Path integral methods in string theory.

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Outlook

1. **Functional methods in bosonic string theory**
   - Introduction
   - Equations of motion and solutions
   - Canonical quantization
   - Polyakov path integral
   - FP ghosts
   - BRST symmetry
2. **Functional methods in superstring theory**
   - Introduction
   - R and NS sectors
   - Path integrals and the gravitino
   - Quantization of ghosts

Basics of string theory

Think on an analogy with particles (0 spatial dimensions).

$$S_0 = m \int ds,$$

$$\boxed{ds^2 = g_{\mu\nu}(X) dx^\mu dx^\nu.}$$

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We can generalize this for Dp-branes.

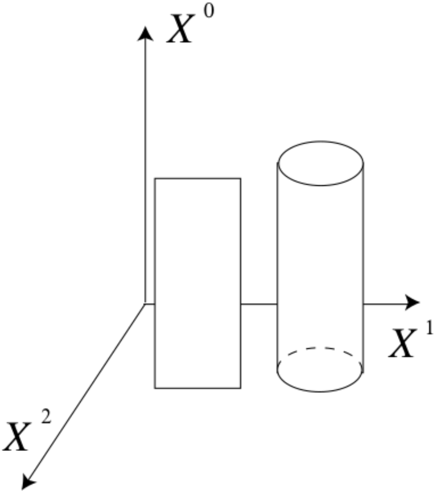
$$S_p = T_p \int d\mu_p,$$

$$d\mu_p^2 = \det(g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu) d^{p+1}\sigma.$$

For a string ($p = 1$) we have the Polyakov action:

$$S = \frac{T}{2} \int d\sigma^2 \sqrt{-h} h^{\alpha\beta} g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$$

The world sheet



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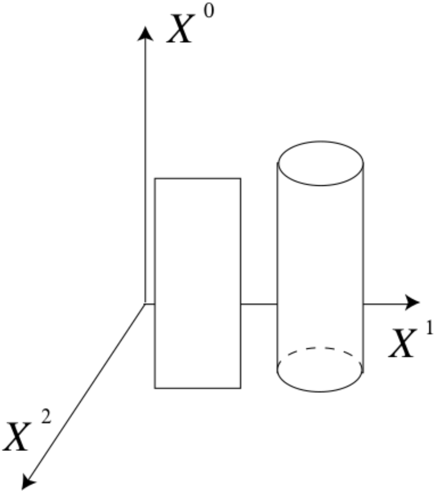

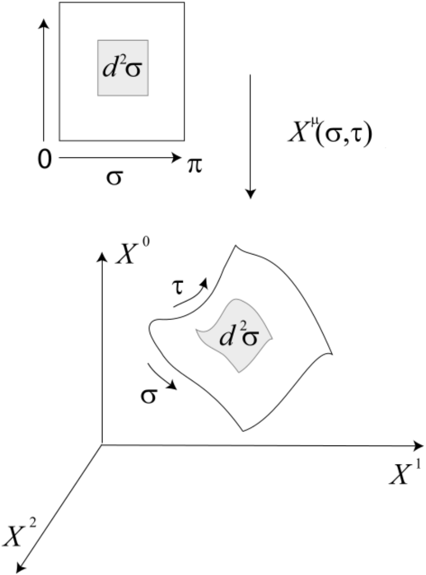

The action is invariant under the transformations:

- Poincaré transformations.

$$\delta X_\mu = a_{\mu\nu} X^\nu + b_\mu \quad \text{and} \quad \delta h_{ab} = 0.$$

$a_{\mu\nu}$ describes infinitesimal Lorentz transformations and b_μ for space-time translations.

- Reparametrizations (diffeomorphisms):

$$\sigma^a \to f^a(\sigma) = \sigma'^a, \quad \text{and} \quad h_{ab}(\sigma) = \frac{\partial f^c}{\partial \sigma^a} \frac{\partial f^d}{\partial \sigma^b} h_{cd}.$$

- Weyl transformations:

$$h_{ab} \to e^{\phi(\sigma,\tau)} h_{ab} \quad \text{and} \quad \delta X_\mu = 0,$$

Equations of motion

By reparametrization invariance we may choose $h_{\alpha\beta} = \eta_{\alpha\beta}$;

$$S = \frac{T}{2}\int d^2\sigma \left(\left[\frac{\partial X}{\partial \tau}\right]^2 - \left[\frac{\partial X}{\partial \sigma}\right]^2 \right)$$

$$\delta S = 0 \rightarrow \left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2}\right) X^\mu = 0$$

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as long as the boundary terms vanish:

$$-T \int d\tau \, \frac{\mathrm{d}X_\mu}{\mathrm{d}\sigma} \bigg|_{\sigma=0}^{\sigma=\pi} = 0,$$

where we have choosen $\sigma \in [0, \pi]$

Boundary conditions

- PERIODIC CONDITIONS: They generate **Closed strings**

$$X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau)$$

The graviton appears as a massless mode of this spectrum.

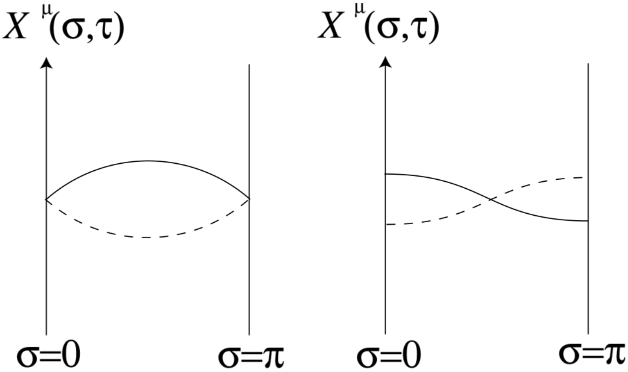
- NEUMANN CONDITIONS: The ends of the **open string** can be anywhere in the space-time

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}X^\mu(\sigma=0,\tau)=0, \qquad \frac{\mathrm{d}}{\mathrm{d}\sigma}X^\mu(\sigma=\pi,\tau)=0.$$

- DIRICHLET CONDITIONS: When the **open string** ends on a space-time hypersurface (D-branes).

$$X^\mu(\sigma=0,\tau)=X_0, \qquad X^\mu(\sigma=\pi,\tau)=X_\pi.$$

Open string



Dirichlet and Neumann strings.

Light cone fixing

Light-cone coordinates:

$$\sigma^{\pm} = \tau \pm \sigma, \quad \partial_{\pm} = \frac{1}{2}[\partial_{\tau} \pm \partial_{\sigma}].$$

$$\begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The equation of motion becomes

$$\partial_+\partial_- X^{\mu} = \partial_+\partial_-\left[X_R^{\mu}(\sigma^-) + X_L^{\mu}(\sigma^+)\right] = 0$$

where $X_R(\sigma^+)$ is a right- and $X_L(\sigma^+)$ a left-mover, obeying:

$$(\partial_- X_R)^2 = (\partial_+ X_L)^2 = 0.$$

Solutions

For closed strings

$$\partial_- X_R^\mu = l_s \sum_{m=-\infty}^{+\infty} \alpha_m^\mu e^{-2im\sigma^-}, \quad \partial_- X_L^\mu = l_s \sum_{m=-\infty}^{+\infty} \tilde{\alpha}_m^\mu e^{-2im\sigma^+}.$$

For an open Neumann string:

$$X^\mu(\tau, \sigma) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m \neq 0} \frac{\alpha_m^\mu}{m} e^{-im\tau} \cos m\sigma.$$

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Canonical quantization:

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad [\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0.$$

- α_m, $m > 0$ destroys particles.
- α_m, $m < 0$ creates particles.

Energy momentum tensor

$$T_{ab} \equiv -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S[X,h]}{\delta h^{ab}} = \partial_a X \cdot \partial_b X - \frac{1}{2}h_{ab}h^{cd}\partial_c X \cdot \partial_d X = 0$$

because there is no kinetic term for h_{ab}. In light cone coord:

$$T_{++} = \partial_+ X^\mu \partial_+ X^\mu, \qquad T_{--} = \partial_- X^\mu \partial_- X^\mu,$$

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Substituting the Fourier expansions for X^μ

$$T_{--} = 2l_s^2 \sum_m L_m e^{-2im\sigma_-}, \qquad T_{++} = 2l_s^2 \sum_m \tilde{L}_m e^{-2im\sigma_+}$$

The Fourier coefficients are Virasoro generators

$$L_m = \frac{1}{2}\sum_n \alpha_{m-n} \cdot \alpha_n, \qquad \tilde{L}_m = \frac{1}{2}\sum_m \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n.$$

In quantum theory these operators are defined to be normal-ordered,

$$L_m = \frac{1}{2}\sum_n : \alpha_{m-n} \cdot \alpha_n :$$

such that in the lowering op. appear to the right of the raising op.,

$$: \alpha_m \alpha_n := \begin{cases} \alpha_m \alpha_n, & \text{if } m \leq n \\ \alpha_n \alpha_m, & \text{if } n < m. \end{cases} .$$

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$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.$$

where $c = d$ is the space-time dimensions. In the classical theory

$$[L_m, L_n]_{P.B.} = i(m-n)L_{m+n}.$$

The term proportional to c is a quantum effect, called *central charge*.

Polyakov path integral

$$Z = \int \mathcal{D}h(\sigma,\, \tau)\, \mathcal{D}X(\sigma,\, \tau)\, e^{iS[h,\, X]}$$

which runs over all metrics h_{ab} and over all embeddings $X^\mu(\sigma,\, \tau)$. As always we will run into problems if we overcount states.

Transformation

Consider a general diffeomorphism $\sigma^a \to \sigma^a - \epsilon^a(\sigma,\, \tau)$ and local Weyl rescalings $h_{ab} \to e^{2\Lambda(\sigma,\, \tau)} h_{ab}$,

$$X^\mu \to X^\mu + \epsilon^a \partial_a X^\mu,$$

$$h_{ab} \to h_{ab} + (P \cdot \epsilon)_{ab} + \left(2\Lambda + \frac{1}{2} D \cdot \epsilon\right) h_{ab}$$

$$\text{with} \quad (P \cdot \epsilon)_{ab} = D_a \epsilon_b + D_b \epsilon_a - h_{ab}(D \cdot \epsilon)$$

The operator P maps vectors to symmetric traceless 2-tensors:

$$(P\cdot\epsilon)_{ab} = P^c_{ab}\epsilon_c, \qquad \text{with } P^c_{ab} = \delta^c_{(b}D_{a)} - h_{ab}D^c$$

where $f_{(a}g_{b)} = f_a g_b + f_b g_a$. ϵ_a are called conformal Killing vectors.

Fix $\hat{h}_{ab}$. If $\zeta = (\epsilon^a, \Lambda)$ runs over all diffeomorphisms and Weyl rescalings then $\hat{h}^\zeta = \hat{h} + \delta h$ runs over all metrics, then

$$\int \mathcal{D}h\ F[h] = \int \mathcal{D}(P\cdot\epsilon)\mathcal{D}\tilde{\Lambda}\ F[h^g]\ \det\frac{\partial P\cdot\epsilon,\ \tilde{\Lambda}}{\partial\epsilon,\ \Lambda}.$$

$$\text{where} \quad \det\frac{\partial P\cdot\epsilon,\ \tilde{\Lambda}}{\partial\epsilon,\ \Lambda} = \begin{vmatrix} P & 0 \\ * & 1 \end{vmatrix} = \det P$$

$$Z = \int \mathcal{D}\zeta\, \mathcal{D}X \;\det P \exp\left[iS[X,\; \hat{h}^{\zeta}]\right]$$

where $S[X, \hat{h}^{\zeta}] = S[X^{\zeta^{-1}}, \hat{h}]$. **If and only if** the functional measure is also invariant:

$$\int \mathcal{D}\zeta \mathcal{D}X^{\zeta^{-1}}\; e^{iS\left[X^{\zeta^{-1}},\, \hat{h}\right]} \det P \to \int \mathcal{D}X\; e^{iS[X,\, \hat{h}]} \det P.$$

with $X^{\zeta^{-1}} \to X$.

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with $X^{\zeta^{-1}} \to X$.
In general it's invariant only under diffeomorphisms, not under Weyl rescalings. Later we will derive the conditions for the absence of total Weyl anomaly in the quantum measure. Using the FP method:

$$\det P = \int \mathcal{D}b_{(ab)} \mathcal{D}c^{d} \exp\left(\frac{1}{4\pi} \int d^2\sigma \sqrt{-\hat{h}} b^{ab} P^{d}_{ab} c_d\right)$$

FP ghosts

Here $b_{(ab)}(\sigma^a)$ transforms as a symmetric traceless tensor on the world sheet and $c^d(\sigma^a)$ as a vector. As result

$$Z = \int \mathcal{D}X\mathcal{D}b\mathcal{D}c\; e^{i(S_X+S_g)}$$

where:

$$S_g = -\frac{i}{2\pi}\int d^2\sigma\sqrt{-\hat{h}}\hat{h}^{ab}c^d D_a b_{(ab)}.$$

The equation of motion $P\cdot c = 0$ means that c^a correspond to conformal Killing vectors.

$$\partial_+ b_{--} = \partial_- b_{++} = 0, \qquad \partial_+ c^- = \partial_- c^+ = 0$$

provided the boundary terms obey

$$\int d\tau\,(c^+\delta b_{++} - c^-\delta b_{--})\Big|_{\sigma=0}^{\sigma=\pi} = 0.$$

- **Closed string**: $b(\sigma + \pi) = b(\sigma)$ and $c(\sigma + \pi) = c(\sigma)$. The most general solution:

$$b_{++} = 4 \sum_n \tilde{b}_n e^{-2in\sigma_+}, \qquad b_{--} = 4 \sum_n b_n e^{-2in\sigma_-},$$

$$c^+ = \frac{1}{2} \sum_n \tilde{c}_n e^{-2in\sigma_+}, \qquad c^- = \frac{1}{2} \sum_n c_n e^{-2in\sigma_-}.$$

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- **Open string**: Boundary terms must vanish at $\sigma = 0$ and $\sigma = \pi$ separately:

$$c^+(\sigma^+)\Big|_{\sigma=0,\,\pi} = c^-(\sigma^-)\Big|_{\sigma=0,\,\pi} \qquad b_{++}(\sigma^+)\Big|_{\sigma=0,\,\pi} = b_{--}(\sigma^-)\Big|_{\sigma=0,\,\pi}$$

  The most general solution:

$$c^\pm = \sum_n c_n e^{-in\sigma^\pm}, \qquad b^{\pm\pm} = \sum_n b_n e^{-in\sigma^\pm}$$

Canonical quantization:

$$\{c_m,\ b_n\} = \delta_{m+n,0}, \quad \{c_m,\ c_n\} = \{b_m,\ b_n\} = 0$$

with the same relations obeyed in addition by $\tilde{c}'_n$, $\tilde{b}_n$ for the closed string.

$$T^{(g)}_{++} = -i\big(2b_{++}\partial_+ c^+ + (\partial_+ b_{++})c^+\big) = \frac{1}{2}\sum_n \tilde{L}^{(g)}_n e^{-2in\sigma},$$

$$T^{(g)}_{--} = -i\big(2b_{--}\partial_- c^- + \partial_- b_{--}\big)c^-\big) = \frac{1}{2}\sum_n L^{(g)}_n e^{-2in\sigma},$$

with Virasoro generators for the closed string:

$$L^{(g)}_n = -\frac{1}{4\pi}\int_0^\pi d\sigma e^{-2in\sigma}\, T_{--}, \quad \tilde{L}^{(g)}_n = -\frac{1}{4\pi}\int_0^\pi d\sigma e^{2in\sigma}\, T_{++}$$

and similar formulae for the open string.

$$L_m^{(g)} = \sum_n (m-n) b_{m+n} c_{-n}$$

is valid classically. Due to the anti-commuting nature of the modes, we pick up a minus sign in this process.

$$: b_m b_n := \begin{cases} b_m b_n, & \text{if } m \leq n \\ -b_n b_m, & \text{if } n < m. \end{cases}$$

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$$L_m^{(g)} = \sum_n (m-n) : b_{m+n} c_{-n} :$$

At the quantum level:

$$\left[L_m^{(g)}, L_n^{(g)}\right] = (m-n) L_{m+n}^{(g)} + \frac{1}{6}(m - 13m^3)\delta_{m+n,0}.$$

The generators of the conformal transformations of $S = S_X + S_g$:

$$L_m^{tot} = L_m^{(X)} + L_m^{(g)} - a^{tot}\delta_{m,0},$$

with a^{tot} as the total normal ordering constant in the definition of L_m^{tot},

$$a^{tot} = a^{(X)} + a^{(g)} = a.$$

$$c^{tot} = c^{(X)} + c^{(g)} = d - 26$$

$$[L_m^{tot}, L_n^{tot}] = (m-n)L_{m+n}^{tot} + \delta_{m+n,0}\left(\frac{c^{tot}}{12}(m^3 - m) + 2m(a-1)\right)$$

To eliminate the anomaly in the Weyl invariance:

$$d = 26, \qquad a = 1.$$

Thus criticality arises as a self-consistency requirement of the Faddeev-Popov treatment.

BRST symmetry

Definition

$S_X + S_g$ after gauge fixing $h_{ab} = \eta_{ab}$ enjoys a global fermionic symmetry. Let ϵ be a constant Grassmann parameter. There is invariance under

$$\begin{aligned} \delta_\epsilon X^\mu &= \epsilon(c^+\partial_+ + c^-\partial_-)X^\mu, \\ \delta_\epsilon c^\pm &= \epsilon(c^+\partial_+ + c^-\partial_-)c^\pm, \\ \delta_\epsilon b_{\pm\pm} &= i\epsilon(T^{(x)}_{\pm\pm} + T^{(g)}_{\pm\pm}). \end{aligned}$$

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A conserved charge Q_B is associated the symmetry, it satisfies:

$$\delta_\epsilon X^\mu = \epsilon[Q_B,\ X^\mu], \quad \delta_\epsilon c^\pm = \epsilon\{Q_B,\ c^\pm\}, \quad \delta_\epsilon b_{\pm\pm} = \epsilon\{Q_B,\ b_{\pm\pm}\}.$$

One can show explicitly that (for open strings):

$$Q_B = \sum_m : (L_{-m}^{(X)} + L_{-m}^{(g)} - a\delta_{m,0})c_m :$$

does the job.

$$\delta_\epsilon \delta_{\epsilon'} \Phi = 0 \quad \text{for } \Phi \in \{X^\mu,\ b,\ c\},$$

which means that the charge is nilpotent $Q_B^2 = 0$. By an explicit computation:

$$Q_B^2 = \frac{1}{2}\{Q_B,\ Q_B\} = \frac{1}{2}\sum_{m,n}([L_m^{tot}, L_n^{tot}] + (m-n)L_{m+n}^{tot})c_{-m}c_{-n},$$

which vanishes if and only if the full Virasoro algebra is non-anomalous.

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Consistency of the BRST symmetry is equivalent to absence of the total Weyl anomaly.

Superstring theories

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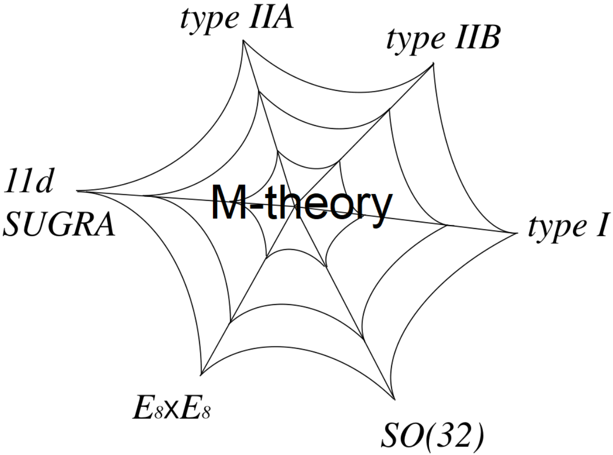

RNS formalism: World-sheet SUSY

In the RNS formalism we study supersymmetry on the world-sheet. The simpliest case is with $\mathcal{N} = 1$ supersymmetry.

$$S = -\frac{1}{2\pi}\int d^2\sigma \left[h^{ab}\partial_a X^\mu \partial_b X^\nu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right]$$

where ψ^μ_A is a D-plet of Majorana spinors, ρ^a are 2-d Dirac matrices.

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where ψ^μ_A is a D-plet of Majorana spinors, ρ^a are 2-d Dirac matrices. In light cone coordinates as

$$S=\frac{1}{\pi}\int d^2\sigma(2\partial_+ X\cdot\partial_- X + i\psi_-\cdot\partial_+\psi_- + i\psi_+\cdot\partial_-\psi_+)$$

We may derive equations of motion for the fermionic coordinates:

$$\partial_+\psi_-=0 \quad \text{and} \quad \partial_-\psi_+.$$

Boundary conditions

$$\delta S \sim \int d\tau \left(\psi_+ \delta\psi_+ - \psi_- \delta\psi_- \Big|_{\sigma=\pi} - \psi_+ \delta\psi_+ - \psi_- \delta\psi_- \Big|_{\sigma=0} \right) = 0.$$

For open strings $\psi_+^\mu = \pm\psi_-^\mu$ at each end of the string.

- **Ramond sector**: space-time fermions for open strings

$$\psi_+^\mu = \psi_-^\mu \quad \text{at } \sigma = \pi.$$

$$\psi_-^\mu(\sigma,\, \tau) = \frac{1}{\sqrt{2}} \sum_n d_n^\mu e^{-in(\tau-\sigma)}, \quad \psi_+^\mu(\sigma,\, \tau) = \frac{1}{\sqrt{2}} \sum_n d_n^\mu e^{-in(\tau+\sigma)}.$$

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- **Neveu-Schwarz sector**: Space-time bosons.

$$\psi_+^\mu = -\psi_-^\mu \quad \text{at } \sigma = \pi.$$

$$\psi_-^\mu(\sigma,\, \tau) = \frac{1}{\sqrt{2}} \sum_r b_r^\mu e^{-ir(\tau-\sigma)}, \quad \psi_+^\mu(\sigma,\, \tau) = \frac{1}{\sqrt{2}} \sum_r b_r^\mu e^{-ir(\tau+\sigma)}.$$

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu + \frac{1}{4}\bar{\bar{\psi}}^\mu \rho_a \partial_b \psi_\mu + \frac{1}{4}\bar{\bar{\psi}}^\mu \rho_\beta \partial_a \psi_\mu - \text{trace}.$$

The action is also invariant under a transformation

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \quad \text{and} \quad \delta\psi^\mu = \rho^a \partial_a X^\mu \epsilon$$

where ϵ is a constant infinitesimal Majorana spinor. There:

$$\delta S \sim \int d^2\sigma \ (\partial_a \bar{\epsilon}) J^a, \quad \text{where} \quad J_A^a = -\frac{1}{2}(\rho^b \rho^a \psi_\mu)_A \partial_\beta X^\mu.$$

$$T_{++} = \partial_+ X_\mu \partial_+ X^\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\mu},$$

$$T_{--} = \partial_- X_\mu \partial_- X^\mu + i\frac{i}{2}\psi_-^\mu \partial_- \psi_{-\mu},$$

$$J_+ = \psi_+^\mu \partial_+ X_\mu \quad \text{and} \quad J_- = \psi_-^\mu \partial_- X_\mu.$$

The super-Virasoro generators are modes of the energy momentum tensor T_{ab} and supercurrent J^a_A. For the open string:

$$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma} T_{++} = L_m^{(b)} + L_m^{(f)}$$

- The bosonic mode contribution is given by:

$$L_m^{(b)} = \frac{1}{2}\sum_r : \alpha_{-n} \cdot \alpha_{m+n} : \ .$$

- The contribution from the NS fermionic modes:

$$L_m^{(f)} = \frac{1}{2}\sum_r (r + \frac{m}{2}) : b_{-r} \cdot b_{m+r} : \ .$$

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma \ e^{ir\sigma} J_+ = \sum_n \alpha_{-n} \cdot b_{n+r}.$$

This results in a super-Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m+n,0},$$
$$[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r},$$
$$\{G_r,\ G_s\} = 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.$$

In the *NS* sector the physical-state conditions are:

$$G_r\,|\phi\rangle = 0, \quad L_m\,|\phi\rangle = 0, \quad (L_0 - a_{NS})\,|\phi\rangle = 0.$$

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$$L_m^{(f)} = \frac{1}{2}\sum_n (n + \frac{m}{2}) : d_{-n} \cdot d_{m+n} : \ .$$

$$F_m = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma \;\, e^{im\sigma} J_+ = \sum_n \alpha_{-n} \cdot d_{n+m}$$

with super-Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8} m^3 \delta_{m+n,0},$$
$$[L_m, F_n] = \left(\frac{m}{2} - n\right) F_{m+n},$$
$$\{F_m,\;\, F_n\} = 2L_{m+n} + \frac{D}{8} m^2 \delta_{m+n,0}.$$

with physical states determined from:

$$F_n \left|\phi\right\rangle = 0, \quad L_m \left|\phi\right\rangle = 0, \quad (L_0 - a_R) \left|\phi\right\rangle = 0.$$

R-S field

Invariance under local SUSY transformations and local superconformal transformations requires the Rarita-Schwinger field

$$\chi_a = i\rho_a\eta + D_a\epsilon$$

ϵ is an infinitesimal Majorana spinor (SUSY transf.), and η an arbitrary Majorana spinor (superconformal transf.).

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Transformations of the gravitino

χ_{aA} has four components of spin ± 1, $\pm 1/2$, i.e., $3/2$, $1/2$, $-1/2$ and $-3/2$ respectively. The gauge parameters η and ϵ are spinors with two components of spin $\pm 1/2$. The derivative operator D_a is a vector with components of spin ± 1.

$$\delta\chi_{3/2} = D_1\epsilon_{1/2}, \quad \delta\chi_{1/2} = D_1\epsilon_{-1/2} + \eta_{1/2};$$
$$\delta\chi_{-3/2} = D_{-1}\epsilon_{-1/2}, \quad \delta\chi_{-1/2} = D_{-1}\epsilon_{1/2} + \eta_{-1/2}.$$

Path integrals from the gravitino

The change of variables from $\chi_{\pm 1/2}$ to $\eta_{\pm 1/2}$ introduces a non trivial Jacobian, for example from $\chi_{3/2}$ to $\epsilon_{1/2}$ the Jacobian is:

$$\text{Jacobian}_{3/2} = 1/\det\left[D_1^{1/2\to 3/2}\right],$$

where the superscript $1/2 \to 3/2$ represents the operator mapping spin $1/2$ to spin $3/2$. We can represent this determinant by ghost fields $\gamma_{1/2}$ and $\beta_{-3/2}$:

$$\text{Jacobian}_{3/2} = \int \mathcal{D}\gamma_{1/2}\mathcal{D}\beta_{-3/2} \exp\left[-\frac{1}{\pi}\int d^2\sigma \beta_{-3/2} D_1 \gamma_{1/2}\right].$$

γ and β must be commuting fields since they are ghosts for an anticommuting symmetry.

Ghosts

Likewise a change of variables from $\chi_{-3/2}$ to $\epsilon_{-1/2}$ gives

$$\text{Jacobian}_{-3/2} = \int \mathcal{D}\gamma_{-1/2}\mathcal{D}\beta_{3/2} \exp\left[-\frac{1}{\pi}\int d^2\sigma \beta_{3/2} D_1 \gamma_{-1/2}\right].$$

The fields γ and β are called superconformal ghosts. The components of $\gamma_{\pm 1/2}$ make up a spinor γ_A. The components of $\beta_{\pm 3/2}$ make up a vector-spinor β_{aA} which is subject to the constraints $\rho^{aAB}\beta_{aB} = 0$.

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The ghost action can be thus expressed as:

$$S_{FP} = -\frac{i}{2\pi}\int d^2\sigma h^{ab}\bar{\gamma}\partial_a\beta_b.$$

and also

$$T_{++} = \frac{i}{2}\gamma\partial_+\beta + \frac{3i}{2}\beta\partial_+\gamma, \qquad J_+^{(g)} = \beta\gamma.$$

In terms of mode expansions:

$$\gamma(\tau) = \frac{1}{\sqrt{2}} \sum_n \gamma_n e^{-2in\tau}, \qquad \beta(\tau) = \frac{1}{\sqrt{2}} \sum_n \beta_n e^{-2in\tau}.$$

The commutation relations implied by S_{FP} are:

$$[\gamma_m, \beta_n] = \delta_{m+n,0}, \qquad [\gamma_m, \gamma_n] = [\beta_m, \beta_n] = 0.$$

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Until now we have considered the R sector of open strings. For the NS sector we require half integer modes γ_r, β_s.

$$L_m^{(g)} = \sum_n \left[(m+n) : b_{m-n} c_n : + \left(\frac{1}{2} m + n \right) : \beta_{m-n} \gamma_n : \right]$$

$$F_m^{(g)} = -2 \sum_n \left[b_{-n} \gamma_{m+n} + \left(\frac{1}{2} n - m \right) c_{-n} \gamma_{m+n} \right]$$

The relation

$$\left\{F_m^{(g)},\ F_n^{(g)}\right\} = 2L_{m+n}^{(g)} - 5m^2\delta_{m+n,0}$$

implies the ghost anomaly $-5m^2$. We found before that the anomaly in the R sector is given by $\frac{1}{2}Dm^2 + 2a_R$. Therefore the the total anomaly

$$c_R^{tot} = -5m^2 + \frac{1}{2}Dm^2 + 2a_R$$

cancels for $D = 10$ and $a_R = 0$. In the NS sector one has

$$c_{NS}^{tot} = c^{(g)} + c^{(X,\psi)} = \left[\frac{1}{4} - 5r^2\right] + \left[\frac{1}{2}D\left(r^2 - \frac{1}{4}\right) + 2a_{NS}\right]$$

which vanishes only for $D = 10$ and $a_{NS} = 1/2$.

Summary

Now we have ten X^μ boson, ten ψ^μ fermions, conformal ghosts b, c and superconformal ghost γ and β.

The conditions for eliminating the Weyl anomaly in the theory is when $D = 10$, $a_R = 0$ and $a_{NS} = \frac{1}{2}$, and this allows the BRST symmetry with conserved supercharge Q_B.