. \tag{56}$$

Evaluating the action of the ghost modes on the vacuum yilds

$$(L_0^{(X)} - 1)\,|\chi\rangle = 0 \quad \text{and} \quad L^{(X)}\,|\chi\rangle = 0,\ \forall\, m > 0.$$

This recovers the correct constraints. By contrast, case 1.) with vacuum $|0,\uparrow\rangle$ does not allow us to recover the known constraints in this simple fashion. This suggests that only $|0,\downarrow\rangle$ is a meaningful vacuum. Note that the two vacua are distinguished by the defining property $b_0\,|0,\downarrow\rangle = 0$.

This means that positive norm physical states are the states $Q_B\,|\Psi\rangle = 0$ modulo $|\Psi_i\rangle = Q_B\,|\chi\rangle$ built on $|0^{tot}\rangle = |0,p\rangle^{(X)} \otimes |\downarrow\rangle^{(g)}$ that satisfy in addition $b_0\,|\psi\rangle = 0$.

Now we can deduce the number of physical states in the bosonic open string spectrum. For the first level we make the ansatz

$$|\Psi\rangle = (\xi_\mu \alpha_{-1}^\mu + \beta b_{-1} + \gamma c_{-1})\,\left|0^{tot}\right\rangle.$$

This gives us 26 + 2 states to begin with.

- From $b_0\,|\Psi\rangle = 0$ we deduce $0 = \{Q_B, b_0\}\,|\Psi\rangle = L_0^{tot}\,|\Psi\rangle$. This yields the mass shell condition $p^2 = 0$.
- $Q_B\,|\Psi\rangle = 0$ leads to

  $$0 = ((p\cdot\xi)c_{-1} + \beta p\cdot\alpha_{-1})\,|0\rangle^{tot}$$

  which is satisfied by $p\cdot\xi = 0$ and $\beta = 0$. Thus requiring Q-closedness therefore removes the unphysical anti-ghost excitations as well as all polarizations that are not orthogonal to the momentum, thereby eliminating 2 out the 26+2 original states.
- To analyze $|\Psi\rangle \equiv |\chi\rangle + Q_B\,|\chi\rangle$ we observe that for a general state $|\chi\rangle = (\chi\cdot\alpha_{-1} + \beta' b_{-1} + \gamma' c_{-1}) ket0^{tot}$ at level n=1 we have:

  $$Q_B\,|\chi\rangle = ((p\cdot\chi')c_{-1} + \beta' p\cdot\alpha_{-1})\,\left|o^{tot}\right\rangle.$$

  This shows that $c^{-1}\,|0^{tot}\rangle$ is BRST exact and the polarisation vector is only defined up to the equivalence

  $$\chi^\mu \equiv \chi^\mu + \beta' p^\mu, \quad \beta' \in \mathbb{C}.$$

Thus we are left with 24 physical positive norm states. This is the exact same number of states that we would have obtained in the gauge fixed canonical quantization of X^μ without ghost, as required.

III. FUNCTIONAL METHODS IN THE RNS FORMALISM

A. Superstring theory in the RNS formalism

Untill now we have consider a theory that only has bosonic coordinates, described by an action that contains kinetic terms for bosons propagating on the worldsheet. It seems natural that, in order to produce fermions in the spectrum as we desire in a theory that is supposed to be able to describe our world, we should add the kinetic terms of some fermion. This is one of the motivations for writing a supersymmetric action. Also it can be shown that tachyons appear in the bosonic string spectrum (we have not explored for tachyons in the string spectrum since it's not relevant for developing path integral methods), which are unphysical since they imply an instability of the vacuum. We may generalize the bosonic string action to include supersymmetry which can be shown to elimination tachyons in a consistent theory. The most common approaches to superstrings:

- Ramond-Neveu-Schwarz (RNS) formalism which is supersymmetric on the string world-sheet.
- Green-Schwarz (GS) formalism, supersymmetric in ten-dimensional Minkowski space-time or in other background space-time geometries.
- The pure spinors formalism that is both supersymmetric in the world-sheet and space-time.

Let's study the RNS formalism in the simplest case of $\mathcal{N}=1$ supersymmetry in the world-sheet. The desired action is obtained by adding the Dirac action for D free massless fermions to the free theory of D massless bosons.

$$S = -\frac{1}{2\pi}\int d^2\sigma\big[h^{ab}\partial_a X^\mu \partial_b X^\nu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\big] \tag{57}$$

where we are considering $\alpha' = 1/2$, and ψ^μ_A is a D-plet of Majorana spinors (which means they are two-component real spinors ψ^μ_- and ψ^μ_+) transforming in the vector representation of the Lorentz group $SO(D-1,1)$, and ρ^a are two-dimensional Dirac matrices.

It may seem counterintuitive to introduce an anticommuting field ψ^μ that transforms as a vector (bosonic representation) of $SO(D-1,\ 1)$. This choice simply *maps in space-time, bosons to bosons and fermions to fermions*. The Lorentz group $SO(D-1,\ 1)$ is an internal symmetry in the world sheet view, and the spin and statistics theorem says nothing whether anticommuting fields should transform as vectors or spinors under an internal symmetry.

The action can be expressed in light cone coordinates as

$$S = \frac{1}{\pi}\int d^2\sigma(2\partial_+ X\cdot\partial_- X + i\psi_-\cdot\partial_+\psi_- + i\psi_+\cdot\partial_-\psi_+) \tag{58}$$

We may derive equations of motion in the light cone coordinates, for bosonic coordinates we would recover the previous results in (7), and for the fermionic coordinates:

$$\partial_+\psi_- = 0 \quad \text{and} \quad \partial_-\psi_+. \tag{59}$$

The energy momentum tensor of the RNS string follows from the action:

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu + \frac{1}{4}\bar{\psi}^\mu \rho_a \partial_b \psi_\mu + \frac{1}{4}\bar{\psi}^\mu \rho_\beta \partial_a \psi_\mu - \text{trace}. \tag{60}$$

The action is also invariant under a transformation

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \quad \text{and} \quad \delta\psi^\mu = \rho^a \partial_a X^\mu \epsilon$$

where ϵ is a constant infinitesimal Majorana spinor. There is a conserved current that can be constructed by taking the ϵ parameter to be non-constant, so we may find the total variation of the action under such variation:

$$\delta S \sim \int d^2\sigma\ (\partial_a\bar{\epsilon})J^a, \quad \text{where} \quad J^a_A = -\frac{1}{2}(\rho^b\rho^a\psi_\mu)_A \partial_\beta X^\mu. \tag{61}$$

Written in terms of world sheet light-cone coordinates, the non-zero components of the energy-momentum tensor and supercurrent are:

$$T_{++} = \partial_+ X_\mu \partial_+ X^\mu + \frac{i}{2}\psi^\mu_+ \partial_+ \psi_{+\mu}, \quad T_{--} = \partial_- X_\mu \partial_- X^\mu + i\frac{i}{2}\psi^\mu_- \partial_- \psi_{-\mu}, \tag{62}$$

$$J_+ = \psi^\mu_+ \partial_+ X_\mu \quad \text{and} \quad J_- = \psi^\mu_- \partial_- X_\mu. \tag{63}$$

From equations of motion in the world-sheet metric, it can be shown that analogous to the bosonic case we have $J_+ = J_- = T_{++} = T_{--} = 0$. By considering variations of the fields $\psi_\pm$ one requires the vanishing of the boundary terms in the variation of the action,

$$\delta S \sim \int d\tau\Big(\psi_+\delta\psi_+ - \psi_-\delta\psi_-\Big|_{\sigma=\pi} - \psi_+\delta\psi_+ - \psi_-\delta\psi_-\Big|_{\sigma=0}\Big) = 0. \tag{64}$$

B. R and NS sectors

Let's focus on the **open string** case with Neumann conditions for the embeddings X^μ. In the opens string case the two terms in (64) must vanish separately. This is satisfied if $\psi_+^\mu = \pm\psi_-^\mu$ at each end of the string. The relative sign between ψ_+^μ and ψ_-^μ is a matter of convention. We may choose $\psi_+^\mu\big|_{\sigma=0} = \psi_-^\mu\big|_{\sigma=0}$, and the sign at the other end of the string is the one physically meaningful.

- **Ramond boundary condition**: Choosing both ends of the string with the same sign:

$$\psi_+^\mu\Big|_{\sigma=\pi} = \psi_-^\mu\Big|_{\sigma=\pi} \tag{65}$$

  This boundary condition gives rise to space-time fermions in the case of open strings. The mode expansion of the fermionic field in the R sector:

$$\psi_-^\mu(\sigma,\ \tau) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)}, \quad \psi_+^\mu(\sigma,\ \tau) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)} \tag{66}$$

  The Majorana condition requires these expansion to be real, hence $d_{-n}^\mu = d_n^{\mu\dagger}$. The normalization factor is just a matter of convenience.

- **Neveu-Schwarz boundary condition**: Choosing an relative minus sign:

$$\psi_+^\mu\Big|_{\sigma=\pi} = -\,\psi_-^\mu\Big|_{\sigma=\pi} \tag{67}$$

  This NS boundary condition gives rise to space-time bosons. The mode expansions:

$$\psi_-^\mu(\sigma,\ \tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau-\sigma)}, \quad \psi_+^\mu(\sigma,\ \tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau+\sigma)} \tag{68}$$

From now on we follow the convention $m,\ n \in \mathbb{Z}$ while $r,\ s \in \mathbb{Z}+\frac{1}{2}$.

In the case of **closed strings** (which is not much relevant for the rest of the review) we would need to choose between the possible periodic boundary conditions that make the boundary term to vanish:

$$\psi_\pm(\sigma,\ \tau) = \pm\psi_\pm(\sigma+\pi,\ \tau). \tag{69}$$

The positive sign gives periodic boundary conditions while the negative on gives antiperiodic boundary conditions. We may impose periodicity (R) or antiperiodicity (NS) of the right- and left-movers separately. For the right movers one can choose:

$$\psi_-^\mu(\sigma,\ \tau) = \sum_n d_n^\mu e^{-2in\sigma_-}, \quad \text{or} \quad \psi_-^\mu(\sigma,\ \tau) = \sum_r b_r^\mu e^{-2ir\sigma_-} \tag{70}$$

$$\psi_+^\mu(\sigma,\ \tau) = \sum_n \tilde{d}_n^\mu e^{-2in\sigma_+}, \quad \text{or} \quad \psi_+^\mu(\sigma,\ \tau) = \sum_r \tilde{b}_r^\mu e^{-2ir\sigma_+} \tag{71}$$

Therefore we have 4 possibilities for pairing the left- and right-movers for the closed strings. The NS-NS and R-R sectors are space-time bosons, while NS-R and R-NS sectors are space-time fermions.

For now on the discussion focus solely on open strings, the extension for open strings is straightforward. The Fourier modes for the space-time coordinates X^μ have the same commutation relations as with bosonic strings,

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu} \tag{72}$$

For closed strings there would be a second set with modes $\tilde{\alpha}_m^\mu$. The fermionic coordinates obey the Dirac equation on the world sheet, with canonical anticommutation relation given by:

$$\{\psi_A^\mu(\sigma,\ \tau),\ \psi_B^\nu(\sigma',\ \tau)\} = \pi\delta_{AB}\delta(\sigma-\sigma') \tag{73}$$

so that inserting the mode expansions we would find:

$$\{b_r^\mu,\ b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \quad \text{and} \quad \{d_m^\mu,\ d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}. \tag{74}$$

There is an important discussion on how to construct the superstring spectrum using these operators on the states of the R or NS sectors, the reader is refereed to any of the recommended textbooks for this discussion. We shall concentrate on finding the central charges for each sector. The super-Virasoro generators are modes of the energy momentum tensor T_{ab} and supercurrent J_A^a. For the open string:

$$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma} T_{++} = L_m^{(b)} + L_m^{(f)} \tag{75}$$

- The bosonic mode contribution is given by:

$$L_m^{(b)} = \frac{1}{2}\sum_r :\alpha_{-n}\cdot\alpha_{m+n}: \ . \tag{76}$$

- The contribution from the fermionic modes in the NS sector:

$$L_m^{(f)} = \frac{1}{2}\sum_r (r+\frac{m}{2}) :b_{-r}\cdot b_{m+r}: \ . \tag{77}$$

  The modes of the supercurrent can be written as:

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma\ e^{ir\sigma} J_+ = \sum_n \alpha_{-n}\cdot b_{n+r}. \tag{78}$$

- The contribution from the fermionic modes in the R sector:

$$L_m^{(f)} = \frac{1}{2}\sum_n (n+\frac{m}{2}) :d_{-n}\cdot d_{m+n}: \ . \tag{79}$$

  The modes of the supercurrent can be written as:

$$F_m = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma\ e^{im\sigma} J_+ = \sum_n \alpha_{-n}\cdot d_{n+m} \tag{80}$$

Now we can determine the super-Virasoro algebra for the modes of the energy momentum tensor and the supercurrent.

- For the R sector:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n,0}, \tag{81}$$

$$[L_m, F_n] = \Big(\frac{m}{2} - n\Big)F_{m+n}, \tag{82}$$

$$\{F_m,\ F_n\} = 2L_{m+n} + \frac{D}{8}m^2\delta_{m+n,0}. \tag{83}$$

- For the NS sector:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m+n,0}, \tag{84}$$

$$[L_m, G_r] = \Big(\frac{m}{2} - r\Big)G_{m+r}, \tag{85}$$

$$\{G_r,\ G_s\} = 2L_{r+s} + \frac{D}{2}\Big(r^2 - \frac{1}{4}\Big)\delta_{r+s,0}. \tag{86}$$

When quantizing the RNS string one can only require that the positive modes of the Virasoro generators annihilate the physical state. In the NS sector the physical-state conditions are:

$$G_r \left|\phi\right\rangle = 0, \quad L_m \left|\phi\right\rangle = 0, \quad (L_0 - a_{NS}) \left|\phi\right\rangle = 0. \tag{87}$$

where $r > 0$, $m > 0$, and a_{NS} is a constant introduced to allow for the normal-ordering ambiguity. Similarly, the physical state conditions in the R sector:

$$F_n \left|\phi\right\rangle = 0, \quad L_m \left|\phi\right\rangle = 0, \quad (L_0 - a_R) \left|\phi\right\rangle = 0. \tag{88}$$

From the expressions we can identify the central charges for each sector. By an analogous development respect with bosonic string theory, we will be able to identify the critical values for a_{NS}, a_R and D for the theory to be Lorentz. It can be found with such results that no tachyons would be present [4]. However that is a whole other discussion, we will focus on developing the Faddeev–Popov ghosts using path integrals and and derive its central charges, as in the bosonic case. Futhermore we classify from the chirality of the string spectrum 3 of the 5 types of superstring theories, namely type I, type IIA and type IIB, however this detail is not relevant for the present discussion.

C. Local symmetries of the action

Consider that the world-sheet metric h_{ab} depends on the σ, τ coordinates; so that we can explore the local symmetries in the gauge fixed action (57). Reparametrization invariance and 2-dimensional local Lorentz invariance on the world sheet can be implemented by replacing (57) for an action:

$$S_1 = -\frac{1}{2\pi} \int d^2\sigma \sqrt{-h} \big\{ h^{ab} \partial_a X^\mu \partial_\mu X^\mu - i \bar{\psi}^\mu \rho^a D_a \psi_\mu \big\}. \tag{89}$$

where we are suppressing the spinor indices, and $h = \det\{h_{ab}\}$. The reader might wonder what do we even mean by covariant derivative of an spinor.

For this purpose it's useful to write the metric h_{ab} of a D-dimensional manifold M (in contrast with the world sheet case that we have studied so far that is only 2 dimensional) in terms of orthogonal tangent vectors $e_a^{a'}$ chosen respect to some local inertial frame at each point of M. The index a' is the name of each one of these tangent vectors, and a is a vector index. $e_a^{a'}$ is called vielbein, where bein is the suffix indicating frame and the prefix is a german word indicating the dimension. We may express:

$$h_{ab} = \eta_{a'b'} e_a^{a'} e_b^{b'}, \qquad \eta^{a'b'} = h^{ab} e_a^{a'} e_b^{b'}. \tag{90}$$

From now on we are free to represent $e = \sqrt{-h}$ for the determinant of the vielbein, assuming a Minkowski signature.

The index a of $e_a^{a'}$ transforms like any vector index under diffeomorphism of M, while the index a' is simply a name so it doesn't change under diffeomorphism of M. However, since the introduction of $e_a^{a'}$ involves arbitrary choices at each space-time point, we are free to make local $SO(D-1,\ 1)$ transformations on the index a', like local Lorentz indices. Also $SO(D-1,\ 1)$ admits spinor representations, so spinor indices may be regarded as local Lorentz indices. Analogous to an ordinary Yang-Mills potential field (or connection) A_μ one can introduce a spin connection $\omega_a^{a'b'}$ as a gauge field for local Lorentz transformations. Under an infinitesimal Lorentz transformation by a parameter $\Theta^{a'b'}$, the variation of the spin connection becomes:

$$\delta\omega_a^{a'b'} = \partial_a \Theta^{a'b'} + [\omega_a,\ \Theta]^{a'b'} = (D_a \Theta)^{a'b'}. \tag{91}$$

The rules for the local Lorentz transformations of a spinor ψ are given by $\delta\psi = -\frac{1}{8}\Theta^{a'b'}[\rho_{a'}, \rho_{b'}]\psi$, where we are considering the Dirac matrices in D-dimensions. The covariant derivative of the spinor is:

$$D_a \psi = (\partial_a + \frac{1}{8} \omega_a^{a'b'} \Gamma_{a'b'}) \psi \tag{92}$$

In the special case of 2 dimensions the spin connection doesn't contribute to covariant derivative of a spin term, so it D_a can be replaced by ∂_a when it acts on a spin term. Also, in a curved world sheet we introduce $\rho^\mu(\sigma) = e_{a'}^{a}(\sigma)\rho^{a'}$.

Returning to the action in (89), it needs to be invariant under supersymmetry transformations, and because of the term $e = \sqrt{-h}$ this may be difficult. A clear way out is to introduce a supersymmetry field χ_a that is related with the infinitesimal variation of the zweibein (now 2 dimensional frame). Defining:

$$S_2 = -\frac{1}{\pi}\int d^2\sigma \; e\bar{\chi}_a \rho^b \rho^a \psi^\mu \partial_b X_\mu, \tag{93}$$

so the total action $(S_1 + S_2)$ is now invariant under supersymmetry transformations (which the readers should check by themselves, by verifying the variation of the total action is zero):

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \quad \delta\psi^\mu = -i\rho^a \epsilon\big(\partial_a X^\mu - \bar{\psi}^\mu \chi_a\big) \tag{94}$$

$$\delta e_a^{a'} = -2i\bar{\epsilon}\rho^{a'}\chi_a, \quad \delta\chi_a = D_a\epsilon, \tag{95}$$

where ϵ is an infinitesimal Majorana spinor function of $\sigma,\ \tau$. Here the term $\chi_{Aa}(\sigma,\ \tau)$ is a two component Majorana spinor which is a world-sheet vector, called Rarita-Schwinger field (which is a 3/2-spin space-time fermion).

Futhermore, the action has 2 superconformal bosonic symmetries under the transformation:

$$\delta X^\mu, \quad \delta\psi^\mu = -\frac{1}{2}\Lambda\psi^\mu, \quad \delta e_a^{a'} = \Lambda e_a^{a'}, \quad \delta\chi_a = \frac{1}{2}\Lambda\chi_a. \tag{96}$$

with $\Lambda(\sigma,\ \tau)$ is a scalar function (the same scaling e^ϕ factor that we encountered in bosonic string theory). There are also 2 superconformal symmetries under the transformation

$$\delta\chi_a = i\rho_a\eta, \qquad \delta e_a^{a'} = \delta\psi^\mu = \delta X^\mu = 0, \tag{97}$$

where $\eta(\sigma,\ \tau)$ is an arbitrary Majorana spinor. The theory now has 4 local bosonic symmetries these allow us to gauge the four components of $e_a^{a'} = \delta_a^{a'}$, and also 4 local fermionic symmetries, which implies that we can locally set the four components of $\chi_a = 0$.

D. Path integrals in RNS

The classical statement that χ_a can be gauge away means that it can always be expressed in the form:

$$\chi_a = i\rho_a\eta + D_a\epsilon, \tag{98}$$

and with reasonable boundary conditions this expression is unique. So we can change variables in the path integral from χ_a to η and ϵ. After the change of variables, the integral from η and ϵ can be dropped, since they are symmetry parameters and the action does not depend on them (this is an oversimplification, but the formal treatment leads to the same conclusion). The change of variable lead to a Jacobian that will end up as a ghost path integral.

In 1+1 dimension, the Lorentz group is $SO(1,1)$ in the Minkowski world-sheet, and $SO(2)$ in Euclidean world sheet. $SO(1,1)$ has a single generator, which we might call W, and an $SO(1,1)$ representation is specified by its eigenvalue, which is the spin, so in this case the eigenvalues of W will be integers or half-integers. The gravitino field χ_{aA} has a vector index a corresponding to spin ±1, and a spinor index A that carries spin $\pm1/2$. Altogether, χ_{aA} has four components of spin ±1, $\pm1/2$, i.e., 3/2, 1/2, -1/2 and -3/2 respectively. The gauge parameters η and ϵ are spinors with two components each of spin $\pm1/2$. The derivative operator D_a is a vector with components of spin ±1. Given a field V with components of various spin, let's denote the spin q component of V as V_q, then (98) becomes:

$$\delta\chi_{3/2} = D_1\epsilon_{1/2}, \quad \delta\chi_{1/2} = D_1\epsilon_{-1/2} + \eta_{1/2}; \tag{99}$$

$$\delta\chi_{-3/2} = D_{-1}\epsilon_{-1/2}, \quad \delta\chi_{-1/2} = D_{-1}\epsilon_{1/2} + \eta_{-1/2}. \tag{100}$$

The change of variables from $\chi_{\pm1/2}$ to $\eta_{\pm1/2}$ introduces a non trivial Jacobian, for example from $\chi_{3/2}$ to $\epsilon_{1/2}$ the Jacobian is:

$$\text{Jacobian}_{3/2} = 1/\det\left[D_1^{1/2\to3/2}\right], \tag{101}$$

where the superscript $1/2 \to 3/2$ represents the operator mapping spin 1/2 to spin 3/2. We can represent this determinant by ghost fields $\gamma_{1/2}$ and $\beta_{-3/2}$:

$$\text{Jacobian}_{3/2} = \int \mathcal{D}\gamma_{1/2}\mathcal{D}\beta_{-3/2} \exp\left[-\frac{1}{\pi}\int d^2\sigma \beta_{-3/2} D_1 \gamma_{1/2}\right]. \tag{102}$$

γ and β must be commuting fields since they are ghosts for an anticommuting symmetry. Likewise a change of variables from $\chi_{-3/2}$ to $\epsilon_{-1/2}$ gives a Jacobian with commuting ghosts $\gamma_{-1/2}$ and $\beta_{3/2}$:

$$\text{Jacobian}_{-3/2} = \int \mathcal{D}\gamma_{-1/2}\mathcal{D}\beta_{3/2} \exp\left[-\frac{1}{\pi}\int d^2\sigma \beta_{3/2} D_1 \gamma_{-1/2}\right]. \tag{103}$$

The fields γ and β are called superconformal ghosts. The components of $\gamma_{\pm 1/2}$ make up a spinor γ_A. The components of $\beta_{\pm 3/2}$ make up a vector-spinor β_{aA} which is subject to the constraints $\rho^{aAB}\beta_{aB} = 0$.

The ghost action implied by the determinants (102) and (103) can be thus expressed as:

$$S_{FP} = -\frac{i}{2\pi}\int d^2\sigma h^{ab}\bar{\gamma}\partial_a\beta_b. \tag{104}$$

In the gauge $h_{ab} = \delta_{ab}$, the equations of motion imply that $\beta_{3/2}$ and $\gamma_{-1/2}$ are right-moving while the other components are left-moving. By varying the action (104) respect to the world-sheet metric, we can compute the energy momentum tensor and current for the ghosts:

$$T_{++} = \frac{i}{2}\gamma\partial_+\beta + \frac{3i}{2}\beta\partial_+\gamma, \qquad J_+^{(g)} = \beta\gamma. \tag{105}$$

In terms of mode expansions in the R-sector:

$$\gamma(\tau) = \frac{1}{\sqrt{2}}\sum_n \gamma_n e^{-2in\tau}, \qquad \beta(\tau) = \frac{1}{\sqrt{2}}\sum_n \beta_n e^{-2in\tau}. \tag{106}$$

The commutation relations implied by S_{FP} are:

$$[\gamma_m, \beta_n] = \delta_{m+n,0}, \qquad [\gamma_m, \gamma_n] = [\beta_m, \beta_n] = 0. \tag{107}$$

As defined here, γ is herminitian and β is antihermitian. β could be redefined by a factor i if we wanted. The coefficients γ_m, β_n are moded by integer numbers; for the bosonic sector we require half integer modes γ_r, β_s. Altogether the ghost contributions are:

$$L_m^{(g)} = \sum_n \left[(m+n) : b_{m-n}c_n : + \left(\frac{1}{2}m + n\right) : \beta_{m-n}\gamma_n :\right] \tag{108}$$

$$F_m^{(g)} = -2\sum_n \left[b_{-n}\gamma_{m+n} + \left(\frac{1}{2}n - m\right)c_{-n}\gamma_{m+n}\right] \tag{109}$$

Also:

$$\left\{F_m^{(g)},\, F_n^{(g)}\right\} = 2L_{m+n}^{(g)} - 5m^2\delta_{m+n,0} \tag{110}$$

which implies the ghost anomaly $-5m^2$. We found before that the anomaly in the R sector is given by $\frac{1}{2}Dm^2 + 2a_R$. Therefore the total anomaly

$$c_R^{tot} = -5m^2 + \frac{1}{2}Dm^2 + 2a_R$$

cancels for $D = 10$ and $a_R = 0$. In the NS sector one has

$$c_{NS}^{tot} = c^{(g)} + c^{(X,\psi)} = \left[\frac{1}{4} - 5r^2\right] + \left[\frac{1}{2}D\left(r^2 - \frac{1}{4}\right) + 2a_{NS}\right]$$

which vanishes only for $D = 10$ and $a_{NS} = 1/2$. As in the bosonic case, the quantum action has global fermionic symmetry, namely BRST symmetry, which we could apply to obtain the same results of consistency of the theory when $D = 10$. This procedure is very standard and it can be found in any of the references.

To summarize, now we have ten X^μ boson, ten ψ^μ fermions, conformal ghosts b, c and superconformal ghost γ and β. The conditions for eliminating the Weyl anomaly in the theory is when $D = 10$, $a_R = 0$ and $a_{NS} = \frac{1}{2}$, and this allows the BRST symmetry with conserved supercharge Q_B.

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