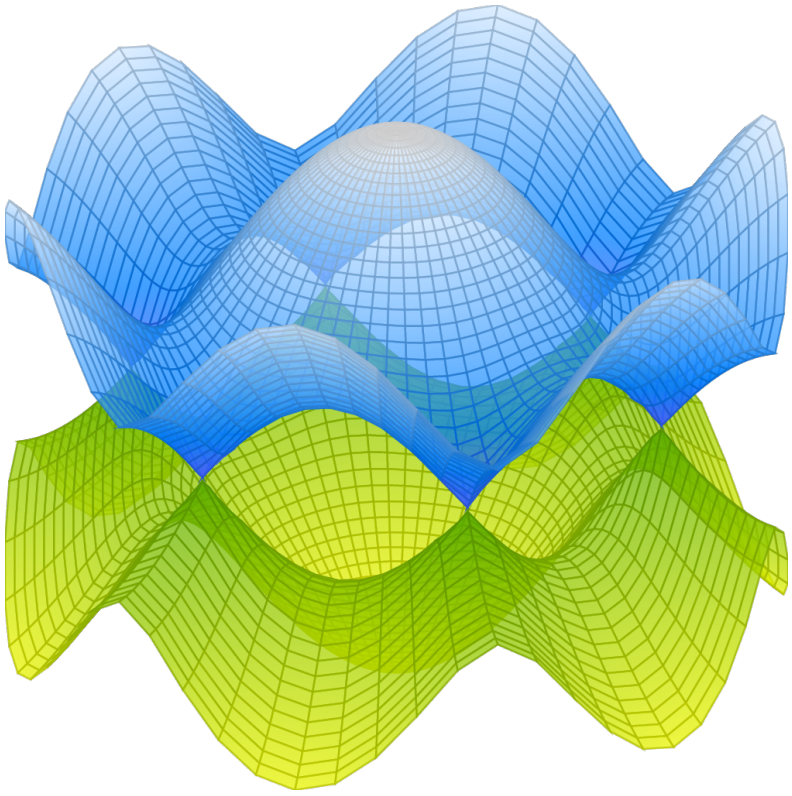
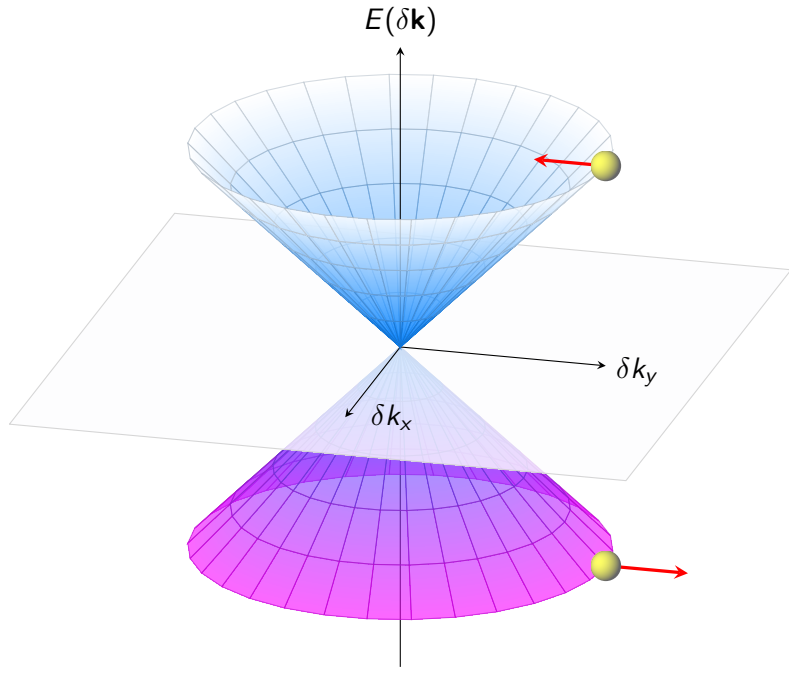
Dirac Spinors in Graphene: A Good Analogy?

Lars Dehlwes | IFT-UNESP





Graphene?

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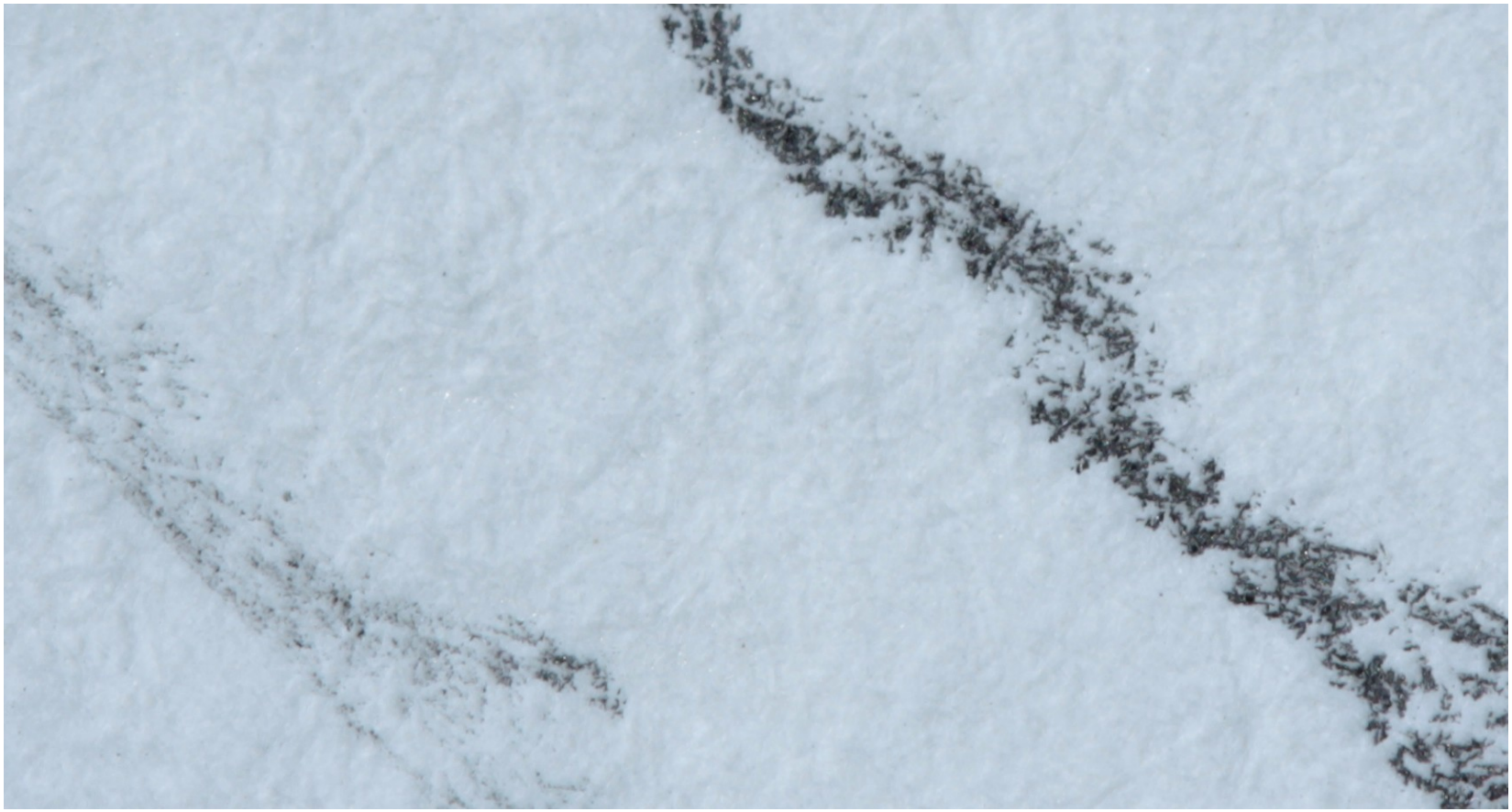

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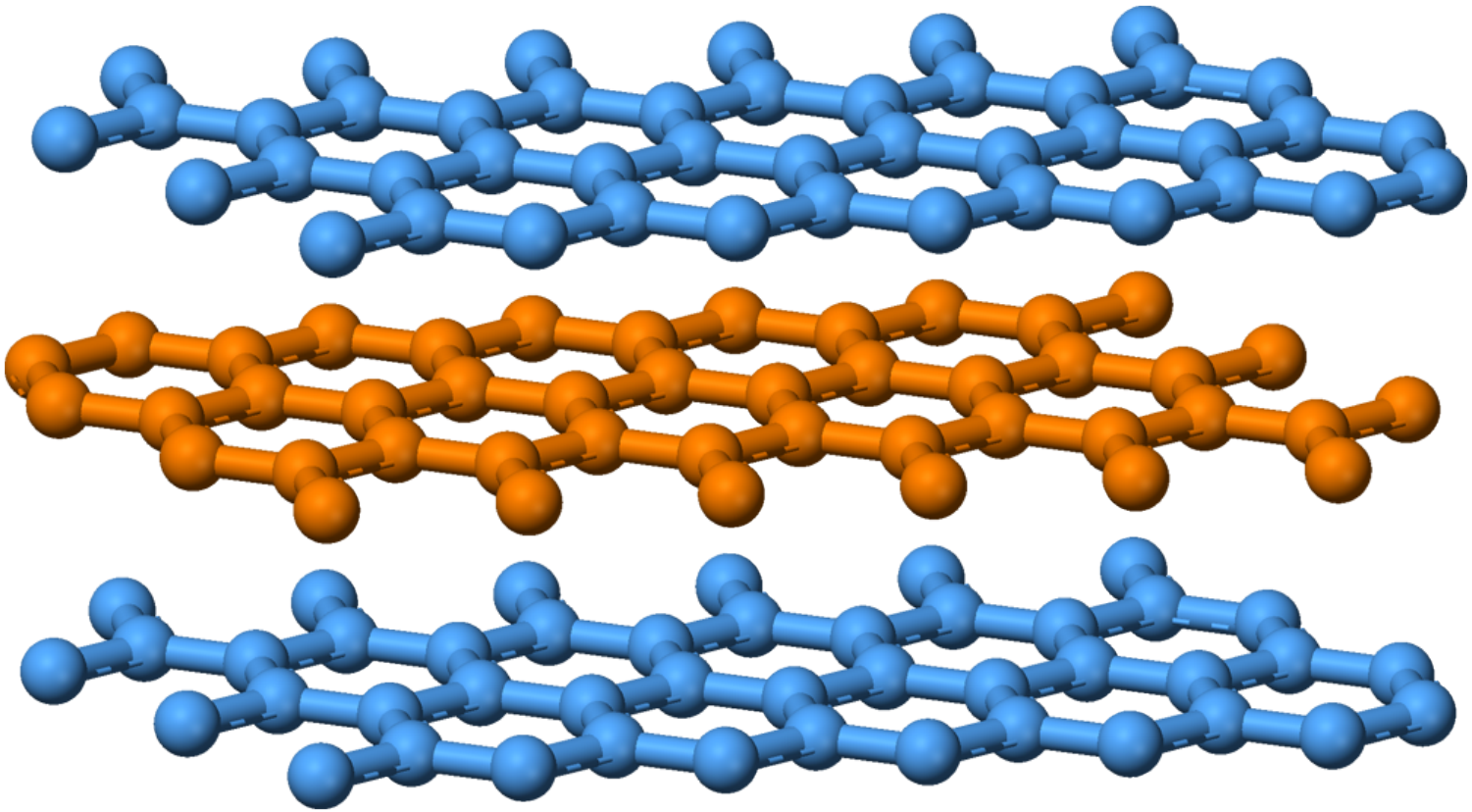
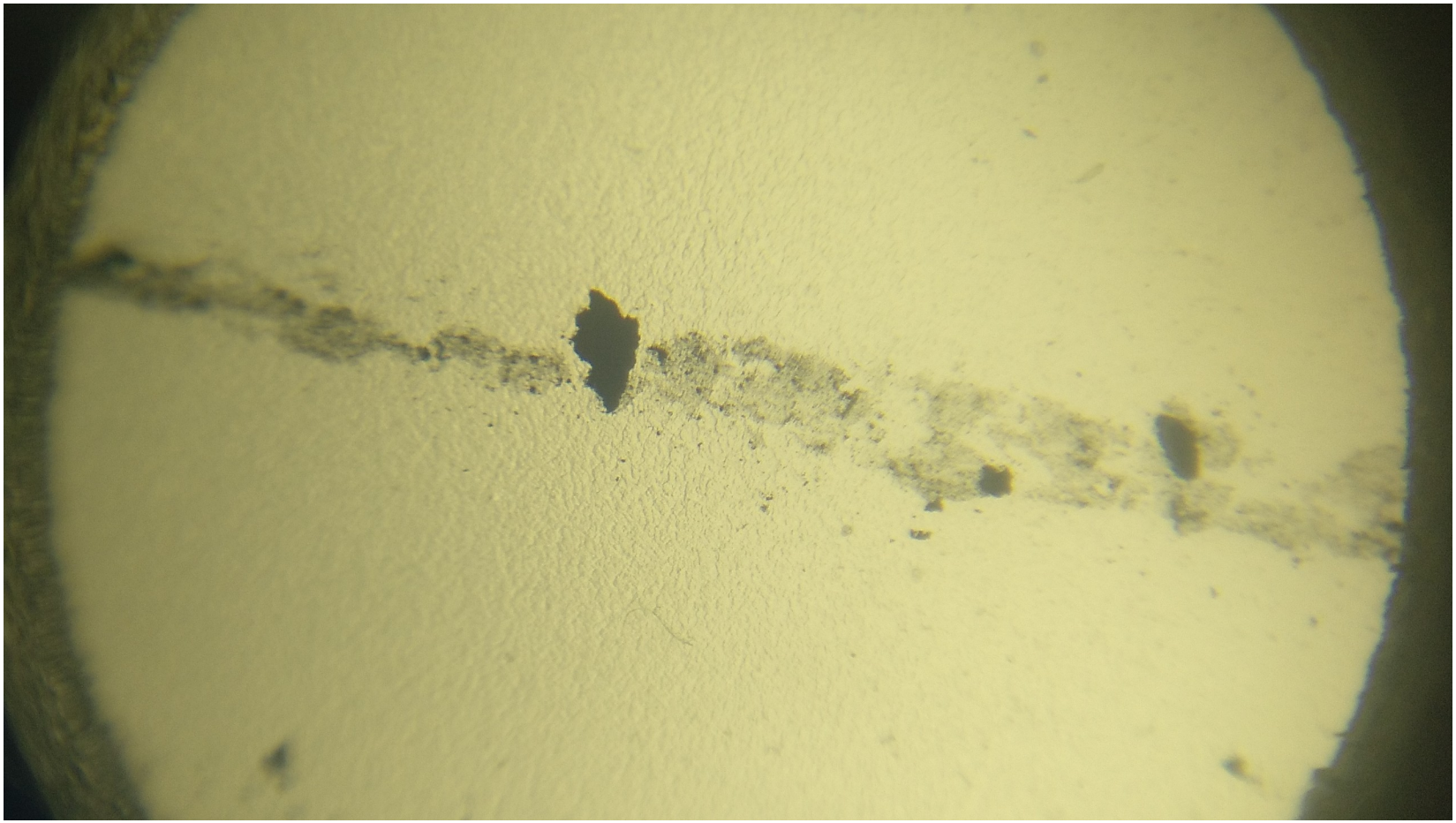


[1]

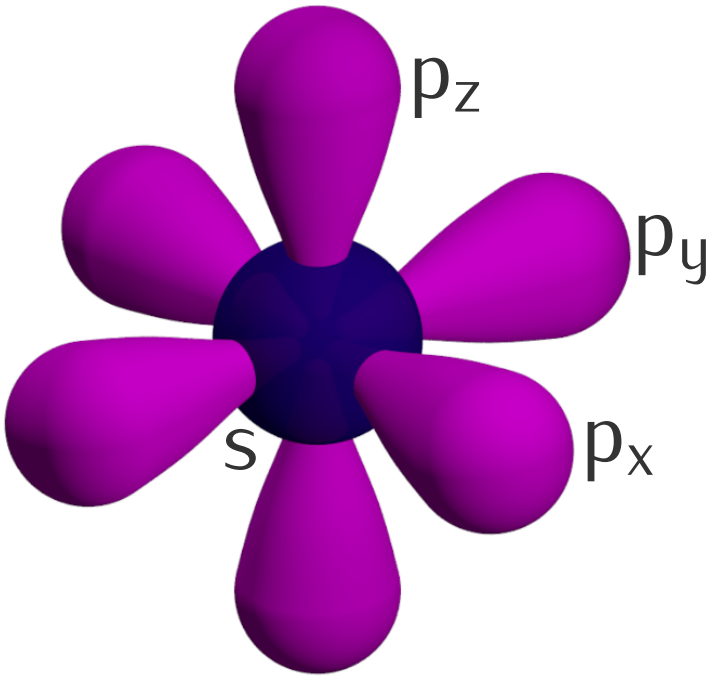
[1] https://w.wiki/q4k

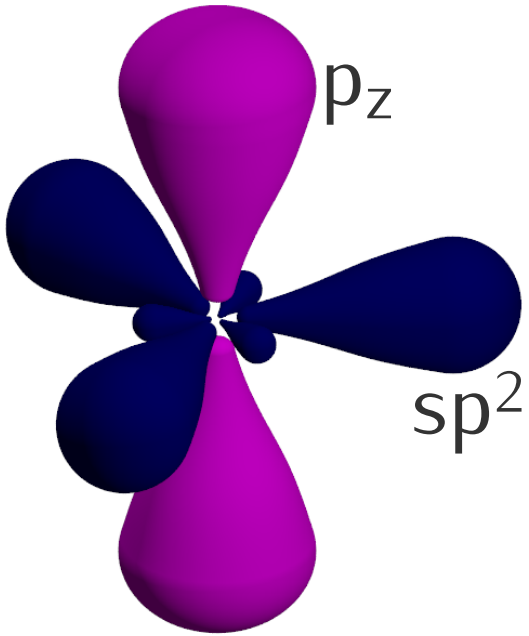
Graphene?

Graphene?



Why in layers?

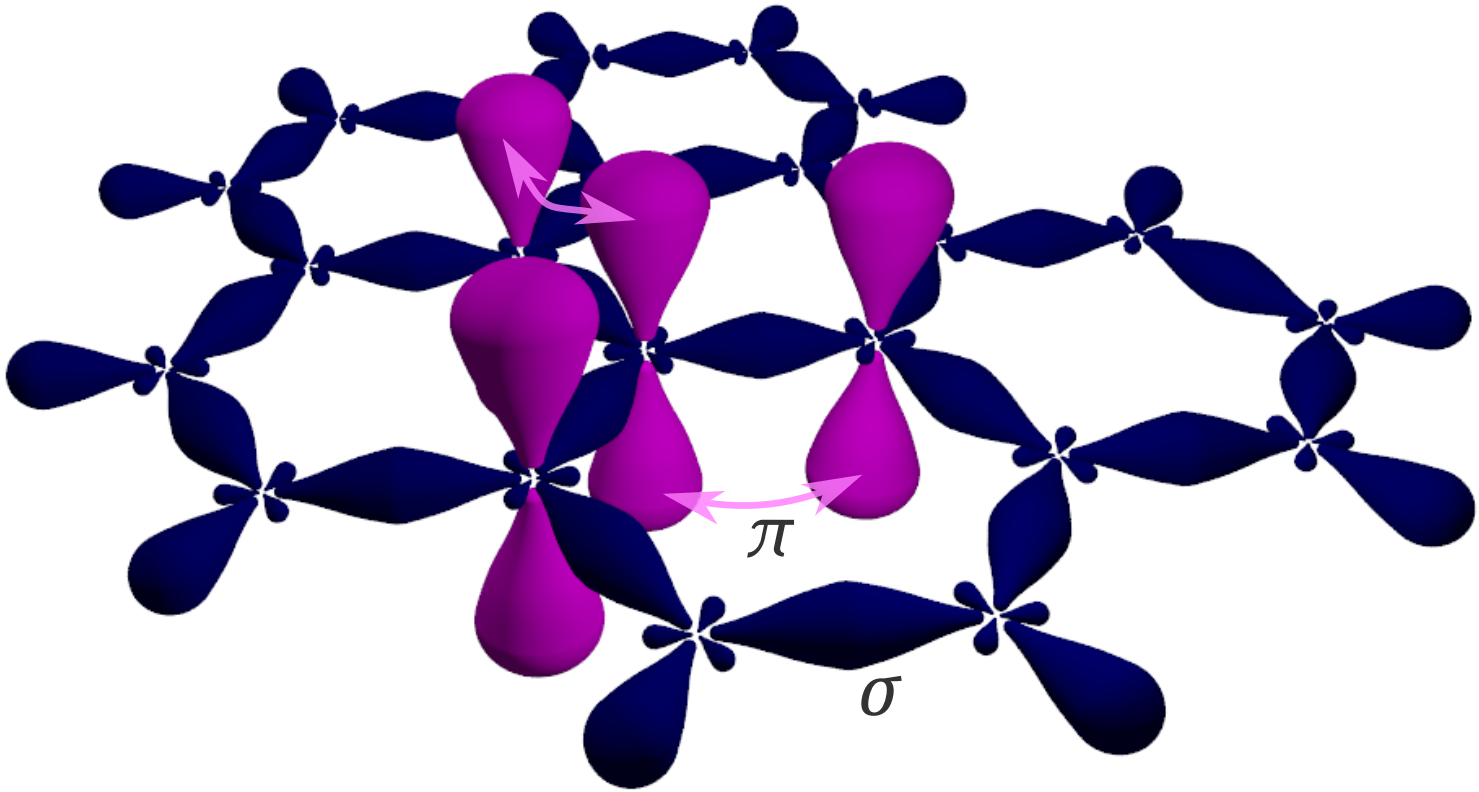




[2]

[2] https://w.wiki/q4C

Why in layers?



[3]

[3] https://w.wiki/q4E

Overview

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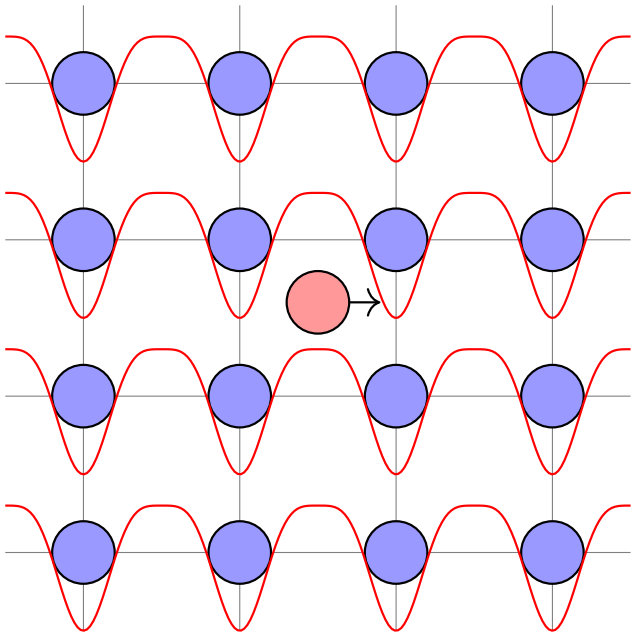
Review of Periodic Potentials

Revisiting: Non-interacting electrons in a periodic potential
$\rightarrow$ "Bloch" states

- Description:

$$H = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m} + V_{\text{periodic}}(\boldsymbol{r}) \quad (1)$$

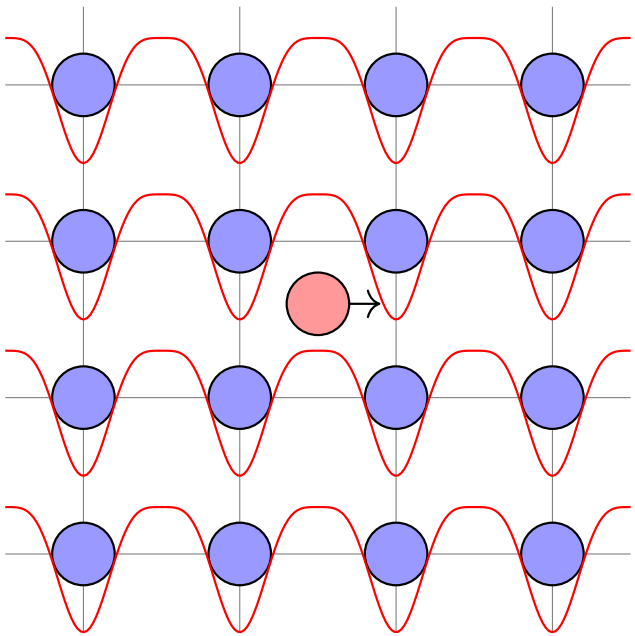
- Reduces to single-particle problem



Assume infinte 2D crystal lattice with ions at positions

$$\boldsymbol{R_n} = n_1 \boldsymbol{a}_1 + n_2 \boldsymbol{a}_2 \qquad \boldsymbol{n} \in \mathbb{Z}^2 \tag{2}$$

- Potential invariant w.r.t. crystal symmetry group; in particular: Translation.
- Thus $[T_{\boldsymbol{R_n}}, H] = 0$, and we can strive for simultaneous eigenstates of all $T_{\boldsymbol{R_n}}$ and H.



Bloch waves

Result

Any solution $\psi(\boldsymbol{r})$ of

$$H\psi(\boldsymbol{r}) = E_\psi \psi(\boldsymbol{r}) \qquad \forall \boldsymbol{n} \in \mathbb{N}^3 \colon T_{\boldsymbol{R_n}} \psi(\boldsymbol{r}) = t_{\boldsymbol{n},\psi} \psi(\boldsymbol{r}) \tag{3}$$

can be written as

$$\psi(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ u(\boldsymbol{r}), \tag{4}$$

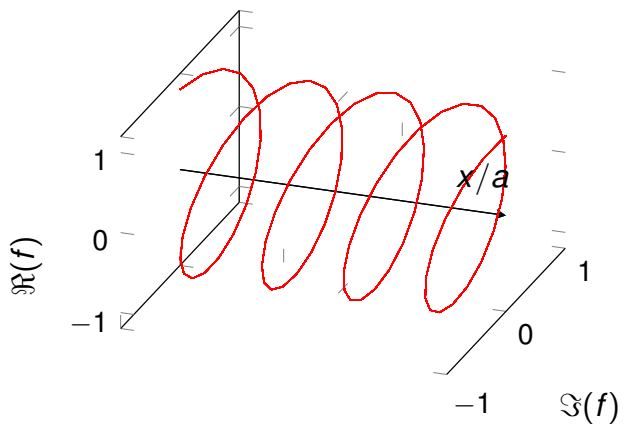
where $u(\boldsymbol{r})$ exhibits the full translational symmetry of the crystal, i.e.

$$\forall \boldsymbol{n} \in \mathbb{N}^3 \colon T_{\boldsymbol{R_n}} u(\boldsymbol{r}) = u(\boldsymbol{r}) \tag{5}$$

and $\boldsymbol{k}$ such that $e^{ik_j L_j} = 0, j = 1, 2, 3.$ (periodic boundary conditions)

Ambiguity of of quasi-momentum $\boldsymbol{k}$: An 1D Example

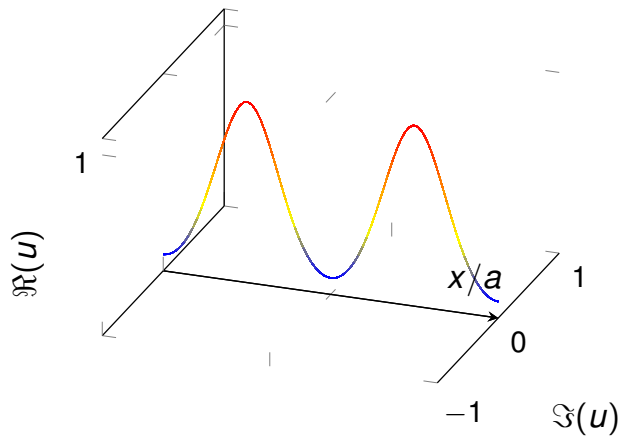
$f(x) = e^{-ikx}$



times

$\times$

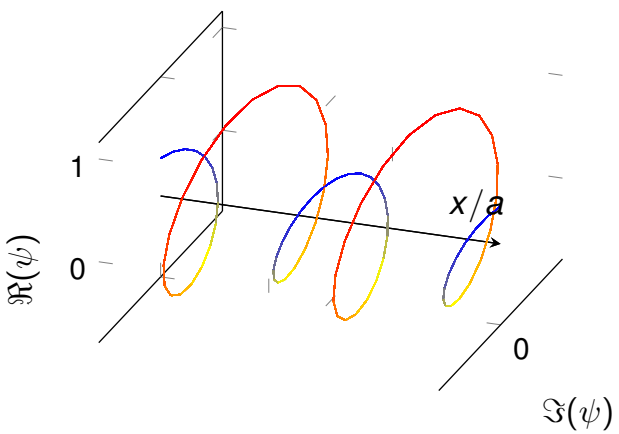
$u(x) = e^{-cos^2\left(\frac{\pi}{2}\frac{x}{a}\right)}$



equals

$=$

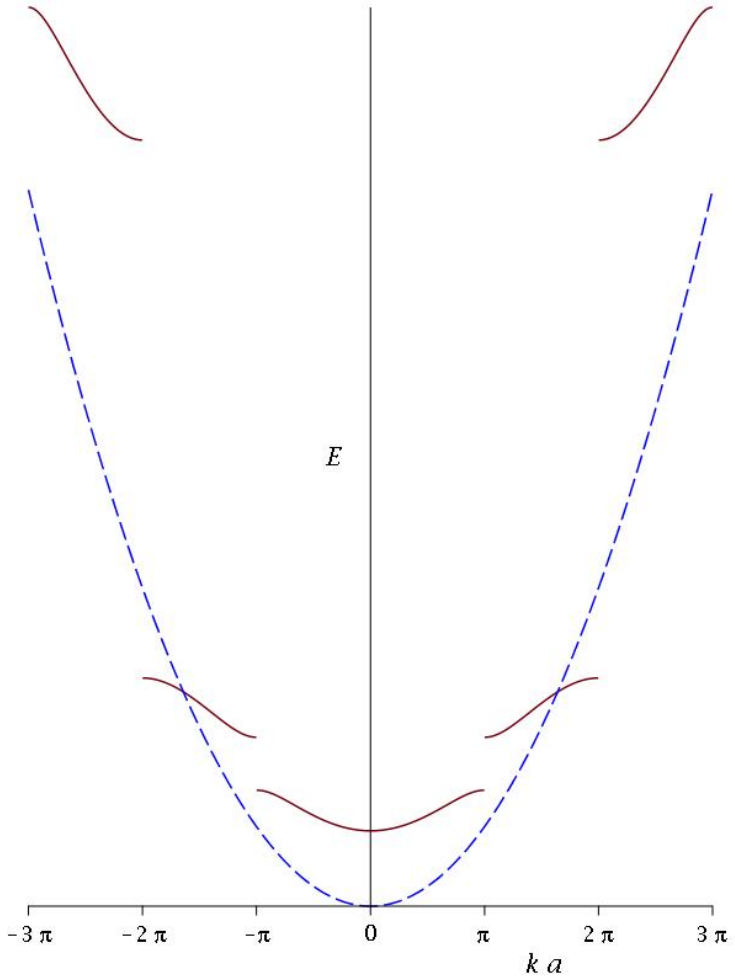
$\psi(x) = f(x) \cdot u(x)$



- also satisfies a-periodicity!
- choice of k and $u(x)$ ambigious!

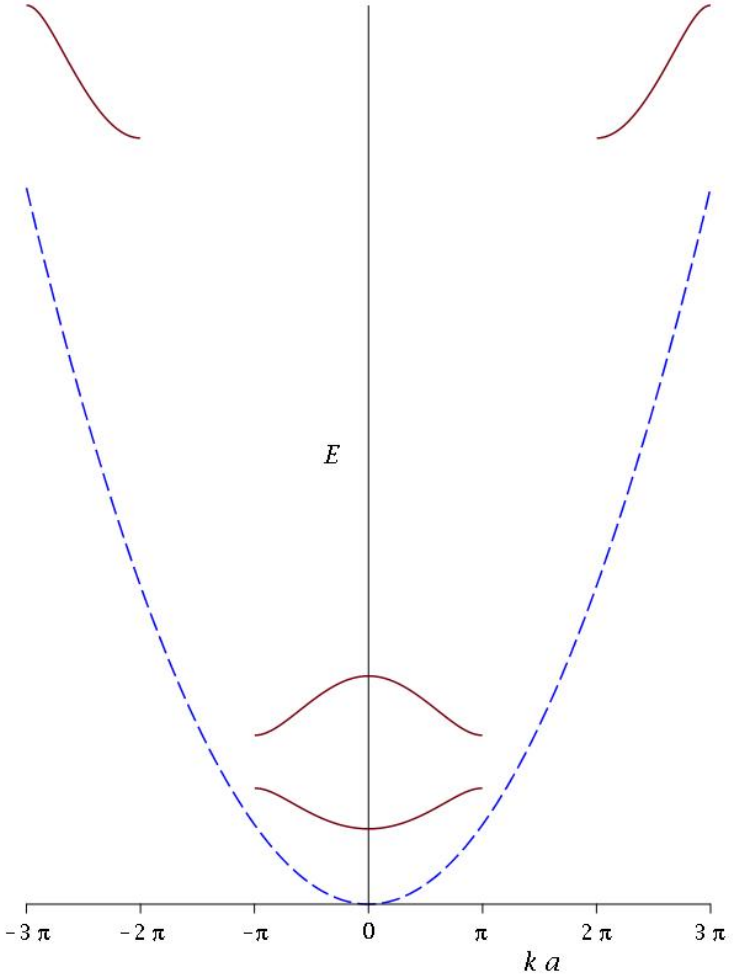
An one-dimensional visualization

- $H\psi_k = E_k\psi_k$ gives rise to spectrum
- fixing number of revolutions of $u(x)$ to 0
- obtain a discontinuous, deformed free-particle spectrum



An one-dimensional visualization

- $H\psi_k = E_k\psi_k$ gives rise to spectrum
- ~~fixing number of revolutions of $u(x)$ to 0~~
- ~~obtain a discontinuous, deformed free-particle spectrum~~
- due to the equivalence on reciprocal space, possible to attribute each E_k to some $k \in (-\pi/a, \pi/a)$ (first Brillouin zone).



An one-dimensional visualization

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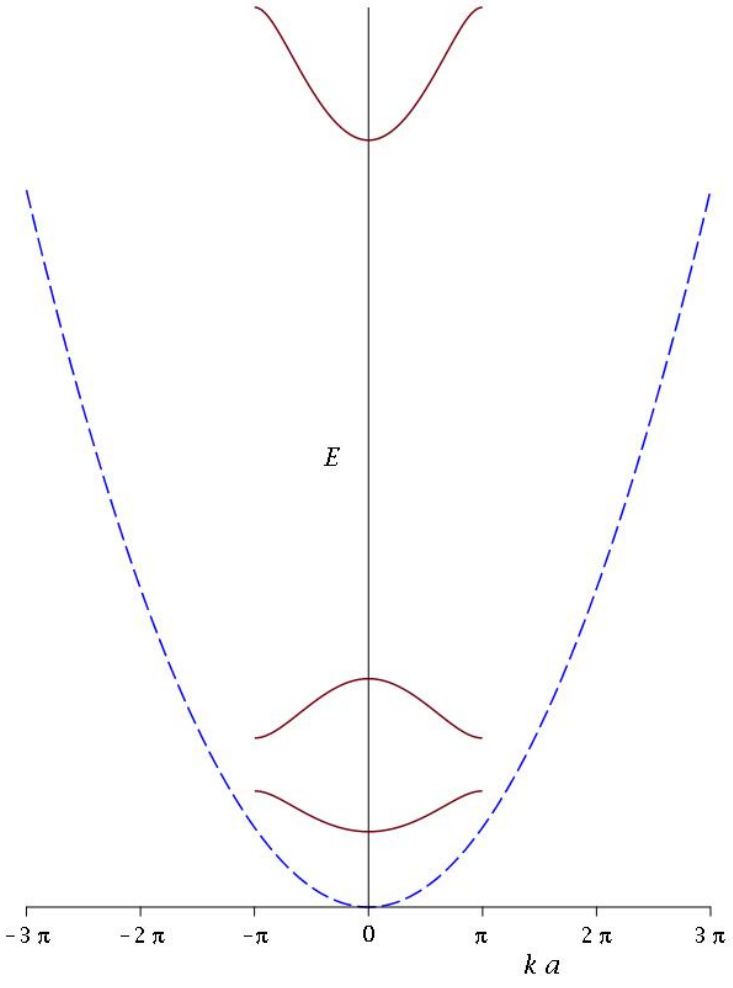


$\Rightarrow$ Energy bands

Reciprocal space and Brillouin zones

- We have seen: Ambiguity in one-dimensional case
- In general, introduce Reciprocal Lattice

$$\mathrm{RL} := \left\{ \boldsymbol{K} \in \mathbb{R}^2 \,\middle|\, \forall \boldsymbol{n} \in \mathbb{Z}^2 : e^{i\boldsymbol{K}\cdot\boldsymbol{R_n}} = 1 \right\} \tag{6}$$

- Declare equivalence relation on Reciprocal Space

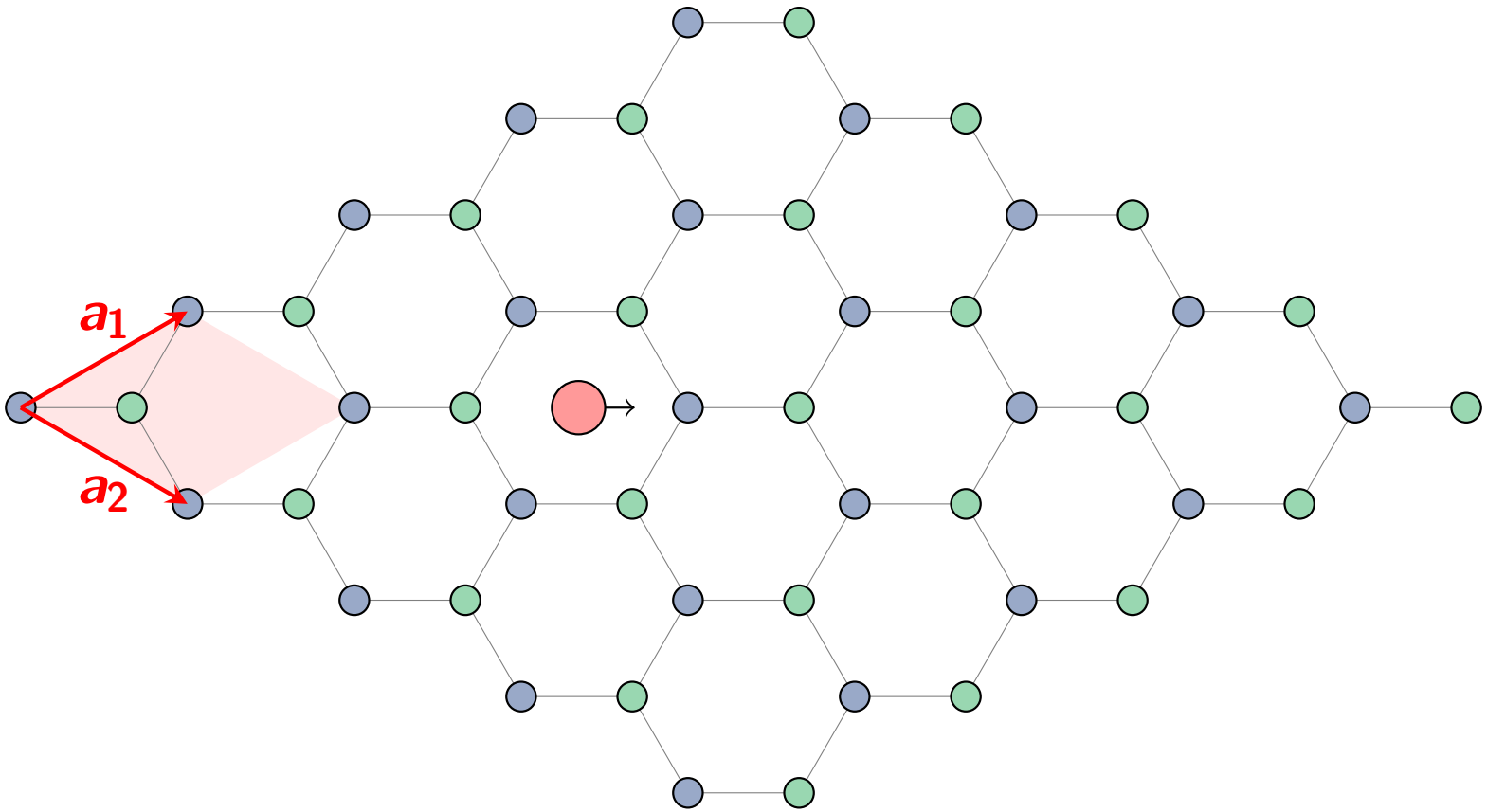
$$\forall \boldsymbol{k}, \boldsymbol{k}' \in \mathbb{R}^2 : \left[\boldsymbol{k} \sim \boldsymbol{k}' \iff \exists \boldsymbol{K} \in \mathrm{RL} \colon \boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{K} \right] \tag{7}$$

- We can reduce everything to $\mathbb{R}^2/_\sim = \mathbb{R}^2/\mathrm{RL}$
- One possibility: Brillouin zone

$$\mathrm{BZ} := \left\{ \boldsymbol{k} \in \mathbb{R}^2 \mid \forall \boldsymbol{K} \in \mathrm{RL} \setminus \{0\} \colon |\boldsymbol{k}| \leq |\boldsymbol{K} - \boldsymbol{k}| \right\} \tag{8}$$

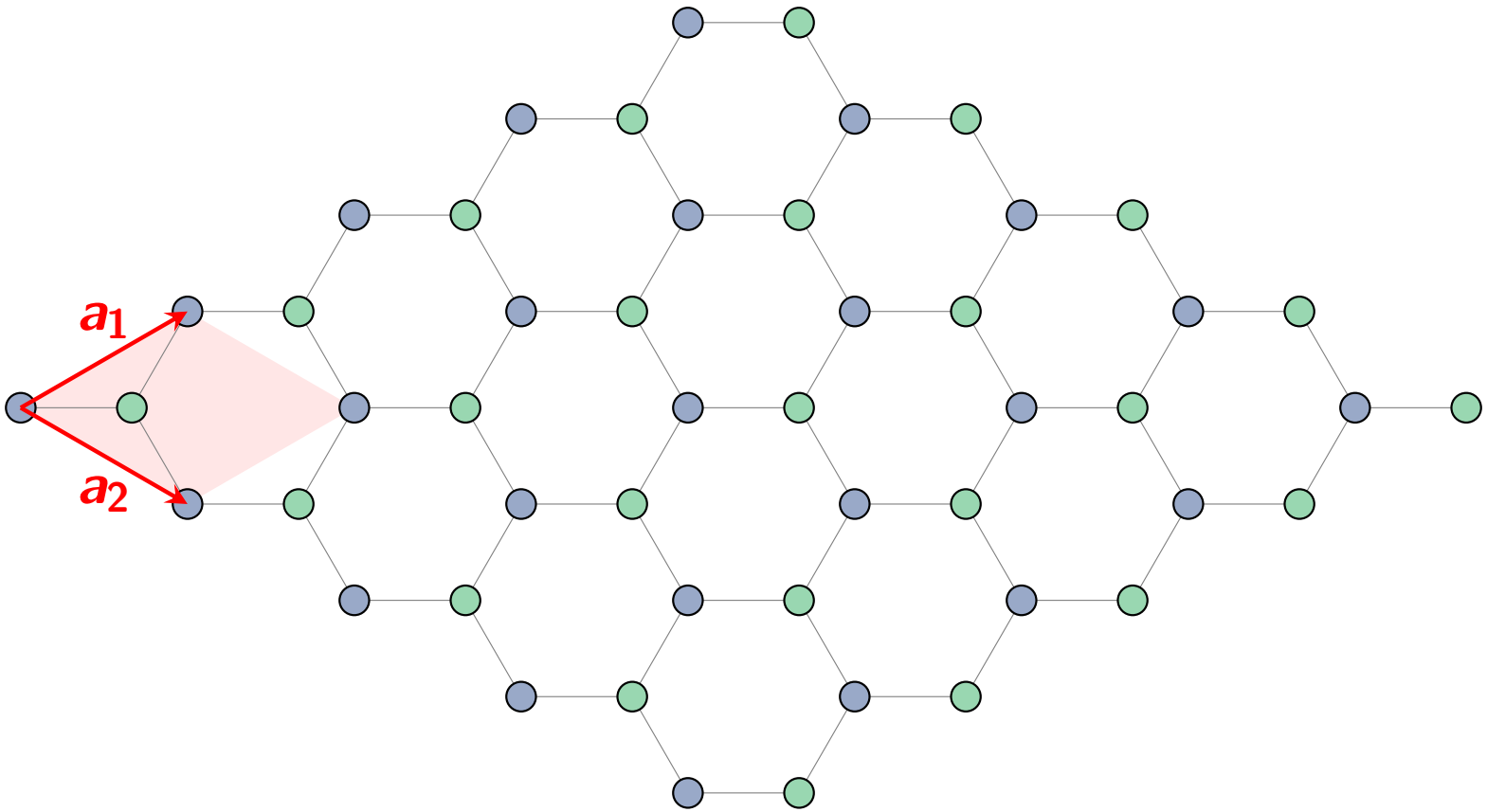
In the case of Graphene...

Note: The unit cell contains two (identical) carbon atoms.



In the case of Graphene...

Constructing the reciprocal lattice...



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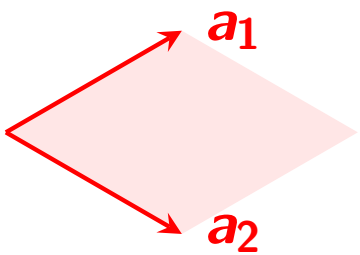

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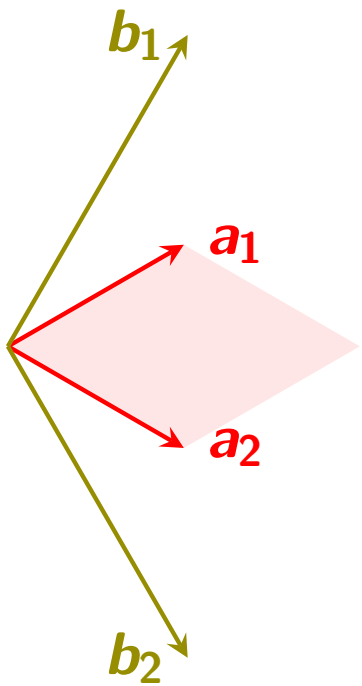

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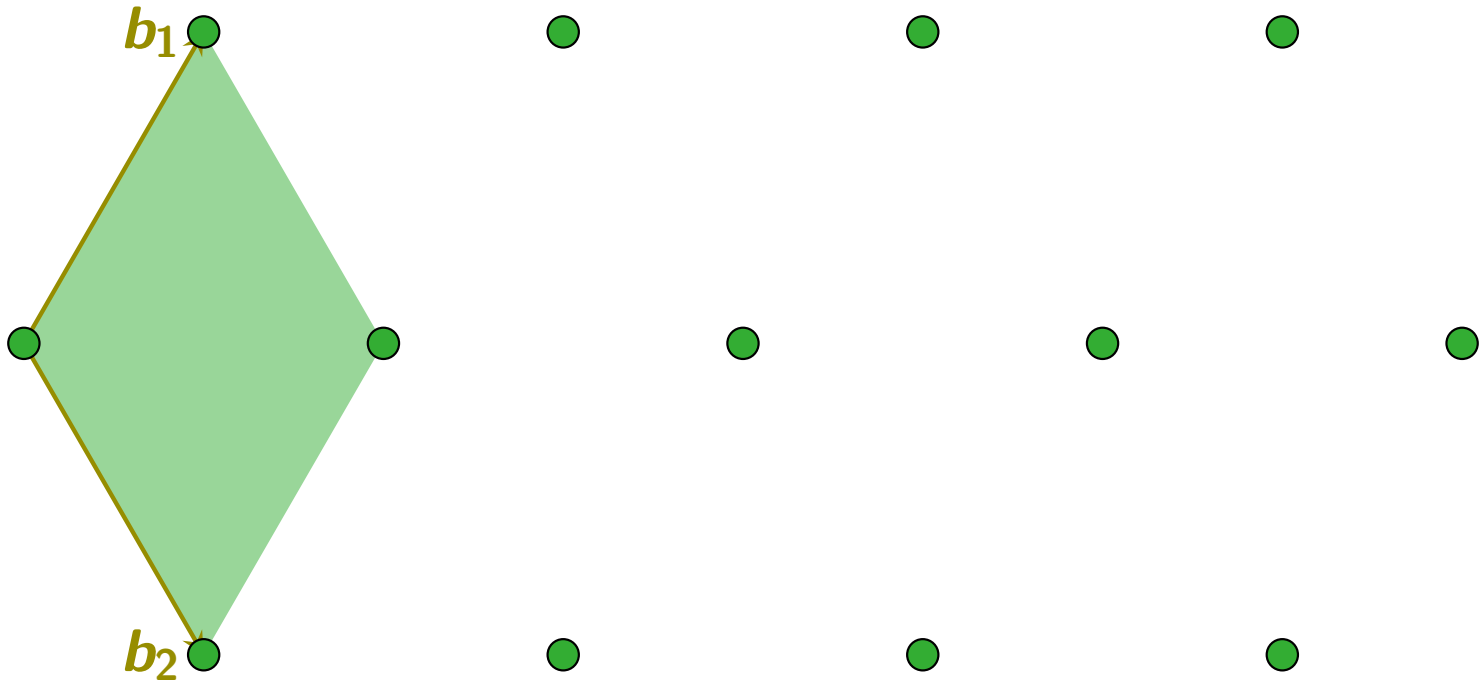

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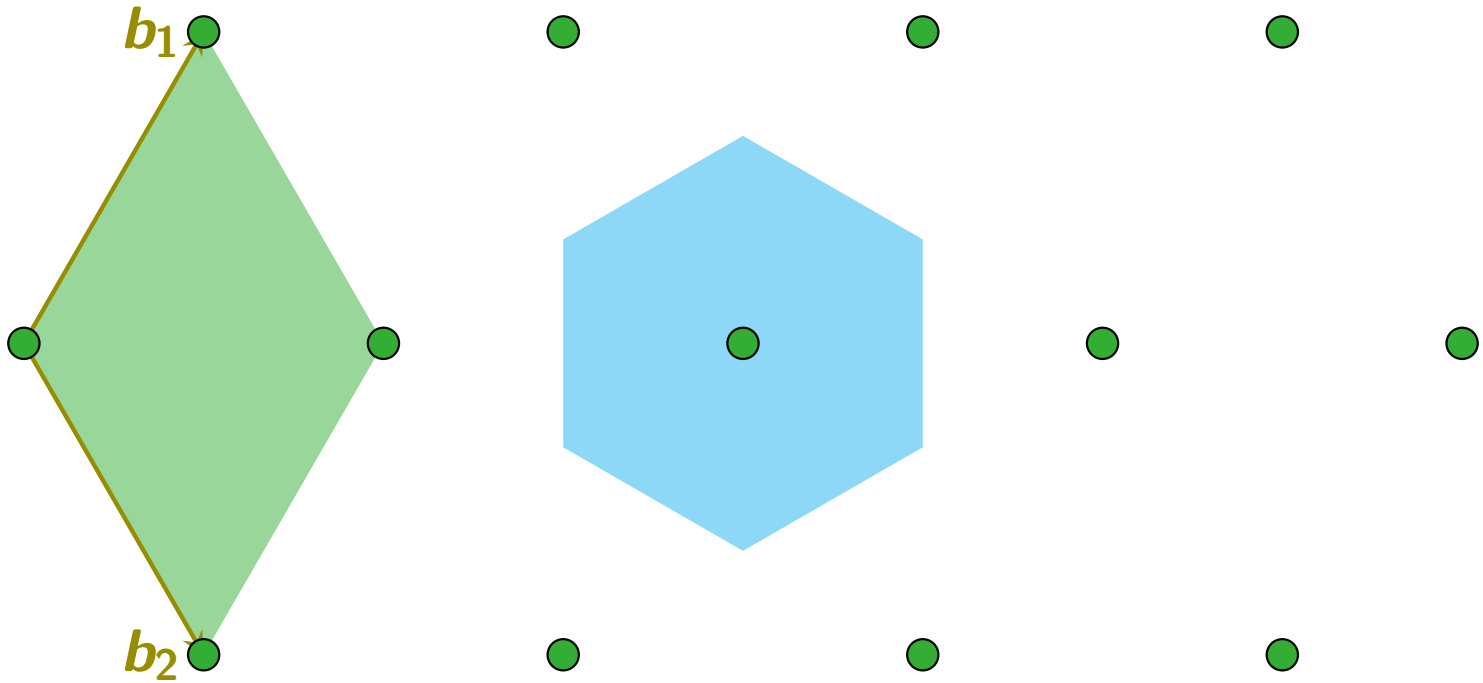

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Tight-Binding Ansatz for Graphene

Suppose $p_z(\boldsymbol{r})$ represents the p_z-orbital of a carbon atom, i.e.,

$$H_{\mathrm{C}}(\boldsymbol{r})p_z(\boldsymbol{r}) = \underbrace{E_z}_{\text{wlog. } = 0} p_z(\boldsymbol{r}). \tag{9}$$

The total Hamiltonian is given by

$$H = \frac{\boldsymbol{p}^2}{2m} + \sum_{\boldsymbol{n}\in\left(\mathbb{Z}_1^L\right)^2} \left(V_{\mathrm{C}}\left(\boldsymbol{r} - \boldsymbol{r}_A - \boldsymbol{R}_{\boldsymbol{n}}\right) + V_{\mathrm{C}}\left(\boldsymbol{r} - \boldsymbol{r}_B - \boldsymbol{R}_{\boldsymbol{n}}\right)\right). \tag{10}$$

We make an approximate ansatz

$$\begin{aligned}\Psi_{\boldsymbol{k}}(\boldsymbol{r}) = c_A(\boldsymbol{k}) &\sum_{\boldsymbol{n}\in\left(\mathbb{Z}_1^L\right)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_{\boldsymbol{n}}} p_z(\boldsymbol{r} - \boldsymbol{r}_A - \boldsymbol{R}_{\boldsymbol{n}}) \\ + c_B(\boldsymbol{k}) &\sum_{\boldsymbol{n}\in\left(\mathbb{Z}_L^1\right)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_{\boldsymbol{n}}} p_z(\boldsymbol{r} - \boldsymbol{r}_B - \boldsymbol{R}_{\boldsymbol{n}}).\end{aligned} \tag{11}$$

Tight-Binding Ansatz for Graphene

We basically restricted ourselves to the subspace spanned by the functions

$$\varphi^A_{\boldsymbol{k}}(\boldsymbol{r}) = \frac{1}{L^2} \sum_{\boldsymbol{n}\in\left(\mathbb{Z}^L_1\right)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{R_n}} p_z(\boldsymbol{r} - \boldsymbol{r}_A - \boldsymbol{R_n}) \tag{12}$$

and

$$\varphi^B_{\boldsymbol{k}}(\boldsymbol{r}) = \frac{1}{L^2} \sum_{\boldsymbol{n}\in\left(\mathbb{Z}^L_1\right)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{R_n}} p_z(\boldsymbol{r} - \boldsymbol{r}_B - \boldsymbol{R_n}). \tag{13}$$

Note that $\langle \varphi^\alpha_{\boldsymbol{k}} \mid \varphi^\beta_{\boldsymbol{k}'} \rangle = 0$ and $\langle \varphi^\alpha_{\boldsymbol{k}} \mid H \mid \varphi^\beta_{\boldsymbol{k}'} \rangle = 0$ whenever $\boldsymbol{k} \neq \boldsymbol{k}'$. The subspaces spanned by $\left\{\varphi^A_{\boldsymbol{k}}, \varphi^B_{\boldsymbol{k}}\right\}$ for each $\boldsymbol{k}$ are thus orthogonal to each other and the Hamiltonian does not couple these subspaces. Thanks to this fact our ansatz is a good one.

Tight-Binding Ansatz for Graphene

We then look for the minimum and maximum of the equation

$$\langle E \rangle_{\Psi_{\boldsymbol{k}}} = \frac{\langle \Psi_{\boldsymbol{k}} \mid H \mid \Psi_{\boldsymbol{k}} \rangle}{\langle \Psi_{\boldsymbol{k}} \mid \Psi_{\boldsymbol{k}} \rangle}, \tag{14}$$

and the corresponding coefficients $c_A(\boldsymbol{k})$ and $c_B(\boldsymbol{k})$. It follows

$$\begin{aligned} \langle E \rangle_{\Psi_{\boldsymbol{k}}} &= \frac{\begin{pmatrix} c^A(\boldsymbol{k})^* & c^B(\boldsymbol{k})^* \end{pmatrix} \begin{pmatrix} \langle \varphi^A_{\boldsymbol{k}} | H | \varphi^A_{\boldsymbol{k}} \rangle & \langle \varphi^A_{\boldsymbol{k}} | H | \varphi^B_{\boldsymbol{k}} \rangle \\ \langle \varphi^B_{\boldsymbol{k}} | H | \varphi^A_{\boldsymbol{k}} \rangle & \langle \varphi^B_{\boldsymbol{k}} | H | \varphi^B_{\boldsymbol{k}} \rangle \end{pmatrix} \begin{pmatrix} c^A(\boldsymbol{k}) \\ c^B(\boldsymbol{k}) \end{pmatrix}}{\begin{pmatrix} c^A(\boldsymbol{k})^* & c^B(\boldsymbol{k})^* \end{pmatrix} \begin{pmatrix} \langle \varphi^A_{\boldsymbol{k}} | \varphi^A_{\boldsymbol{k}} \rangle & \langle \varphi^A_{\boldsymbol{k}} | \varphi^B_{\boldsymbol{k}} \rangle \\ \langle \varphi^B_{\boldsymbol{k}} | \varphi^A_{\boldsymbol{k}} \rangle & \langle \varphi^B_{\boldsymbol{k}} | \varphi^B_{\boldsymbol{k}} \rangle \end{pmatrix} \begin{pmatrix} c^A(\boldsymbol{k}) \\ c^B(\boldsymbol{k}) \end{pmatrix}} \\ &= \frac{c(\boldsymbol{k})^\dagger H(\boldsymbol{k}) c(\boldsymbol{k})}{c(\boldsymbol{k})^\dagger S(\boldsymbol{k}) c(\boldsymbol{k})} \approx \frac{c(\boldsymbol{k})^\dagger H(\boldsymbol{k}) c(\boldsymbol{k})}{c(\boldsymbol{k})^\dagger c(\boldsymbol{k})}. \end{aligned} \tag{15}$$

Simplification: Suppose $S(\boldsymbol{k}) = \mathbb{1}_2$, i.e $\langle \varphi^A_{\boldsymbol{k}} \mid \varphi^B_{\boldsymbol{k}} \rangle = 0$.[4]

[4] The real deal is technically a little bit more elaborate. The result is the same in our case.

Tight-Binding Ansatz for Graphene

- Now we can use the *min-max-theorem*[5] that states that the minimum and maximum correspond to the eigenvalues of $H(\boldsymbol{k})$ and is attained precisely by the eigenvectors.

$\Rightarrow$ Found a new eigenvalue problem!

- It remains to calculate $H(\boldsymbol{k})$.
- Tight-binding $\equiv$ "take only nearest neighbour interactions into account"

$$\langle \varphi_{\boldsymbol{k}}^{A} \mid H \mid \varphi_{\boldsymbol{k}}^{A} \rangle \approx 0 \approx \langle \varphi_{\boldsymbol{k}}^{B} \mid H \mid \varphi_{\boldsymbol{k}}^{B} \rangle \tag{16}$$

$$\langle \varphi_{\boldsymbol{k}}^{A} \mid H \mid \varphi_{\boldsymbol{k}}^{B} \rangle \approx -\gamma_0 \alpha(\boldsymbol{k}) \approx \overline{\langle \varphi_{\boldsymbol{k}}^{B} \mid H \mid \varphi_{\boldsymbol{k}}^{A} \rangle} \tag{17}$$

where

$$\alpha(\boldsymbol{k}) = 1 + e^{-i\boldsymbol{k}\cdot\boldsymbol{a}_1} + e^{-i\boldsymbol{k}\cdot\boldsymbol{a}_2}. \tag{18}$$

[5] aka. Courant-Fischer theorem

Tight-Binding Ansatz for Graphene

We found

$$H(\boldsymbol{k}) = \begin{pmatrix} 0 & -\gamma_0 \alpha(\boldsymbol{k}) \\ -\gamma_0 \alpha(\boldsymbol{k})^* & 0 \end{pmatrix}, \tag{19}$$

where

$$\alpha(\boldsymbol{k}) = 1 + e^{-i\boldsymbol{k}\cdot\boldsymbol{a}_1} + e^{-i\boldsymbol{k}\cdot\boldsymbol{a}_2}. \tag{20}$$

Eigenvalues:

$$\begin{aligned} E_\pm(\boldsymbol{k}) &= \pm\gamma_0 |\alpha(\boldsymbol{k})| \\ &= \pm\sqrt{3+2\cos(\boldsymbol{k}\cdot\boldsymbol{a}_1)+2\cos(\boldsymbol{k}\cdot\boldsymbol{a}_2)+2\cos(\boldsymbol{k}\cdot(\boldsymbol{a}_1-\boldsymbol{a}_2))} \end{aligned} \tag{21}$$

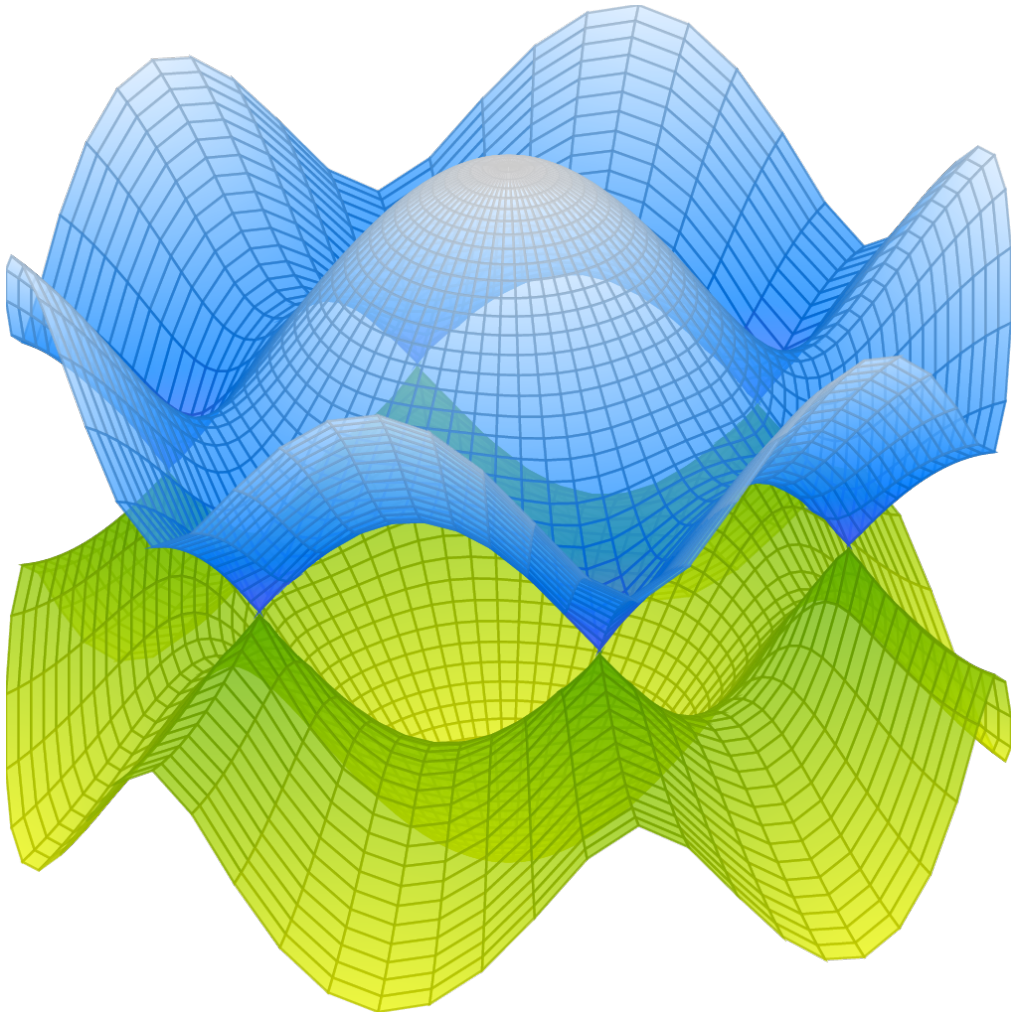
Note that at

$$\boldsymbol{K}_\pm = \frac{1}{2}(\boldsymbol{b}_1 + \boldsymbol{b}_2) \pm \frac{1}{6}(\boldsymbol{b}_2 - \boldsymbol{b}_1) \tag{22}$$

we have

$$E_+(\boldsymbol{K}_\pm) = E_-(\boldsymbol{K}_\pm) = 0. \tag{23}$$

Band Structure for the p_z-Orbitals in Graphene



$$E_{\pm}(\boldsymbol{k}) = \pm\sqrt{3+2\cos(\boldsymbol{k}\cdot\boldsymbol{a}_1)+2\cos(\boldsymbol{k}\cdot\boldsymbol{a}_2)+2\cos(\boldsymbol{k}\cdot(\boldsymbol{a}_1-\boldsymbol{a}_2))} \tag{24}$$

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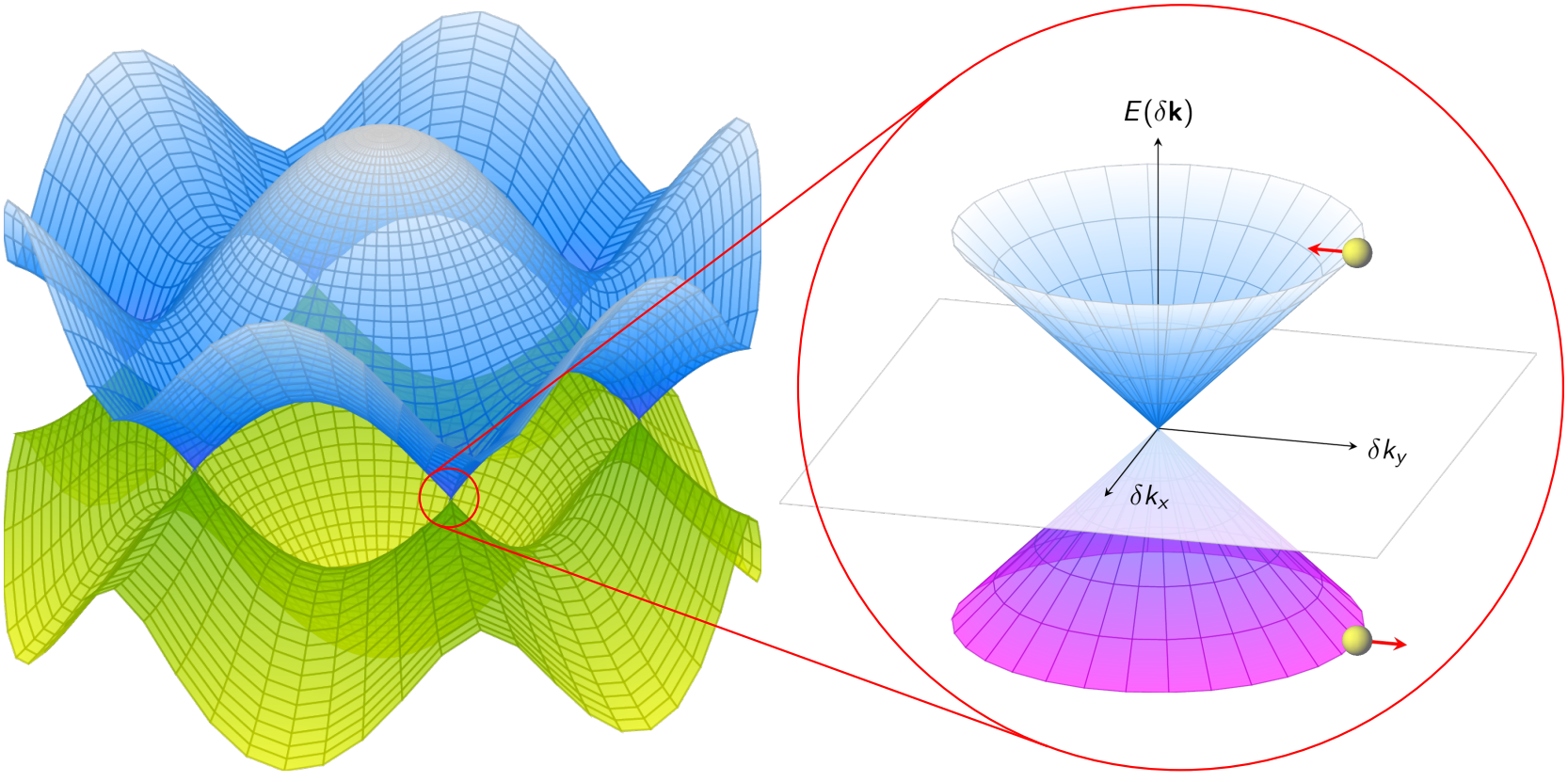

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Dispersion at the Dirac Points

Indeed, for $\boldsymbol{k} = \boldsymbol{K}_{\pm} + \delta\boldsymbol{k}$, we have

$$E_{\pm}(\delta\boldsymbol{k}) \approx \pm\frac{\sqrt{3}\gamma_0 a}{2}|\delta\boldsymbol{k}| = \pm\hbar v_F|\delta\boldsymbol{k}|, \tag{25}$$

where we defined the Fermi velocity

$$v_F = \frac{\sqrt{3}\gamma_0 a}{2\hbar} \approx 8.5\times10^5\,\mathrm{m\,s^{-1}} \cong 3‰ \times \text{speed of light} \tag{26}$$

that is the electronic group velocity of wave packets.

Next step:

What is the form of the Hamiltonian in the promixity of the points $\boldsymbol{K}_{\pm}$?

Dispersion at the Dirac Points

$$H(\boldsymbol{K}_- + \delta\boldsymbol{k}) \approx \underbrace{\frac{\sqrt{3}\gamma_0 a}{2}}_{\hbar v_F} \begin{pmatrix} 0 & -\delta\boldsymbol{k}_y - i\delta\boldsymbol{k}_x \\ -\delta\boldsymbol{k}_y + i\delta\boldsymbol{k}_x & 0 \end{pmatrix} \tag{27}$$

$$H(\boldsymbol{K}_+ + \delta\boldsymbol{k}) \approx \underbrace{\frac{\sqrt{3}\gamma_0 a}{2}}_{\hbar v_F} \begin{pmatrix} 0 & \delta\boldsymbol{k}_y - i\delta\boldsymbol{k}_x \\ \delta\boldsymbol{k}_y + i\delta\boldsymbol{k}_x & 0 \end{pmatrix} \tag{28}$$

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Let us modify our ansatz:

$$\Psi_{\boldsymbol{k}}(\boldsymbol{r}) = c_A(\boldsymbol{k}) \sum_{\boldsymbol{n}\in\left(\mathbb{Z}_1^L\right)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_{\boldsymbol{n}}} p_z(\boldsymbol{r} - \boldsymbol{r}_A - \boldsymbol{R}_{\boldsymbol{n}}) \tag{29}$$

$$+ \quad c_B(\boldsymbol{k}) \sum_{\boldsymbol{n}\in\left(\mathbb{Z}_L^1\right)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_{\boldsymbol{n}}} p_z(\boldsymbol{r} - \boldsymbol{r}_B - \boldsymbol{R}_{\boldsymbol{n}}).$$

Dispersion at the Dirac Points

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$$+ \, i \, c_B(\boldsymbol{k}) \sum_{\boldsymbol{n}\in\left(\mathbb{Z}_L^1\right)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_{\boldsymbol{n}}} p_z(\boldsymbol{r} - \boldsymbol{r}_B - \boldsymbol{R}_{\boldsymbol{n}}).$$

Dispersion at the Dirac Points

$$H(\boldsymbol{K}_- + \delta\boldsymbol{k}) \approx \underbrace{\frac{\sqrt{3}\gamma_0 a}{2}}_{\hbar v_F} \begin{pmatrix} 0 & \delta\boldsymbol{k}_x - i\delta\boldsymbol{k}_y \\ \delta\boldsymbol{k}_x + i\delta\boldsymbol{k}_y & 0 \end{pmatrix} \tag{27}$$

$$H(\boldsymbol{K}_+ + \delta\boldsymbol{k}) \approx \underbrace{\frac{\sqrt{3}\gamma_0 a}{2}}_{\hbar v_F} \begin{pmatrix} 0 & \delta\boldsymbol{k}_x + i\delta\boldsymbol{k}_y \\ \delta\boldsymbol{k}_x - i\delta\boldsymbol{k}_y & 0 \end{pmatrix} \tag{28}$$

Let us modify our ansatz:

$$\Psi_{\boldsymbol{k}}(\boldsymbol{r}) = c_A(\boldsymbol{k}) \sum_{\boldsymbol{n}\in\left(\mathbb{Z}_1^L\right)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{R_n}} p_z(\boldsymbol{r} - \boldsymbol{r}_A - \boldsymbol{R_n}) \tag{29}$$

$$+ \, i \, c_B(\boldsymbol{k}) \sum_{\boldsymbol{n}\in\left(\mathbb{Z}_L^1\right)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{R_n}} p_z(\boldsymbol{r} - \boldsymbol{r}_B - \boldsymbol{R_n}).$$

Dispersion at the Dirac Points

$$H(\boldsymbol{K}_- + \delta\boldsymbol{k}) \approx \underbrace{\frac{\sqrt{3}\gamma_0 a}{2}}_{\hbar v_F} \delta\boldsymbol{k}\cdot\boldsymbol{\sigma} \tag{27}$$

$$H(\boldsymbol{K}_+ + \delta\boldsymbol{k}) \approx \underbrace{\frac{\sqrt{3}\gamma_0 a}{2}}_{\hbar v_F} \delta\boldsymbol{k}\cdot\boldsymbol{\sigma}^\dagger \tag{28}$$

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Weyl Spinors?

$$H(\boldsymbol{K}_- + \delta\boldsymbol{k}) \approx \underbrace{\frac{\sqrt{3}\gamma_0 a}{2}}_{\hbar v_F} \delta\boldsymbol{k}\cdot\boldsymbol{\sigma} \tag{30}$$

$$H(\boldsymbol{K}_+ + \delta\boldsymbol{k}) \approx \underbrace{\frac{\sqrt{3}\gamma_0 a}{2}}_{\hbar v_F} \delta\boldsymbol{k}\cdot\boldsymbol{\sigma}^\dagger \tag{31}$$

It definitively "looks like" the Weyl equation $\sigma^\mu p_\mu \psi = 0$!

A word of warning

The states $\Psi_{\boldsymbol{k}}$ are not eigenvectors of the momentum operator $\hat{p}$! Even worse, $\delta\boldsymbol{k}$ is not even remotely related to the expectation value of $\hat{p}$ in the state $\Psi_{\boldsymbol{K}_\pm+\delta\boldsymbol{k}}$!

Anyway, as long as we do not take the analogy too literally, it is a valid comparison between wave phenomena!

Massless Dirac fermions?

We can build a four-component object:

$$\begin{aligned} |\,\psi_{\delta\boldsymbol{k}}\rangle = c_A^-(\delta\boldsymbol{k})\,|\,\varphi^A_{\boldsymbol{K}_-+\delta\boldsymbol{k}}\rangle - ic_B^-(\delta\boldsymbol{k})\,|\,\varphi^B_{\boldsymbol{K}_-+\delta\boldsymbol{k}}\rangle \\ +ic_B^+(\delta\boldsymbol{k})\,|\,\varphi^B_{\boldsymbol{K}_+ +\delta\boldsymbol{k}}\rangle + c_A^+(\delta\boldsymbol{k})\,|\,\varphi^A_{\boldsymbol{K}_+ +\delta\boldsymbol{k}}\rangle \end{aligned} \tag{32}$$

With respect to the ordered basis

$$\left(|\,\varphi^A_{\boldsymbol{K}_-+\delta\boldsymbol{k}}\rangle, -i\,|\,\varphi^B_{\boldsymbol{K}_-+\delta\boldsymbol{k}}\rangle, i\,|\,\varphi^B_{\boldsymbol{K}_+ +\delta\boldsymbol{k}}\rangle, |\,\varphi^A_{\boldsymbol{K}_+ +\delta\boldsymbol{k}}\rangle\right), \tag{33}$$

the joint Hamiltonian is given by

$$H(\delta\boldsymbol{k}) = \hbar v_F \begin{pmatrix} -\delta\boldsymbol{k}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \delta\boldsymbol{k}\cdot\boldsymbol{\sigma} \end{pmatrix}. \tag{34}$$

This looks like the massless Fermi-Dirac Hamiltonian in the Weyl representation!

$$H_W^{\text{F.D.}} = \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix} \tag{35}$$

Energy eigenstates

Suppose $\delta \boldsymbol{k} = |\delta \boldsymbol{k}| \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$ and denote by $\xi = \pm 1$ the valley $\boldsymbol{K}_\xi$. The positive ($s = +1$) and negative ($s = -1$) energy eigenstates for the valley $\boldsymbol{K}_\xi$ are given by

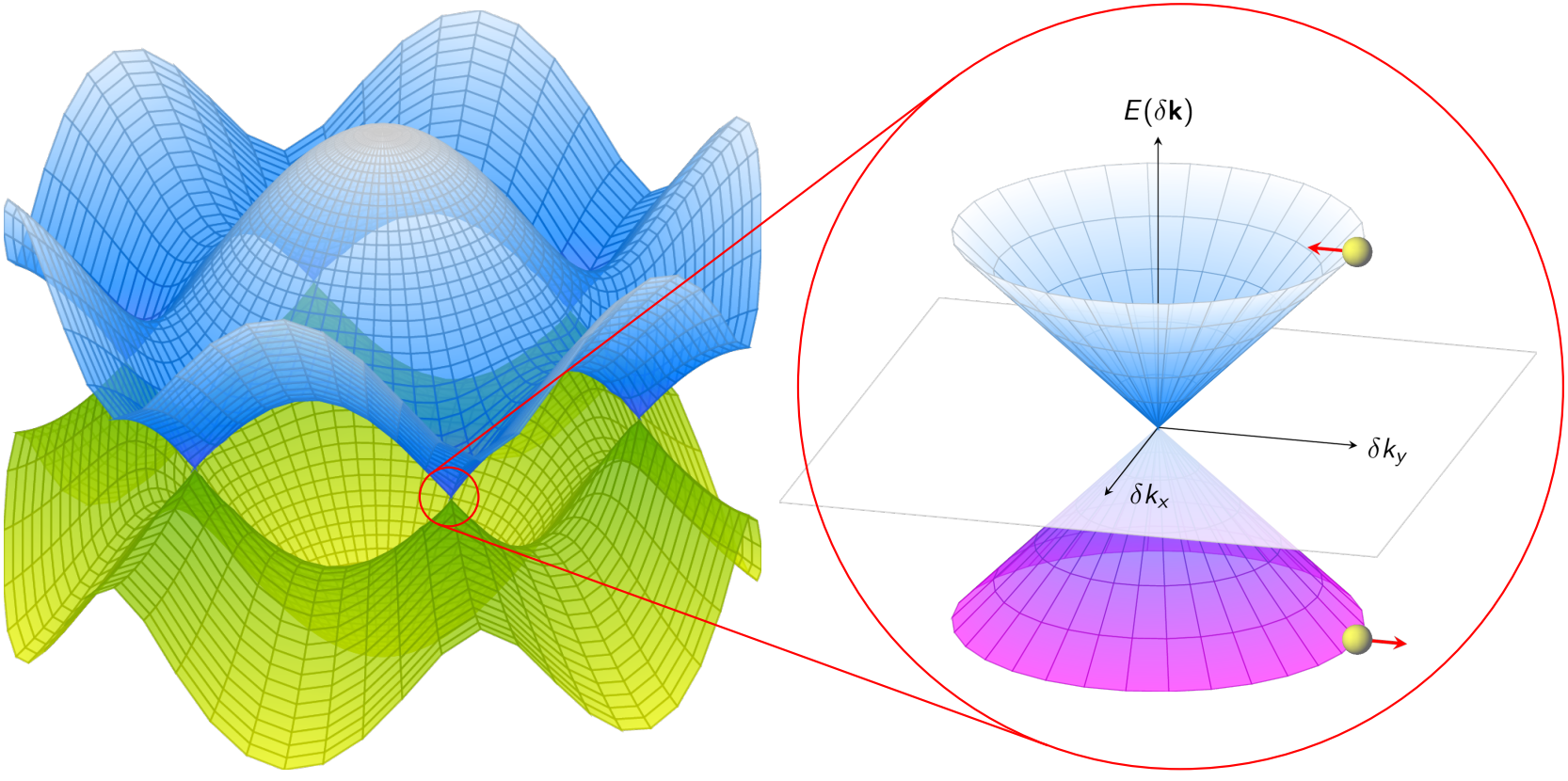
$$\psi_s^\xi = \begin{pmatrix} 1 \\ \xi\, s\, e^{i\theta} \end{pmatrix}. \tag{36}$$

Check that:

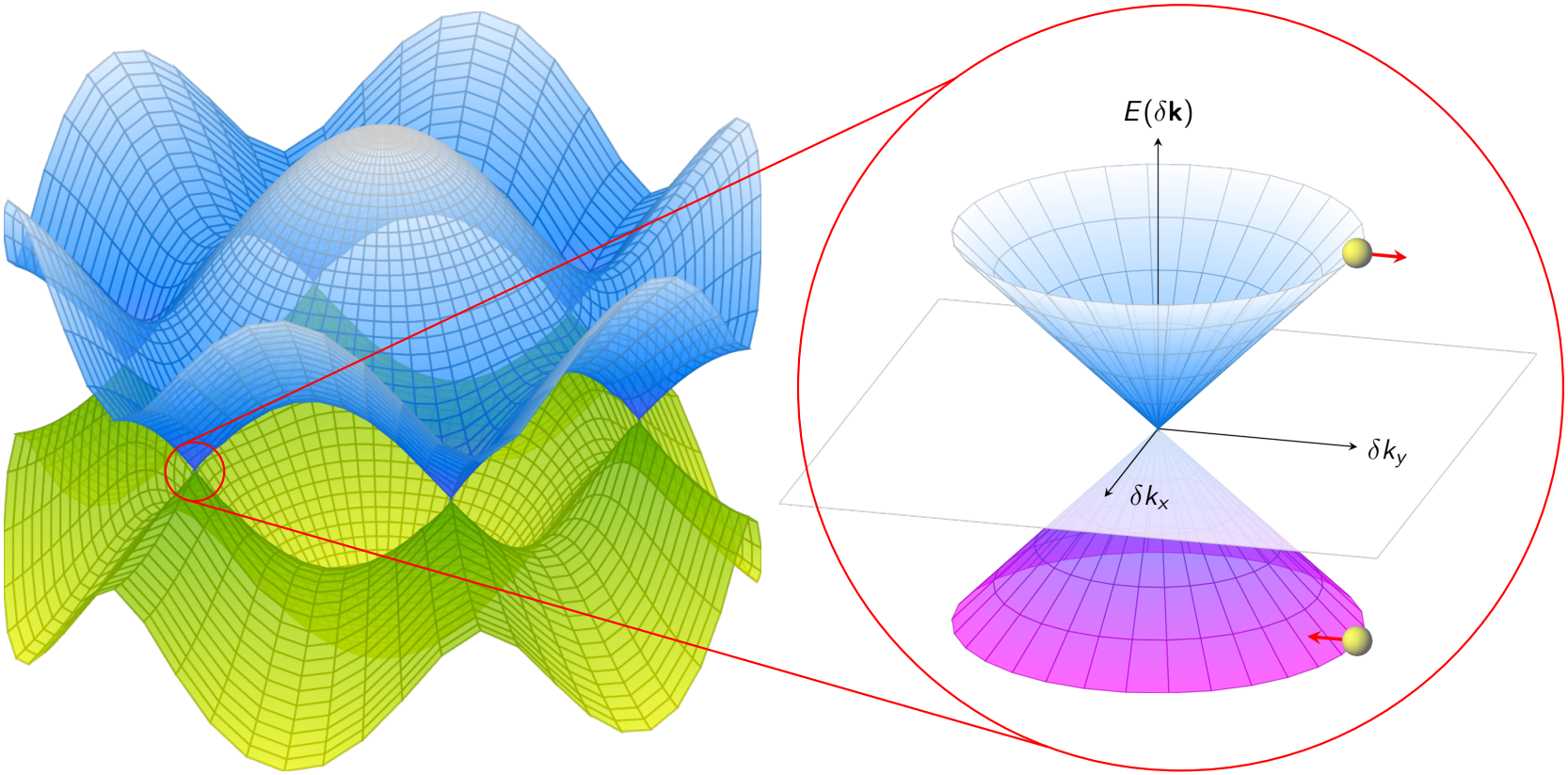
$$H(\boldsymbol{K}_\xi + \delta \boldsymbol{k})\psi_s^\xi = \hbar v_F\, \xi\, (\delta \boldsymbol{k} \cdot \boldsymbol{\sigma})\, \psi_s^\xi = s \hbar v_F |\delta \boldsymbol{k}| \psi_s^\xi \tag{37}$$

$s = +1$ for the "conduction" band and $s = -1$ for the "valence band". Note that the pseudohelicity eigenvalue of ψ_s^ξ is ξs. This introduces an artificial distinction of the two valles $\boldsymbol{K}_+$ and $\boldsymbol{K}_-$.

Pseudohelicity at the Dirac Point K_-



Pseudohelicity at the Dirac Point K_+



Conclusion

Similarities:

Differences:

Conclusion

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- linear dispersion

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- only in the rest frame!

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Curiosity

Dirac fermions only exist in $1+3$ dimensions due to the double cover $\mathsf{SL}(2,\mathbb{C}) \to \mathsf{SO}(1,3)$ of the Lorentz group.
The "massless Dirac fermions" of Graphene appear in a $1+2$ dimensional system, with "z-momentum" always equal to zero.

Conclusion

Leads to:

- no intravalley backscattering ($\Rightarrow$ high mobility)
- Klein tunneling (we cannot electrostatically confine charge carriers)
- half integer quantum hall effect

Thank you for your attention!