# Quantum Computational Complexity and the Vacuum of Free Scalar QFT

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# Main Topics

- Complexity
- Interpreting Free Scalar QFT as Harmonic Oscillators
- Complexity of the Vacuum in Free Scalar QFT
  (2017 - *Circuit Complexity in Quantum Field Theory*)
- Applications of Quantum Complexity
- Holography, Quantum Gravity and Complexity

# Complexity

In computer science, it's useful to classify problems according to the computational resources required for their solution.

The complexity of a problem is, roughly speaking, the minimum number of operations required to solve the problem.

Nielsen has developed an approach to evaluate quantum complexity based on the continuous evolution of quantum states.

# Nielsen's Approach

We define an unitary operator to represent a circuit which performs the transformation from reference to target:

$$\hat{U}\,|\phi_r\rangle = |\phi_t\rangle$$

It's analogous to a time evolution operator:

$$\hat{U}(\tau) = \overleftarrow{\mathcal{P}}\ \exp \int_0^\tau d\tilde{\tau}\hat{H}(\tilde{\tau})\ ;\ \hat{U}(1) \equiv \hat{U}$$

The Hamiltonian can be expanded in terms of a basis of "elementary generators" $\hat{\mathcal{O}}$, with coefficients $Y$ called "control functions":

$$\hat{H}(\tau) = \sum_I Y_I(\tau)\hat{\mathcal{O}}_I$$

We define the “cost" of a circuit in terms of a “cost function" $\mathcal{F}$:

$$\textit{Cost} = \int_0^1 d\tau \mathcal{F}(\hat{U}(\tau), \dot{\hat{U}}(\tau))$$

The complexity of a target state is the cost of the most efficient circuit that generates it from the reference state.

Finding the complexity means minimizing the cost integral.

The complexity:

- Is relative to the reference state
- Depends on the choice of $\mathcal{F}$

There are desirable restrictions which can be imposed on $\mathcal{F}$ in a way that it ends up describing the metric of a Finsler manifold. The problem of minimizing the cost then becomes a problem of finding geodesics.

Still, one can pick between various possibilities, such as:

$$\mathcal{F}_1(\tau) = \sum_I |Y_I(\tau)|$$
$$\mathcal{F}_2(\tau) = \sqrt{\sum_I Y_I^2(\tau)}$$

With $\mathcal{F}_2$, the problem of minimizing the cost becomes the problem of finding a geodesic in a Riemann manifold.

# Scalar QFT as Harmonic Oscillators

Classical Lagrangian:

$$L = \int d^4x \, \frac{1}{2} \left[ \partial_\mu \phi \partial^\mu \phi - m^2 \phi \right]$$

Hamiltonian after quantization (mostly minus metric):

$$\hat{H} = \int d^3x \, \frac{1}{2} \left[ \hat{\Pi}^2 + (\nabla \hat{\phi})^2 + m^2 \hat{\phi} \right]$$

We can discretize by separating points by a distance $\delta$:

$$\hat{H} = \sum_{\substack{n_1, n_2, n_3 \\ = -\infty}}^{\infty} \delta^3 \, \frac{1}{2} \left[ \hat{\Pi}^2(\vec{n}) + m^2 \hat{\phi}^2(\vec{n}) + \delta^{-2} \sum_{i=1}^{3} (\hat{\phi}(\vec{n} + \vec{e}_i \delta) - \hat{\phi}(\vec{n}))^2 \right]$$

By making the following definitions:

$$\hat{x}(\vec{n}) = \delta^2 \hat{\phi}(\vec{n}) \ ; \ \hat{p}(\vec{n}) = \delta \hat{\Pi}(\vec{n})$$

$$m = \omega \ ; \ \Omega = \delta^{-1} \ ; \ M = \delta^{-1}$$

We obtain the following Hamiltonian:

$$\hat{H} = \sum_{\substack{n_1, n_2, n_3 \\ = -\infty}}^{\infty} \frac{1}{2} \left[ \frac{1}{M} \hat{p}^2(\vec{n}) + M\omega^2 \hat{x}^2(\vec{n}) + M\Omega^2 \sum_{i=1}^{3} (\hat{x}(\vec{n} + \vec{e}_i \delta) - \hat{x}(\vec{n}))^2 \right]$$

Infinite coupled harmonic oscillators in each direction!!

# Complexity for a Pair of Coupled Oscillators

Let's start with a pair of oscillators. The Hamiltonian is:

$$\hat{H} = \frac{1}{2}\left[\hat{p}_1^2 + \hat{p}_2^2 + \omega^2(\hat{x}_1^2 + \hat{x}_2^2) + \Omega^2(\hat{x}_1 - \hat{x}_2)^2\right]$$

Defining normal modes:

$$\hat{\tilde{x}}_\pm = \frac{1}{\sqrt{2}}(\hat{x}_1 \pm \hat{x}_2)\ ;\ \hat{\tilde{p}}_\pm = \frac{1}{\sqrt{2}}(\hat{p}_1 \pm \hat{p}_2)$$

$$\tilde{\omega}_+^2 = \omega^2\ ;\ \tilde{\omega}_-^2 = \omega^2 + 2\Omega^2$$

We obtain:

$$\hat{H} = \frac{1}{2}\left[\hat{\tilde{p}}_+^2 + \hat{\tilde{p}}_-^2 + \tilde{\omega}_+^2\hat{\tilde{x}}_+^2 + \tilde{\omega}_-^2\hat{\tilde{x}}_-^2\right]$$

Decoupled oscillators!!

The ground ground state is our target:

$$\phi_t(\tilde{x}_+, \tilde{x}_-) = \frac{(\tilde{\omega}_+\tilde{\omega}_-)^{1/4}}{\sqrt{\pi}} \exp\left[ -\frac{1}{2}(\tilde{\omega}_+^2 \tilde{x}_+^2 + \tilde{\omega}_-^2 \tilde{x}_-^2) \right]$$

We pick an arbitrary reference state:

$$\phi_r(x_1, x_2) = \sqrt{\frac{\mu}{\pi}} \exp\left[ -\frac{\mu}{2}(x_1^2 + x_2^2) \right]$$

$$\phi_r(\tilde{x}_+, \tilde{x}_-) = \sqrt{\frac{\mu}{\pi}} \exp\left[ -\frac{\mu}{2}(\tilde{x}_+^2 + \tilde{x}_-^2) \right]$$

We can define elementary gates using position and momentum as generators:

$$\begin{aligned}
\hat{T}_a(\epsilon) \equiv \exp i\epsilon\hat{p}_a &\longrightarrow \text{Shift } \langle\hat{x}_a\rangle \text{ by } \epsilon \\
\hat{K}_a(\epsilon) \equiv \exp i\epsilon\hat{x}_a &\longrightarrow \text{Shift } \langle\hat{p}_a\rangle \text{ by } \epsilon \\
\hat{Q}_{ab}(\epsilon) \equiv \exp i\epsilon\hat{x}_a\hat{p}_b &\longrightarrow \text{Shift } \langle\hat{x}_b\rangle \text{ by } \epsilon\langle\hat{x}_a\rangle \text{ ; } a \neq b \\
\hat{Q}_a(\epsilon) \equiv \exp i\epsilon(\hat{x}_a\hat{p}_a + \hat{p}_a\hat{x}_a) &\longrightarrow \text{Scale } \langle\hat{x}_a\rangle \text{ by } \exp(2\epsilon)
\end{aligned}$$

The same can be done for the normal modes:

$$\hat{\tilde{Q}}_\pm(\epsilon) \equiv \exp i\epsilon(\hat{\tilde{x}}_\pm\hat{\tilde{p}}_\pm + \hat{\tilde{p}}_\pm\hat{\tilde{x}}_\pm) \longrightarrow \text{Scale } \langle\hat{\tilde{x}}_\pm\rangle \text{ by } \exp(2\epsilon)$$

To go from reference:

$$\phi_r(\tilde{x}_+, \tilde{x}_-) = \sqrt{\frac{\mu}{\pi}} \exp\left[ -\frac{\mu}{2}(\tilde{x}_+^2 + \tilde{x}_-^2) \right]$$

To target:

$$\phi_t(\tilde{x}_+, \tilde{x}_-) = \frac{(\tilde{\omega}_+ \tilde{\omega}_-)^{1/4}}{\sqrt{\pi}} \exp\left[ -\frac{1}{2}(\tilde{\omega}_+^2 \tilde{x}_+^2 + \tilde{\omega}_-^2 \tilde{x}_-^2) \right]$$

We only need the scaling operator $\hat{\hat{Q}}_\pm$:

$$|\phi_r\rangle \xrightarrow{\hat{\hat{Q}}_+(\epsilon_+)\hat{\hat{Q}}_-(\epsilon_-)} |\phi_t\rangle$$

$$\epsilon_\pm = \frac{1}{2} \ln\left( \frac{\tilde{\omega}_\pm}{\mu} \right)$$

It has been proved that this is the most efficient circuit for $\mathcal{F}_1$ and $\mathcal{F}_2$!

To obtain the complexity, we go back to the definitions:

$$\hat{U}(1) = \overleftarrow{\mathcal{P}} \exp \int_0^1 d\tilde{\tau} \hat{H}(\tilde{\tau}) = \hat{\tilde{Q}}_+(\epsilon_+)\hat{\tilde{Q}}_-(\epsilon_-)$$

$$\therefore \overleftarrow{\mathcal{P}} \exp \int_0^1 d\tilde{\tau} (Y_+ \hat{\mathcal{O}}_+ + Y_- \hat{\mathcal{O}}_-) = \exp\Big(\epsilon_+ \hat{\mathcal{O}}_+ + \epsilon_- \hat{\mathcal{O}}_-\Big)$$

Since there's no path dependency:

$$Y_\pm = \epsilon_\pm$$

By using the cost functions $\mathcal{F}_1$ or $\mathcal{F}_2$:

$$\mathcal{F}_1(\tau) = \sum_l |\mathsf{Y}_l(\tau)|$$

$$\mathcal{F}_2(\tau) = \sqrt{\sum_l \mathsf{Y}_l^2(\tau)}$$

We obtain:

$$\mathcal{C}_1 = \frac{1}{2}\left|\ln\left(\frac{\tilde{\omega}_+}{\mu}\right)\right| + \frac{1}{2}\left|\ln\left(\frac{\tilde{\omega}_-}{\mu}\right)\right|$$

$$\mathcal{C}_2 = \frac{1}{2}\sqrt{\ln^2\left(\frac{\tilde{\omega}_+}{\mu}\right) + \ln^2\left(\frac{\tilde{\omega}_-}{\mu}\right)}$$

# Complexity of the Vacuum in Free Scalar QFT

To study the complexity of the vacuum in free scalar QFT, let's consider a system of N one dimensional coupled harmonic oscillators in a lattice with periodic boundary conditions.
This is a way of regularizing divergences:

$$\hat{H} = \sum_{\substack{n_1,n_2,n_3 \\ =-\infty}}^{\infty} \frac{1}{2} \left[ \frac{1}{M}\hat{p}^2(\vec{n}) + Mm^2\hat{x}^2(\vec{n}) + M\Omega^2 \sum_{i=1}^{3}(\hat{x}(\vec{n}+\vec{e_i}\delta) - \hat{x}(\vec{n}))^2 \right]$$

$$\downarrow$$

$$\hat{H} = \sum_{n=0}^{N-1} \frac{1}{2} \left[ \hat{p}_n^2 + \omega^2 \hat{x}_n^2 + \Omega^2(\hat{x}_{n+1} - \hat{x}_n)^2 \right] \; ; \; M = 1$$

To find the normal modes, we employ a discrete Fourier transform:

$$\hat{\tilde{x}}_k \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \exp\left(\frac{-2\pi i k n}{N}\right) \hat{x}_n$$

$$\hat{\tilde{p}}_k \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \exp\left(\frac{2\pi i k n}{N}\right) \hat{p}_n$$

We obtain a Hamiltonian for N decoupled oscillators:

$$\hat{H} = \sum_{k=0}^{N-1} \frac{1}{2} \left[ \hat{\tilde{p}}_k \hat{\tilde{p}}_k^\dagger + \tilde{\omega}_k^2 \hat{\tilde{x}}_k \hat{\tilde{x}}_k^\dagger \right]$$

$$\tilde{\omega}_k^2 \equiv \omega^2 + 4\Omega^2 \sin^2\left(\frac{\pi k}{N}\right)$$

The excitations of these oscillators are the so-called particles.

The vacuum is our target:

$$\phi_t(\tilde{x}_0, \tilde{x}_1, \ldots) = \prod_{k=0}^{N-1} \left( \frac{\tilde{\omega}_k}{\sqrt{\pi}} \right)^{1/4} \exp\left[ -\frac{1}{2} \tilde{\omega}_k |\tilde{x}_k|^2 \right]$$

And we pick the reference as:

$$\phi_r(x_0, x_1, \ldots) = \prod_{n=0}^{N-1} \left( \frac{\mu}{\sqrt{\pi}} \right)^{1/4} \exp\left[ -\frac{1}{2} \mu x_n^2 \right]$$

$$\phi_r(\tilde{x}_0, \tilde{x}_1, \ldots) = \prod_{k=0}^{N-1} \left( \frac{\mu}{\sqrt{\pi}} \right)^{1/4} \exp\left[ -\frac{1}{2} \mu |\tilde{x}_k|^2 \right]$$

Like in the case with two oscillators, the only operator we need to go from reference to target is:

$$\hat{\tilde{Q}}_k(\epsilon) \equiv \exp i\epsilon(\hat{\tilde{x}}_k\hat{\tilde{p}}_k + \hat{\tilde{p}}_k\hat{\tilde{x}}_k) \quad \longrightarrow \quad \text{Scale } \langle\hat{\tilde{x}}_k\rangle \text{ by } \exp(2\epsilon)$$

$$|\phi_r\rangle \xrightarrow{\prod_{k=0}^{N-1}\hat{\tilde{Q}}_k(\epsilon_k)} |\phi_t\rangle$$

$$\epsilon_k = \frac{1}{2}\ln\left(\frac{\tilde{\omega}_k}{\mu}\right)$$

At last, we obtain the complexity by employing the definitions:

$$\hat{U}(1) = \overleftarrow{\mathcal{P}} \exp \int_0^1 d\tilde{\tau} \hat{H}(\tilde{\tau}) = \prod_{k=0}^{N-1} \hat{\tilde{Q}}_k(\epsilon_k)$$

$$\therefore \overleftarrow{\mathcal{P}} \exp \int_0^1 d\tilde{\tau} \left( \sum_{k=0}^{N-1} Y_k \hat{\mathcal{O}}_k \right) = \exp\left( \sum_{k=0}^{N-1} \epsilon_k \hat{\mathcal{O}}_k \right)$$

$$\therefore Y_k = \epsilon_k$$

$$\therefore \mathcal{C}_1 = \frac{1}{2} \sum_{k=0}^{N-1} \left| \ln \frac{\tilde{\omega}_k}{\mu} \right| \; ; \; \mathcal{C}_2 = \frac{1}{2} \sqrt{\sum_{k=0}^{N-1} \ln^2 \left( \frac{\tilde{\omega}_k}{\mu} \right)}$$

Let's analyze $\mathcal{C}_1$.

We can rewrite $\mathcal{C}_1$ as:

$$\mathcal{C}_1 = \frac{1}{4} \sum_{k=0}^{N-1} \left| \ln \frac{\tilde{\omega}_k^2}{\mu^2} \right|$$

We must remember that the normal mode frequencies were defined as:

$$\tilde{\omega}_k^2 \equiv \omega^2 + 4\Omega^2 \sin^2 \left( \frac{\pi k}{N} \right)$$

Where $\omega \equiv m$ and $\Omega \equiv \delta^{-1}$.
Also, we can write the number of oscillators as:

$$N = \frac{L}{\delta}$$

Where L is the length of our lattice.

Therefore, we obtain:

$$\mathcal{C}_1 = \frac{1}{4}\sum_{k=0}^{N-1}\left| ln\left(\frac{m^2}{\mu^2} + \frac{4\sin^2\left(\frac{\pi k\delta}{L}\right)}{\delta^2\mu^2}\right)\right|$$

To recover QFT, two limits must be taken:

$$L \to \infty \text{ (or } N \to \infty) \; ; \; \delta \to 0$$

Each of them causes $\mathcal{C}_1$ to diverge.

We know that, in the QFT limit, the normal mode frequencies respect the following:

$$\tilde{\omega}_k^2 = m^2 + p_k^2$$

The momentum originates from the second term inside the *ln*. Let's identify this term as $p_k^2$:

$$\frac{1}{4}\sum_{k=0}^{N-1}\left| ln\left(\frac{m^2}{\mu^2} + \frac{4\sin^2\left(\frac{\pi k}{N}\right)}{\delta^2\mu^2}\right)\right| = \frac{1}{4}\sum_{k=0}^{N-1}\left| ln\left(\frac{m^2}{\mu^2} + \frac{p_k^2}{\mu^2}\right)\right|$$

When we take the limits and sum over all $k$, we also end up with $k \to \infty$. This is the so-called UV divergence:

$$\lim_{\substack{\delta\to 0\\ N\to\infty\\ k\to\infty}} p_k^2 = \infty$$

Then, it's possible to argue that, in order to obtain the leading contribution to $\mathcal{C}_1$, we only need to consider the UV contribution:

$$\lim_{\substack{\delta\to 0\\ N\to\infty}} \mathcal{C}_1 = \lim_{\substack{\delta\to 0\\ N\to\infty}} \frac{1}{4} \sum_{k=0}^{N-1} \left| ln\left(\frac{m^2}{\mu^2} + \frac{p_k^2}{\mu^2}\right)\right| \sim \lim_{\substack{\delta\to 0\\ N\to\infty}} \frac{1}{4} \sum_{k>IR}^{N-1} \left| ln\frac{p_k^2}{\mu^2}\right|$$

But we can find an even simpler expression:

$$ln\left[\frac{p_k^2}{\mu^2}\right] = \ln\left[\frac{4\sin^2\left(\frac{\pi k}{N}\right)}{\delta^2\mu^2}\right] = 2\ln\left[\frac{2}{\mu\delta}\right] + \ln\left[\sin^2\left(\frac{\pi k}{N}\right)\right]$$

In the UV, $2\ln\left[\frac{2}{\mu\delta}\right]$ dominates.

We have, finally:

$$\mathcal{C}_1 \sim \lim_{\substack{\delta\to 0\\ N\to\infty}} \frac{1}{2}\sum_{k=0}^{N-1} ln\left(\frac{2}{\mu\delta}\right) = \lim_{\substack{\delta\to 0\\ N\to\infty}} \frac{N}{2} ln\left(\frac{2}{\mu\delta}\right)$$

$$\therefore \mathcal{C}_1 \sim \frac{1}{2}\left(\frac{L}{\delta}\right) ln\left(\frac{2}{\mu\delta}\right)$$

The leading contribution to $\mathcal{C}_1$.

A similar analysis gives the leading contribution to $\mathcal{C}_2$:

$$\mathcal{C}_2 \sim \frac{1}{2}\sqrt{\frac{L}{\delta}}\, ln\left(\frac{2}{\mu\delta}\right)$$

# Applications of Quantum Complexity

- Quantum computing
- Quantum chaos
- Topological phase transitions
- Holography and quantum gravity

# Holography, Quantum Gravity and Complexity

- AdS/CFT

Mathematical equivalence between a 4 dimensional CFT (boundary) and a gravitational theory in 5 dimensional AdS space (bulk).

A special case of the holographic principle.

- 2001 - *Eternal Black Holes in Anti-de Sitter*
  Description of ER bridges as entangled CFTs.

- 2006 - *Holographic Derivation of Entanglement Entropy from AdS/CFT*
  Relation between entanglement and spacetime geometry.

- 2010 - *Building up Spacetime with Quantum Entanglement*
  Relation between spacetime connectivity and entanglement.

- 2013 - *Cool Horizons for Entangled Black Holes*
  (ER = EPR) Description of ER bridges as a manifestation of EPR correlations.

- 2014 - *Entanglement is not Enough*

- 2014 - *Entanglement is not Enough*

"Entanglement is not enough to understand the rich geometric structures that exist behind the horizon and which are predicted by general relativity. Entanglement entropy only grows for a very short time, but the growth of Einstein-Rosen bridges is expected to last for a very long time. Encoding that growth in the quantum state requires quantum complexity."

*Leonard Susskind*

Conjectures relating complexity and the geometry of spacetime allow us to obtain the so-called “holographic complexity".

The leading contributions to holographic complexity obtained from CV and CA conjectures:

$$\mathcal{C}_V \sim \frac{L}{\delta} \quad ; \quad \mathcal{C}_A \sim \left(\frac{L}{\delta}\right) \ln\left(\frac{L_{AdS}}{\alpha\delta}\right)$$

For the vacuum of free scalar QFT, we have found:

$$\mathcal{C}_1 \sim \frac{1}{2}\left(\frac{L}{\delta}\right) ln\left(\frac{2}{\mu\delta}\right) \quad ; \quad \mathcal{C}_2 \sim \frac{1}{2}\sqrt{\frac{L}{\delta}} ln\left(\frac{2}{\mu\delta}\right)$$

Just a few days ago!

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# Traversable wormhole dynamics on a quantum processor

Daniel Jafferis, Alexander Zlokapa, Joseph D. Lykken, David K. Kolchmeyer, Samantha I. Davis, Nikolai Lauk, Hartmut Neven & Maria Spiropulu

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“Where there is quantum mechanics, there is also gravity."

*Leonard Susskind*

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