Quantum Field Theory in Curved Spacetimes

Fundamentals, representations and a little on the notions of vacuum and particles

Eduardo Amâncio Barbosa Oliveira
eduardo.amancio.oliveira@usp.br

Instituto de Física Teórica - Universidade Estadual Paulista

June 30, 2023

ICTP SAIFR
International Centre for Theoretical Physics
South American Institute for Fundamental Research

IFT - UNESP
INSTITUTO DE FÍSICA TEÓRICA

Contents

1. A Brief Review of QFT in Minkowski Spaces
2. Quantum Field Theory in Curved Spaces
3. Particle Detectors
4. Particle Creation
5. Adiabatic Vacuum
6. Conclusions

Introduction

Over this course, we have worked with QFT in Minkowski spacetime. On its standard formulation, QFT relies heavily on Poincaré symmetries to pick a preferred representation and vacuum state (as well as to implement calculations, such as spectral expansions, expected values, etc.).

However, in more general curved spacetimes, one does not have so many symmetries at hand. It happens that, for any system with infinitely many DOFs, we actually have an infinite number of unitarily inequivalent representations.Generally, in curved spacetimes (and also in flat ones), *all representations can be physically meaningful*, and one generally needs more than one to compute many physical effects of the theory.

In this seminar, we will show how to generalize some QFT notions into curved spacetimes, and use this framework to compute a few nontrivial physical effects in both flat and curved spaces.

Introduction

Over this course, we have worked with QFT in Minkowski spacetime. On its standard formulation, QFT relies heavily on Poincaré symmetries to pick a preferred representation and vacuum state (as well as to implement calculations, such as spectral expansions, expected values, etc.).

However, in more general curved spacetimes, one does not have so many symmetries at hand. It happens that, for any system with infinitely many DOFs, we actually have an infinite number of unitarily inequivalent representations.Generally, in curved spacetimes (and also in flat ones), *all representations can be physically meaningful*, and one generally needs more than one to compute many physical effects of the theory.

In this seminar, we will show how to generalize some QFT notions into curved spacetimes, and use this framework to compute a few nontrivial physical effects in both flat and curved spaces.

Introduction

Over this course, we have worked with QFT in Minkowski spacetime. On its standard formulation, QFT relies heavily on Poincaré symmetries to pick a preferred representation and vacuum state (as well as to implement calculations, such as spectral expansions, expected values, etc.).

However, in more general curved spacetimes, one does not have so many symmetries at hand. It happens that, for any system with infinitely many DOFs, we actually have an infinite number of unitarily inequivalent representations.Generally, in curved spacetimes (and also in flat ones), *all representations can be physically meaningful*, and one generally needs more than one to compute many physical effects of the theory.

In this seminar, we will show how to generalize some QFT notions into curved spacetimes, and use this framework to compute a few nontrivial physical effects in both flat and curved spaces.

Introduction

Over this course, we have worked with QFT in Minkowski spacetime. On its standard formulation, QFT relies heavily on Poincaré symmetries to pick a preferred representation and vacuum state (as well as to implement calculations, such as spectral expansions, expected values, etc.).

However, in more general curved spacetimes, one does not have so many symmetries at hand. It happens that, for any system with infinitely many DOFs, we actually have an infinite number of unitarily inequivalent representations. Generally, in curved spacetimes (and also in flat ones), *all representations can be physically meaningful*, and one generally needs more than one to compute many physical effects of the theory.

In this seminar, we will show how to generalize some QFT notions into curved spacetimes, and use this framework to compute a few nontrivial physical effects in both flat and curved spaces.

Introduction

Over this course, we have worked with QFT in Minkowski spacetime. On its standard formulation, QFT relies heavily on Poincaré symmetries to pick a preferred representation and vacuum state (as well as to implement calculations, such as spectral expansions, expected values, etc.).

However, in more general curved spacetimes, one does not have so many symmetries at hand. It happens that, for any system with infinitely many DOFs, we actually have an infinite number of unitarily inequivalent representations.Generally, in curved spacetimes (and also in flat ones), *all representations can be physically meaningful*, and one generally needs more than one to compute many physical effects of the theory.

In this seminar, we will show how to generalize some QFT notions into curved spacetimes, and use this framework to compute a few nontrivial physical effects in both flat and curved spaces.

Introduction

Over this course, we have worked with QFT in Minkowski spacetime. On its standard formulation, QFT relies heavily on Poincaré symmetries to pick a preferred representation and vacuum state (as well as to implement calculations, such as spectral expansions, expected values, etc.).

However, in more general curved spacetimes, one does not have so many symmetries at hand. It happens that, for any system with infinitely many DOFs, we actually have an infinite number of unitarily inequivalent representations.Generally, in curved spacetimes (and also in flat ones), *all representations can be physically meaningful*, and one generally needs more than one to compute many physical effects of the theory.

In this seminar, we will show how to generalize some QFT notions into curved spacetimes, and use this framework to compute a few nontrivial physical effects in both flat and curved spaces.

A Brief Review of QFT in Minkowski Spaces

Classical Fields and the Hamilton Principle

We can describe the dynamics of a classical field ϕ_a in Minkowski spacetime, armed with the flat metric η_{ab}, through an extreme action principle:

$$S = S_M[\phi_a] = \int d^4x \sqrt{-\eta(x)} \mathscr{L}(\phi_a(x), \partial_b \phi_a(x); x)$$

where $\eta = \det(\eta_{\mu\nu})$. Extremizing S with respect to ϕ_a, we obtain the Euler-Lagrange equations of the field:

$$\frac{\delta S}{\delta \phi_a} = \partial_b \left(\frac{\partial \mathscr{L}}{\partial(\partial_b \phi_a)} \right) - \frac{\partial \mathscr{L}}{\partial \phi_a} = 0$$

Classical Fields and the Hamilton Principle

We can describe the dynamics of a classical field ϕ_a in Minkowski spacetime, armed with the flat metric η_{ab}, through an extreme action principle:

$$S = S_M[\phi_a] = \int d^4x\, \sqrt{-\eta(x)}\mathscr{L}(\phi_a(x), \partial_b\phi_a(x); x)$$

where $\eta = \det(\eta_{\mu\nu})$. Extremizing S with respect to ϕ_a, we obtain the Euler-Lagrange equations of the field:

$$\frac{\delta S}{\delta \phi_a} = \partial_b\left(\frac{\partial \mathscr{L}}{\partial(\partial_b\phi_a)}\right) - \frac{\partial \mathscr{L}}{\partial \phi_a} = 0$$

Classical Fields and the Hamilton Principle

We can describe the dynamics of a classical field ϕ_a in Minkowski spacetime, armed with the flat metric η_{ab}, through an extreme action principle:

$$S = S_M[\phi_a] = \int d^4x \sqrt{-\eta(x)} \mathscr{L}(\phi_a(x), \partial_b\phi_a(x); x)$$

where $\eta = \det(\eta_{\mu\nu})$. Extremizing S with respect to ϕ_a, we obtain the Euler-Lagrange equations of the field:

$$\frac{\delta S}{\delta \phi_a} = \partial_b \left(\frac{\partial \mathscr{L}}{\partial(\partial_b \phi_a)} \right) - \frac{\partial \mathscr{L}}{\partial \phi_a} = 0$$

The Free Scalar Field and Plane-Wave Modes

A pivotal example will be the free real scalar field, whose Lagrangian reads:

$$\mathscr{L} = \frac{1}{2}\eta^{ab}(\partial_a\phi)(\partial_b\phi) - \frac{m^2}{2}\phi^2$$

It yields the linear Klein-Gordon equation:

$$\left[\Box_x + m^2\right]\phi(x) = 0 \qquad \Rightarrow \qquad \begin{cases} u_{\mathbf{k}}(\mathbf{x},t) = \dfrac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t} e^{i\mathbf{k}\cdot\mathbf{x}} \\ \phi(\mathbf{x},t) = \displaystyle\sum_{\mathbf{k}} \alpha_{\mathbf{k}} u_{\mathbf{k}}(\mathbf{x},t) + \alpha^*_{\mathbf{k}} u^*_{\mathbf{k}}(\mathbf{x},t) \end{cases}$$

The Free Scalar Field and Plane-Wave Modes

A pivotal example will be the free real scalar field, whose Lagrangian reads:

$$\mathscr{L} = \frac{1}{2}\eta^{ab}(\partial_a\phi)(\partial_b\phi) - \frac{m^2}{2}\phi^2$$

It yields the linear Klein-Gordon equation:

$$\left[\Box_x + m^2\right]\phi(x) = 0 \qquad \Rightarrow \qquad \begin{cases} u_{\mathbf{k}}(\mathbf{x},t) = \dfrac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t} e^{i\mathbf{k}\cdot\mathbf{x}} \\ \phi(\mathbf{x},t) = \displaystyle\sum_{\mathbf{k}} \alpha_{\mathbf{k}} u_{\mathbf{k}}(\mathbf{x},t) + \alpha^*_{\mathbf{k}} u^*_{\mathbf{k}}(\mathbf{x},t) \end{cases}$$

The Free Scalar Field and Plane-Wave Modes

A pivotal example will be the free real scalar field, whose Lagrangian reads:

$$\mathscr{L} = \frac{1}{2}\eta^{ab}(\partial_a\phi)(\partial_b\phi) - \frac{m^2}{2}\phi^2$$

It yields the linear Klein-Gordon equation:

$$[\Box_x + m^2]\phi(x) = 0 \qquad \Rightarrow \qquad \begin{cases} u_{\mathbf{k}}(\mathbf{x},t) = \dfrac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t} e^{i\mathbf{k}\cdot\mathbf{x}} \\ \phi(\mathbf{x},t) = \sum_{\mathbf{k}} \alpha_{\mathbf{k}} u_{\mathbf{k}}(\mathbf{x},t) + \alpha^*_{\mathbf{k}} u^*_{\mathbf{k}}(\mathbf{x},t) \end{cases}$$

The Free Scalar Field and Plane-Wave Modes

We define the conserved Klein-Gordon product:

$$(\phi, \psi) \equiv i \int_t d^3\mathbf{x}\, \phi^* \overleftrightarrow{\partial_t} \psi = i \int_t d^3\mathbf{x}\, \phi^* \partial_t \psi - (\partial_t \phi^*)\psi$$

in terms of which plane-waves are orthornormal:

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta_{\mathbf{k}\mathbf{k}'} = -(u^*_{\mathbf{k}}, u^*_{\mathbf{k}'}), \quad (u_{\mathbf{k}}, u^*_{\mathbf{k}'}) = 0$$

With this product, we can easily obtain the field solutions from some initial conditions at t_0, obtaining the field coefficients as projections:

$$\alpha_{\mathbf{k}} = (u_{\mathbf{k}}, \phi),$$
$$\alpha^*_{\mathbf{k}} = -(u^*_{\mathbf{k}}, \phi)$$

The Free Scalar Field and Plane-Wave Modes

We define the conserved Klein-Gordon product:

$$(\phi, \psi) \equiv i \int_t d^3\mathbf{x}\; \phi^* \overleftrightarrow{\partial_t} \psi = i \int_t d^3\mathbf{x}\; \phi^* \partial_t \psi - (\partial_t \phi^*)\psi$$

in terms of which plane-waves are orthornormal:

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta_{\mathbf{k}\mathbf{k}'} = -(u^*_{\mathbf{k}}, u^*_{\mathbf{k}'}), \quad (u_{\mathbf{k}}, u^*_{\mathbf{k}'}) = 0$$

With this product, we can easily obtain the field solutions from some initial conditions at t_0, obtaining the field coefficients as projections:

$$\alpha_{\mathbf{k}} = (u_{\mathbf{k}}, \phi),$$
$$\alpha^*_{\mathbf{k}} = -(u^*_{\mathbf{k}}, \phi)$$

The Free Scalar Field and Plane-Wave Modes

We define the conserved Klein-Gordon product:

$$(\phi, \psi) \equiv i \int_t d^3\mathbf{x}\ \phi^* \overleftrightarrow{\partial_t} \psi = i \int_t d^3\mathbf{x}\ \phi^* \partial_t \psi - (\partial_t \phi^*)\psi$$

in terms of which plane-waves are orthonormal:

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta_{\mathbf{k}\mathbf{k}'} = -(u^*_{\mathbf{k}}, u^*_{\mathbf{k}'}), \quad (u_{\mathbf{k}}, u^*_{\mathbf{k}'}) = 0$$

With this product, we can easily obtain the field solutions from some initial conditions at t_0, obtaining the field coefficients as projections:

$$\alpha_{\mathbf{k}} = (u_{\mathbf{k}}, \phi),$$
$$\alpha^*_{\mathbf{k}} = -(u^*_{\mathbf{k}}, \phi)$$

Field Mode Quantization

In terms of these plane-wave modes, we can quantize our field by promoting the field amplitudes $\alpha_{\mathbf{k}}, \alpha^*_{\mathbf{k}}$ to field operators $a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}$, imposing the commutation relations:

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = 0, \qquad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}.$$

such that the quantized field reads:

$$\phi(x) = \sum_{\mathbf{k}} a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x)$$

A particularly convenient C.S.C.O is the number operators, $N_{\mathbf{k}} \equiv a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$, which allow us to construct the Fock space, starting with the vacuum state $|0\rangle$:

$$a_{\mathbf{k}} |0\rangle = 0 \qquad |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots\rangle = \frac{1}{\sqrt{(n_{\mathbf{k}_1})!(n_{\mathbf{k}_2})! \ldots}} (a^\dagger_{\mathbf{k}_1})^{n_{\mathbf{k}_1}} (a^\dagger_{\mathbf{k}_2})^{n_{\mathbf{k}_2}} \ldots |0\rangle$$

Field Mode Quantization

In terms of these plane-wave modes, we can quantize our field by promoting the field amplitudes $\alpha_{\mathbf{k}}, \alpha^*_{\mathbf{k}}$ to field operators $a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}$, imposing the commutation relations:

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = 0, \qquad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}.$$

such that the quantized field reads:

$$\phi(x) = \sum_{\mathbf{k}} a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x)$$

A particularly convenient C.S.C.O is the number operators, $N_{\mathbf{k}} \equiv a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$, which allow us to construct the Fock space, starting with the vacuum state $|0\rangle$:

$$a_{\mathbf{k}} |0\rangle = 0 \qquad |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots\rangle = \frac{1}{\sqrt{(n_{\mathbf{k}_1})!(n_{\mathbf{k}_2})! \ldots}} (a^\dagger_{\mathbf{k}_1})^{n_{\mathbf{k}_1}} (a^\dagger_{\mathbf{k}_2})^{n_{\mathbf{k}_2}} \ldots |0\rangle$$

Field Mode Quantization

In terms of these plane-wave modes, we can quantize our field by promoting the field amplitudes $\alpha_{\mathbf{k}}, \alpha^*_{\mathbf{k}}$ to field operators $a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}$, imposing the commutation relations:

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = 0, \qquad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}.$$

such that the quantized field reads:

$$\phi(x) = \sum_{\mathbf{k}} a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x)$$

A particularly convenient C.S.C.O is the number operators, $N_{\mathbf{k}} \equiv a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$, which allow us to construct the Fock space, starting with the vacuum state $|0\rangle$:

$$a_{\mathbf{k}} |0\rangle = 0 \qquad |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots\rangle = \frac{1}{\sqrt{(n_{\mathbf{k}_1})!(n_{\mathbf{k}_2})! \ldots}} (a^\dagger_{\mathbf{k}_1})^{n_{\mathbf{k}_1}} (a^\dagger_{\mathbf{k}_2})^{n_{\mathbf{k}_2}} \ldots |0\rangle$$

Quantum Field Theory in Curved Spaces

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)} \{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b \phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}} \qquad \frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab} \qquad \nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)} \{ \mathscr{L}_G(x) + \mathscr{L}_M(x) \}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b \phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}} \qquad \frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab} \qquad \nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)} \{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b \phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}} \qquad \frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab} \qquad \nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial (\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)} \{ \mathscr{L}_G(x) + \mathscr{L}_M(x) \}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b \phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}} \qquad \frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab} \qquad \nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial (\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x\sqrt{-g(x)}\{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b\phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}} \qquad \frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab} \qquad \nabla_b\left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b\phi_a)}\right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)} \{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b \phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}} \qquad \frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab} \qquad \nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x\sqrt{-g(x)}\{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad\qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b\phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}} \qquad\qquad \frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab} \qquad\qquad \nabla_b\left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b\phi_a)}\right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Gravitational Coupling

The most direct prescription to obtain a general-covariant matter term from a special-covariant one in flat space is the so-called "minimal substitution" $(\partial_a, \eta_{ab}) \to (\nabla_a, g_{ab})$.

In this framework, matter will necessarily be coupled to gravity, even if only indirectly.

Further, one may consider other covariant interaction terms between matter and space-time. The most common in the literature takes the form:

$$\mathscr{L}_I = -\xi R \phi^2$$

Special values of ξ are $\xi = 0$ (minimal coupling) and $\xi = \xi(n)$ (conformal coupling in n dimensions:

$$\xi(n) = \frac{n-2}{4(n-1)}, \qquad n = 4 \Rightarrow \xi(n) = 1/6$$

Gravitational Coupling

The most direct prescription to obtain a general-covariant matter term from a special-covariant one in flat space is the so-called "minimal substitution" $(\partial_a, \eta_{ab}) \to (\nabla_a, g_{ab})$.

In this framework, matter will necessarily be coupled to gravity, even if only indirectly.

Further, one may consider other covariant interaction terms between matter and space-time. The most common in the literature takes the form:

$$\mathscr{L}_I = -\xi R \phi^2$$

Special values of ξ are $\xi = 0$ (minimal coupling) and $\xi = \xi(n)$ (conformal coupling in n dimensions:

$$\xi(n) = \frac{n-2}{4(n-1)}, \qquad n = 4 \Rightarrow \xi(n) = 1/6$$

Gravitational Coupling

The most direct prescription to obtain a general-covariant matter term from a special-covariant one in flat space is the so-called "minimal substitution" $(\partial_a, \eta_{ab}) \rightarrow (\nabla_a, g_{ab})$.

In this framework, matter will necessarily be coupled to gravity, even if only indirectly.

Further, one may consider other covariant interaction terms between matter and space-time. The most common in the literature takes the form:

$$\mathscr{L}_I = -\xi R \phi^2$$

Special values of ξ are $\xi = 0$ (minimal coupling) and $\xi = \xi(n)$ (conformal coupling in n dimensions:

$$\xi(n) = \frac{n-2}{4(n-1)}, \qquad n = 4 \Rightarrow \xi(n) = 1/6$$

Gravitational Coupling

The most direct prescription to obtain a general-covariant matter term from a special-covariant one in flat space is the so-called "minimal substitution" $(\partial_a, \eta_{ab}) \to (\nabla_a, g_{ab})$.

In this framework, matter will necessarily be coupled to gravity, even if only indirectly.

Further, one may consider other covariant interaction terms between matter and space-time. The most common in the literature takes the form:

$$\mathscr{L}_I = -\xi R \phi^2$$

Special values of ξ are $\xi = 0$ (minimal coupling) and $\xi = \xi(n)$ (conformal coupling in n dimensions:

$$\xi(n) = \frac{n-2}{4(n-1)}, \qquad n = 4 \Rightarrow \xi(n) = 1/6$$

Mode Quantization in Curved Spacetimes

The outline of our quantization procedure for noninteracting fields in curved spaces follows very similar lines to what we did in flat space. Generally:

- Classical field: $$\phi(x) = \sum_i \alpha_i u_i(x) + \alpha_i^* u_i^*(x)$$
- $$\alpha_i, \alpha_i^* \in \mathbb{C} \longrightarrow a_i, a_i^\dagger \in GL(\mathcal{H}), \qquad [a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger]$$
- Quantized field: $$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

Mode Quantization in Curved Spacetimes

The outline of our quantization procedure for noninteracting fields in curved spaces follows very similar lines to what we did in flat space. Generally:

- Classical field: $$\phi(x) = \sum_i \alpha_i u_i(x) + \alpha_i^* u_i^*(x)$$
- $$\alpha_i, \alpha_i^* \in \mathbb{C} \longrightarrow a_i, a_i^\dagger \in GL(\mathcal{H}), \qquad [a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger]$$
- Quantized field: $$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

Mode Quantization in Curved Spacetimes

The outline of our quantization procedure for noninteracting fields in curved spaces follows very similar lines to what we did in flat space. Generally:

- Classical field: $$\phi(x) = \sum_i \alpha_i u_i(x) + \alpha_i^* u_i^*(x)$$
- $$\alpha_i, \alpha_i^* \in \mathbb{C} \longrightarrow a_i, a_i^\dagger \in GL(\mathcal{H}), \qquad [a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger]$$
- Quantized field: $$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

Mode Quantization in Curved Spacetimes

The outline of our quantization procedure for noninteracting fields in curved spaces follows very similar lines to what we did in flat space. Generally:

- Classical field: $$\phi(x) = \sum_i \alpha_i u_i(x) + \alpha_i^* u_i^*(x)$$
- $$\alpha_i, \alpha_i^* \in \mathbb{C} \longrightarrow a_i, a_i^\dagger \in GL(\mathcal{H}), \qquad [a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger]$$
- Quantized field: $$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

Quantization of the Free Scalar Field

We consider a scalar field with a Lagrangian:

$$\mathscr{L} = \tfrac{1}{2} g^{ab} (\nabla_a \phi)(\nabla_b \phi) - \tfrac{1}{2}[m^2 + \xi R]\phi^2$$

which yields linear dynamical equations:

$$\frac{\delta S}{\delta \phi(x)} = 0 \quad \Rightarrow \quad [\Box_x + m^2 + \xi R(x)]\,\phi(x) = 0$$

The general field solution can then be written as an expansion in a complete set of modes $\{u_i, u_i^*\}$:

$$\phi(x) = \sum_i \alpha_i u_i(x) + \alpha_i^* u_i^*(x)$$

Quantization of the Free Scalar Field

We consider a scalar field with a Lagrangian:

$$\mathscr{L} = \tfrac{1}{2} g^{ab}(\nabla_a \phi)(\nabla_b \phi) - \tfrac{1}{2}[m^2 + \xi R]\phi^2$$

which yields linear dynamical equations:

$$\frac{\delta S}{\delta \phi(x)} = 0 \quad \Rightarrow \quad \left[\Box_x + m^2 + \xi R(x)\right]\phi(x) = 0$$

The general field solution can then be written as an expansion in a complete set of modes $\{u_i, u_i^*\}$:

$$\phi(x) = \sum_i \alpha_i u_i(x) + \alpha_i^* u_i^*(x)$$

Quantization of the Free Scalar Field

We consider a scalar field with a Lagrangian:

$$\mathscr{L} = \tfrac{1}{2}g^{ab}(\nabla_a\phi)(\nabla_b\phi) - \tfrac{1}{2}[m^2 + \xi R]\phi^2$$

which yields linear dynamical equations:

$$\frac{\delta S}{\delta\phi(x)} = 0 \quad \Rightarrow \quad \left[\Box_x + m^2 + \xi R(x)\right]\phi(x) = 0$$

The general field solution can then be written as an expansion in a complete set of modes $\{u_i, u_i^*\}$:

$$\phi(x) = \sum_i \alpha_i u_i(x) + \alpha_i^* u_i^*(x)$$

Scalar Product

As in flat space, we are interested in a notion of orthornormal modes, in terms of which it is simple to obtain field solutions from initial conditions. Then, we define the scalar product:

$$(\phi, \psi) \equiv i \int_\Sigma d^3x |g_\Sigma(x)|^{\frac{1}{2}} \, n^\mu(x) \phi^*(x) \overleftrightarrow{\partial_\mu} \psi(x)$$
$$= i \int_\Sigma d^3x |g_\Sigma(x)|^{\frac{1}{2}} \, n^\mu(x) \big(\phi^*(x) \partial_\mu \psi(x) - (\partial_\mu \phi^*(x))\psi(x)\big)$$

Orthornormal modes:

$$(u_i, u_j) = \delta_{ij} = -(u_i^*, u_j^*)$$
$$(u_i, u_j^*) = 0$$

Coefficients:

$$\alpha_i = (u_i, \phi)$$
$$\alpha_i^* = -(u_i^*, \phi)$$

Scalar Product

As in flat space, we are interested in a notion of orthornormal modes, in terms of which it is simple to obtain field solutions from initial conditions. Then, we define the scalar product:

$$
\begin{aligned}
(\phi, \psi) &\equiv i \int_\Sigma d^3x |g_\Sigma(x)|^{\frac{1}{2}} \, n^\mu(x) \phi^*(x) \overleftrightarrow{\partial_\mu} \psi(x) \\
&= i \int_\Sigma d^3x |g_\Sigma(x)|^{\frac{1}{2}} \, n^\mu(x) \big(\phi^*(x) \partial_\mu \psi(x) - (\partial_\mu \phi^*(x)) \psi(x)\big)
\end{aligned}
$$

Orthornormal modes:

$$
\begin{aligned}
(u_i, u_j) = \delta_{ij} = -(u_i^*, u_j^*) \\
(u_i, u_j^*) = 0
\end{aligned}
$$

Coefficients:

$$
\begin{aligned}
\alpha_i &= (u_i, \phi) \\
\alpha_i^* &= -(u_i^*, \phi)
\end{aligned}
$$

As in flat space, we are interested in a notion of orthornormal modes, in terms of which it is simple to obtain field solutions from initial conditions. Then, we define the scalar product:

$$(\phi,\psi) \equiv i\int_\Sigma d^3x|g_\Sigma(x)|^{\frac{1}{2}}\, n^\mu(x)\phi^*(x)\overleftrightarrow{\partial_\mu}\psi(x)$$
$$= i\int_\Sigma d^3x|g_\Sigma(x)|^{\frac{1}{2}}\, n^\mu(x)\big(\phi^*(x)\partial_\mu\psi(x) - (\partial_\mu\phi^*(x))\psi(x)\big)$$

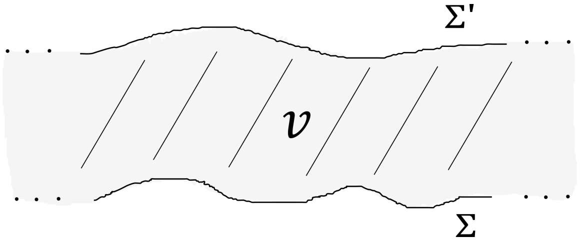
Orthornormal modes:

$$(u_i,u_j) = \delta_{ij} = -(u_i^*,u_j^*)$$
$$(u_i,u_j^*) = 0$$

Coefficients:

$$\alpha_i = (u_i,\phi)$$
$$\alpha_i^* = -(u_i^*,\phi)$$

For arbitrary solutions of the field equations u and v, the product (u, v) will be independent of the choice of Cauchy surface:

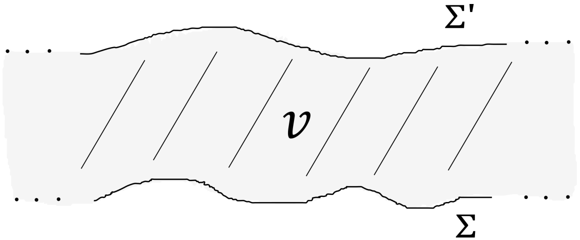


$$(u, w)_{\Sigma'} - (u, w)_{\Sigma} = \int_{\Sigma'} d\mu_{g_{\Sigma'}}(x)\, n^{\mu}(x)\, u^*(x)\overleftrightarrow{\partial}_{\mu} w(x) - \int_{\Sigma} d\mu_{g_{\Sigma}}(x)\, n^{\mu}(x)\, u^*(x)\overleftrightarrow{\partial}_{\mu} w(x)$$

$$= \int_{v} d\mu_{g}(x)\, \nabla^{\mu}(u^*(x)\overleftrightarrow{\partial}_{\mu} w(x))$$

$$= \int_{v} d\mu_{g}(x)(u^*(x)\nabla^{\mu}\nabla_{\mu} w(x) - w(x)\nabla^{\mu}\nabla_{\mu} u^*(x)) = 0$$

For arbitrary solutions of the field equations u and v, the product (u, v) will be independent of the choice of Cauchy surface:



$$\begin{aligned}(u,w)_{\Sigma'} - (u,w)_{\Sigma} &= \int_{\Sigma'} d\mu_{g_{\Sigma'}}(x)\; n^{\mu}(x)\; u^{*}(x)\overleftrightarrow{\partial}_{\mu} w(x) - \int_{\Sigma} d\mu_{g_{\Sigma}}(x)\; n^{\mu}(x)\; u^{*}(x)\overleftrightarrow{\partial}_{\mu} w(x) \\ &= \int_{v} d\mu_{g}(x)\; \nabla^{\mu}(u^{*}(x)\overleftrightarrow{\partial}_{\mu} w(x)) \\ &= \int_{v} d\mu_{g}(x)(u^{*}(x)\nabla^{\mu}\nabla_{\mu} w(x) - w(x)\nabla^{\mu}\nabla_{\mu} u^{*}(x)) = 0\end{aligned}$$

Quantized Field and Fock Space

For an expansion in orthornormal modes, one can consistently perform a quantization by promoting classical field coefficents α_i, α_i^* to field operators $a_i, a_i^\dagger$ with the usual commutation relations:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

in terms of which we obtain a field expansion:

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

and define a Fock Space based on the operators $N_i = a_i^\dagger a_i$:

$$|0\rangle : \ a_i |0\rangle = 0, \quad \forall i \qquad |n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}, \ldots |0\rangle \, .$$

Quantized Field and Fock Space

For an expansion in orthornormal modes, one can consistently perform a quantization by promoting classical field coefficents α_i, α_i^* to field operators $a_i, a_i^\dagger$ with the usual commutation relations:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

in terms of which we obtain a field expansion:

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

and define a Fock Space based on the operators $N_i = a_i^\dagger a_i$:

$$|0\rangle : \ a_i |0\rangle = 0, \quad \forall i \qquad |n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}, \ldots |0\rangle \, .$$

Quantized Field and Fock Space

For an expansion in orthornormal modes, one can consistently perform a quantization by promoting classical field coeffcents α_i, α_i^* to field operators $a_i, a_i^\dagger$ with the usual commutation relations:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

in terms of which we obtain a field expansion:

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

and define a Fock Space based on the operators $N_i = a_i^\dagger a_i$:

$$|0\rangle : \; a_i |0\rangle = 0, \quad \forall i \qquad |n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}, \ldots |0\rangle \, .$$

Quantized Field and Fock Space

For an expansion in orthornormal modes, one can consistently perform a quantization by promoting classical field coefficents α_i, α_i^* to field operators $a_i, a_i^\dagger$ with the usual commutation relations:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

in terms of which we obtain a field expansion:

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

and define a Fock Space based on the operators $N_i = a_i^\dagger a_i$:

$$|0\rangle : \; a_i |0\rangle = 0, \quad \forall i \qquad |n_1, n_2, \dots\rangle = \frac{1}{\sqrt{n_1! n_2! \dots}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}, \dots |0\rangle \, .$$

Quantized Field and Fock Space

For an expansion in orthornormal modes, one can consistently perform a quantization by promoting classical field coefficents α_i, α_i^* to field operators $a_i, a_i^\dagger$ with the usual commutation relations:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

in terms of which we obtain a field expansion:

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

and define a Fock Space based on the operators $N_i = a_i^\dagger a_i$:

$$|0\rangle : \; a_i |0\rangle = 0, \quad \forall i \qquad |n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}, \ldots |0\rangle .$$

Quantized Field and Fock Space

For an expansion in orthornormal modes, one can consistently perform a quantization by promoting classical field coeffcents α_i, α_i^* to field operators $a_i, a_i^\dagger$ with the usual commutation relations:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

in terms of which we obtain a field expansion:

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x)$$

and define a Fock Space based on the operators $N_i = a_i^\dagger a_i$:

$$|0\rangle : \ a_i |0\rangle = 0, \quad \forall i \qquad\qquad |n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}, \ldots |0\rangle \,.$$

Different mode representations

In the lack of spacetime symmetries to distiguish particular modes, we should consider on an equal footing the expansions for a different set of normal modes $\{\bar{u}_i, \bar{u}_i^*\}$:

$$\phi(x) = \sum_i \bar{\alpha}_i \bar{u}_i(x) + \bar{\alpha}_i^* \bar{u}_i^*(x)$$

Then, we could impose a similar quantization procedure using the coefficients $\bar{\alpha}_i, \bar{\alpha}_i^*$:

$$[\bar{a}_i, \bar{a}_j] = [\bar{a}_i^\dagger, \bar{a}_j^\dagger] = 0, \quad [\bar{a}_i, \bar{a}_j^\dagger] = \delta_{ij}.$$

yielding the quantized field expansions and Fock space:

$$\phi(x) = \sum_i \bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x) \qquad \begin{cases} |\bar{0}\rangle : \ \bar{a}_i |\bar{0}\rangle = 0 \\ |\bar{n}_1, \bar{n}_2 \ldots\rangle = \dfrac{1}{\sqrt{\bar{n}_1! \bar{n}_2! \ldots}} (\bar{a}_1^\dagger)^{n_1} (\bar{a}_2^\dagger)^{n_2}, \ldots |\bar{0}\rangle \end{cases}$$

Different mode representations

In the lack of spacetime symmetries to distiguish particular modes, we should consider on an equal footing the expansions for a different set of normal modes $\{\bar{u}_i, \bar{u}_i^*\}$:

$$\phi(x) = \sum_i \bar{\alpha}_i \bar{u}_i(x) + \bar{\alpha}_i^* \bar{u}_i^*(x)$$

Then, we could impose a similar quantization procedure using the coefficients $\bar{\alpha}_i, \bar{\alpha}_i^*$:

$$[\bar{a}_i, \bar{a}_j] = [\bar{a}_i^\dagger, \bar{a}_j^\dagger] = 0, \quad [\bar{a}_i, \bar{a}_j^\dagger] = \delta_{ij}.$$

yielding the quantized field expansions and Fock space:

$$\phi(x) = \sum_i \bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x) \qquad \begin{cases} |\bar{0}\rangle : \ \bar{a}_i |\bar{0}\rangle = 0 \\ |\bar{n}_1, \bar{n}_2 ...\rangle = \frac{1}{\sqrt{\bar{n}_1! \bar{n}_2! ...}} (\bar{a}_1^\dagger)^{n_1} (\bar{a}_2^\dagger)^{n_2}, \ldots |\bar{0}\rangle \end{cases}$$

Different mode representations

In the lack of spacetime symmetries to distiguish particular modes, we should consider on an equal footing the expansions for a different set of normal modes $\{\bar{u}_i, \bar{u}_i^*\}$:

$$\phi(x) = \sum_i \bar{\alpha}_i \bar{u}_i(x) + \bar{\alpha}_i^* \bar{u}_i^*(x)$$

Then, we could impose a similar quantization procedure using the coefficients $\bar{\alpha}_i, \bar{\alpha}_i^*$:

$$[\bar{a}_i, \bar{a}_j] = [\bar{a}_i^\dagger, \bar{a}_j^\dagger] = 0, \quad [\bar{a}_i, \bar{a}_j^\dagger] = \delta_{ij}.$$

yielding the quantized field expansions and Fock space:

$$\phi(x) = \sum_i \bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x) \qquad \begin{cases} |\bar{0}\rangle : \ \bar{a}_i |\bar{0}\rangle = 0 \\ |\bar{n}_1, \bar{n}_2 ...\rangle = \dfrac{1}{\sqrt{\bar{n}_1! \bar{n}_2! ...}} (\bar{a}_1^\dagger)^{n_1} (\bar{a}_2^\dagger)^{n_2}, \ldots |\bar{0}\rangle \end{cases}$$

Different mode representations

In the lack of spacetime symmetries to distiguish particular modes, we should consider on an equal footing the expansions for a different set of normal modes $\{\bar{u}_i, \bar{u}_i^*\}$:

$$\phi(x) = \sum_i \bar{\alpha}_i \bar{u}_i(x) + \bar{\alpha}_i^* \bar{u}_i^*(x)$$

Then, we could impose a similar quantization procedure using the coefficients $\bar{\alpha}_i, \bar{\alpha}_i^*$:

$$[\bar{a}_i, \bar{a}_j] = [\bar{a}_i^\dagger, \bar{a}_j^\dagger] = 0, \quad [\bar{a}_i, \bar{a}_j^\dagger] = \delta_{ij}.$$

yielding the quantized field expansions and Fock space:

$$\phi(x) = \sum_i \bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x)$$

$$\begin{cases} |\bar{0}\rangle : \ \bar{a}_i |\bar{0}\rangle = 0 \\ |\bar{n}_1, \bar{n}_2 \ldots\rangle = \frac{1}{\sqrt{\bar{n}_1! \bar{n}_2! \ldots}} (\bar{a}_1^\dagger)^{n_1} (\bar{a}_2^\dagger)^{n_2}, \ldots |\bar{0}\rangle \end{cases}$$

Different mode representations

In the lack of spacetime symmetries to distiguish particular modes, we should consider on an equal footing the expansions for a different set of normal modes $\{\bar{u}_i, \bar{u}_i^*\}$:

$$\phi(x) = \sum_i \bar{\alpha}_i \bar{u}_i(x) + \bar{\alpha}_i^* \bar{u}_i^*(x)$$

Then, we could impose a similar quantization procedure using the coefficients $\bar{\alpha}_i, \bar{\alpha}_i^*$:

$$[\bar{a}_i, \bar{a}_j] = [\bar{a}_i^\dagger, \bar{a}_j^\dagger] = 0, \quad [\bar{a}_i, \bar{a}_j^\dagger] = \delta_{ij}.$$

yielding the quantized field expansions and Fock space:

$$\phi(x) = \sum_i \bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x) \qquad \begin{cases} |\bar{0}\rangle : \ \bar{a}_i |\bar{0}\rangle = 0 \\ |\bar{n}_1, \bar{n}_2 \ldots\rangle = \dfrac{1}{\sqrt{\bar{n}_1! \bar{n}_2! \ldots}} (\bar{a}_1^\dagger)^{n_1} (\bar{a}_2^\dagger)^{n_2}, \ldots |\bar{0}\rangle \end{cases}$$

Bogolubov Coefficients

Different sets of modes are related by the Bogolubov coeffiecients:

$$\begin{cases} \alpha_{ij} \equiv (u_j, \bar{u}_i) = (\bar{u}_i, u_j)^* \\ \beta_{ij} \equiv -(u_j^*, \bar{u}_i) = -(\bar{u}_i, u_j^*)^* \end{cases}$$

in terms of which we can write modes and operator transformations:

$$\begin{cases} u_i = \sum_j \alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^* \\ \bar{u}_i = \sum_j \alpha_{ij} u_j + \beta_{ij} u_j^* \end{cases}$$

$$\begin{cases} a_i = \sum_j \alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger \\ \bar{a}_i = \sum_j \alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger \end{cases}$$

Bogolubov Coefficients

Different sets of modes are related by the Bogolubov coeffiecients:

$$\begin{cases} \alpha_{ij} \equiv (u_j, \bar{u}_i) = (\bar{u}_i, u_j)^* \\ \beta_{ij} \equiv -(u_j^*, \bar{u}_i) = -(\bar{u}_i, u_j^*)^* \end{cases}$$

in terms of which we can write modes and operator transformations:

$$\begin{cases} u_i = \sum_j \alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^* \\ \bar{u}_i = \sum_j \alpha_{ij} u_j + \beta_{ij} u_j^* \end{cases} \qquad \begin{cases} a_i = \sum_j \alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger \\ \bar{a}_i = \sum_j \alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger \end{cases}$$

• Ambiguity in the mode vacuum concept: in general $|0\rangle \neq |\bar{0}\rangle$:

$$a_i |\bar{0}\rangle = \sum_j \beta^*_{ji} |\bar{1}_j\rangle \not\equiv 0 \qquad \Rightarrow \qquad \langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \not\equiv 0$$

• A special case, for which distingushed families of modes arises is stationary spacetimes. Their time translation symmetry (along a Killing field ξ^a) allows for a split in positive and negative frequencies:

$$\begin{cases} i\partial_t u_{\mathbf{k}} = +\omega_k u_{\mathbf{k}} \\ i\partial_t u^*_{\mathbf{k}} = -\omega_k u^*_{\mathbf{k}} \end{cases} \qquad \begin{cases} i\pounds_\xi u_j = +\omega_j u_j \\ i\pounds_\xi u^*_j = -\omega_j u^*_j \end{cases}$$

which allows for a dintinguished definition of creation and annihilation operators, and of a vacuum state.

• Ambiguity in the mode vacuum concept: in general $|0\rangle \neq |\bar{0}\rangle$:

$$a_i\,|\bar{0}\rangle = \sum_j \beta^*_{ji}\,|\bar{1}_j\rangle \not\equiv 0 \qquad \Rightarrow \qquad \langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \not\equiv 0$$

• A special case, for which distingushed families of modes arises is stationary spacetimes. Their time translation symmetry (along a Killing field ξ^a) allows for a split in positive and negative frequencies:

$$\begin{cases} i\partial_t u_{\mathbf{k}} = +\omega_k u_{\mathbf{k}} \\ i\partial_t u^*_{\mathbf{k}} = -\omega_k u^*_{\mathbf{k}} \end{cases} \qquad \begin{cases} i\pounds_\xi u_j = +\omega_j u_j \\ i\pounds_\xi u^*_j = -\omega_j u^*_j \end{cases}$$

which allows for a dintinguished definition of creation and annihilation operators, and of a vacuum state.

• Ambiguity in the mode vacuum concept: in general $|0\rangle \neq |\bar{0}\rangle$:

$$a_i\,|\bar{0}\rangle = \sum_j \beta^*_{ji}\,|\bar{1}_j\rangle \not\equiv 0 \qquad \Rightarrow \qquad \langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \not\equiv 0$$

• A special case, for which distingushed families of modes arises is stationary spacetimes. Their time translation symmetry (along a Killing field ξ^a) allows for a split in positive and negative frequencies:

$$\begin{cases} i\partial_t u_{\mathbf{k}} = +\omega_k u_{\mathbf{k}} \\ i\partial_t u^*_{\mathbf{k}} = -\omega_k u^*_{\mathbf{k}} \end{cases} \qquad \begin{cases} i\pounds_\xi u_j = +\omega_j u_j \\ i\pounds_\xi u^*_j = -\omega_j u^*_j \end{cases}$$

which allows for a dintinguished definition of creation and annihilation operators, and of a vacuum state.

• Ambiguity in the mode vacuum concept: in general $|0\rangle \neq |\bar{0}\rangle$:

$$a_i\,|\bar{0}\rangle = \sum_j \beta^*_{ji}\,|\bar{1}_j\rangle \not\equiv 0 \qquad \Rightarrow \qquad \langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \not\equiv 0$$

• A special case, for which distingushed families of modes arises is stationary spacetimes. Their time translation symmetry (along a Killing field ξ^a) allows for a split in positive and negative frequencies:

$$\begin{cases} i\partial_t u_{\mathbf{k}} = +\omega_k u_{\mathbf{k}} \\ i\partial_t u^*_{\mathbf{k}} = -\omega_k u^*_{\mathbf{k}} \end{cases} \qquad \begin{cases} i\pounds_\xi u_j = +\omega_j u_j \\ i\pounds_\xi u^*_j = -\omega_j u^*_j \end{cases}$$

which allows for a dintinguished definition of creation and annihilation operators, and of a vacuum state.

Particle Detectors

So far, we have discussed the concepts of vacuum and particles in terms of occupation numbers of field modes. How do they relate to the experience of particles by localized observers?

To better investigate this question, we analyze an idealized model of a pointlike particle detector, with proper time τ, worldline $x^\mu(\tau)$ and internal DOFs $\{E\}$:

$$H\,|E\rangle = E\,|E\rangle\,, \qquad |\psi(\tau)\rangle = e^{-iH\tau}\,|\psi_0\rangle$$

We denote detector and field joint states $|E, \Psi\rangle$, and we consider a simple coupling given by a *local monopole interaction* with monopole moment (charge) $m(\tau)$:

$$\mathscr{L}_I = c\, m(\tau)\phi(x^\mu(\tau))$$

So far, we have discussed the concepts of vacuum and particles in terms of occupation numbers of field modes. How do they relate to the experience of particles by localized observers?

To better investigate this question, we analyze an idealized model of a pointlike particle detector, with proper time τ, worldline $x^\mu(\tau)$ and internal DOFs $\{E\}$:

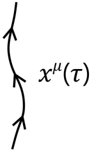
$$H\,|E\rangle = E\,|E\rangle\,, \qquad |\psi(\tau)\rangle = e^{-iH\tau}\,|\psi_0\rangle$$

We denote detector and field joint states $|E, \Psi\rangle$, and we consider a simple coupling given by a *local monopole interaction* with monopole moment (charge) $m(\tau)$:

$$\mathscr{L}_I = c\,m(\tau)\phi(x^\mu(\tau))$$

So far, we have discussed the concepts of vacuum and particles in terms of occupation numbers of field modes. How do they relate to the experience of particles by localized observers?

To better investigate this question, we analyze an idealized model of a pointlike particle detector, with proper time τ, worldline $x^\mu(\tau)$ and internal DOFs $\{E\}$:



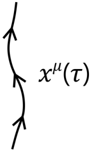
$$H\,|E\rangle = E\,|E\rangle\,, \qquad |\psi(\tau)\rangle = e^{-iH\tau}\,|\psi_0\rangle$$

We denote detector and field joint states $|E, \Psi\rangle$, and we consider a simple coupling given by a *local monopole interaction* with monopole moment (charge) $m(\tau)$:

$$\mathscr{L}_I = c\, m(\tau)\phi(x^\mu(\tau))$$

So far, we have discussed the concepts of vacuum and particles in terms of occupation numbers of field modes. How do they relate to the experience of particles by localized observers?

To better investigate this question, we analyze an idealized model of a pointlike particle detector, with proper time τ, worldline $x^\mu(\tau)$ and internal DOFs $\{E\}$:



$$H\,|E\rangle = E\,|E\rangle\,, \qquad |\psi(\tau)\rangle = e^{-iH\tau}\,|\psi_0\rangle$$

We denote detector and field joint states $|E, \Psi\rangle$, and we consider a simple coupling given by a *local monopole interaction* with monopole moment (charge) $m(\tau)$:

$$\mathscr{L}_I = c\, m(\tau)\phi(x^\mu(\tau))$$

Detection Amplitudes

These interactions will generally lead to transitions in field and detector states $|E, \Psi\rangle \to |E', \Psi'\rangle$. We say we have a detection when we excite the detector from its ground state $|E_0\rangle \to |E\rangle$ $(E > E_0)$. Then, we want to evaluate the probabilities of obtaining any detection starting from a vacuum state $|E_0, 0\rangle$. Perturbatively, we find the amplitudes:

$$\mathcal{A}(|E_0, 0_M\rangle \to |E, \Psi\rangle) = ic \langle E, \Psi| \int_{-\infty}^{+\infty} m(\tau)\phi(x^\mu(\tau))d\tau \, |E_0, 0_M\rangle \, .$$

Also, in the Interaction Picture $m(\tau) = e^{iH\tau}m(0)e^{-iH\tau}$, so we obtain:

$$\mathcal{A}(|E_0, 0_M\rangle \to |E, \Psi\rangle) = ic\langle E| \, m(0) \, |E_0\rangle \int_{-\infty}^{+\infty} e^{i(E-E_0)\tau} \langle \Psi| \, \phi(x^\mu(\tau)) \, |0_M\rangle d\tau.$$

Note that they depend on the detector's trajectory.

Detection Amplitudes

These interactions will generally lead to transitions in field and detector states $|E, \Psi\rangle \to |E', \Psi'\rangle$. We say we have a detection when we excite the detector from its ground state $|E_0\rangle \to |E\rangle$ $(E > E_0)$. Then, we want to evaluate the probabilities of obtaining any detection starting from a vacuum state $|E_0, 0\rangle$. Perturbatively, we find the amplitudes:

$$\mathcal{A}\big(|E_0, 0_M\rangle \to |E, \Psi\rangle\big) = ic\,\langle E, \Psi| \int_{-\infty}^{+\infty} m(\tau)\phi(x^\mu(\tau))d\tau\,|E_0, 0_M\rangle\,.$$

Also, in the Interaction Picture $m(\tau) = e^{iH\tau}m(0)e^{-iH\tau}$, so we obtain:

$$\mathcal{A}(|E_0, 0_M\rangle \to |E, \Psi\rangle) = ic\langle E|\,m(0)\,|E_0\rangle \int_{-\infty}^{+\infty} e^{i(E-E_0)\tau}\langle\Psi|\,\phi(x^\mu(\tau))\,|0_M\rangle d\tau.$$

Note that they depend on the detector's trajectory.

Detection Amplitudes

These interactions will generally lead to transitions in field and detector states $|E, \Psi\rangle \to |E', \Psi'\rangle$. We say we have a detection when we excite the detector from its ground state $|E_0\rangle \to |E\rangle$ $(E > E_0)$. Then, we want to evaluate the probabilities of obtaining any detection starting from a vacuum state $|E_0, 0\rangle$. Perturbatively, we find the amplitudes:

$$\mathcal{A}\big(|E_0, 0_M\rangle \to |E, \Psi\rangle\big) = ic\,\langle E, \Psi| \int_{-\infty}^{+\infty} m(\tau)\phi(x^\mu(\tau))d\tau\,|E_0, 0_M\rangle\,.$$

Also, in the Interaction Picture $m(\tau) = e^{iH\tau}m(0)e^{-iH\tau}$, so we obtain:

$$\mathcal{A}(|E_0, 0_M\rangle \to |E, \Psi\rangle) = ic\langle E|\,m(0)\,|E_0\rangle \int_{-\infty}^{+\infty} e^{i(E-E_0)\tau}\langle\Psi|\,\phi(x^\mu(\tau))\,|0_M\rangle d\tau.$$

Note that they depend on the detector's trajectory.

The only transitions that may occur *in first perturbative order* are those to one-particle states: $|\Psi\rangle = |1_{\mathbf{k}}\rangle$:

$$\langle 1_{\mathbf{k}}| \phi(x) |0\rangle = \int d^3\mathbf{k}' (16\pi^3\omega_{k'})^{-1/2} \langle 1_{\mathbf{k}}| a^\dagger_{\mathbf{k}'} |0\rangle e^{i\omega' t - i\mathbf{k}'\cdot\mathbf{x}} = (16\pi^3\omega_k)^{-1/2} e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$$

For an inertial world-line $\mathbf{x} = \mathbf{x_0} + \mathbf{v}t = \mathbf{x_0} + \mathbf{v}\gamma_v\tau$:

$$\begin{aligned}
\mathcal{A}(|E_0, 0_M\rangle \to |E, 1_{\mathbf{k}}\rangle) &= \frac{ic\langle E| m(0) |E_0\rangle}{16\pi^3\omega} e^{-i\mathbf{k}\cdot\mathbf{x_0}} \int_{-\infty}^{+\infty} e^{i(E-E_0)\tau} e^{i(\omega - \mathbf{k}\cdot\mathbf{v})\gamma_v\tau} d\tau \\
&= \frac{ic\langle E| m(0) |E_0\rangle}{4\pi\omega} e^{-i\mathbf{k}\cdot\mathbf{x_0}} \delta(E - E_0 + [\omega - \mathbf{k}\cdot\mathbf{v}]\gamma_v). \\
&= 0
\end{aligned}$$

Since $E > E_0$ and $\omega > |\mathbf{k}\cdot\mathbf{v}|$. As dictated by energy conservation – a direct consequence of time translation symmetry.

The only transitions that may occur *in first perturbative order* are those to one-particle states: $|\Psi\rangle = |1_{\mathbf{k}}\rangle$:

$$\langle 1_{\mathbf{k}}|\,\phi(x)\,|0\rangle = \int d^3\mathbf{k}'(16\pi^3\omega_{k'})^{-1/2}\,\langle 1_{\mathbf{k}}|\,a^\dagger_{\mathbf{k}'}\,|0\rangle\, e^{i\omega' t - i\mathbf{k}'\cdot\mathbf{x}} = (16\pi^3\omega_k)^{-1/2}e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$$

For an inertial world-line $\mathbf{x} = \mathbf{x_0} + \mathbf{v}t = \mathbf{x_0} + \mathbf{v}\gamma_v\tau$:

$$\begin{aligned}\mathcal{A}(|E_0, 0_M\rangle \to |E, 1_{\mathbf{k}}\rangle) &= \frac{ic\,\langle E|\,m(0)\,|E_0\rangle}{16\pi^3\omega}e^{-i\mathbf{k}\cdot\mathbf{x_0}}\int_{-\infty}^{+\infty} e^{i(E-E_0)\tau}e^{i(\omega-\mathbf{k}\cdot\mathbf{v})\gamma_v\tau}d\tau \\ &= \frac{ic\,\langle E|\,m(0)\,|E_0\rangle}{4\pi\omega}e^{-i\mathbf{k}\cdot\mathbf{x_0}}\delta(E - E_0 + [\omega - \mathbf{k}\cdot\mathbf{v}]\gamma_v). \\ &= 0\end{aligned}$$

Since $E > E_0$ and $\omega > |\mathbf{k}\cdot\mathbf{v}|$. As dictated by energy conservation – a direct consequence of time translation symmetry.

The only transitions that may occur *in first perturbative order* are those to one-particle states: $|\Psi\rangle = |1_{\mathbf{k}}\rangle$:

$$\langle 1_{\mathbf{k}}|\,\phi(x)\,|0\rangle = \int d^3\mathbf{k}'(16\pi^3\omega_{k'})^{-1/2}\,\langle 1_{\mathbf{k}}|\,a^\dagger_{\mathbf{k}'}\,|0\rangle\,e^{i\omega' t - i\mathbf{k}'\cdot\mathbf{x}} = (16\pi^3\omega_k)^{-1/2}e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$$

For an inertial world-line $\mathbf{x} = \mathbf{x_0} + \mathbf{v}t = \mathbf{x_0} + \mathbf{v}\gamma_v\tau$:

$$\begin{aligned}\mathcal{A}(|E_0, 0_M\rangle \to |E, 1_{\mathbf{k}}\rangle) &= \frac{ic\,\langle E|\,m(0)\,|E_0\rangle}{16\pi^3\omega}e^{-i\mathbf{k}\cdot\mathbf{x_0}}\int_{-\infty}^{+\infty} e^{i(E-E_0)\tau}e^{i(\omega-\mathbf{k}\cdot\mathbf{v})\gamma_v\tau}d\tau \\ &= \frac{ic\,\langle E|\,m(0)\,|E_0\rangle}{4\pi\omega}e^{-i\mathbf{k}\cdot\mathbf{x_0}}\delta(E - E_0 + [\omega - \mathbf{k}\cdot\mathbf{v}]\gamma_v). \\ &= 0\end{aligned}$$

Since $E > E_0$ and $\omega > |\mathbf{k}\cdot\mathbf{v}|$. As dictated by energy conservation – a direct consequence of time translation symmetry.

Detection Probabilities

For generic trajectories, however, the amplitudes will be nonzero. We sum over all possible excited final states to obtain the total probability that *any* transition may occur:

$$P = \sum_{E,\Psi} \left| \mathcal{A}(|E_0, 0_M\rangle \to |E, \Psi\rangle) \right|^2$$

$$= c^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 \mathscr{F}(E - E_0),$$

where we defined the response function of the detector $\mathscr{F}(E)$:

$$\mathscr{F}(E) \equiv \iint d\tau \, d\tau' e^{-iE(\tau-\tau')} G^+(x(\tau), x(\tau'))$$

with $G^+(x, x') = \langle 0_M|\phi(x)\phi(x')|0_M\rangle$ the vacuum 2-point correlation function.

Detection Probabilities

For generic trajectories, however, the amplitudes will be nonzero. We sum over all possible excited final states to obtain the total probability that *any* transition may occur:

$$P = \sum_{E,\Psi} \left| \mathcal{A}(|E_0, 0_M\rangle \to |E, \Psi\rangle) \right|^2$$
$$= c^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 \mathscr{F}(E - E_0),$$

where we defined the response function of the detector $\mathscr{F}(E)$:

$$\mathscr{F}(E) \equiv \iint d\tau\, d\tau' e^{-iE(\tau-\tau')} G^+(x(\tau), x(\tau'))$$

with $G^+(x, x') = \langle 0_M|\phi(x)\phi(x')|0_M\rangle$ the vacuum 2-point correlation function.

Detection Probabilities

For generic trajectories, however, the amplitudes will be nonzero. We sum over all possible excited final states to obtain the total probability that *any* transition may occur:

$$P = \sum_{E,\Psi} \left| \mathcal{A}(|E_0, 0_M\rangle \to |E, \Psi\rangle) \right|^2$$
$$= c^2 \sum_{E} |\langle E|m(0)|E_0\rangle|^2 \mathscr{F}(E - E_0),$$

where we defined the response function of the detector $\mathscr{F}(E)$:

$$\mathscr{F}(E) \equiv \iint d\tau\, d\tau' e^{-iE(\tau-\tau')} G^+(x(\tau), x(\tau'))$$

with $G^+(x, x') = \langle 0_M|\phi(x)\phi(x')|0_M\rangle$ the vacuum 2-point correlation function.

Detection Probabilities

For generic trajectories, however, the amplitudes will be nonzero. We sum over all possible excited final states to obtain the total probability that *any* transition may occur:

$$P = \sum_{E,\Psi} \left| \mathcal{A}(|E_0, 0_M\rangle \to |E, \Psi\rangle) \right|^2$$
$$= c^2 \sum_{E} |\langle E|m(0)|E_0\rangle|^2 \mathscr{F}(E - E_0),$$

where we defined the response function of the detector $\mathscr{F}(E)$:

$$\mathscr{F}(E) \equiv \iint d\tau \, d\tau' e^{-iE(\tau-\tau')} G^+(x(\tau), x(\tau'))$$

with $G^+(x, x') = \langle 0_M|\phi(x)\phi(x')|0_M\rangle$ the vacuum 2-point correlation function.

Stationary detectors – Response Rates

Particularly, for a stationary trajectory, that is, $G^+(x(\tau), x(\tau')) = G^+(\Delta\tau)$, we obtain trivially separable integrals:

$$\mathscr{F}(E) = \left(\int_{-\infty}^{\infty} d\bar{\tau}\right)\left(\int_{-\infty}^{\infty} d(\Delta\tau) e^{-iE\Delta\tau} G^+(\Delta\tau)\right),$$

which can be immediately interpreted as a (constant) *transition rate* multiplied by the (infinite) time interval of the interactions $T \equiv \int d\bar{\tau}$.

In this stationary case, it is more convenient to work directly with transition rates. Thus, we define the *response function per unit time*:

$$\mathscr{F}'(E) = \frac{\mathscr{F}(E)}{T} = \int_{-\infty}^{\infty} d(\Delta\tau) e^{-iE\Delta\tau} G^+(\Delta\tau).$$

Stationary detectors – Response Rates

Particularly, for a stationary trajectory, that is, $G^+(x(\tau), x(\tau')) = G^+(\Delta\tau)$, we obtain trivially separable integrals:

$$\mathscr{F}(E) = \left(\int_{-\infty}^{\infty} d\bar{\tau}\right)\left(\int_{-\infty}^{\infty} d(\Delta\tau) e^{-iE\Delta\tau} G^+(\Delta\tau)\right),$$

which can be immediately interpreted as a (constant) *transition rate* multiplied by the (infinite) time interval of the interactions $T \equiv \int d\bar{\tau}$.

In this stationary case, it is more convenient to work directly with transition rates. Thus, we define the *response function per unit time*:

$$\mathscr{F}'(E) = \frac{\mathscr{F}(E)}{T} = \int_{-\infty}^{\infty} d(\Delta\tau) e^{-iE\Delta\tau} G^+(\Delta\tau).$$

Inertial detectors

Let us consider a massless field in Minkowski spacetime. The vacuum correlations read:

$$G_\epsilon^+(x, x') = \frac{1}{4\pi^2} \frac{1}{(\Delta t - i\epsilon)^2 - |\Delta \mathbf{x}|^2}$$

Then, for an inertial worldline we find the correlations:

$$\begin{cases} t = \gamma_v \tau \\ \mathbf{x} = \mathbf{x_0} + \mathbf{v}t = \mathbf{x_0} + \mathbf{v}\gamma_v \tau \end{cases} \Rightarrow \quad G_\epsilon^+(x(\tau), x(\tau')) = \frac{1}{4\pi^2(\Delta\tau - i\epsilon)^2}$$

which yield null response rates:

$$\mathscr{F}'(E - E_0) = \int_{-\infty}^{\infty} d(\Delta\tau) e^{-i(E-E_0)\Delta\tau} G^+(\Delta\tau) = 0, \quad \forall\, E > E_0$$

Inertial detectors

Let us consider a massless field in Minkowski spacetime. The vacuum correlations read:

$$G^+_\epsilon(x,x') = \frac{1}{4\pi^2}\frac{1}{(\Delta t - i\epsilon)^2 - |\Delta \mathbf{x}|^2}$$

Then, for an inertial worldline we find the correlations:

$$\begin{cases} t = \gamma_v \tau \\ \mathbf{x} = \mathbf{x_0} + \mathbf{v}t = \mathbf{x_0} + \mathbf{v}\gamma_v\tau \end{cases} \quad \Rightarrow \quad G^+_\epsilon\big(x(\tau), x(\tau')\big) = \frac{1}{4\pi^2(\Delta\tau - i\epsilon)^2}$$

which yield null response rates:

$$\mathscr{F}'(E - E_0) = \int_{-\infty}^{\infty} d(\Delta\tau) e^{-i(E-E_0)\Delta\tau} G^+(\Delta\tau) = 0, \quad \forall\, E > E_0$$

Inertial detectors

Let us consider a massless field in Minkowski spacetime. The vacuum correlations read:

$$G_\epsilon^+(x,x') = \frac{1}{4\pi^2}\frac{1}{(\Delta t - i\epsilon)^2 - |\Delta \mathbf{x}|^2}$$

Then, for an inertial worldline we find the correlations:

$$\begin{cases} t = \gamma_v \tau \\ \mathbf{x} = \mathbf{x_0} + \mathbf{v}t = \mathbf{x_0} + \mathbf{v}\gamma_v\tau \end{cases} \quad \Rightarrow \quad G_\epsilon^+\big(x(\tau), x(\tau')\big) = \frac{1}{4\pi^2(\Delta\tau - i\epsilon)^2}$$

which yield null response rates:

$$\mathscr{F}'(E-E_0) = \int_{-\infty}^{\infty} d(\Delta\tau) e^{-i(E-E_0)\Delta\tau} G^+(\Delta\tau) = 0, \quad \forall\, E > E_0$$

Furthermore, for a many-particle state $|\Psi\rangle = |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots\rangle$, we recover an intuitive response rate in Minkowski space:

$$\langle\Psi|\phi(x)\phi(x')|\Psi\rangle = G^{+}(x,x') + \int d^3\mathbf{k}\, n(\mathbf{k}) u_{\mathbf{k}}(x) u^*_{\mathbf{k}}(x') + \int d^3\mathbf{k}\, n(\mathbf{k}) u^*_{\mathbf{k}}(x) u_{\mathbf{k}}(x')$$

Since our detector is insentitive to particle directions, let us consider the detection rates for isotropic states $n(\mathbf{k}) = n(k)$:

$$\mathscr{F}'(E) = \frac{1}{4\pi^2}\int_m^\infty d\omega \sqrt{\omega^2 - m^2}\, \bar{n}(\omega)\delta(E-\omega)$$
$$= \frac{1}{4\pi^2}\sqrt{E^2 - m^2}\, \bar{n}(E)\Theta(E-m)$$

In such a case, we find response rates proportional to the occupation numbers, so that we can ascribe a clear physical meaning for occupation numbers in terms of detection rates of inertial observers.

Furthermore, for a many-particle state $|\Psi\rangle = |n_{\mathbf{k_1}}, n_{\mathbf{k_2}}, \ldots\rangle$, we recover an intuitive response rate in Minkowski space:

$$\langle\Psi|\phi(x)\phi(x')|\Psi\rangle = G^{+}(x,x') + \int d^3\mathbf{k}\, n(\mathbf{k}) u_{\mathbf{k}}(x) u^{*}_{\mathbf{k}}(x') + \int d^3\mathbf{k}\, n(\mathbf{k}) u^{*}_{\mathbf{k}}(x) u_{\mathbf{k}}(x')$$

Since our detector is insentitive to particle directions, let us consider the detection rates for isotropic states $n(\mathbf{k}) = n(k)$:

$$\begin{aligned}\mathscr{F}'(E) &= \frac{1}{4\pi^2}\int_m^\infty d\omega \sqrt{\omega^2 - m^2}\, \bar{n}(\omega)\delta(E-\omega) \\ &= \frac{1}{4\pi^2}\sqrt{E^2 - m^2}\, \bar{n}(E)\Theta(E-m)\end{aligned}$$

In such a case, we find response rates proportional to the occupation numbers, so that we can ascribe a clear physical meaning for occupation numbers in terms of detection rates of inertial observers.

Furthermore, for a many-particle state $|\Psi\rangle = |n_{\mathbf{k_1}}, n_{\mathbf{k_2}}, \ldots\rangle$, we recover an intuitive response rate in Minkowski space:

$$\langle\Psi|\phi(x)\phi(x')|\Psi\rangle = G^{+}(x,x') + \int d^3\mathbf{k}\, n(\mathbf{k}) u_{\mathbf{k}}(x) u^*_{\mathbf{k}}(x') + \int d^3\mathbf{k}\, n(\mathbf{k}) u^*_{\mathbf{k}}(x) u_{\mathbf{k}}(x')$$

Since our detector is insentitive to particle directions, let us consider the detection rates for isotropic states $n(\mathbf{k}) = n(k)$:

$$\begin{aligned} \mathscr{F}'(E) &= \frac{1}{4\pi^2} \int_m^\infty d\omega \sqrt{\omega^2 - m^2}\, \bar{n}(\omega)\delta(E-\omega) \\ &= \frac{1}{4\pi^2} \sqrt{E^2 - m^2}\, \bar{n}(E)\Theta(E-m) \end{aligned}$$

In such a case, we find response rates proportional to the occupation numbers, so that we can ascribe a clear physical meaning for occupation numbers in terms of detection rates of inertial observers.

Uniformly accelerated detector

Now, let us consider an uniformly accelerated detector with proper acceleration a:

$$\begin{cases} x(\tau) = a^{-1}\cosh(a\tau) \\ t(\tau) = a^{-1}\sinh(a\tau) \end{cases} \quad \Rightarrow \quad G^+_\epsilon(\Delta\tau) = \left[16\pi^2\alpha^2\sinh^2\left(\frac{\Delta\tau - 2i\epsilon}{2\alpha}\right)\right]^{-1}$$

Instead of null detection rates, we actually find a thermal spectrum with temperature $T = a/2\pi$. The total detection probability reads:

$$P' = \lim_{\epsilon\to 0^+} c^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 I_\epsilon = \frac{c^2}{2\pi}\sum_E \frac{|\langle E|m(0)|E_0\rangle|^2}{e^{2\pi(E-E_0)/a} - 1}$$

Uniformly accelerated detector

Now, let us consider an uniformly accelerated detector with proper acceleration a:

$$\begin{cases} x(\tau) = a^{-1}\cosh(a\tau) \\ t(\tau) = a^{-1}\sinh(a\tau) \end{cases} \quad \Rightarrow \quad G^{+}_{\epsilon}(\Delta\tau) = \left[16\pi^2\alpha^2\sinh^2\left(\frac{\Delta\tau - 2i\epsilon}{2\alpha}\right)\right]^{-1}$$

Instead of null detection rates, we actually find a thermal spectrum with temperature $T = a/2\pi$. The total detection probability reads:

$$P' = \lim_{\epsilon\to 0^+} c^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 I_\epsilon = \frac{c^2}{2\pi}\sum_E \frac{|\langle E|m(0)|E_0\rangle|^2}{e^{2\pi(E-E_0)/a}-1}$$

Uniformly accelerated detector

Now, let us consider an uniformly accelerated detector with proper acceleration a:

$$\begin{cases} x(\tau) = a^{-1}\cosh(a\tau) \\ t(\tau) = a^{-1}\sinh(a\tau) \end{cases} \quad \Rightarrow \quad G^{+}_{\epsilon}(\Delta\tau) = \left[16\pi^2\alpha^2\sinh^2\left(\frac{\Delta\tau - 2i\epsilon}{2\alpha}\right)\right]^{-1}$$

Instead of null detection rates, we actually find a thermal spectrum with temperature $T = a/2\pi$. The total detection probability reads:

$$P' = \lim_{\epsilon\to 0^+} c^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 I_\epsilon = \frac{c^2}{2\pi}\sum_E \frac{|\langle E|m(0)|E_0\rangle|^2}{e^{2\pi(E-E_0)/a} - 1}$$

Now, let us consider an uniformly accelerated detector with proper acceleration a:

$$\begin{cases} x(\tau) = a^{-1}\cosh(a\tau) \\ t(\tau) = a^{-1}\sinh(a\tau) \end{cases} \quad \Rightarrow \quad G_\epsilon^+(\Delta\tau) = \left[16\pi^2\alpha^2\sinh^2\left(\frac{\Delta\tau - 2i\epsilon}{2\alpha}\right)\right]^{-1}$$

Instead of null detection rates, we actually find a thermal spectrum with temperature $T = a/2\pi$. The total detection probability reads:

$$P' = \lim_{\epsilon\to0^+} c^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 I_\epsilon = \frac{c^2}{2\pi}\sum_E \frac{|\langle E|m(0)|E_0\rangle|^2}{e^{2\pi(E-E_0)/a} - 1}$$

Particle Creation

We have seen that response rates can relate simply with occupation numbers in special cases. A special case relates to (asymptotically) stationary spacetimes:

$$\begin{cases} u_i^{(p)}(x) \simeq \dfrac{e^{-i\omega_i t}}{\sqrt{2\omega_i}}\psi_i^{(p)}(\mathbf{x}), & x \in \Omega_p \\ u_j^{(f)}(x) \simeq \dfrac{e^{-i\omega_j t}}{\sqrt{2\omega_j}}\psi_j^{(f)}(\mathbf{x}), & x \in \Omega_f \end{cases}$$

We can expand the field operators in both sets of modes:

$$\phi(x) = \sum_i a_i^{(p)} u_i^{(p)}(x) + a_i^{\dagger(p)} u_i^{*(p)}(x) = \sum_j a_j^{(f)} u_j^{(f)}(x) + a_j^{\dagger(f)} u_j^{*(f)}(x)$$

A state that was initially perceived as vacuum will have particles in the future according to stationary observers:

$$\langle 0_p | N_j^{(f)} | 0_p \rangle = \sum_i |\beta_{ji}|^2$$

We have seen that response rates can relate simply with occupation numbers in special cases. A special case relates to (asymptotically) stationary spacetimes:

$$\begin{cases} u_i^{(p)}(x) \simeq \dfrac{e^{-i\omega_i t}}{\sqrt{2\omega_i}}\psi_i^{(p)}(\mathbf{x}), & x \in \Omega_p \\ u_j^{(f)}(x) \simeq \dfrac{e^{-i\omega_j t}}{\sqrt{2\omega_j}}\psi_j^{(f)}(\mathbf{x}), & x \in \Omega_f \end{cases}$$

We can expand the field operators in both sets of modes:

$$\phi(x) = \sum_i a_i^{(p)} u_i^{(p)}(x) + a_i^{\dagger(p)} u_i^{*(p)}(x) = \sum_j a_j^{(f)} u_j^{(f)}(x) + a_j^{\dagger(f)} u_j^{*(f)}(x)$$

A state that was initially perceived as vacuum will have particles in the future according to stationary observers:

$$\langle 0_p | N_j^{(f)} | 0_p \rangle = \sum_i |\beta_{ji}|^2$$

We have seen that response rates can relate simply with occupation numbers in special cases. A special case relates to (asymptotically) stationary spacetimes:

$$\begin{cases} u_i^{(p)}(x) \simeq \dfrac{e^{-i\omega_i t}}{\sqrt{2\omega_i}}\psi_i^{(p)}(\mathbf{x}), & x \in \Omega_p \\ u_j^{(f)}(x) \simeq \dfrac{e^{-i\omega_j t}}{\sqrt{2\omega_j}}\psi_j^{(f)}(\mathbf{x}), & x \in \Omega_f \end{cases}$$

We can expand the field operators in both sets of modes:

$$\phi(x) = \sum_i a_i^{(p)} u_i^{(p)}(x) + a_i^{\dagger(p)} u_i^{*(p)}(x) = \sum_j a_j^{(f)} u_j^{(f)}(x) + a_j^{\dagger(f)} u_j^{*(f)}(x)$$

A state that was initially perceived as vacuum will have particles in the future according to stationary observers:

$$\langle 0_p | N_j^{(f)} | 0_p \rangle = \sum_i |\beta_{ji}|^2$$

We have seen that response rates can relate simply with occupation numbers in special cases. A special case relates to (asymptotically) stationary spacetimes:

$$\begin{cases} u_i^{(p)}(x) \simeq \dfrac{e^{-i\omega_i t}}{\sqrt{2\omega_i}}\psi_i^{(p)}(\mathbf{x}), & x \in \Omega_p \\ u_j^{(f)}(x) \simeq \dfrac{e^{-i\omega_j t}}{\sqrt{2\omega_j}}\psi_j^{(f)}(\mathbf{x}), & x \in \Omega_f \end{cases}$$

We can expand the field operators in both sets of modes:

$$\phi(x) = \sum_i a_i^{(p)} u_i^{(p)}(x) + a_i^{\dagger(p)} u_i^{*(p)}(x) = \sum_j a_j^{(f)} u_j^{(f)}(x) + a_j^{\dagger(f)} u_j^{*(f)}(x)$$

A state that was initially perceived as vacuum will have particles in the future according to stationary observers:

$$\langle 0_p | N_j^{(f)} | 0_p \rangle = \sum_i |\beta_{ji}|^2$$

Particle creation in FLRW spaces

Let us consider asymptotically static spatially flat FLRW metrics:

$$ds^2 = dt^2 - a^2(t)d\Sigma^2 \qquad\qquad a(t) \longrightarrow \begin{cases} a_1, & t \to -\infty \\ a_2, & t \to +\infty \end{cases}$$

This yields simple separable field equations *at all times*:

$$u_{\mathbf{k}}(x) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{Va^3(t)}} h_k(t)$$

In this case, the Bogolubov coefficients will be quasidiagonal and particle creation on each mode will be simply:

$$\begin{cases} \alpha_{\mathbf{k}\mathbf{k}'} = \alpha_k \delta_{\mathbf{k},\mathbf{k}'} \\ \beta_{\mathbf{k}\mathbf{k}'} = \beta_k \delta_{\mathbf{k},-\mathbf{k}'} \end{cases} \qquad \Rightarrow \qquad \langle 0_p | N_{\mathbf{k}}^{(f)} | 0_p \rangle = \beta_k^2$$

Particle creation in FLRW spaces

Let us consider asymptotically static spatially flat FLRW metrics:

$$ds^2 = dt^2 - a^2(t)d\Sigma^2 \qquad a(t) \longrightarrow \begin{cases} a_1, & t \to -\infty \\ a_2, & t \to +\infty \end{cases}$$

This yields simple separable field equations *at all times*:

$$u_{\mathbf{k}}(x) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{Va^3(t)}} h_k(t)$$

In this case, the Bogolubov coefficients will be quasidiagonal and particle creation on each mode will be simply:

$$\begin{cases} \alpha_{\mathbf{k}\mathbf{k}'} = \alpha_k \delta_{\mathbf{k},\mathbf{k}'} \\ \beta_{\mathbf{k}\mathbf{k}'} = \beta_k \delta_{\mathbf{k},-\mathbf{k}'} \end{cases} \Rightarrow \quad \langle 0_p | N_{\mathbf{k}}^{(f)} | 0_p \rangle = \beta_k^2$$

Particle creation in FLRW spaces

Let us consider asymptotically static spatially flat FLRW metrics:

$$ds^2 = dt^2 - a^2(t)d\Sigma^2 \qquad\qquad a(t) \longrightarrow \begin{cases} a_1, & t \to -\infty \\ a_2, & t \to +\infty \end{cases}$$

This yields simple separable field equations *at all times*:

$$u_{\mathbf{k}}(x) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{Va^3(t)}} h_k(t)$$

In this case, the Bogolubov coefficients will be quasidiagonal and particle creation on each mode will be simply:

$$\begin{cases} \alpha_{\mathbf{k}\mathbf{k}'} = \alpha_k \delta_{\mathbf{k},\mathbf{k}'} \\ \beta_{\mathbf{k}\mathbf{k}'} = \beta_k \delta_{\mathbf{k},-\mathbf{k}'} \end{cases} \qquad \Rightarrow \qquad \langle 0_p | N_{\mathbf{k}}^{(f)} | 0_p \rangle = \beta_k^2$$

The conformally coupled case

For conformally flat FLRW spaces ($g_{ab} = a^2(\eta)\eta_{ab}$), it is simpler to carry an analysis in conformal time η:

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 = a^2(\eta)\left[d\eta^2 - d\mathbf{x}^2\right]$$

Then, for a conformally coupled field, $\xi = \xi(n)$, we obtain simple rescaled equationd:

$$\left[\Box + a^2(\eta)\, m^2\,\right]\tilde{\phi}(\eta, \mathbf{x}) = 0$$

which yield a time-dependent harmonic oscillator (TDHO):

$$\frac{d^2\chi_k}{d\eta^2} + \omega^2(\eta)\chi_k = 0 \qquad\qquad \omega^2(\eta) = k^2 + a^2(\eta)m^2$$

A Simple Model for Particle Creation

Let us consider a 1+1-dimensional space-time, with scale factor:

$$\boxed{a^2(\eta) = A + B\tanh(\rho\eta)}$$

$$\begin{cases} \omega_1 = \sqrt{k^2 + (A-B)m^2} \\ \omega_2 = \sqrt{k^2 + (A+B)m^2} \end{cases}$$

Bogolubov coefficients:

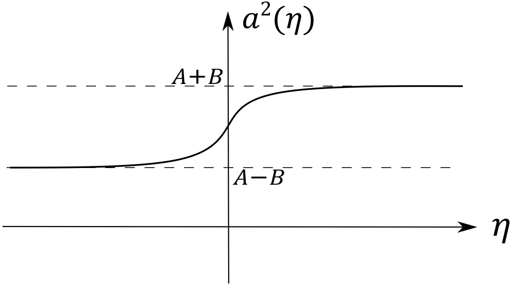
$$\begin{cases} |\alpha_k|^2 = \dfrac{\sinh^2(\pi\omega_+/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \\ |\beta_k|^2 = \dfrac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \end{cases}$$

$$\boxed{\omega_\pm \equiv \tfrac{1}{2}(\omega_2 \pm \omega_1)}$$

A Simple Model for Particle Creation

Let us consider a 1+1-dimensional space-time, with scale factor:

$$a^2(\eta) = A + B\tanh(\rho\eta)$$



$$\begin{cases} \omega_1 = \sqrt{k^2 + (A-B)m^2} \\ \omega_2 = \sqrt{k^2 + (A+B)m^2} \end{cases}$$

Bogolubov coefficients:

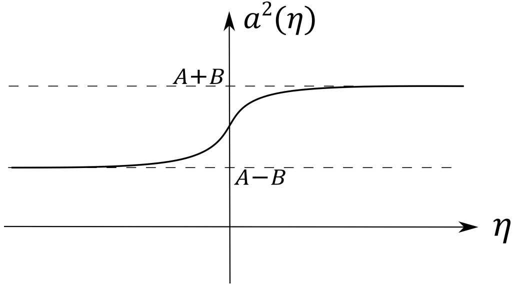
$$\begin{cases} |\alpha_k|^2 = \dfrac{\sinh^2(\pi\omega_+/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \\ |\beta_k|^2 = \dfrac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \end{cases}$$

$$\omega_\pm \equiv \tfrac{1}{2}(\omega_2 \pm \omega_1)$$

A Simple Model for Particle Creation

Let us consider a 1+1-dimensional space-time, with scale factor:

$$a^2(\eta) = A + B\tanh(\rho\eta)$$



$$\begin{cases} \omega_1 = \sqrt{k^2 + (A-B)m^2} \\ \omega_2 = \sqrt{k^2 + (A+B)m^2} \end{cases}$$

Bogolubov coefficients:

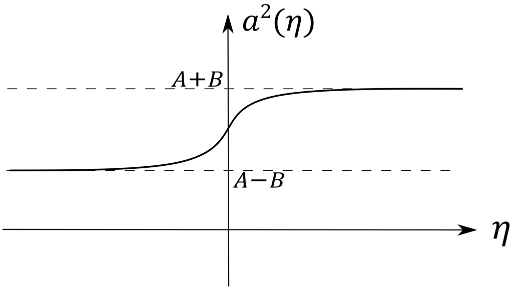
$$\begin{cases} |\alpha_k|^2 = \dfrac{\sinh^2(\pi\omega_+/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \\ |\beta_k|^2 = \dfrac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \end{cases}$$

$$\omega_\pm \equiv \tfrac{1}{2}(\omega_2 \pm \omega_1)$$

A Simple Model for Particle Creation

Let us consider a 1+1-dimensional space-time, with scale factor:

$$a^2(\eta) = A + B\tanh(\rho\eta)$$



$$\begin{cases} \omega_1 = \sqrt{k^2 + (A-B)m^2} \\ \omega_2 = \sqrt{k^2 + (A+B)m^2} \end{cases}$$

Bogolubov coefficients:

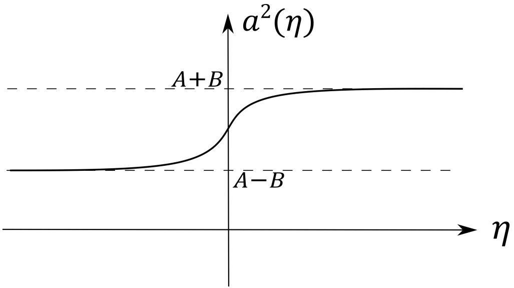
$$\begin{cases} |\alpha_k|^2 = \dfrac{\sinh^2(\pi\omega_+/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \\ |\beta_k|^2 = \dfrac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \end{cases}$$

$$\omega_\pm \equiv \tfrac{1}{2}(\omega_2 \pm \omega_1)$$

A Simple Model for Particle Creation

Let us consider a 1+1-dimensional space-time, with scale factor:

$$a^2(\eta) = A + B\tanh(\rho\eta)$$



$$\begin{cases} \omega_1 = \sqrt{k^2 + (A-B)m^2} \\ \omega_2 = \sqrt{k^2 + (A+B)m^2} \end{cases}$$

Bogolubov coefficients:

$$\begin{cases} |\alpha_k|^2 = \dfrac{\sinh^2(\pi\omega_+/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \\ |\beta_k|^2 = \dfrac{\sinh^2(\pi\omega_-/\rho)}{\sinh(\pi\omega_1/\rho)\sinh(\pi\omega_2/\rho)} \end{cases}$$

$$\omega_\pm \equiv \tfrac{1}{2}(\omega_2 \pm \omega_1)$$

Adiabatic Vacuum

We want to appropriately extend the concepts of particle and vacuum to fully dynamical regions of spacetime, but such regions present many difficulties. Particularly, if we have a particle creation rate A, we find a fundamental limit for the uncertainty in particle numbers:

$$\Delta N_{min} \sim 2(|A|/m)^{1/2}$$

Nonetheless, we can see from our previous model that particle creation will be suppressed in certain limits:

$$B \to 0: \ |\beta_k|^2 \propto B^2 \to 0 \qquad \rho \to 0: \ |\beta_k|^2 \propto e^{-2\pi\omega_1/\rho} \to 0$$

Therefore, we turn to the limit of an infinitely slow expansion (adiabatic limit).

We want to appropriately extend the concepts of particle and vacuum to fully dynamical regions of spacetime, but such regions present many difficulties. Particularly, if we have a particle creation rate A, we find a fundamental limit for the uncertainty in particle numbers:

$$\Delta N_{min} \sim 2(|A|/m)^{1/2}$$

Nonetheless, we can see from our previous model that particle creation will be suppressed in certain limits:

$$B \to 0 : \; |\beta_k|^2 \propto B^2 \to 0 \qquad\qquad \rho \to 0 : \; |\beta_k|^2 \propto e^{-2\pi\omega_1/\rho} \to 0$$

Therefore, we turn to the limit of an infinitely slow expansion (adiabatic limit).

We want to appropriately extend the concepts of particle and vacuum to fully dynamical regions of spacetime, but such regions present many difficulties. Particularly, if we have a particle creation rate A, we find a fundamental limit for the uncertainty in particle numbers:

$$\Delta N_{min} \sim 2(|A|/m)^{1/2}$$

Nonetheless, we can see from our previous model that particle creation will be suppressed in certain limits:

$$B \to 0 : \ |\beta_k|^2 \propto B^2 \to 0 \qquad\qquad \rho \to 0 : \ |\beta_k|^2 \propto e^{-2\pi\omega_1/\rho} \to 0$$

Therefore, we turn to the limit of an infinitely slow expansion (adiabatic limit).

We want to appropriately extend the concepts of particle and vacuum to fully dynamical regions of spacetime, but such regions present many difficulties. Particularly, if we have a particle creation rate A, we find a fundamental limit for the uncertainty in particle numbers:

$$\Delta N_{min} \sim 2(|A|/m)^{1/2}$$

Nonetheless, we can see from our previous model that particle creation will be suppressed in certain limits:

$$B \to 0: \ |\beta_k|^2 \propto B^2 \to 0 \qquad \rho \to 0: \ |\beta_k|^2 \propto e^{-2\pi\omega_1/\rho} \to 0$$

Therefore, we turn to the limit of an infinitely slow expansion (adiabatic limit).

WKB Solutions

To extend the notion of positive-frequency modes, we turn to WKB soulutions of our TDHO:

$$\chi_k = \frac{1}{\sqrt{2W_k(\eta)}} e^{-i\int^{\eta} d\eta' W_k(\eta')}$$

The dynamic equation for χ_k then leads us to the nonlinear equation for W_k:

$$W_k^2(\eta) = \omega_k^2(\eta) - \frac{1}{2}\left(\frac{\ddot{W}_k}{W_k} - \frac{3}{2}\frac{\dot{W}_k^2}{W_k^2}\right)$$

WKB Solutions

To extend the notion of positive-frequency modes, we turn to WKB soulutions of our TDHO:

$$\chi_k = \frac{1}{\sqrt{2W_k(\eta)}} e^{-i\int^{\eta} d\eta' W_k(\eta')}$$

The dynamic equation for χ_k then leads us to the nonlinear equation for W_k:

$$W_k^2(\eta) = \omega_k^2(\eta) - \frac{1}{2}\left(\frac{\ddot{W}_k}{W_k} - \frac{3}{2}\frac{\dot{W}_k^2}{W_k^2}\right)$$

Adiabatic Expansions

To analyze the limit of an arbitrarily slow expansion, we introduce the adiabatic parameter T:

$$a_T(\eta) \equiv a(\eta/T) = a(\eta_1) \qquad \Rightarrow \qquad \frac{d^n}{d\eta^n} a\left(\frac{\eta}{T}\right) = \frac{1}{T^n} a^{(n)}(\eta_1)$$

With this parameter, we can organize the terms in our solution by their adiabatic order A ($\propto T^{-A}$). Taking $T \to \infty$, we obtain a zeroth order adiabatic solution:

$$\left((W_k)^{(0)}(\eta_1)\right)^2 = \omega_k^2(\eta_1)$$

Iterating it, we obtain the 2nd order solution for W_k:

$$\left(W_k^{(2)}(\eta_1)\right)^2 = \omega_k^2(\eta_1) - \frac{1}{2T^2}\left[\frac{\ddot{\omega}_k(\eta_1)}{\omega_k(\eta_1)} - \frac{3}{2}\frac{\dot{\omega}_k^2(\eta_1)}{\omega_k^2(\eta_1)}\right]$$

Adiabatic Expansions

To analyze the limit of an arbitrarily slow expansion, we introduce the adiabatic parameter T:

$$a_T(\eta) \equiv a(\eta/T) = a(\eta_1) \qquad \Rightarrow \qquad \frac{d^n}{d\eta^n} a\left(\frac{\eta}{T}\right) = \frac{1}{T^n} a^{(n)}(\eta_1)$$

With this parameter, we can organize the terms in our solution by their adiabatic order A ($\propto T^{-A}$). Taking $T \to \infty$, we obtain a zeroth order adiabatic solution:

$$\left((W_k)^{(0)}(\eta_1)\right)^2 = \omega_k^2(\eta_1)$$

Iterating it, we obtain the 2nd order solution for W_k:

$$\left(W_k^{(2)}(\eta_1)\right)^2 = \omega_k^2(\eta_1) - \frac{1}{2T^2}\left[\frac{\ddot{\omega}_k(\eta_1)}{\omega_k(\eta_1)} - \frac{3}{2}\frac{\dot{\omega}_k^2(\eta_1)}{\omega_k^2(\eta_1)}\right]$$

Adiabatic Expansions

To analyze the limit of an arbitrarily slow expansion, we introduce the adiabatic parameter T:

$$a_T(\eta) \equiv a(\eta/T) = a(\eta_1) \qquad \Rightarrow \qquad \frac{d^n}{d\eta^n} a\left(\frac{\eta}{T}\right) = \frac{1}{T^n} a^{(n)}(\eta_1)$$

With this parameter, we can organize the terms in our solution by their adiabatic order A ($\propto T^{-A}$). Taking $T \to \infty$, we obtain a zeroth order adiabatic solution:

$$\left((W_k)^{(0)}(\eta_1)\right)^2 = \omega_k^2(\eta_1)$$

Iterating it, we obtain the 2nd order solution for W_k:

$$\left(W_k^{(2)}(\eta_1)\right)^2 = \omega_k^2(\eta_1) - \frac{1}{2T^2}\left[\frac{\ddot{\omega}_k(\eta_1)}{\omega_k(\eta_1)} - \frac{3}{2}\frac{\dot{\omega}_k^2(\eta_1)}{\omega_k^2(\eta_1)}\right]$$

Adiabatic Expansions

To analyze the limit of an arbitrarily slow expansion, we introduce the adiabatic parameter T:

$$a_T(\eta) \equiv a(\eta/T) = a(\eta_1) \qquad \Rightarrow \qquad \frac{d^n}{d\eta^n} a\left(\frac{\eta}{T}\right) = \frac{1}{T^n} a^{(n)}(\eta_1)$$

With this parameter, we can organize the terms in our solution by their adiabatic order A ($\propto T^{-A}$). Taking $T \to \infty$, we obtain a zeroth order adiabatic solution:

$$\left((W_k)^{(0)}(\eta_1)\right)^2 = \omega_k^2(\eta_1)$$

Iterating it, we obtain the 2nd order solution for W_k:

$$\left(W_k^{(2)}(\eta_1)\right)^2 = \omega_k^2(\eta_1) - \frac{1}{2T^2}\left[\frac{\ddot{\omega}_k(\eta_1)}{\omega_k(\eta_1)} - \frac{3}{2}\frac{\dot{\omega}_k^2(\eta_1)}{\omega_k^2(\eta_1)}\right]$$

Further iteration and exact solutions

We can repeat the iteration process up to the desired adiabatic order. For exemple, the next (4th order) term is given by:

$$\left(W_k^{(4)}\right)^2 = (W_k^{(2)})^2 - \frac{1}{2T^2}\left[\frac{\ddot{W}_k^{(2)}}{W_k^{(2)}} - \frac{3}{2}\frac{(\dot{W}_k^{(2)})^2}{(W_k^{(2)})^2}\right]$$

And so on for higher orders. We can match exact solutions with an adiabatic expansion up to the desired order at a given time η_0, $u_{\mathbf{k}}(\mathbf{x}, \eta_0) = u_{\mathbf{k}}^{(A)}(\mathbf{x}, \eta_0)$, to obtain an approximate notion of positive-frequency modes at η_0

$$u_{\mathbf{k}}(\mathbf{x}, \eta) = \alpha_{\mathbf{k}}^{(A)}(\eta) u_{\mathbf{k}}^{(A)}(\mathbf{x}, \eta) + \beta_{\mathbf{k}}^{(A)}(\eta) (u_{\mathbf{k}}^{(A)})^*(\mathbf{x}, \eta)$$

With these modes, we can obtain a corresponding adiabatic vacuum state $|0^{(A)}\rangle$ up to the desired order A, for any time η_0.

Further iteration and exact solutions

We can repeat the iteration process up to the desired adiabatic order. For exemple, the next (4th order) term is given by:

$$\left(W_k^{(4)}\right)^2 = (W_k^{(2)})^2 - \frac{1}{2T^2}\left[\frac{\ddot{W}_k^{(2)}}{W_k^{(2)}} - \frac{3}{2}\frac{(\dot{W}_k^{(2)})^2}{(W_k^{(2)})^2}\right]$$

And so on for higher orders. We can match exact solutions with an adiabatic expansion up to the desired order at a given time η_0, $u_{\mathbf{k}}(\mathbf{x}, \eta_0) = u_{\mathbf{k}}^{(A)}(\mathbf{x}, \eta_0)$, to obtain an approximate notion of positive-frequency modes at η_0

$$u_{\mathbf{k}}(\mathbf{x}, \eta) = \alpha_{\mathbf{k}}^{(A)}(\eta) u_{\mathbf{k}}^{(A)}(\mathbf{x}, \eta) + \beta_{\mathbf{k}}^{(A)}(\eta) (u_{\mathbf{k}}^{(A)})^*(\mathbf{x}, \eta)$$

With these modes, we can obtain a corresponding adiabatic vacuum state $|0^{(A)}\rangle$ up to the desired order A, for any time η_0.

Conclusions

Conclusions

- Vacuum is complex. Even classically, it can give rise to nontrivial dynamical behaviour.
- At a quantum level, one finds an even richer multitude of phenomena, like the Casimir Effect, particle detection and particle creation.
- Generally, in the absence of very strict symmetries, one is obliged to consider multiple unitarily inequivelent representations to account for physical phenomena.
- In curved spaces, one may still find suitable extensions to a physical notion of vacuum, e.g. through the adiabatic condition. (Among other things, this allows one to carry renormalization and speak meaningfully of vacuum energy).

Conclusions

- Vacuum is complex. Even classically, it can give rise to nontrivial dynamical behaviour.
- At a quantum level, one finds an even richer multitude of phenomena, like the Casimir Effect, particle detection and particle creation.
- Generally, in the absence of very strict symmetries, one is obliged to consider multiple unitarily inequivelent representations to account for physical phenomena.
- In curved spaces, one may still find suitable extensions to a physical notion of vacuum, e.g. through the adiabatic condition. (Among other things, this allows one to carry renormalization and speak meaningfully of vacuum energy).

Conclusions

- Vacuum is complex. Even classically, it can give rise to nontrivial dynamical behaviour.
- At a quantum level, one finds an even richer multitude of phenomena, like the Casimir Effect, particle detection and particle creation.
- Generally, in the absence of very strict symmetries, one is obliged to consider multiple unitarily inequivelent representations to account for physical phenomena.
- In curved spaces, one may still find suitable extensions to a physical notion of vacuum, e.g. through the adiabatic condition. (Among other things, this allows one to carry renormalization and speak meaningfully of vacuum energy).

Conclusions

- Vacuum is complex. Even classically, it can give rise to nontrivial dynamical behaviour.
- At a quantum level, one finds an even richer multitude of phenomena, like the Casimir Effect, particle detection and particle creation.
- Generally, in the absence of very strict symmetries, one is obliged to consider multiple unitarily inequivelent representations to account for physical phenomena.
- In curved spaces, one may still find suitable extensions to a physical notion of vacuum, e.g. through the adiabatic condition. (Among other things, this allows one to carry renormalization and speak meaningfully of vacuum energy).

Conclusions

- Vacuum is complex. Even classically, it can give rise to nontrivial dynamical behaviour.
- At a quantum level, one finds an even richer multitude of phenomena, like the Casimir Effect, particle detection and particle creation.
- Generally, in the absence of very strict symmetries, one is obliged to consider multiple unitarily inequivelent representations to account for physical phenomena.
- In curved spaces, one may still find suitable extensions to a physical notion of vacuum, e.g. through the adiabatic condition. (Among other things, this allows one to carry renormalization and speak meaningfully of vacuum energy).

Thank you!

Detection amplitudes

To first perturbative order, we find the transition amplitudes $\mathcal{A}$ between two states $|E, \Psi\rangle$ and $|E', \Psi'\rangle$:

$$\mathcal{A}\big(|E, \Psi\rangle \to |E', \Psi'\rangle\big) = ic \langle E', \Psi'| \int_{-\infty}^{+\infty} m(\tau)\phi(x^{\mu}(\tau))d\tau \, |E, \Psi\rangle \,. \tag{1}$$

Particularly, starting from the usual Minkowski vacuum $|0_M\rangle$:

$$\mathcal{A}\big(|E_0, 0_M\rangle \to |E, \Psi\rangle\big) = ic \langle E, \Psi| \int_{-\infty}^{+\infty} m(\tau)\phi(x^{\mu}(\tau))d\tau \, |E_0, 0_M\rangle \,. \tag{2}$$

Also, in the Interaction Picture $m(\tau) = e^{iH_0\tau} m(0) e^{-iH_0\tau}$, so we obtain:

$$\mathcal{A}(|E_0, 0_M\rangle \to |E, \Psi\rangle) = ic \langle E| m(0) |E_0\rangle \int_{-\infty}^{+\infty} e^{i(E-E_0)\tau} \langle \Psi| \phi(x^{\mu}(\tau)) |0_M\rangle \, d\tau. \tag{3}$$

The only transitions that may occur *in first perturbative order* are those to one-particle states: $|\Psi\rangle = |1_{\mathbf{k}}\rangle$:

$$\langle 1_{\mathbf{k}}|\,\phi(x)\,|0\rangle = \int d^3\mathbf{k}'(16\pi^3\omega_{k'})^{-1/2}\,\langle 1_{\mathbf{k}}|\,a^\dagger_{\mathbf{k}'}\,|0\rangle\, e^{i\omega' t - i\mathbf{k}'\cdot\mathbf{x}}$$
$$= (16\pi^3\omega_k)^{-1/2} e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}. \qquad (4)$$

We must specify trajectory $x^\mu(\tau)$ for the detector. For an inertial world-line $\mathbf{x} = \mathbf{x_0} + \mathbf{v}t = \mathbf{x_0} + \mathbf{v}\gamma_v\tau$:

$$\mathcal{A}(|E_0, 0_M\rangle \to |E, 1_{\mathbf{k}}\rangle) = \frac{ic\,\langle E|\,m(0)\,|E_0\rangle}{16\pi^3\omega} e^{-i\mathbf{k}\cdot\mathbf{x_0}} \int_{-\infty}^{+\infty} e^{i(E-E_0)\tau} e^{i(\omega - \mathbf{k}\cdot\mathbf{v})\gamma_v\tau} d\tau$$
$$= \frac{ic\,\langle E|\,m(0)\,|E_0\rangle}{4\pi\omega} e^{-i\mathbf{k}\cdot\mathbf{x_0}} \delta(E - E_0 + [\omega - \mathbf{k}\cdot\mathbf{v}]\gamma_v). \qquad (5)$$

But since $E > E_0$ and $\omega > |\mathbf{k}\cdot\mathbf{v}|$ (as $v < 1$ for any timelike trajectory and $\omega = \sqrt{k^2 + m^2} \geq k$) there are no roots in the arguments of the δ distribution in (5), and the

For generic trajectories, however, the amplitudes will be nonzero. In such cases, we shall sum over all possible final states $|\Psi\rangle$ and $|E\rangle$ ($\neq |E_0\rangle$) to obtain the total probability that *any* transition (detection) may occur:

$$\sum_{E,\Psi}\left|\mathcal{A}(|E_0,0_M\rangle \rightarrow |E,\Psi\rangle)\right|^2 = c^2\sum_{E}\Big\{|\langle E|m(0)|E_0\rangle|^2 \times \iint d\tau\, d\tau' e^{i(E-E_0)(\tau-\tau')}\,\langle 0_M|\phi(\tau')[\textstyle\sum_\Psi|\Psi\rangle\langle\Psi|]\phi(\tau)|0_M\rangle\Big\}. \quad (6)$$

Using the completeness relation $\sum_\Psi|\Psi\rangle\langle\Psi| = \mathbb{1}$, and recognizing the vacuum two-point correlation $G^+(x,x') = \langle 0_M|\phi(\tau')\phi(\tau)|0_M\rangle$, we have:

$$\begin{aligned} P &= c^2\sum_E |\langle E|m(0)|E_0\rangle|^2 \iint d\tau d\tau' e^{-i(E-E_0)(\tau-\tau')}G^+(x(\tau),x(\tau')) \\ &= c^2\sum |\langle E|m(0)|E_0\rangle|^2 \mathscr{F}(E-E_0), \end{aligned} \quad (7)$$

Detection amplitudes

Performing a change of variables: $(\tau, \tau') \to (\bar{\tau}, \Delta\tau)$, we have:

$$\mathscr{F}(E) = \iint d\bar{\tau}\, d(\Delta\tau) e^{-iE\Delta\tau} \tilde{G}^{+}(\bar{\tau}, \Delta\tau), \tag{9}$$

where $\tilde{G}^{+}(\bar{\tau}, \Delta\tau) \equiv G(\tau, \tau')$.

Particularly, for a stationary trajectory, that is, $G^{+}(\tau, \tau') = G^{+}(\Delta\tau)$, we obtain trivially separable integrals:

$$\mathscr{F}(E) = \left(\int_{-\infty}^{\infty} d\bar{\tau}\right)\left(\int_{-\infty}^{\infty} d(\Delta\tau) e^{-iE\Delta\tau} G^{+}(\Delta\tau)\right), \tag{10}$$

which can be immediately interpreted as a (constant) *transition rate* multiplied by the (infinite) time interval of the interactions $T \equiv \int d\bar{\tau}$.

In this stationary case, it is more convenient to work directly with transition rates. Thus, A convenient trick to work directly with G^+ (*i.e.* is to introduce the regularizer $e^{-\epsilon|\mathbf{k}|}$ ($\epsilon > 0$), making (12) absolutely convergent.

$$G^+_\epsilon(x, x') = \frac{1}{4\pi^2} \frac{1}{(\Delta t - i\epsilon)^2 - |\Delta \mathbf{x}|^2} \tag{13}$$

In the case of an inertial detector, we have:

$$\frac{1}{(\Delta t - i\epsilon)^2 - |\Delta \mathbf{x}|^2} = \frac{1}{(\gamma_v \Delta\tau - i\epsilon)^2 - (\gamma_v v \Delta\tau)^2} = \frac{1}{\Delta\tau^2 - 2i\Delta\tau\gamma_v\epsilon + \mathcal{O}(\epsilon^2)}.$$

We then absorb the positive factor γ into ϵ and ignore any higher order ($\mathcal{O}(\epsilon^2)$) corrections to write:

$$G^+_\epsilon(x, x') = \frac{1}{4\pi^2(\Delta\tau - i\epsilon)^2}. \tag{14}$$

The Hamiltonian Formalism

Defining the canonically conjugated momenta $\pi^a \equiv \frac{\partial \mathscr{L}}{\partial \dot{\phi}_a}$, we can perform a Legendre transformation to obtain the hamiltonian:

$$\mathcal{H}(\phi_a, \pi^a, x) = \pi^a(x)\dot{\phi}_a(x) - \mathscr{L}(\phi_a, \dot{\phi}_a, x)$$

A very important geometrical structure in phase space are the Poisson brackets:

$$\{F, G\} = \int d^3\mathbf{x}\, \frac{\delta F}{\delta \phi_a(\mathbf{x}, t)} \frac{\delta G}{\delta \pi^a(\mathbf{x}, t)} - \frac{\delta G}{\delta \phi_a(\mathbf{x}, t)} \frac{\delta F}{\delta \pi^a(\mathbf{x}, t)} \tag{15}$$

Particularly, the fundamental canonical Poisson Brackets:

$$\{\phi_a(\mathbf{x}, t), \pi^b(\mathbf{y}, t)\} = \delta_a^{\ b}\, \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{16}$$

which play a key role in canonical quantization.

The Hamiltonian Formalism

Defining the canonically conjugated momenta $\pi^a \equiv \frac{\partial \mathscr{L}}{\partial \dot{\phi}_a}$, we can perform a Legendre transformation to obtain the hamiltonian:

$$\mathcal{H}(\phi_a, \pi^a, x) = \pi^a(x)\dot{\phi}_a(x) - \mathscr{L}(\phi_a, \dot{\phi}_a, x)$$

A very important geometrical structure in phase space are the Poisson brackets:

$$\{F, G\} = \int d^3\mathbf{x} \, \frac{\delta F}{\delta \phi_a(\mathbf{x}, t)} \frac{\delta G}{\delta \pi^a(\mathbf{x}, t)} - \frac{\delta G}{\delta \phi_a(\mathbf{x}, t)} \frac{\delta F}{\delta \pi^a(\mathbf{x}, t)} \tag{15}$$

Particularly, the fundamental canonical Poisson Brackets:

$$\{\phi_a(\mathbf{x}, t), \pi^b(\mathbf{y}, t)\} = \delta_a^{\ b} \, \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{16}$$

which play a key role in canonical quantization.

The Hamiltonian Formalism

Defining the canonically conjugated momenta $\pi^a \equiv \frac{\partial \mathscr{L}}{\partial \dot{\phi}_a}$, we can perform a Legendre transformation to obtain the hamiltonian:

$$\mathcal{H}(\phi_a, \pi^a, x) = \pi^a(x)\dot{\phi}_a(x) - \mathscr{L}(\phi_a, \dot{\phi}_a, x)$$

A very important geometrical structure in phase space are the Poisson brackets:

$$\{F, G\} = \int d^3\mathbf{x} \, \frac{\delta F}{\delta \phi_a(\mathbf{x}, t)} \frac{\delta G}{\delta \pi^a(\mathbf{x}, t)} - \frac{\delta G}{\delta \phi_a(\mathbf{x}, t)} \frac{\delta F}{\delta \pi^a(\mathbf{x}, t)} \tag{15}$$

Particularly, the fundamental canonical Poisson Brackets:

$$\{\phi_a(\mathbf{x}, t), \pi^b(\mathbf{y}, t)\} = \delta_a{}^b \, \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{16}$$

which play a key role in canonical quantization.

Canonical Commutation Relations:

$$[x_i, x_j] = [p_i, p_j] = 0, \qquad [x_i, p_j] = i\delta_{ij}$$

For fields we have:

$$[\phi_a(\mathbf{x}, t), \phi_b(\mathbf{y}, t)] = [\pi^a(\mathbf{x}, t), \pi^b(\mathbf{y}, t)] = 0, \qquad [\phi_a(\mathbf{x}, t), \pi^b(\mathbf{y}, t)] = i\delta_a^{\ b}\,\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

$$\Rightarrow [\phi(x), \phi(y)] = 0, \ \forall x, y \text{ spacelike separated}$$

Vacuum Energy divergences

In the continuum, we find:

$$\langle 0|H|0\rangle = \frac{1}{2}\sum_{\mathbf{k}} \omega_k \longrightarrow \lim_{L\to\infty} \frac{L^3}{2(2\pi)^3}\int d^3\mathbf{k}\,\omega_k = \left(\lim_{V\to\infty}\frac{V}{4\pi^2}\right)\int_0^\infty dk\, k^2\omega_k$$

Infinite volume + UV divergences in the energy density.

For noninteracting fields in flat space, one is only interested in energy differences, so a trivial procedure to get rid of the divergent vacuum energies is normal ordering:

$$:H: = \sum_{\mathbf{k}} \omega_k a^\dagger_{\mathbf{k}} a_{\mathbf{k}} = \sum_{\mathbf{k}} \omega_k N_{\mathbf{k}} \qquad \Rightarrow \qquad \langle 0|:H:|0\rangle = 0$$

Vacuum Energy divergences

In the continuum, we find:

$$\langle 0|H|0\rangle = \frac{1}{2}\sum_{\mathbf{k}} \omega_k \longrightarrow \lim_{L\to\infty} \frac{L^3}{2(2\pi)^3}\int d^3\mathbf{k}\,\omega_k = \left(\lim_{V\to\infty}\frac{V}{4\pi^2}\right)\int_0^\infty dk\, k^2\omega_k$$

Infinite volume + UV divergences in the energy density.

For noninteracting fields in flat space, one is only interested in energy differences, so a trivial procedure to get rid of the divergent vacuum energies is normal ordering:

$$:H: = \sum_{\mathbf{k}} \omega_k a^\dagger_{\mathbf{k}} a_{\mathbf{k}} = \sum_{\mathbf{k}} \omega_k N_{\mathbf{k}} \qquad \Rightarrow \qquad \langle 0| :H: |0\rangle = 0$$

To handle the divergent vacuum energy, we write it as the limit of a convergent sum:

$$\rho_0(a) = \frac{1}{V}\langle 0_a|H|0_a\rangle = \frac{1}{2aL^2}\sum_{\mathbf{k}}\omega_k = -\frac{1}{2aL^2}\lim_{\alpha\to 0^+}\left[\frac{d}{d\alpha}\sum_{\mathbf{k}} e^{-\alpha\omega_k}\right]$$

and we define the auxiliary function:

$$S(\alpha,a) = \frac{1}{(2\pi)^2}\sum_{l=-\infty}^{+\infty}\int d^2\mathbf{k}_\perp \exp[-\alpha(\mathbf{k}_\perp^2 + (\tfrac{2\pi}{a})^2 l^2)^{1/2}]$$
$$= \frac{1}{2\pi}\left[\, F(0) + 2\textstyle\sum_l F(l)\,\right]$$

$$F(l) \equiv F(l) \equiv \int_0^\infty dk_\perp k_\perp e^{-\alpha\left[k_\perp^2+\left(\frac{2\pi}{a}\right)^2 l^2\right]^{1/2}} = \left[\frac{1}{\alpha^2} + \frac{1}{\alpha}\frac{2\pi l}{a}\right]e^{-\frac{2\pi l}{a}\alpha}$$

The strategy then is to isolate the divergent contributions by means of a convenient series expansion (Euler-MacLauren formula):

$$\frac{1}{2}F(b) + \sum_{l=1}^{\infty} F(b+l) = \int_b^{\infty} dl\, F(l) - \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} F^{(2m-1)}(b)$$

$$F^{(1)}(0) = 0, \quad F^{(3)}(0) = 2\alpha\left(\frac{2\pi}{a}\right)^3, \quad \text{and} \quad F^{(j)}(0) = \mathcal{O}(\alpha^2), j \geq 5$$

We then arrive at:

$$\rho_0(a) = -\frac{1}{2a}\lim_{\alpha\to 0^+}\left\{\frac{d}{d\alpha}S(\alpha,a)\right\} = -\frac{1}{2\pi}\lim_{\alpha\to 0^+} G'(\alpha) - \frac{\pi^2}{90a^4}$$

The strategy then is to isolate the divergent contributions by means of a convenient series expansion (Euler-MacLauren formula):

$$\tfrac{1}{2}F(b) + \sum_{l=1}^{\infty} F(b+l) = \int_b^{\infty} dl\, F(l) - \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} F^{(2m-1)}(b)$$

$$F^{(1)}(0) = 0, \quad F^{(3)}(0) = 2\alpha\left(\frac{2\pi}{a}\right)^3, \quad \text{and} \quad F^{(j)}(0) = \mathcal{O}(\alpha^2), j \geq 5$$

We then arrive at:

$$\rho_0(a) = -\frac{1}{2a}\lim_{\alpha\to 0^+}\left\{\frac{d}{d\alpha}S(\alpha,a)\right\} = -\frac{1}{2\pi}\lim_{\alpha\to 0^+} G'(\alpha) - \frac{\pi^2}{90a^4}$$

The strategy then is to isolate the divergent contributions by means of a convenient series expansion (Euler-MacLauren formula):

$$\tfrac{1}{2}F(b)+\sum_{l=1}^{\infty}F(b+l)=\int_b^{\infty}dl\,F(l)-\sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}F^{(2m-1)}(b)$$

$$F^{(1)}(0)=0,\quad F^{(3)}(0)=2\alpha\left(\frac{2\pi}{a}\right)^3,\quad \text{and}\quad F^{(j)}(0)=\mathcal{O}(\alpha^2),j\geq 5$$

We then arrive at:

$$\rho_0(a)=-\frac{1}{2a}\lim_{\alpha\to 0^+}\left\{\frac{d}{d\alpha}S(\alpha,a)\right\}=-\frac{1}{2\pi}\lim_{\alpha\to 0^+}G'(\alpha)-\frac{\pi^2}{90a^4}$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)}\{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b \phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}}$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab}$$

$$T_{ab} \equiv \frac{2}{|g|^{\frac{1}{2}}} \frac{\delta S_M}{\delta g^{ab}}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$\nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)} \{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b \phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}}$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab}$$

$$T_{ab} \equiv \frac{2}{|g|^{\frac{1}{2}}} \frac{\delta S_M}{\delta g^{ab}}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$\nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)} \{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b \phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}}$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab}$$

$$T_{ab} \equiv \frac{2}{|g|^{\frac{1}{2}}} \frac{\delta S_M}{\delta g^{ab}}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$\nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)} \{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b \phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}}$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab}$$

$$T_{ab} \equiv \frac{2}{|g|^{\frac{1}{2}}} \frac{\delta S_M}{\delta g^{ab}}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$\nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial (\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)} \{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b\phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}}$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab}$$

$$T_{ab} \equiv \frac{2}{|g|^{\frac{1}{2}}} \frac{\delta S_M}{\delta g^{ab}}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$\nabla_b \left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b \phi_a)} \right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x\sqrt{-g(x)}\{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b\phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}}$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab}$$

$$T_{ab} \equiv \frac{2}{|g|^{\frac{1}{2}}}\frac{\delta S_M}{\delta g^{ab}}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$\nabla_b\left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b\phi_a)}\right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x\sqrt{-g(x)}\{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b\phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}}$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab}$$

$$T_{ab} \equiv \frac{2}{|g|^{\frac{1}{2}}}\frac{\delta S_M}{\delta g^{ab}}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$\nabla_b\left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b\phi_a)}\right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x \sqrt{-g(x)}\{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b\phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}}$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab}$$

$$T_{ab} \equiv \frac{2}{|g|^{\frac{1}{2}}}\frac{\delta S_M}{\delta g^{ab}}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$\nabla_b\left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b\phi_a)}\right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

Once again, the starting point for us to obtain a quantized field theory is a Lagrangian formulation, within a minimal action principle. In curved spaces, we write the total action with a purely geometrical contribution and a (general-covariant) matter contribution:

$$S = S_G[g^{ab}] + S_M[\phi_a, g^{ab}] = \int d^4x\sqrt{-g(x)}\{\mathscr{L}_G(x) + \mathscr{L}_M(x)\}$$

$$\mathscr{L}_G = \frac{R - 2\Lambda}{8\pi G} \qquad \mathscr{L}_M = \mathscr{L}_M(\phi_a, \nabla_b\phi_a)$$

This joint formulation yields both Einstein and Euler-Largrange equations:

$$\frac{\delta S}{\delta g^{ab}} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta g^{ab}} = -\frac{\delta S_M}{\delta g^{ab}}$$

$$G_{ab} - \Lambda g_{ab} = -8\pi G T_{ab}$$

$$\boxed{T_{ab} \equiv \frac{2}{|g|^{\frac{1}{2}}}\frac{\delta S_M}{\delta g^{ab}}}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\delta S_M}{\delta \phi_a} = 0$$

$$\nabla_b\left(\frac{\partial \mathscr{L}_M}{\partial(\nabla_b\phi_a)}\right) - \frac{\partial \mathscr{L}_M}{\partial \phi_a} = 0$$

We consider a smooth, n-dimensional, globally hyperbolic spacetime, so that we can derive any predictions from information a "simultaneity surface":

Domains of dependence:

Global Hiperbolicity and Cauchy Surfaces

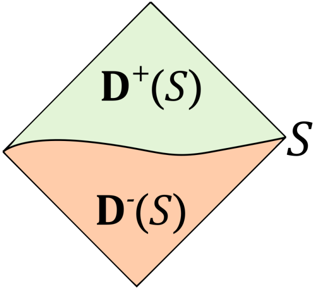
We consider a smooth, n-dimensional, globally hyperbolic spacetime, so that we can derive any predictions from information a “simultaneity surface”:

Domains of dependence:

Global Hiperbolicity and Cauchy Surfaces

We consider a smooth, n-dimensional, globally hyperbolic spacetime, so that we can derive any predictions from information a “simultaneity surface”:

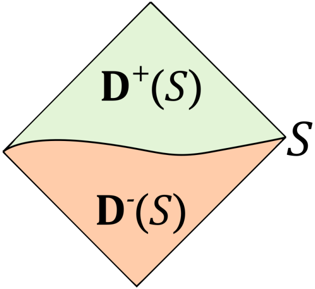
Domains of dependence:



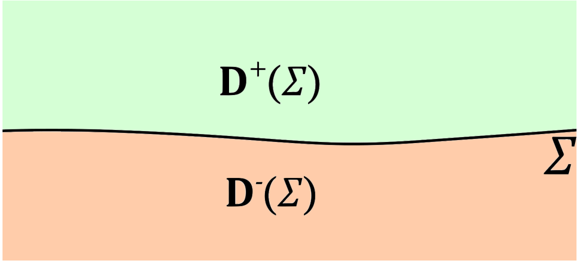
Global Hiperbolicity and Cauchy Surfaces

We consider a smooth, n-dimensional, globally hyperbolic spacetime, so that we can derive any predictions from information a "simultaneity surface":

Domains of dependence:



Cauchy Surfaces:



Technical Remarks on Adiabatic Subtraction

- Each adiabatic order will contain divergent and convergent terms. One must subtract *all* terms in a given order.
- Some special values of the theory parameters (masses, coupling constants...) can make the leading coefficients in a given adiabatic term vanish. One must subtract divergent terms for *generic paramenters*.
- Classically positive-definite observers can present negative expected values upon renormalization, due to the subtractions.

Technical Remarks on Adiabatic Subtraction

- Each adiabatic order will contain divergent and convergent terms. One must subtract *all* terms in a given order.
- Some special values of the theory parameters (masses, coupling constants...) can make the leading coefficients in a given adiabatic term vanish. One must subtract divergent terms for *generic paramenters*.
- Classically positive-definite observers can present negative expected values upon renormalization, due to the subtractions.

Technical Remarks on Adiabatic Subtraction

- Each adiabatic order will contain divergent and convergent terms. One must subtract *all* terms in a given order.
- Some special values of the theory parameters (masses, coupling constants...) can make the leading coefficients in a given adiabatic term vanish. One must subtract divergent terms for *generic paramenters*.
- Classically positive-definite observers can present negative expected values upon renormalization, due to the subtractions.

Technical Remarks on Adiabatic Subtraction

- Each adiabatic order will contain divergent and convergent terms. One must subtract *all* terms in a given order.
- Some special values of the theory parameters (masses, coupling constants...) can make the leading coefficients in a given adiabatic term vanish. One must subtract divergent terms for *generic paramenters*.
- Classically positive-definite observers can present negative expected values upon renormalization, due to the subtractions.

WKB frequencies

Similarly, we write WKB solutions in proper-time:

$$h_k(t) = \frac{1}{\sqrt{2W_k}} e^{-i\int^t dt' W_k(t')} \qquad \Rightarrow \qquad W_k^2 = \Omega_k^2 + \omega_k^{\frac{1}{2}} \frac{d^2}{dt^2} \omega_k^{-\frac{1}{2}}$$

Then, iteractively, we can compute an expansion for W_k in successive adiabatic orders:

$$W_k \sim \omega_k + \omega_k^{(2)} + \omega_k^{(4)} + \dots$$

Similarly, any functions of W_k are expanded as:

$$f(W_k) \sim f^{(0)}(W_k) + f^{(2)}(W_k) + f^{(4)}(W_k) + \dots$$

Expansions of WKB Functions

In applying this method, one must calculate expansions of various functions of W_k to any desired adiabatic order in terms of $\omega^{(n)}$. Parlticularly, we find power-laws:

$$W^\alpha \sim \left[\omega + \omega^{(2)} + \omega^{(4)} + \ldots\right]^\alpha$$
$$= \omega^\alpha \left[1 + \alpha\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right)^2 + \ldots\right].$$

$$(W^\alpha)^{(0)} = \omega^\alpha, \quad (W^\alpha)^{(2)} = \alpha\frac{\omega^{(2)}}{\omega}\omega^\alpha, \quad (W^\alpha)^{(4)} = \left[\alpha\frac{\omega^{(4)}}{\omega} + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega}\right)^2\right]\omega^\alpha$$

$$(W^{-1})^{(0)} = \omega^{-1}, \quad (W^{-1})^{(2)} = \frac{-\omega^{(2)}}{\omega^2}, \quad (W^{-1})^{(4)} = \frac{-\omega^{(4)}}{\omega^2} + \frac{(\omega^{(2)})^2}{\omega^3}$$

Expansions of WKB Functions

In applying this method, one must calculate expansions of various functions of W_k to any desired adiabatic order in terms of $\omega^{(n)}$. Parlticularly, we find power-laws:

$$W^\alpha \sim \left[\omega + \omega^{(2)} + \omega^{(4)} + ...\right]^\alpha$$
$$= \omega^\alpha \left[1 + \alpha\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + ...\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \dots\right)^2 + ...\right].$$

$$(W^\alpha)^{(0)} = \omega^\alpha, \quad (W^\alpha)^{(2)} = \alpha\frac{\omega^{(2)}}{\omega}\omega^\alpha, \quad (W^\alpha)^{(4)} = \left[\alpha\frac{\omega^{(4)}}{\omega} + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega}\right)^2\right]\omega^\alpha$$

$$(W^{-1})^{(0)} = \omega^{-1}, \quad (W^{-1})^{(2)} = \frac{-\omega^{(2)}}{\omega^2}, \quad (W^{-1})^{(4)} = \frac{-\omega^{(4)}}{\omega^2} + \frac{(\omega^{(2)})^2}{\omega^3}$$

Expansions of WKB Functions

In applying this method, one must calculate expansions of various functions of W_k to any desired adiabatic order in terms of $\omega^{(n)}$. Parlticularly, we find power-laws:

$$W^\alpha \sim \left[\omega + \omega^{(2)} + \omega^{(4)} + \ldots\right]^\alpha$$
$$= \omega^\alpha \left[1 + \alpha\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right)^2 + \ldots\right].$$

$$(W^\alpha)^{(0)} = \omega^\alpha, \quad (W^\alpha)^{(2)} = \alpha\frac{\omega^{(2)}}{\omega}\omega^\alpha, \quad (W^\alpha)^{(4)} = \left[\alpha\frac{\omega^{(4)}}{\omega} + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega}\right)^2\right]\omega^\alpha$$

$$(W^{-1})^{(0)} = \omega^{-1}, \quad (W^{-1})^{(2)} = \frac{-\omega^{(2)}}{\omega^2}, \quad (W^{-1})^{(4)} = \frac{-\omega^{(4)}}{\omega^2} + \frac{(\omega^{(2)})^2}{\omega^3}$$

Expansions of WKB Functions

In applying this method, one must calculate expansions of various functions of W_k to any desired adiabatic order in terms of $\omega^{(n)}$. Parlticularly, we find power-laws:

$$W^\alpha \sim \left[\omega + \omega^{(2)} + \omega^{(4)} + \ldots\right]^\alpha$$
$$= \omega^\alpha \left[1 + \alpha\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right)^2 + \ldots\right].$$

$$(W^\alpha)^{(0)} = \omega^\alpha, \quad (W^\alpha)^{(2)} = \alpha\frac{\omega^{(2)}}{\omega}\omega^\alpha, \quad (W^\alpha)^{(4)} = \left[\alpha\frac{\omega^{(4)}}{\omega} + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega}\right)^2\right]\omega^\alpha$$

$$(W^{-1})^{(0)} = \omega^{-1}, \quad (W^{-1})^{(2)} = \frac{-\omega^{(2)}}{\omega^2}, \quad (W^{-1})^{(4)} = \frac{-\omega^{(4)}}{\omega^2} + \frac{(\omega^{(2)})^2}{\omega^3}$$

Expansions of WKB Functions

In applying this method, one must calculate expansions of various functions of W_k to any desired adiabatic order in terms of $\omega^{(n)}$. Parlticularly, we find power-laws:

$$W^\alpha \sim \left[\omega + \omega^{(2)} + \omega^{(4)} + \ldots\right]^\alpha$$
$$= \omega^\alpha \left[1 + \alpha\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right)^2 + \ldots\right].$$

$$(W^\alpha)^{(0)} = \omega^\alpha, \quad (W^\alpha)^{(2)} = \alpha\frac{\omega^{(2)}}{\omega}\omega^\alpha, \quad (W^\alpha)^{(4)} = \left[\alpha\frac{\omega^{(4)}}{\omega} + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega}\right)^2\right]\omega^\alpha$$

$$(W^{-1})^{(0)} = \omega^{-1}, \quad (W^{-1})^{(2)} = \frac{-\omega^{(2)}}{\omega^2}, \quad (W^{-1})^{(4)} = \frac{-\omega^{(4)}}{\omega^2} + \frac{(\omega^{(2)})^2}{\omega^3}$$

Expansions of WKB Functions

In applying this method, one must calculate expansions of various functions of W_k to any desired adiabatic order in terms of $\omega^{(n)}$. Parlticularly, we find power-laws:

$$W^\alpha \sim \left[\omega + \omega^{(2)} + \omega^{(4)} + \dots\right]^\alpha$$
$$= \omega^\alpha \left[1 + \alpha\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \dots\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \dots\right)^2 + \dots\right].$$

$$(W^\alpha)^{(0)} = \omega^\alpha, \quad (W^\alpha)^{(2)} = \alpha\frac{\omega^{(2)}}{\omega}\omega^\alpha, \quad (W^\alpha)^{(4)} = \left[\alpha\frac{\omega^{(4)}}{\omega} + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega}\right)^2\right]\omega^\alpha$$

$$(W^{-1})^{(0)} = \omega^{-1}, \qquad (W^{-1})^{(2)} = \frac{-\omega^{(2)}}{\omega^2}, \qquad (W^{-1})^{(4)} = \frac{-\omega^{(4)}}{\omega^2} + \frac{(\omega^{(2)})^2}{\omega^3}$$

Expansions of WKB Functions

In applying this method, one must calculate expansions of various functions of W_k to any desired adiabatic order in terms of $\omega^{(n)}$. Parlticularly, we find power-laws:

$$W^{\alpha} \sim \left[\omega + \omega^{(2)} + \omega^{(4)} + \dots\right]^{\alpha}$$
$$= \omega^{\alpha}\left[1 + \alpha\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \dots\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \dots\right)^{2} + \dots\right].$$

$$(W^{\alpha})^{(0)} = \omega^{\alpha}, \quad (W^{\alpha})^{(2)} = \alpha\frac{\omega^{(2)}}{\omega}\omega^{\alpha}, \quad (W^{\alpha})^{(4)} = \left[\alpha\frac{\omega^{(4)}}{\omega} + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega}\right)^{2}\right]\omega^{\alpha}$$

$$(W^{-1})^{(0)} = \omega^{-1}, \qquad (W^{-1})^{(2)} = \frac{-\omega^{(2)}}{\omega^{2}}, \qquad (W^{-1})^{(4)} = \frac{-\omega^{(4)}}{\omega^{2}} + \frac{(\omega^{(2)})^{2}}{\omega^{3}}$$

Expansions of WKB Functions

In applying this method, one must calculate expansions of various functions of W_k to any desired adiabatic order in terms of $\omega^{(n)}$. Parlticularly, we find power-laws:

$$W^\alpha \sim \left[\omega + \omega^{(2)} + \omega^{(4)} + \ldots\right]^\alpha$$
$$= \omega^\alpha \left[1 + \alpha\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega} + \frac{\omega^{(4)}}{\omega} + \ldots\right)^2 + \ldots\right].$$

$$(W^\alpha)^{(0)} = \omega^\alpha, \quad (W^\alpha)^{(2)} = \alpha\frac{\omega^{(2)}}{\omega}\omega^\alpha, \quad (W^\alpha)^{(4)} = \left[\alpha\frac{\omega^{(4)}}{\omega} + \frac{\alpha(\alpha-1)}{2}\left(\frac{\omega^{(2)}}{\omega}\right)^2\right]\omega^\alpha$$

$$(W^{-1})^{(0)} = \omega^{-1}, \qquad (W^{-1})^{(2)} = \frac{-\omega^{(2)}}{\omega^2}, \qquad (W^{-1})^{(4)} = \frac{-\omega^{(4)}}{\omega^2} + \frac{(\omega^{(2)})^2}{\omega^3}$$

A very convenient way to visualize these spaces is by considering a 4-dimensional hyperboloid embedded in a 5-dimensional flat Lorentzian space. If this embedding space is covered with Cartesian coordinates (T, X, Y, Z, W), such that its line element is:

$$dS^2 = dT^2 - dX^2 - dY^2 - dZ^2 - dW^2, \tag{17}$$

the hypersurface that represents a de Sitter space can be written by the equation:

$$T^2 - X^2 - Y^2 - Z^2 - W^2 = -H^{-2}. \tag{18}$$

Among the many coordinates we can use to cover (portions of) de Sitter spaces, we can use exponentially expanding ones:

$$ds^2 = dt^2 - e^{2Ht}(dx^2 + dy^2 + dz^2)$$

Field equations

We can cover de Sitter spaces with exponentially expanding FLRW coordinates:

$$ds^2 = dt^2 - e^{2Ht} d\mathbf{x}^2$$

which results in simple field equations:

$$\partial_t^2 \phi + 3H\partial_t \phi - e^{-2Ht}\nabla^2 \phi + M^2 \phi = 0$$

with effective mass $M^2 = m^2 + 12\xi H^2$.

General field solutions:

$$h_k(t) = \sqrt{\frac{\pi}{2H}} \left(E(k) H_\nu^{(2)}(v) + F(k) H_\nu^{(1)}(v) \right), \qquad \nu \equiv \left(\frac{9}{4} - \frac{M^2}{H^2} \right)^{\frac{1}{2}}$$

Field equations

We can cover de Sitter spaces with exponentially expanding FLRW coordinates:

$$ds^2 = dt^2 - e^{2Ht}d\mathbf{x}^2$$

which results in simple field equations:

$$\partial_t^2\phi + 3H\partial_t\phi - e^{-2Ht}\nabla^2\phi + M^2\phi = 0$$

with effective mass $M^2 = m^2 + 12\xi H^2$.

General field solutions:

$$h_k(t) = \sqrt{\frac{\pi}{2H}}\left(E(k)H_\nu^{(2)}(v) + F(k)H_\nu^{(1)}(v)\right), \qquad \nu \equiv \left(\frac{9}{4} - \frac{M^2}{H^2}\right)^{\frac{1}{2}}$$

Adiabatic Field Modes and *Bunch-Davies* Vacuum

Then imposing the adiabatic condition and requiring invariance under de Sitter symmetries, we arrive at positive-frequency solutions:

$$h_k \sim (2ke^{-Ht})^{-1/2} \exp\left(-i\int^t ke^{-Ht'} dt'\right) \quad \Rightarrow \quad h_k(t) = \sqrt{\frac{\pi}{2H}} H_\nu^{(1)}\left(\frac{k}{H}e^{-Ht}\right)$$

These solutions allow us to construct a distinguished notion of positive-frequency modes:

$$f_{\mathbf{k}}(x) = \sqrt{\frac{\pi}{2H}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2(2\pi)^3 e^{3Ht}}} H_\nu^{(1)}\left(\frac{k}{H}e^{-Ht}\right)$$

in terms of which we define a distinguished vacuum state: the *Bunch-Davies vacuum*.

Adiabatic Field Modes and *Bunch-Davies* Vacuum

Then imposing the adiabatic condition and requiring invariance under de Sitter symmetries, we arrive at positive-frequency solutions:

$$h_k \sim (2ke^{-Ht})^{-1/2} \exp\left(-i \int^t k e^{-Ht'} dt'\right) \qquad \Rightarrow \qquad h_k(t) = \sqrt{\frac{\pi}{2H}} H_\nu^{(1)}\left(\frac{k}{H} e^{-Ht}\right)$$

These solutions allow us to construct a distinguished notion of positive-frequency modes:

$$f_{\mathbf{k}}(x) = \sqrt{\frac{\pi}{2H}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2(2\pi)^3 e^{3Ht}}} H_\nu^{(1)}\left(\frac{k}{H} e^{-Ht}\right)$$

in terms of which we define a distinguished vacuum state: the *Bunch-Davies vacuum*.

Adiabatic Field Modes and *Bunch-Davies* Vacuum

Then imposing the adiabatic condition and requiring invariance under de Sitter symmetries, we arrive at positive-frequency solutions:

$$h_k \sim (2ke^{-Ht})^{-1/2} \exp\left(-i\int^t ke^{-Ht'}dt'\right) \qquad \Rightarrow \qquad h_k(t) = \sqrt{\frac{\pi}{2H}} H_\nu^{(1)}\left(\frac{k}{H}e^{-Ht}\right)$$

These solutions allow us to construct a distinguished notion of positive-frequency modes:

$$f_{\mathbf{k}}(x) = \sqrt{\frac{\pi}{2H}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2(2\pi)^3 e^{3Ht}}} H_\nu^{(1)}\left(\frac{k}{H}e^{-Ht}\right)$$

in terms of which we define a distinguished vacuum state: the *Bunch-Davies vacuum*.

Pathological parameters

We have well behaved parameters only for $1/2 \leq \nu \leq 3/2$.

Images/mathematica/PSu+2(-0,5;0,2).png

The Renormalized Trace

Numerically integrating the the power spectrum we can obtain the stress tensor trace for various parameters:

Images/mathematica/Trace_Table(ugly-line).png

Equilibrium renormalized energy

Starting a definite integral from $\rho = 0$ we find both a decaying transient amplitude $\propto a^{-4}$ and an equilibrium value:

Images/mathematica/time_rhop(0,0;0,1).png Images/mathematica/time_rhop(1

Initial Conditions

We know that the observable universe was highly homogeneous (with tiny density fluctuations at very early times). Then a seemingly natural choice would be to consider a homogeneous ϕ.

However, this choice is extremely particular. Besides, we cannot safely extrapolate our observations earlier than $\sim 10^{-13}s$. More generally, we start by considering limits for a classical description of spacetime:

$$R^2 \lesssim M_p^4 \qquad \Leftrightarrow \qquad (\partial_0\phi)(\partial^0\phi),\, (\partial_i\phi)(\partial^i\phi),\, V(\phi) \lesssim M_p^4$$

- Ignorance and "generic initial conditions" around Planck time $t_P \sim 10^{-43}s$:

$$R^2 \sim M_p^4 \qquad (\partial_0\phi)(\partial^0\phi) \sim (\partial_i\phi)(\partial^i\phi) \sim V(\phi) \sim M_p^4$$

Initial Conditions

We know that the observable universe was highly homogeneous (with tiny density fluctuations at very early times). Then a seemingly natural choice would be to consider a homogeneous ϕ.

However, this choice is extremely particular. Besides, we cannot safely extrapolate our observations earlier than $\sim 10^{-13}s$. More generally, we start by considering limits for a classical description of spacetime:

$$R^2 \lesssim M_p^4 \qquad \Leftrightarrow \qquad (\partial_0\phi)(\partial^0\phi),\ (\partial_i\phi)(\partial^i\phi),\ V(\phi) \lesssim M_p^4$$

- Ignorance and "generic initial conditions" around Planck time $t_P \sim 10^{-43}s$:

$$R^2 \sim M_p^4 \qquad (\partial_0\phi)(\partial^0\phi) \sim (\partial_i\phi)(\partial^i\phi) \sim V(\phi) \sim M_p^4$$

Initial Conditions

We know that the observable universe was highly homogeneous (with tiny density fluctuations at very early times). Then a seemingly natural choice would be to consider a homogeneous ϕ.

However, this choice is extremely particular. Besides, we cannot safely extrapolate our observations earlier than $\sim 10^{-13}s$. More generally, we start by considering limits for a classical description of spacetime:

$$R^2 \lesssim M_p^4 \qquad \Leftrightarrow \qquad (\partial_0\phi)(\partial^0\phi),\, (\partial_i\phi)(\partial^i\phi),\, V(\phi) \lesssim M_p^4$$

- Ignorance and "generic initial conditions" around Planck time $t_P \sim 10^{-43}s$:

$$R^2 \sim M_p^4 \qquad (\partial_0\phi)(\partial^0\phi) \sim (\partial_i\phi)(\partial^i\phi) \sim V(\phi) \sim M_p^4$$

Initial Conditions

We know that the observable universe was highly homogeneous (with tiny density fluctuations at very early times). Then a seemingly natural choice would be to consider a homogeneous ϕ.

However, this choice is extremely particular. Besides, we cannot safely extrapolate our observations earlier than $\sim 10^{-13}s$. More generally, we start by considering limits for a classical description of spacetime:

$$R^2 \lesssim M_p^4 \qquad \Leftrightarrow \qquad (\partial_0\phi)(\partial^0\phi),\, (\partial_i\phi)(\partial^i\phi),\, V(\phi) \lesssim M_p^4$$

- Ignorance and "generic initial conditions" around Planck time $t_P \sim 10^{-43}s$:

$$R^2 \sim M_p^4 \qquad (\partial_0\phi)(\partial^0\phi) \sim (\partial_i\phi)(\partial^i\phi) \sim V(\phi) \sim M_p^4$$

Initial Conditions

We know that the observable universe was highly homogeneous (with tiny density fluctuations at very early times). Then a seemingly natural choice would be to consider a homogeneous ϕ.

However, this choice is extremely particular. Besides, we cannot safely extrapolate our observations earlier than $\sim 10^{-13}s$. More generally, we start by considering limits for a classical description of spacetime:

$$R^2 \lesssim M_p^4 \qquad \Leftrightarrow \qquad (\partial_0\phi)(\partial^0\phi),\ (\partial_i\phi)(\partial^i\phi),\ V(\phi) \lesssim M_p^4$$

- Ignorance and "generic initial conditions" around Planck time $t_P \sim 10^{-43}s$:

$$R^2 \sim M_p^4 \qquad (\partial_0\phi)(\partial^0\phi) \sim (\partial_i\phi)(\partial^i\phi) \sim V(\phi) \sim M_p^4$$

Initial Conditions

We know that the observable universe was highly homogeneous (with tiny density fluctuations at very early times). Then a seemingly natural choice would be to consider a homogeneous ϕ.

However, this choice is extremely particular. Besides, we cannot safely extrapolate our observations earlier than $\sim 10^{-13}s$. More generally, we start by considering limits for a classical description of spacetime:

$$R^2 \lesssim M_p^4 \qquad \Leftrightarrow \qquad (\partial_0\phi)(\partial^0\phi),\, (\partial_i\phi)(\partial^i\phi),\, V(\phi) \lesssim M_p^4$$

- Ignorance and "generic initial conditions" around Planck time $t_P \sim 10^{-43}s$:

$$R^2 \sim M_p^4 \qquad (\partial_0\phi)(\partial^0\phi) \sim (\partial_i\phi)(\partial^i\phi) \sim V(\phi) \sim M_p^4$$

Initial Conditions and Chaotic Inflation

For roughly random initial conditions at $t \sim t_p$, we can obtain an inflationary region from a single roughly homogeneous region of initial diameter $\sim 2l_p$, since:

- In de Sitter spaces, every observer is surrounded by an event horizon at distance H^{-1}; inhomogeneities falling out of the horizon lose causal contact from an inflating region. ("No hair" theorems.)
- Particularly, for $V(\phi) \sim M_p^4$, the Hubble radius becomes extremely small $H^{-1}(\phi) \sim M_p^{-1} \sim l_p$.
- Then, in a region where ϕ has small spacetime variations, $V(\phi) \approx cte$, and $T_{ab} \approx V(\phi) g_{ab} \approx \Lambda g_{ab}$.

Initial Conditions and Chaotic Inflation

For roughly random initial conditions at $t \sim t_p$, we can obtain an inflationary region from a single roughly homogeneous region of initial diameter $\sim 2l_p$, since:

- In de Sitter spaces, every observer is surrounded by an event horizon at distance H^{-1}; inhomogeneities falling out of the horizon lose causal contact from an inflating region. ("No hair" theorems.)
- Particularly, for $V(\phi) \sim M_p^4$, the Hubble radius becomes extremely small $H^{-1}(\phi) \sim M_p^{-1} \sim l_p$.
- Then, in a region where ϕ has small spacetime variations, $V(\phi) \approx cte$, and $T_{ab} \approx V(\phi) g_{ab} \approx \Lambda g_{ab}$.

Initial Conditions and Chaotic Inflation

For roughly random initial conditions at $t \sim t_p$, we can obtain an inflationary region from a single roughly homogeneous region of initial diameter $\sim 2l_p$, since:

- In de Sitter spaces, every observer is surrounded by an event horizon at distance H^{-1}; inhomogeneities falling out of the horizon lose causal contact from an inflating region. ("No hair" theorems.)
- Particularly, for $V(\phi) \sim M_p^4$, the Hubble radius becomes extremely small $H^{-1}(\phi) \sim M_p^{-1} \sim l_p$.
- Then, in a region where ϕ has small spacetime variations, $V(\phi) \approx cte$, and $T_{ab} \approx V(\phi)g_{ab} \approx \Lambda g_{ab}$.

Initial Conditions and Chaotic Inflation

For roughly random initial conditions at $t \sim t_p$, we can obtain an inflationary region from a single roughly homogeneous region of initial diameter $\sim 2l_p$, since:

- In de Sitter spaces, every observer is surrounded by an event horizon at distance H^{-1}; inhomogeneities falling out of the horizon lose causal contact from an inflating region. ("No hair" theorems.)
- Particularly, for $V(\phi) \sim M_p^4$, the Hubble radius becomes extremely small $H^{-1}(\phi) \sim M_p^{-1} \sim l_p$.
- Then, in a region where ϕ has small spacetime variations, $V(\phi) \approx cte$, and $T_{ab} \approx V(\phi) g_{ab} \approx \Lambda g_{ab}$.

Symmetry Breaking?

Out of league.

We can draw rough estimates for power-law potentials:

$$V(\phi) = \frac{\lambda_n \phi^n}{n M_p^{n-4}}, \quad \lambda_n \ll M_p^{n-4} \qquad V(\phi_0) \sim M_p^4 \quad \Rightarrow \quad \phi_0 \gg M_p^4$$

Slow-roll scale factor:

$$a(t) = a_0 e^{\frac{4\pi}{n M_p^2}\left(\phi_0^2 - \phi^2(t)\right)}$$

Beamer Junk

$$a = b$$
$$E = mc^2 + \int_a^a x\,dx$$

$$a = b$$
$$E = mc^2 + \int_a^a x\,dx$$

$$a = b$$
$$E = mc^2 + \int_a^a x\,dx$$

$$a = b$$
$$E = mc^2 + \int_a^a x\,dx$$

Pestana

CPF Cancelado

Text

Now Let's write something with a litte more content to it. We want to see multiple lines in a paragraph.

And we want to see multiple paragraphs.

Deus é Deus e as planta cura. Graças a Jesus, que ama todas as famílias.

Let us also add some small text to this paragraph. This is so small I may have to get repetitive. Let us also add some small text to this paragraph. This is so small I may have to get repetitive. Let us also add some small text to this paragraph. This is so small I may have to get repetitive. Let us also add some small text to this paragraph. This is so small I may have to get repetitive.

Lists

1. Item 1
2. Item 2
3. Item 3

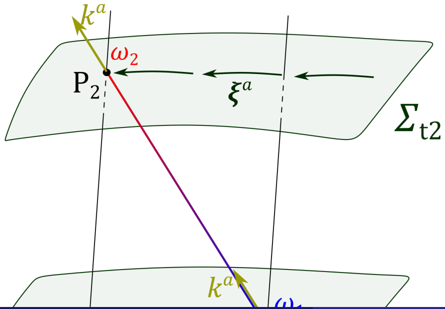
- Item 1
- Item 2
- Item 3

Columned Slide

1. Item 1
2. Item 2
3. Item 3

1. Item 1
2. Item 2
3. Item 3

Text?


k^a
ω_2
P_2
ξ^a
Σ_{t2}
k^a

$$\int_0^1 f(x)dx = \infty. \tag{19}$$