Calculation of the Anomalous Magnetic Moment of the Electron

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The S-Matrix

- Scattering processes in QFT are described by the S-Matrix, which is a time ordered perturbative series given by:

$$S = \underbrace{I}_{S^{(0)}} \underbrace{-i\int_{-\infty}^{\infty} \mathcal{H}_I^I(x_1)d^4x_1}_{S^{(1)}} \underbrace{-\frac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T\left\{\mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2)\right\}d^4x_1 d^4x_2}_{S^{(2)}} + \ldots\ldots = \sum_{n=0}^{\infty} S^{(n)}$$

- Imposing conservation of momentum between in and out states allows one to identify each term in the expansion as a partial transition amplitude.

$$\langle p_1 p_2 \ldots | S | k_A k_B \rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f) \cdot iM(k_A, k_B \to p_f)$$

Feynman Diagrams

- Wick's theorem states that the time ordering of operators can be written as a normal ordering plus all possible contractions:

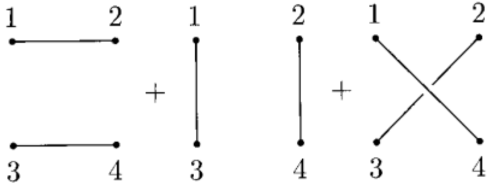
$$T\{\phi(x_1)\phi(x_2)...\phi(x_m)\} = N\{\phi(x_1)\phi(x_2)...\phi(x_m) + \text{contractions}\}$$

- Where:

$$\overbrace{\phi(x)\phi}(y) = [\phi^+(x), \phi^-(y)] =$$

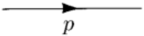
- Example:

$$\langle 0| T\{\phi_1\phi_2\phi_3\phi_4\} |0\rangle =$$

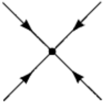


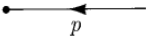
Feynman Rules

- For ϕ^4 theory ($H_I = \lambda \frac{\phi^4}{4!}$) the Feynman rules are given by:

1. For each propagator, 

$= \dfrac{i}{p^2 - m^2 + i\epsilon}$;

2. For each vertex,  $= -i\lambda$;

3. For each external point, x 

$= e^{-ip\cdot x}$;

4. Impose momentum conservation at each vertex;

5. Integrate over each undetermined momentum: $\displaystyle\int \frac{d^4p}{(2\pi)^4}$;

6. Divide by the symmetry factor.

Quantum Electrodynamics (QED)

- We combine the Dirac and EM Lagrangian, and introduce an interaction term, obtaining:

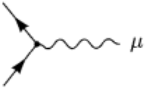
$$\mathcal{L}^{1/2,1} = \underbrace{-\frac{1}{2}\left(\partial_\nu A_{\underset{state}{\mu}}\right)\left(\partial^\nu A^\mu_{state}\right)}_{\mathcal{L}_0^1 = \mathcal{L}_0^{e/m}} + \underbrace{\bar{\psi}_{state}\left(i\gamma^\mu\partial_\mu - m\right)\psi_{state}}_{\mathcal{L}_0^{1/2}} + \underbrace{e\bar{\psi}_{state}\gamma^\mu\psi_{state}A_{\underset{state}{\mu}}}_{\mathcal{L}_I^{1/2,1} = \mathcal{L}_I^{e/m}}$$

$$\Longrightarrow \quad \left(i\gamma^\mu\partial_\mu - m\right)\psi_{state} = -e\gamma^\mu\psi_{state}A_{\underset{state}{\mu}}$$

- The interaction term is motivated by the local U(1) symmetry requirement of the theory.

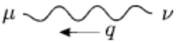
- The Feynman rules in this case are as following:

New vertex:



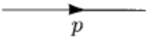
$$= -ie\gamma^\mu$$

Photon propagator:



$$= \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$$

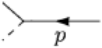
Fermion propagator:



$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

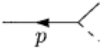
External leg contractions:

$\underbrace{\psi\,|\mathbf{p}, s\rangle}_{\text{fermion}} =$



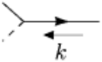
$= u^s(p)$

$\underbrace{\langle \mathbf{p}, s|\,\bar{\psi}}_{\text{fermion}} =$



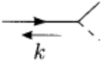
$= \bar{u}^s(p)$

$\underbrace{\bar{\psi}\;\;|\mathbf{k}, s\rangle}_{\text{antifermion}} =$



$= \bar{v}^s(k)$

$\underbrace{\langle \mathbf{k}, s|\;\;\psi}_{\text{antifermion}} =$



$= v^s(k)$

Non-relativistic Limit

- In the non-relativistic limit, the Dirac equation in the presence of an external magnetic field is given by the Hamiltonian:

$$H = \frac{\vec{p}^2}{2m} + V(r) + \frac{e}{2m}\vec{B} \cdot (\vec{L} + g\vec{S})$$

- Pauli's equation in the weak magnetic field regime implies g = 2.

Non-relativistic limit

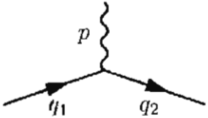
- Experiments disagree with the exact value of $g = 2$.
- Measurement of the hyperfine structure of hydrogen evidentiate the fact:

  "There is clearly an important difference between the measured and calculated values of ν_H and ν_D of about 0.26 percent compared with the probable error of the calculated value of 0.05 percent. The difference is five times greater than the claimed probable error in the natural constants. Whether the failure of theory and experiment to agree is because of some unknown factor in the theory of the hydrogen atom or simply an error in the estimate of one of the natural constants such as α, only further experiment can decide." [Nafe et al., 1947]

The QFT Solution

- We can deduce the value of g from Feynman diagrams in QED. Consider the tree level diagram below:

$$i\mathcal{M}_0^\mu =$$



$$= -ie\bar{u}(q_2)\gamma^\mu u(q_1)$$

- Applying Gordon's identity we reach:

$$M_0^\mu = -e\left(\frac{q_1^\mu + q_2^\mu}{2m}\right)\bar{u}(q_2)u(q_1) - \frac{e}{2m}i\bar{u}(q_2)p_\nu\sigma^{\mu\nu}u(q_1)$$

The QFT Solution

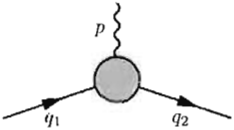
- Comparing the second term to the term that couples to the magnetic field in the previous Hamiltonian we note that if they are to be equivalent g must equal 2.

$$\frac{e}{2m} g \vec{B} \cdot \vec{S} = \frac{e}{2m} g B_z \cdot \frac{\sigma^{\mu\nu}}{2} \quad \Longleftrightarrow \quad \frac{e}{2m} i \bar{u}(q_2) p_\nu \sigma^{\mu\nu} u(q_1)$$

The QFT Solution

- We're interested in higher order corrections to g. The most general diagram should be something of the form:

$$i\mathcal{M}^\mu =$$



$$= \bar{u}(q_2)\left(f_1\gamma^\mu + f_2 p^\mu + f_3 q_1^\mu + f_4 q_2^\mu\right)u(q_1)$$

- This in turn can be reduced to:

$$iM^\mu = (-ie)\bar{u}(q_2)\left[F_1\left(\frac{p^2}{m^2}\right)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m}p_\nu F_2\left(\frac{p^2}{m^2}\right)\right]$$

The QFT Solution

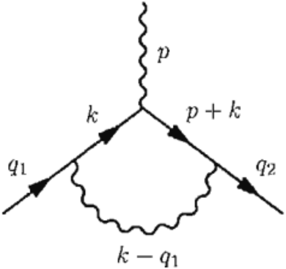
- The g dependance lies on the second form factor evaluated in the non-relativistic limit. The problem is then reduced to evaluating:

$$g = 2 + 2F_2(0)$$

The QFT Solution

- In order to evaluate this term we'll analyze the following diagram:

$$i\mathcal{M}_2^{\mu} =$$



The QFT Solution

- Applying the Feynman rules we reach:

$$iM_2^\mu = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\nu\alpha}}{(k-q_1)^2 + i\epsilon} \bar{u}(q_2)\gamma^\nu$$

$$\times \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i\epsilon} \gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} \gamma^\alpha u(q_1)$$

$$= -e^3 \bar{u}(q_2) \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu(\not{p} + \not{k} + m)\gamma^\mu(\not{k} + m)\gamma_\nu}{[(k-q_1)^2 + i\epsilon][(p+k)^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon]} u(q_1)$$

The QFT Solution

- We can simplify the denominator, by applying Feynman's parametrization trick:

$$\frac{1}{ABC} = 2\int_0^1 dx\; dy\; dz\; \delta(x+y+z-1)\frac{1}{[xA+yB+zC]^3}$$

- With:

$$A = k^2 - m^2 + i\epsilon$$

$$B = (p+k)^2 - m^2 + i\epsilon$$

$$C = (k-q_1)^2 + i\epsilon$$

The QFT Solution

- So that:

$$xA + yB + zC = k^2 + 2k(yp - zq_1) + yp^2 + zq_1^2 - (x+y)m^2 + i\epsilon$$

$$= (k + yp^\mu - zq_1^\mu)^2 - \Delta + i\epsilon$$

- Where:

$$\Delta = -xyp^2 + (1-z)^2 m^2$$

- And the denominator becomes:

$$= 2\int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{1}{[(k + yp^\mu - zq_1^\mu)^2 - \Delta + i\epsilon]^3}$$

The QFT Solution

- The numerator in turn, through the shifting $k^\mu \to k^\mu - yp^\mu + zq_1^\mu$ and parametrization $x + y + z = 1$ can be shown to reduce to:

$$\frac{-1}{2}N^\mu = \left[-\frac{1}{2}k^2 + (1-x)(1-y)p^2 + (1-4z+z^2)m^2\right]\bar{u}(q_2)\gamma^\mu u(q_1)$$

$$+imz(1-z)p_\nu \bar{u}(q_2)\sigma^{\mu\nu}u(q_1) + m(z-2)(x-y)p^\mu \bar{u}(q_2)u(q_1)$$

- We are only interested in the term involving $\sigma^{\mu\nu}$ so:

$$F_2(p^2) = \frac{2m}{e}(4ie^3 m)\int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\int \frac{d^4k}{(2\pi)^4}\frac{z(1-z)}{(k^2-\Delta+i\epsilon)^3}$$

The QFT Solution

- To evaluate it we use the result that:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^3} = \frac{-i}{32\pi^2 \Delta}$$

- So that:

$$F_2(p^2) = \frac{\alpha}{\pi} m^2 \int_0^1 dx\, dy\, dz\, \delta(x + y + z - 1) \frac{z(1-z)}{(1-z)^2 m^2 - xyp^2}$$

The QFT Solution

- In the non-relativistic limit we have:

$$F_2(0) = \frac{\alpha}{\pi} \int_0^1 dz \int_0^1 dy \int_0^1 dx \, \delta(x + y + z - 1) \frac{z}{(1 - z)}$$

- So that:

$$F_2(0) = \frac{\alpha}{\pi} \int_0^1 dz \int_0^{1-z} dy \frac{z}{(1 - z)}$$

$$= \frac{\alpha}{2\pi}$$

The QFT Solution

- Finally, we reach:

$$g = 2 + \frac{\alpha}{\pi} = 2.00232$$

- This result was first obtained by Schwinger in 1948, on the occasion of the APS meeting.

References

Matthew D. Schwartz (2014)
Quantum Field Theory and the Standard Model
Cambridge University Press - 850 pages.

Silvan S. Schweber (2012)
QED and the Men who Made it
University Press - 732 pages.