Sigma Model

Notation

In[1]:=
```
<< Notation`
```

Notation: The defined current default input format StandardForm differs from the current default output format TraditionalForm. The WorkingForm option will default to the current default output format, but the Notations, Symbolizations, and InfixNotations may behave differently than expected.

In[2]:=
```
Dag[Σ_] := ConjugateTranspose[Σ] // Simplify
```

In[3]:=
```
Notation[ π_{i_} ⟺ pi[i_] ]
Notation[ π'_{i_} ⟺ pip[i_] ]
Notation[ Σ_^† ⟺ Dag[Σ_] ]
```

In[6]:=
```
$Assumptions = {σ[_] ∈ ℝ, pi[_][_] ∈ ℝ, pip[_][_] ∈ ℝ, s[_] ∈ ℝ, ν ∈ ℝ, f_π ∈ ℝ, a ∈ ℝ}
```

Out[6]= $\{\sigma(\_) \in \mathbb{R}, \pi\_(\_) \in \mathbb{R}, \pi'\_(\_) \in \mathbb{R}, s(\_) \in \mathbb{R}, \nu \in \mathbb{R}, f_\pi \in \mathbb{R}, a \in \mathbb{R}\}$

Useful functions

In[7]:=
```
PickℒTerm[expression_, operator_] :=
  (Coefficient[expression, operator] /. _[x] :> 0) operator
```

Σ - Field

In[8]:=
```
Σ = σ[x] (1 0; 0 1) + i (Sum_{i}^{3} π_i(x) PauliMatrix[i]) // Expand
```

Out[8]= $\begin{pmatrix} \sigma(x) + i\,\pi_3(x) & i\,\pi_1(x) + \pi_2(x) \\ i\,\pi_1(x) - \pi_2(x) & \sigma(x) - i\,\pi_3(x) \end{pmatrix}$

In[9]:=
```
Σ†
```

Out[9]= $\begin{pmatrix} \sigma(x) - i\,\pi_3(x) & -i\,\pi_1(x) - \pi_2(x) \\ \pi_2(x) - i\,\pi_1(x) & \sigma(x) + i\,\pi_3(x) \end{pmatrix}$

Linear Sigma Model

In[10]:=
```
ℒkin = (-1/4 Tr[∂{x} Σ.∂{x} Σ†]) // Expand
```
Out[10]=
$$-\frac{1}{2}\sigma'(x)^2-\frac{1}{2}\pi_1'(x)^2-\frac{1}{2}\pi_2'(x)^2-\frac{1}{2}\pi_3'(x)^2$$

In[11]:=
```
ℒmass = (μ^2/4 Tr[Σ.Σ†]) // Expand
```
Out[11]=
$$\frac{1}{2}\mu^2\sigma(x)^2+\frac{1}{2}\mu^2\pi_1(x)^2+\frac{1}{2}\mu^2\pi_2(x)^2+\frac{1}{2}\mu^2\pi_3(x)^2$$

In[12]:=
```
ℒinter = -λ/16 (Tr[Σ.Σ†]^2) // Expand
```
Out[12]=
$$-\frac{1}{2}\lambda\sigma(x)^2\pi_1(x)^2-\frac{1}{2}\lambda\sigma(x)^2\pi_2(x)^2-\frac{1}{2}\lambda\sigma(x)^2\pi_3(x)^2-\frac{1}{4}\lambda\sigma(x)^4-\frac{1}{4}\lambda\pi_1(x)^4-\frac{1}{4}\lambda\pi_2(x)^4-\frac{1}{4}\lambda\pi_3(x)^4-\frac{1}{2}\lambda\pi_1(x)^2\pi_2(x)^2-\frac{1}{2}\lambda\pi_1(x)^2\pi_3(x)^2-\frac{1}{2}\lambda\pi_2(x)^2\pi_3(x)^2$$

Exponential Parametrization

In[13]:=
```
Σ = (v + s[x]) U[x]
```
Out[13]=
$$U(x)\,(v+s(x))$$

In[14]:=
```
ⅈ Sum[pip[i][x] PauliMatrix[i], {i, 3}]
U[x_] = MatrixExp[%]
U[x_] = Sum[MatrixPower[%%, k] / k!, {k, 0, 2}]
```
Out[14]=
$$\begin{pmatrix} i\,\pi_3(x) & i\,(\pi_1(x)-i\,\pi_2(x)) \\ i\,(\pi_1(x)+i\,\pi_2(x)) & -i\,\pi_3(x) \end{pmatrix}$$

Out[15]=
$$\left(\begin{matrix} \frac{e^{i\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}}\left(\pi_3(x)+\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}\right)}{2\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}}-\frac{e^{-i\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}}\left(\pi_3(x)-\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}\right)}{2\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}} & \frac{e^{i\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}}\,(\pi_1}{2\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi}} \\ \frac{e^{i\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}}\,(\pi_1(x)+i\,\pi_2(x))}{2\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}}-\frac{e^{-i\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}}\,(\pi_1(x)+i\,\pi_2(x))}{2\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}} & \frac{e^{i\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}}\left(\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x}\right.}{2\sqrt{\pi_1(x)^2+\pi_2(x)^2+\pi_3(x)^2}} \end{matrix}\right.$$

Out[16]=
$$\left(\begin{matrix} i\,\pi_3(x)+\frac{1}{2}\left(-\pi_3(x)^2-(\pi_1(x)-i\,\pi_2(x))\,(\pi_1(x)+i\,\pi_2(x))\right)+1 & i\,(\pi_1(x)-i\,\pi_2(x)) \\ i\,(\pi_1(x)+i\,\pi_2(x)) & -i\,\pi_3(x)+\frac{1}{2}\left(-\pi_3(x)^2-(\pi_1(x)-i\,\pi_2(x))\,(\pi_1(x)+i\,\pi_2\right. \end{matrix}\right.$$

In[17]:=

$$\mathcal{L}\sigma\mathrm{kinp} = \left(-\frac{1}{4}\,\mathrm{Tr}\left[\partial_{\{x\}}\Sigma.\partial_{\{x\}}\Sigma^{\dagger}\right]\right) \;\text{// Expand}$$

$$\mathrm{Pick}\mathcal{L}\mathrm{term}\left[\mathcal{L}\sigma\mathrm{kinp},\, s'(x)^2\right]$$

$$\mathrm{Pick}\mathcal{L}\mathrm{term}\left[\mathcal{L}\sigma\mathrm{kinp},\, ((\pi_1)'(x))^2\right]$$

$$\mathrm{Pick}\mathcal{L}\mathrm{term}\left[\mathcal{L}\sigma\mathrm{kinp},\, ((\pi_2)'(x))^2\right]$$

$$\mathrm{Pick}\mathcal{L}\mathrm{term}\left[\mathcal{L}\sigma\mathrm{kinp},\, ((\pi_3)'(x))^2\right]$$

Out[17]=

$$-\frac{1}{8} s'(x)^2 \pi_1(x)^4 - \frac{1}{2} \nu\, s'(x)\, (\pi_1)'(x)\, \pi_1(x)^3 - \frac{1}{2} s(x)\, s'(x)\, (\pi_1)'(x)\, \pi_1(x)^3 - \frac{1}{4} \pi_2(x)^2 s'(x)^2 \pi_1(x)^2 -$$

$$\frac{1}{4} \pi_3(x)^2 s'(x)^2 \pi_1(x)^2 - \frac{1}{2} \nu^2 ((\pi_1)'(x))^2 \pi_1(x)^2 - \frac{1}{2} s(x)^2 ((\pi_1)'(x))^2 \pi_1(x)^2 - \nu\, s(x)\, ((\pi_1)'(x))^2 \pi_1(x)^2 -$$

$$\frac{1}{2} \nu\, \pi_2(x)\, s'(x)\, (\pi_2)'(x)\, \pi_1(x)^2 - \frac{1}{2} s(x)\, \pi_2(x)\, s'(x)\, (\pi_2)'(x)\, \pi_1(x)^2 - \frac{1}{2} \nu\, \pi_3(x)\, s'(x)\, (\pi_3)'(x)\, \pi_1(x)^2 -$$

$$\frac{1}{2} s(x)\, \pi_3(x)\, s'(x)\, (\pi_3)'(x)\, \pi_1(x)^2 - \frac{1}{2} \nu\, \pi_2(x)^2 s'(x)\, (\pi_1)'(x)\, \pi_1(x) - \frac{1}{2} s(x)\, \pi_2(x)^2 s'(x)\, (\pi_1)'(x)\, \pi_1(x) -$$

$$\frac{1}{2} \nu\, \pi_3(x)^2 s'(x)\, (\pi_1)'(x)\, \pi_1(x) - \frac{1}{2} s(x)\, \pi_3(x)^2 s'(x)\, (\pi_1)'(x)\, \pi_1(x) - \nu^2 \pi_2(x)\, (\pi_1)'(x)\, (\pi_2)'(x)\, \pi_1(x) -$$

$$s(x)^2 \pi_2(x)\, (\pi_1)'(x)\, (\pi_2)'(x)\, \pi_1(x) - 2\, \nu\, s(x)\, \pi_2(x)\, (\pi_1)'(x)\, (\pi_2)'(x)\, \pi_1(x) - \nu^2 \pi_3(x)\, (\pi_1)'(x)\, (\pi_3)'(x)\, \pi_1(x) -$$

$$s(x)^2 \pi_3(x)\, (\pi_1)'(x)\, (\pi_3)'(x)\, \pi_1(x) - 2\, \nu\, s(x)\, \pi_3(x)\, (\pi_1)'(x)\, (\pi_3)'(x)\, \pi_1(x) - \frac{1}{8} \pi_2(x)^4 s'(x)^2 - \frac{1}{8} \pi_3(x)^4 s'(x)^2 -$$

$$\frac{1}{4} \pi_2(x)^2 \pi_3(x)^2 s'(x)^2 - \frac{1}{2} s'(x)^2 - \frac{1}{2} \nu^2 ((\pi_1)'(x))^2 - \frac{1}{2} s(x)^2 ((\pi_1)'(x))^2 - \nu\, s(x)\, ((\pi_1)'(x))^2 - \frac{1}{2} \nu^2 ((\pi_2)'(x))^2 -$$

$$\frac{1}{2} s(x)^2 ((\pi_2)'(x))^2 - \frac{1}{2} \nu^2 \pi_2(x)^2 ((\pi_2)'(x))^2 - \frac{1}{2} s(x)^2 \pi_2(x)^2 ((\pi_2)'(x))^2 - \nu\, s(x)\, \pi_2(x)^2 ((\pi_2)'(x))^2 -$$

$$\nu\, s(x)\, ((\pi_2)'(x))^2 - \frac{1}{2} \nu^2 ((\pi_3)'(x))^2 - \frac{1}{2} s(x)^2 ((\pi_3)'(x))^2 - \frac{1}{2} \nu^2 \pi_3(x)^2 ((\pi_3)'(x))^2 - \frac{1}{2} s(x)^2 \pi_3(x)^2 ((\pi_3)'(x))^2 -$$

$$\nu\, s(x)\, \pi_3(x)^2 ((\pi_3)'(x))^2 - \nu\, s(x)\, ((\pi_3)'(x))^2 - \frac{1}{2} \nu\, \pi_2(x)^3 s'(x)\, (\pi_2)'(x) - \frac{1}{2} s(x)\, \pi_2(x)^3 s'(x)\, (\pi_2)'(x) -$$

$$\frac{1}{2} \nu\, \pi_2(x)\, \pi_3(x)^2 s'(x)\, (\pi_2)'(x) - \frac{1}{2} s(x)\, \pi_2(x)\, \pi_3(x)^2 s'(x)\, (\pi_2)'(x) - \frac{1}{2} \nu\, \pi_3(x)^3 s'(x)\, (\pi_3)'(x) -$$

$$\frac{1}{2} s(x)\, \pi_3(x)^3 s'(x)\, (\pi_3)'(x) - \frac{1}{2} \nu\, \pi_2(x)^2 \pi_3(x)\, s'(x)\, (\pi_3)'(x) - \frac{1}{2} s(x)\, \pi_2(x)^2 \pi_3(x)\, s'(x)\, (\pi_3)'(x) -$$

$$\nu^2 \pi_2(x)\, \pi_3(x)\, (\pi_2)'(x)\, (\pi_3)'(x) - s(x)^2 \pi_2(x)\, \pi_3(x)\, (\pi_2)'(x)\, (\pi_3)'(x) - 2\, \nu\, s(x)\, \pi_2(x)\, \pi_3(x)\, (\pi_2)'(x)\, (\pi_3)'(x)$$

Out[18]=

$$-\frac{1}{2} s'(x)^2$$

Out[19]=

$$-\frac{1}{2} \nu^2 ((\pi_1)'(x))^2$$

Out[20]=

$$-\frac{1}{2} \nu^2 ((\pi_2)'(x))^2$$

Out[21]=

$$-\frac{1}{2} \nu^2 ((\pi_3)'(x))^2$$

In[22]:=
```
ℒσmassp = (μ^2/4 Tr[Σ.Σ†]) // Expand
PickℒtermCommand[ℒσmassp, s[x]^2]
Pickℒterm[ℒσmassp, π̃1(x)^2]
```

Out[22]=
$$\frac{1}{4}\mu^2 v\, s(x)\, \tilde{\pi}_1(x)^4+\frac{1}{2}\mu^2 v\, s(x)\, \tilde{\pi}_2(x)^2 \tilde{\pi}_1(x)^2+\frac{1}{2}\mu^2 v\, s(x)\, \tilde{\pi}_3(x)^2 \tilde{\pi}_1(x)^2+\frac{1}{4}\mu^2 v\, s(x)\, \tilde{\pi}_2(x)^4+\frac{1}{4}\mu^2 v\, s(x)\, \tilde{\pi}_3(x)^4+\frac{1}{2}\mu^2 v\, s(x)\, \tilde{\pi}_2(x)^2 \tilde{\pi}_3(x)^2+\frac{1}{8}\mu^2 s(x)^2 \tilde{\pi}_1(x)^4+\frac{1}{4}\mu^2 s(x)^2 \tilde{\pi}_2(x)^2 \tilde{\pi}_1(x)^2+\frac{1}{4}\mu^2 s(x)^2 \tilde{\pi}_3(x)^2 \tilde{\pi}_1(x)^2+\frac{1}{8}\mu^2 s(x)^2 \tilde{\pi}_2(x)^4+\frac{1}{8}\mu^2 s(x)^2 \tilde{\pi}_3(x)^4+\frac{1}{4}\mu^2 s(x)^2 \tilde{\pi}_2(x)^2 \tilde{\pi}_3(x)^2+\frac{1}{8}\mu^2 v^2 \tilde{\pi}_1(x)^4+\frac{1}{4}\mu^2 v^2 \tilde{\pi}_2(x)^2 \tilde{\pi}_1(x)^2+\frac{1}{4}\mu^2 v^2 \tilde{\pi}_3(x)^2 \tilde{\pi}_1(x)^2+\frac{1}{8}\mu^2 v^2 \tilde{\pi}_2(x)^4+\frac{1}{8}\mu^2 v^2 \tilde{\pi}_3(x)^4+\frac{1}{4}\mu^2 v^2 \tilde{\pi}_2(x)^2 \tilde{\pi}_3(x)^2+\frac{\mu^2 v^2}{2}+\mu^2 v\, s(x)+\frac{1}{2}\mu^2 s(x)^2$$

Out[23]=
$$\frac{1}{2}\mu^2 s(x)^2$$

Out[24]=
$$0$$

Nonlinear Sigma Model

In[25]:=
```
i/fπ Sum[pip[i][x] PauliMatrix[i], {i, 3}];
U[x_] = Sum[MatrixPower[%, k] / k!, {k, 0, 2}]
```

Out[26]=
$$\begin{pmatrix} \frac{i\,\tilde{\pi}_3(x)}{f_\pi}+\frac{1}{2}\left(-\frac{\tilde{\pi}_3(x)^2}{f_\pi^2}-\frac{(\tilde{\pi}_1(x)-i\,\tilde{\pi}_2(x))(\tilde{\pi}_1(x)+i\,\tilde{\pi}_2(x))}{f_\pi^2}\right)+1 & \frac{i\,(\tilde{\pi}_1(x)-i\,\tilde{\pi}_2(x))}{f_\pi} \\ \frac{i\,(\tilde{\pi}_1(x)+i\,\tilde{\pi}_2(x))}{f_\pi} & -\frac{i\,\tilde{\pi}_3(x)}{f_\pi}+\frac{1}{2}\left(-\frac{\tilde{\pi}_3(x)^2}{f_\pi^2}-\frac{(\tilde{\pi}_1(x)-i\,\tilde{\pi}_2(x))(\tilde{\pi}_1(x)+i\,\tilde{\pi}_2(x))}{f_\pi^2}\right)+1 \end{pmatrix}$$

In[27]:=
```
ℒNLσ = -fπ^2/4 Tr[∂x U[x].∂x U[x]†] // Expand
Pickℒterm[ℒNLσ, ((π̃1)'(x))^2]
Pickℒterm[ℒNLσ, ((π̃2)'(x))^2]
Pickℒterm[ℒNLσ, ((π̃3)'(x))^2]
```

Out[27]=
$$-\frac{\tilde{\pi}_1(x)^2\,((\tilde{\pi}_1)'(x))^2}{2 f_\pi^2}-\frac{\tilde{\pi}_1(x)\,\tilde{\pi}_2(x)\,(\tilde{\pi}_2)'(x)\,(\tilde{\pi}_1)'(x)}{f_\pi^2}-\frac{\tilde{\pi}_1(x)\,\tilde{\pi}_3(x)\,(\tilde{\pi}_3)'(x)\,(\tilde{\pi}_1)'(x)}{f_\pi^2}-\frac{\tilde{\pi}_2(x)^2\,((\tilde{\pi}_2)'(x))^2}{2 f_\pi^2}-\frac{\tilde{\pi}_3(x)^2\,((\tilde{\pi}_3)'(x))^2}{2 f_\pi^2}-\frac{\tilde{\pi}_2(x)\,\tilde{\pi}_3(x)\,(\tilde{\pi}_2)'(x)\,(\tilde{\pi}_3)'(x)}{f_\pi^2}-\frac{1}{2}((\tilde{\pi}_1)'(x))^2-\frac{1}{2}((\tilde{\pi}_2)'(x))^2-\frac{1}{2}((\tilde{\pi}_3)'(x))^2$$

Out[28]=
$$-\frac{1}{2}((\tilde{\pi}_1)'(x))^2$$

Out[29]=
$$-\frac{1}{2}((\tilde{\pi}_2)'(x))^2$$

Out[30]=
$$-\frac{1}{2}((\tilde{\pi}_3)'(x))^2$$

SO(N) vector model

In[31]:=
```
Quit[]
```

Notation

In[1]:=
```
<< Notation`
```

Notation: The defined current default input format StandardForm differs from the current default output format TraditionalForm. The WorkingForm option will default to the current default output format, but the Notations, Symbolizations, and InfixNotations may behave differently than expected.

In[2]:=
```
Notation[ ϕ^i_ ⟺ ϕ[i_] ]
Notation[ ϕ0^i_ ⟺ ϕ0[i_] ]
Notation[ π^i_ ⟺ π[i_] ]
```

In[5]:=
```
$Assumptions = {ϕ[_][_] ∈ ℝ, ν ∈ ℝ, μ ∈ ℝ, λ ∈ ℝ, σ[_] ∈ ℝ, π[_][_] ∈ ℝ}
```

Out[5]= $\{\phi^{\_}(\_) \in \mathbb{R}, \nu \in \mathbb{R}, \mu \in \mathbb{R}, \lambda \in \mathbb{R}, \sigma(\_) \in \mathbb{R}, \pi^{\_}(\_) \in \mathbb{R}\}$

Useful functions

In[6]:=
```
Pickℒterm[expression_, operator_] :=
  (Coefficient[expression, operator] /. _[x] :→ 0) operator
```

Lagrangian

In[7]:=
```
ℒkin = -1/2 Sum[(∂x ϕ[i][x]) (∂x ϕ[i][x]), {i, 1, ℕ}]
```

Out[7]= $-\frac{1}{2}\sum_{i=1}^{\mathbb{N}}((\phi^i)'(x))^2$

In[8]:=
```
ℒpot = μ^2/2 Sum[(ϕ[i][x]) (ϕ[i][x]), {i, 1, ℕ}] -
  λ/4 (Sum[(ϕ[i][x]) (ϕ[i][x]), {i, 1, ℕ}])^2
```

Out[8]= $\frac{1}{2}\mu^2\sum_{i=1}^{\mathbb{N}}\phi^i(x)^2 - \frac{1}{4}\lambda\left(\sum_{i=1}^{\mathbb{N}}\phi^i(x)^2\right)^2$

Vacuum

$\phi_0 = \{0, 0, \ldots, 0, v\}$

In[9]:=
```
ϕ0[i_] := If[i == ℕ, ν, 0]
```

In[10]:=
```
ϕ[i_][x_] := If[i == ℕ, ν + σ[x], π[i][x]]
```

In[11]:=
```
ℒkin //. ℕ → 3
```

Out[11]= $\frac{1}{2}\left(-\left(\left(\pi^{2}\right)'(x)\right)^2 - \left(\left(\pi^{1}\right)'(x)\right)^2 - \sigma'(x)^2\right)$

In[12]:=
```
ℒpot //. ℕ → 3
% // Expand
Pickℒterm[%, σ[x]^2] /. (Solve[ν^2 == μ^2/λ, λ][[1]])
```

Out[12]= $\frac{1}{2}\mu^2\left((\nu+\sigma(x))^2+\pi^{2}(x)^2+\pi^{1}(x)^2\right)-\frac{1}{4}\lambda\left((\nu+\sigma(x))^2+\pi^{2}(x)^2+\pi^{1}(x)^2\right)^2$

Out[13]= $-\frac{\lambda \nu^4}{4}+\frac{\mu^2 \nu^2}{2}-\lambda \nu^3 \sigma(x)-\frac{3}{2}\lambda \nu^2 \sigma(x)^2-\frac{1}{2}\lambda \nu^2 \pi^{2}(x)^2-\frac{1}{2}\lambda \nu^2 \pi^{1}(x)^2-\lambda \nu \sigma(x)\pi^{2}(x)^2-\lambda \nu \sigma(x)\pi^{1}(x)^2-\lambda \nu \sigma(x)^3-\frac{1}{2}\lambda \sigma(x)^2\pi^{2}(x)^2-\frac{1}{2}\lambda \sigma(x)^2 \pi^{1}(x)^2-\frac{1}{4}\lambda \sigma(x)^4-\frac{1}{4}\lambda \pi^{2}(x)^4-\frac{1}{4}\lambda \pi^{1}(x)^4-\frac{1}{2}\lambda \pi^{1}(x)^2\pi^{2}(x)^2+\mu^2 \nu \sigma(x)+\frac{1}{2}\mu^2\sigma(x)^2+\frac{1}{2}\mu^2\pi^{2}(x)^2+\frac{1}{2}\mu^2\pi^{1}(x)^2$

Out[14]= $-\mu^2 \sigma(x)^2$