Derivadas Funcionais

(Peskin pg 289, Ryder 5.4)

Função: $\mathbb{R}^n \longmapsto \mathbb{R}^m$

Funcional: $C^\infty(\underline{M}) \longmapsto \mathbb{R}$ NOTAÇÃO $f(x_n) \longrightarrow F[f]$

$\hookrightarrow$ e.g. : $\mathbb{R}, \mathbb{R}^3, \ldots$

Ex: (1) $F(f) = \int_{-\infty}^{+\infty} f(x)\,dx$

(2) $F_{\underline{a}}[f] = f^2(a)$

$\hookrightarrow$ PARÂMETRO

Derivada Funcional:

podemos definir diferencial de funções por: $df = \lim_{\epsilon \to 0} \frac{f(x+\epsilon h) - f(x)}{\epsilon} = a h \quad \forall h \in \mathbb{R}$

$(\mathbb{R} \longmapsto \mathbb{R})$

$$\lim_{\epsilon \to 0} \frac{f(x+\epsilon h) - f(x)}{\epsilon} = \overbrace{\lim_{\eta \to 0} \frac{f(x+\eta) - f(x)}{\eta}}^{a} h$$

$$h = 1 \Rightarrow df = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} = \boxed{a = \frac{df}{dx}}$$

- Em $\mathbb{R}^n \longmapsto \mathbb{R}$

$$f = f(x_1, x_2, \ldots)$$

DIFERENCIAL: $df = \lim_{\epsilon \to 0} \frac{f(\vec{x} + \epsilon \vec{h}) - f(\vec{x})}{\epsilon} = \vec{a} \cdot \vec{h}$

$$\vec{h} = (0,0,0,\ldots,\underset{i}{1},\ldots) \Rightarrow h_j = \delta_{ij} \Rightarrow \vec{a}\cdot\vec{h} = a_j h_j = = a_i$$

$$\Downarrow$$

$$df = \lim_{\epsilon\to 0} \frac{f(x_1,x_2,\ldots,x_i+\epsilon,\ldots) - f(\vec{x})}{\epsilon} = a_i = \underbrace{\frac{df}{dx_i}}_{\text{DERIVADAS}} \Rightarrow \boxed{\vec{a} = \vec{\nabla} f}$$

- Em $C^\infty(\mathbb{R}) \longmapsto \mathbb{R}$

$$F[f] \Rightarrow \delta F = \lim_{\epsilon\to 0} \frac{F[f+\epsilon h] - F[f]}{\epsilon} = \int a(x) h(x)\, dx$$

LINEAR! (FRÉCHET DERIVATIVE)

$$h = \delta_y \Rightarrow h(x) = \delta(x-y) \Rightarrow \int a(x) h(x)\, dx = a(y)$$

$$\delta F = \lim_{\epsilon\to 0} \frac{F[f+\epsilon\delta_y] - F[f]}{\epsilon} = \boxed{a(y) \equiv \frac{\delta F}{\delta f(y)}}$$

$$\boxed{\frac{\delta F}{\delta f(y)} = \lim_{\epsilon\to 0} \frac{F[f+\epsilon\delta_y] - F[f]}{\epsilon}} \quad \text{(eq. 1)}$$

$$\boxed{\delta F = \int \frac{\delta F}{\delta f(x)} h(x)\, dx = \lim_{\epsilon\to 0} \frac{F[f+\epsilon h] - F[f]}{\epsilon}} \quad \text{(eq. 2)}$$

$\boxed{\text{Ex:}}$ $F[f] = \int_0^1 f^2(x)\, dx$

$$(2) \Rightarrow F[f+\epsilon h] = \int_0^1 \left[f^2(x) + 2\epsilon f(x) h(x) + \cancel{O(\epsilon^2)} \right] dx =$$

$$\delta F = \frac{F[\rho+\epsilon h] - F[\rho]}{\epsilon} = \int_0^1 2\rho(x)\, h(x)\, dx$$

$$\frac{\delta F}{\delta \rho(x)} = 2\rho(x)$$

$$(1) \Rightarrow \frac{\delta F}{\delta \rho(y)} = \lim_{\epsilon \to 0} \frac{F[\rho(x) + \epsilon\, \delta(x-y)] - F[\rho(x)]}{\epsilon} =$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^1 \left[2\epsilon \rho(x)\, \delta(x-y) + O(\epsilon^2) \right] = 2\rho(y)$$

$$\delta^2(x-y)$$

$\boxed{\text{Ex 2:}}$ $F[\rho] = \int \rho(x)\, dx$

$$\frac{\delta F[\rho]}{\delta \rho(y)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \int [\rho(x) + \varepsilon\, \delta(x-y)]\, dx - \int \rho(x)\, dx \right\} =$$

$$= \int \delta(x-y)\, dx = 1$$

$\boxed{\text{Ex 3:}}$ $F_x[\rho] = \int G(x,y)\, \rho(y)\, dy$

$$\frac{\delta F_x[\rho]}{\delta \rho(z)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int G(x,y)\, \varepsilon\, \delta(y-z)\, dy = G(x,z)$$

$\boxed{\text{Ex 4:}}$ $F_a[\rho] = \rho(a)$ a fixo

$$\frac{\delta F_a[\rho]}{\delta \rho(x)} = \lim_{\varepsilon \to 0} \frac{F_a[\rho(y) + \varepsilon\, \delta(y-x)] - F_a[\rho(y)]}{\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \frac{f(a) + \varepsilon\delta(a-x) - f(a)}{\varepsilon} = \delta(a-x)$$

$$f(x) = F_x[f] \Longrightarrow \frac{\delta f(x)}{\delta f(y)} = \delta(x-y)$$

$\boxed{Ex\ 5:}$ $F_x[f] = g(f(x))$

$$\frac{\delta F_a[f]}{\delta f(y)} = \lim_{\varepsilon \to 0} \frac{g(f(x) + \varepsilon\delta(x-y)) - g(f(x))}{\varepsilon}$$

$$g[f(x) + \varepsilon\delta(x-y)] \simeq g(f(x)) + \varepsilon\delta(x-y) \left.\frac{dg}{df}\right|_{f=f(x)} + \dots$$

$$\frac{\delta g(f(x))}{\delta f(y)} = \delta(x-y) \left.\frac{dg}{df}\right|_{f=f(x)}$$

$\boxed{Ex\ 6:}$ $F[y,z] = \int y^2(x) z^4(x)\, dx$

$$\frac{\delta F[y,z]}{\delta y(x)} = 2y(x) z^4(x) \qquad \frac{\delta F[y,z]}{\delta z(x)} = 4y(x) z^3(x)$$

$\boxed{Ex\ 7:}$ $C^\infty(\mathbb{R}^n) \longmapsto \mathbb{R}$

$$\frac{\delta g(f(x_1,x_2,x_3))}{\delta f(y_1,y_2,y_3)} = \delta^3(\vec{x}-\vec{y}) \left.\frac{dg}{df}\right|_{f=f(x_1,x_2,x_3)}$$

$\boxed{\text{Ex 8 :}}$

$$\frac{\delta g(f(k, x_2, x_3))}{\delta f(k, y_2, y_3)} = \delta(x_2 - y_2)\,\delta(x_3 - y_3)\,\frac{\partial g}{\partial f}\bigg|_{f = f(k, x_2, x_3)}$$

$$\therefore\; f(k, x_2, x_3) = f'(x_2, x_3)$$

PARÂMETRO VARIÁVEL

PARÂMETRO FIXO

$$\frac{\delta f(\phi(\vec{x}_i, x^0), \pi(\vec{x}_i, x^0))}{\delta \phi(\vec{x}_j, x^0)} = \delta^3(\vec{x}_i - \vec{x}_j)\,\frac{\partial f(\phi, \pi)}{\partial \phi}\bigg|_{\phi = \phi(\vec{x}_i, x^0)}$$