Casimir Effect in a Nutshell

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“It might come as surprise that both the gecko’s ability to walk across ceilings in apparent defiance of gravity and the evaporation of black holes through Hawking radiation can be understood as arising from zero-point fluctuations of quantum fields”

Steve K. Lamoreaux, Yale Professor in New Haven, Connecticut.

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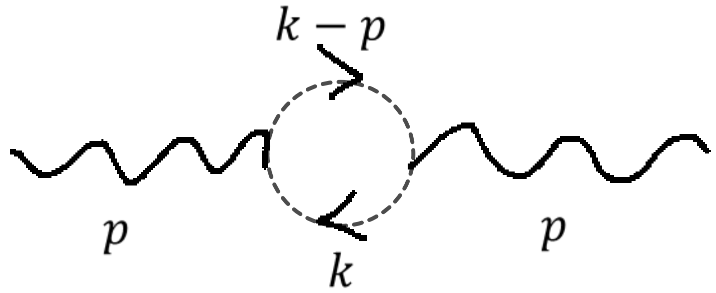
Casimir Effect: Motivation

Let´s consider the vacum polarization diagram in scalar QED

$i\mathcal{M}$ =



$$= e^2 \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu - p^\mu}{(k-p)^2 - m^2 + i\varepsilon} \frac{2k^\nu - p^\nu}{k^2 - m^2 + i\varepsilon}$$

In the region where $k^\mu \gg p^\mu$, we get:

$$i\mathcal{M} \sim 4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{k^4} \sim \int k\,dk = \infty$$

This integral diverges because this loop is not by itself measurable. Measurable quantities must come out finite, so we have to deform the theory in such a way that the integral comes out finite, depending on some regulating parameter, and then after all of them are put together, the answer turns out to be independent of the regultaor!

Casimir Effect: Motivation

To warm up, recall that for a free scalar field, the free Hamiltonian is:

$$H = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\omega_k \left(a_k^\dagger a_k + \frac{1}{2}V \right)$$

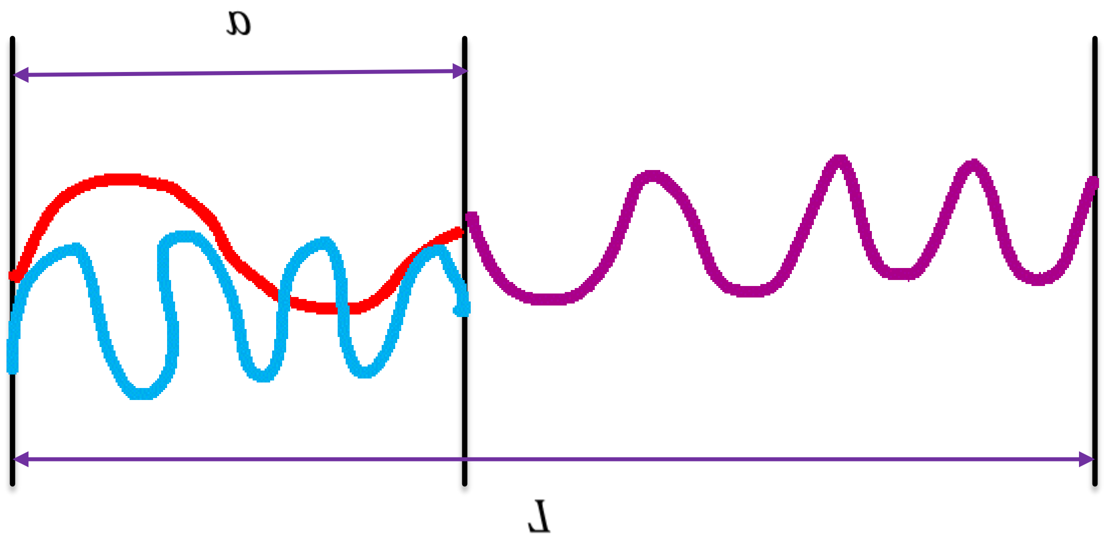
Where $\omega_k = \left|\vec{k}\right|$. So, in QED the contribution to vacuum energy of the photon zero modes are:

$$E =< 0|H|0 > = V \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\omega_k}{2} = \frac{\mathrm{V}}{4\pi^2}\int k^3\,\mathrm{d}k = \infty$$

That’s the zero point energy. This result is not measurable, so we must consider the scalar theory in other contexto than just sitting there in the vacum.

Casimir Effect: Motivation

Consider the following situation:



If energy changes with a, than there is a $F = -\frac{\mathrm{d}E}{\mathrm{d}a}$.

Now we realize that changing “a” the energy inside and outside the box will change, and we have to deal of all space again.

Casimir Effect: Motivation

Working in a one-dimensional box of size r for simplicity, and scalar field instead of photon's, the classicaly quantized frequencies are $\omega_n = n\frac{\pi}{r}$. The integrl on quantum Hamiltonian becomes a discrete series:

$$E(r) = <0|H|0> = \sum_n \frac{\omega_n}{2}$$

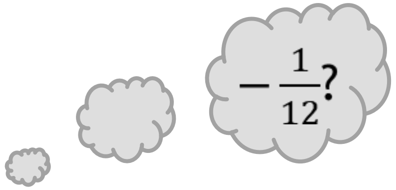
The total energy is the sum of the right side $r = L - a$ plus the left side $r = a$:

$$E_{tot}(a) = E(a) + E(L-a) = \left(\frac{1}{a} + \frac{1}{L-a}\right)\frac{\pi}{2}\sum_{n=1}^{\infty} n$$

We do not expect to get a finite value to the total energy, of course, but we do hope to find a finite value for the force:

$$F(a) = -\frac{dE_{tot}}{da} = \left(\frac{1}{a^2} + \frac{1}{(L-a)^2}\right)\frac{\pi}{2}\sum_{n=1}^{\infty} n$$

Casimir Effect: Motivation



For $L \to \infty$ we get:

$$F(a) = \frac{\pi}{2}\frac{1}{a^2}(1 + 2 + 3 + \cdots) = \infty$$

Sadly, the plates are infinitely repulsive.

Where was the error?

Physics! Whitout interaction nothing can be measured. We are missing the fact that the plates are made of real materials and practically “transparent” to very small wavelengths.

Let’s correct this and proceed.

Hard Cutoff

Instead of dive in the detailed physics of the plates, it is easier to employ effective approximations.

Consider some high-frequency cutoff Λ so that $\omega < \pi\Lambda$. It looks like a good approximation $\Lambda \sim \frac{1}{atomic\ size}$, then:

$$n_{\max}(r) = \Lambda r$$

$$E(r) = \frac{1}{r}\frac{\pi}{2}\sum_{n=1}^{n_{\max}} n = \frac{\pi}{2r}\frac{n_{\max}(n_{\max}+1)}{2} = \frac{\pi}{4\mathrm{r}}(\Lambda r)(\Lambda r + 1) = \frac{\pi}{2}(\Lambda^2 r + \Lambda)$$

Then:

$$E_{tot}(a) = E(a) + E(L-a) = \frac{\pi}{4}(\Lambda^2 L + 2\Lambda)$$

Now the energy continues to be infinite, but independente of a, so $F = -\frac{\mathrm{d}E_{tot}}{\mathrm{d}a} = 0$

Hard Cutoff

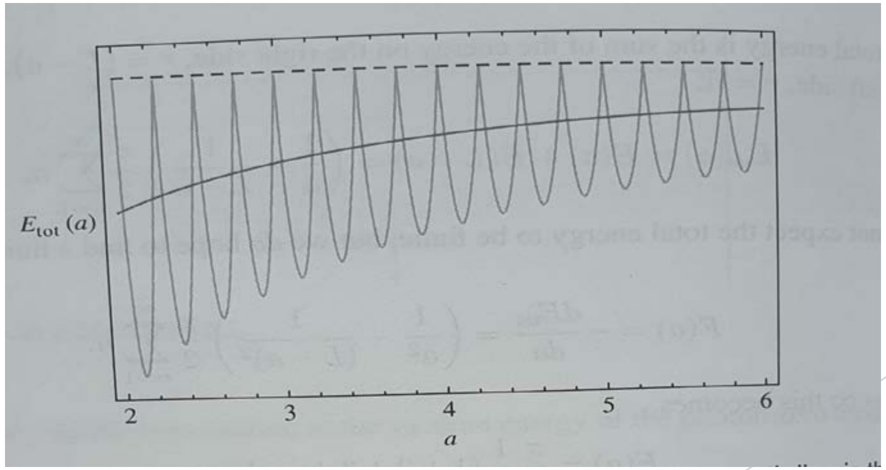
We've not dealt correctly with the calculation i.e. the hard cutoff means a mode is included or not, thus $n_{\max}$ changes discretely, but r is continuous. We can write this fact as a floor function:

$$n_{\max}(r) = \lfloor \Lambda r \rfloor$$

Where $\lfloor x \rfloor$ means the greatest integer less than x. So the sums is:

$$E(r) = \frac{\pi}{4\mathrm{r}} \lfloor \Lambda r \rfloor (\lfloor \Lambda r \rfloor + 1)$$

Now the total energy oscillates with a.



Hard Cutoff

To deal with that, define a number x :

$$x = \Lambda a - \lfloor \Lambda r \rfloor \in [0,1)$$

Which gives:

$$E(a) = \frac{\pi}{4}\left[\Lambda^2 \mathrm{a} + \Lambda - 2\Lambda \mathrm{x} - \frac{x(1-x)}{a}\right]$$

We can also take ΛL to be an integer because L has na arbitrary size that doesn't change when we move the wall in a. Then $\lfloor \Lambda L - \Lambda a \rfloor = \Lambda L - \lceil \Lambda a \rceil$. For simplicity let us assume Λa is not an integer, which let us use $\lceil \Lambda a \rceil = \lfloor \Lambda a \rfloor + 1$, so:

$$E(L-a) = \frac{\pi}{4}\left[\frac{(\Lambda L - \lceil \Lambda a \rceil)(\Lambda L - \lceil \Lambda a \rceil + 1)}{L-a}\right]$$

$$= \frac{\pi}{4}\left[\Lambda^2 (\mathrm{L} - \mathrm{a}) - \Lambda + 2\Lambda \mathrm{x} - \frac{x(1-x)}{L-a}\right]$$

Hard Cutoff

And:

$$E_{tot}(a) = E(a) + E(L-a) = \frac{\pi}{4}\left[\Lambda^2 \mathrm{L} - \frac{x(1-x)}{a} - \frac{x(1-x)}{L-a}\right]$$

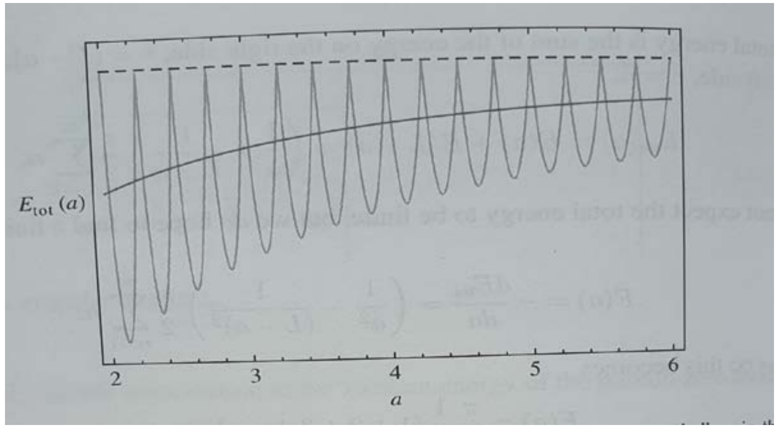
We can see $\Lambda^2\mathrm{L}$ term is the "extrinsic" energy of the whole system, which does not contribute to the force, and other term that oscillates as x goes between 0 and 1. Keeping only the terms up to L, the total energy is:

$$E_{tot}(a) = \frac{\pi}{4}\Lambda^2\mathrm{L} - \frac{\pi}{4a}x(1-x)$$

Since $x = \Lambda a - \lfloor \Lambda r \rfloor$, as $\Lambda \to \infty$ at fixed a, there are more and more oscillations. In the continuum limit ($\Lambda \to \infty$) the plates will experience only the average force, thus, averaging x between 0 and 1 using $\int_0^1 x\,(1-x) = \frac{1}{6}$, so:

$$E_{tot}(a) \approx \frac{\pi}{4}\Lambda^2\mathrm{L} - \frac{\pi}{24a}$$

Hard Cutoff



The result is a finite and non-zero value for the force:

$$F(a) = -\frac{dE_{tot}}{da} = -\frac{\pi}{24a^2}$$

Remembering of the $\hbar$ and c:

$$F(a) = -\frac{\pi\hbar c}{24a^2}$$

In three dimensions, accounting for the two photon polarizations:

$$F(a) = -\frac{\pi^2\hbar c}{240a^4}A$$

Heat-Kernel Regularization

Another reasonable physical assumption besides the hard-cutoff would be the exponential supreession of high energy modes, resembling the classical eletromagnetic case. Let’s try:

$$\mathrm{E(r)} = \frac{1}{2}\sum_{n} \omega_n \mathrm{e}^{-\frac{\omega_n}{\pi \Lambda}}$$

That’s the heat-kernel regularization. Expanding using $\omega_n = \frac{\pi}{r} n$:

$$\mathrm{E(r)} = \frac{\pi}{2}\frac{1}{r}\sum_{n=1}^{\infty} n\mathrm{e}^{-\frac{n}{r\Lambda}} = \frac{\pi}{2}\frac{1}{r}\sum_{n=1}^{\infty} n\mathrm{e}^{-\varepsilon n} \qquad \varepsilon = \frac{1}{r\Lambda} \ll 1$$

Then we calculate:

$$\sum_{n=1}^{\infty} n\mathrm{e}^{-\varepsilon n} = -\partial_\varepsilon \sum_{n=1}^{\infty} \mathrm{e}^{-\varepsilon n} = -\partial_\varepsilon \frac{1}{1 - e^{-\varepsilon}} = \frac{e^{-\varepsilon}}{(1 - e^{-\varepsilon})^2} = \frac{1}{\varepsilon^2} - \underbrace{\frac{1}{12}}_{} + \frac{\varepsilon^2}{240} + \cdots$$

Heat-Kernel Regularization

So we get for the energy:

$$\mathrm{E(r)} = \frac{\pi}{2}\frac{1}{r}\left[\Lambda^2 r^2 - \frac{1}{12} + \frac{1}{240\Lambda^2 r^2} + \cdots\right] = \frac{\pi}{2}\Lambda^2 r - \frac{\pi}{24r} + \cdots$$

And for the force:

$$F(a) = -\frac{d}{da}[E(a) + E(L-a)] = -\frac{d}{da}\left[\frac{\pi}{2}\Lambda^2 L - \frac{\pi}{24}\left(\frac{1}{L-a} + \frac{1}{a}\right) + \cdots\right]$$

$$= \frac{\pi}{24}\left(\frac{1}{(L-a)^2} - \frac{1}{a^2}\right) + \cdots$$

Taking $L \to \infty$ and the old $\hbar$ and c again:

$$F(a) = -\frac{\pi\hbar c}{24a^2}$$

Which is the same thing as before, but the "extrinsic" energy was $\frac{\pi}{4}\Lambda^2\mathrm{L}$ and now is $\frac{\pi}{2}\Lambda^2\mathrm{L}$.

Other Regulators

Let's be creative and try other regulators:

$$E(r) = \frac{1}{2}\sum_n \omega_n e^{-\left(\frac{\omega_n}{\pi\Lambda}\right)^2}$$

Or a ζ −function regulator:

$$E(r) = \frac{1}{2}\sum_n \omega_n \left(\frac{\omega_n}{\mu}\right)^{-s}$$

Where we take $s \to \infty$ instead of $\omega_{\max} \to \infty$ and have added na arbitrary scale μ to keep the dimension correct. Using $\omega_n = \frac{\pi}{r} n$:

$$\sum n^{1-s} = \zeta(s-1) = -\frac{1}{12} - 0.165s + \cdots$$

Which is the definition of the ζ −function.

$$E(r) = \frac{1}{2}\left(\frac{\pi}{r}\right)^{1-s} \mu^s \sum_n n^{1-s}$$

Other Regulators

So we get:

$$E(r) = \frac{1}{r}\frac{\pi}{2}\zeta(s-1)\left[-\frac{1}{12} + O(s)\ldots\right]$$

And the energy come as:

$$E(r) = -\frac{\pi}{24r} + \cdots$$

This is the same as the heat-kernel and floor function gave, although now the “extrinsic” energy is absent. We summarize:

$E(r) = \frac{1}{2}\sum_n \omega_n \theta(\pi\Lambda - \omega_n)$	Hard-cutoff
$E(r) = \frac{1}{2}\sum_n \omega_n e^{-\frac{\omega_n}{\pi\Lambda}}$	Heat Kernel
$\mathrm{E(r)} = \frac{1}{2}\sum_n \omega_n \mathrm{e}^{-\left(\frac{\omega_n}{\pi\Lambda}\right)^2}$	Gaussian
$\mathrm{E(r)} = \frac{1}{2}\sum_n \omega_n \left(\frac{\omega_n}{\mu}\right)^{-s}$	ζ –function

Regulator-Independent Derivation

Casimir showed in his original paper a way to calculate the force in a regulator independente wy. Define the energy as:

$$\mathrm{E(a)} = \frac{\pi}{2}\sum_{n}\frac{n}{a}f\left(\frac{n}{a\Lambda}\right)$$

Where $f(x)$ is some function whose properties we will determine shortly. With this, the energy of the $L - \mathrm{a}$ side is:

$$\mathrm{E}(L-\mathrm{a}) = \frac{\pi}{2}(L-a)\Lambda^2\sum_{n}\frac{n}{(L-a)^2\Lambda^2}f\left(\frac{n}{(L-a)\Lambda}\right)$$

We can take the continuum limit $L \to \infty$ with $x = \frac{n}{(L-a)\Lambda}$. Then:

$$\mathrm{E}(L-\mathrm{a}) = \frac{\pi}{2}L\Lambda^2\int x\,\mathrm{d}x\,f(x) - \frac{\pi}{2}a\Lambda^2\int x\,\mathrm{d}x\,f(x)$$

Regulator-Independent Derivation

The first integral is just the "extrinsic" energy, with energy density:

$$\rho = \frac{\pi}{2}\Lambda^2 \int x \, \mathrm{d}x \, f(x)$$

In the energy expression, we simplify the second integral with the change of variables $\mathrm{x} = \frac{n}{\mathrm{a}\Lambda}$. Adding the discrete sum for the a side with the continuum limite in the $L - a$ one:

$$\mathrm{E(a)} = \frac{\pi}{2}\sum_n \frac{n}{a} f\left(\frac{n}{a\Lambda}\right) \quad + \quad \mathrm{E}(L - \mathrm{a}) = \frac{\pi}{2} L\Lambda^2 \int x \, \mathrm{d}x \, f(x) - \frac{\pi}{2} a\Lambda^2 \int x \, \mathrm{d}x \, f(x)$$

$$E_{tot}(a) = E(a) + E(L-a) = \rho L + \frac{\pi}{2a}\left[\sum_n nf\left(\frac{n}{a\Lambda}\right) - \int ndnf\left(\frac{n}{a\Lambda}\right)\right]$$

This contains the difference between an infinite sum and an infinite integral.

Regulator-Independent Derivation

Taking off the hat the Euler-Maclaurin series:

$$\sum_{n=1}^{N} F(n) - \int_{0}^{N} F(n)\,\mathrm{d}n = \frac{F(0)+F(N)}{2} + \frac{F'(N)-F'(0)}{12} + \cdots + B_j \frac{F^{j-1}(N)-F^{j-1}(0)}{j!} + \cdots$$

Where $F^{(j)}(N) = \frac{\mathrm{d}^j F(N)}{\mathrm{d}N^j}$ and B_j are the Bernoulli numbers. In particular $B_2 = \frac{1}{6}$ and B_j fo odd $j > 1$ vanishes. In our case $F(n) = nf\left(\frac{n}{a\Lambda}\right)$. So we $f(x)$ assume dies sufficiently fast.

$$\lim_{x\to\infty} x f^{(j)}(x) = 0$$

Then:

$$E_{tot} = \rho L - \frac{\pi f(0)}{24a} - \frac{B_4}{4!}\frac{3\pi}{2a^3\Lambda^2} f''(0) + \cdots$$

Regulator-Independent Derivation

For example, if $f(x) = e^{-x}$, then:

$$E_{tot} = \frac{\pi}{2}\Lambda^2 L - \frac{\pi}{24a} + O(\frac{1}{\Lambda a^2})$$

Which we know gives the Casimir force.

After all this talk about regulators, we conclude that any regulator will give the casimir force as long as

$$\lim_{x\to\infty} x f^{(j)}(x) = 0 \qquad \text{and} \qquad f(0) = 1$$

The first requirement ensures that UV (high energy modes) pass right through the box, making the force finite. The second one ensures the regulator does not affect the spectrum in the IR. On physical grounds, only $\frac{\pi}{2}$ modes of size $\frac{1}{a}$ can reach both walls of the box to transmit the force, thus our deformation should not affect those modes.

Casimir force is not depend on any regulator
Casimir force is na infrared effect

Is it a Consensus?

Think twice!

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Proof that Casimir force does not originate from vacuum energy

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ABSTRACT

We present a simple general proof that Casimir force cannot originate from the vacuum energy of electromagnetic (EM) field. The full QED Hamiltonian consists of 3 terms: the pure electromagnetic term $H_{\rm em}$, the pure matter term $H_{\rm matt}$ and the interaction term $H_{\rm int}$. The $H_{\rm em}$-term commutes with all matter fields because it does not have any *explicit* dependence on matter fields. As a consequence, $H_{\rm em}$ cannot generate any forces on matter. Since it is precisely this term that generates the vacuum energy of EM field, it follows that the vacuum energy does not generate the forces. The misleading statements in the literature that vacuum energy generates Casimir force can be boiled down to the fact that $H_{\rm em}$ attains an *implicit* dependence on matter fields by the use of the equations of motion and to the illegitimate treatment of the implicit dependence as if it was explicit. The true origin of the Casimir force is van der Waals force generated by $H_{\rm int}$.

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Casimir effect and the quantum vacuum

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In discussions of the cosmological constant, the Casimir effect is often invoked as decisive evidence that the zero-point energies of quantum fields are "real." On the contrary, Casimir effects can be formulated and Casimir forces can be computed without reference to zero-point energies. They are relativistic, quantum forces between charges and currents. The Casimir force (per unit area) between parallel plates vanishes as α, the fine structure constant, goes to zero, and the standard result, which appears to be independent of α, corresponds to the $\alpha \to \infty$ limit.

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Any doubt?