Quantum Conformal Symmetry

N=4 Super-Yang-Mills theory

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30 de junho de 2023

Content

Content

Formulation of $\mathcal{N} = 4$ Super Yang-Mills

- Begin with a pure Yang-Mills theory:

$$S = \int \mathrm{tr} \left(F_{MN} F^{MN} \right.$$

$A \in \Omega^1 \otimes \mathfrak{su}(N)$ is a bosonic gauge field, and $F = DA = dA - ig[A, A]$

Formulation of $\mathcal{N} = 4$ Super Yang-Mills

- Add a **supersymetric** partner to the boson field :

$$S_{\mathcal{N}=1} = \int \mathrm{Tr}\left(F_{MN}F^{MN} + i\bar{\Psi}\Gamma^M D_M \Psi\right),$$

$A \in \Omega^1 \otimes \mathfrak{su}(N)$ is a bosonic gauge field, and $F = DA = dA - ig[A, A]$ Ψ^a Majorana-Weyl spinor, lie algebra valued; Such that the map:

$$\delta A_M^a = \bar{\epsilon}\Gamma_M \Psi^a \tag{1}$$

$$\delta \Psi^a = \frac{i}{2} F_{MN}^a \Gamma^{MN} \epsilon, \tag{2}$$

is a symmetry (fermionic!)

Formulation of $\mathcal{N} = 4$ Super Yang-Mills

- Put it in 10 dimensions:

$$S_{\mathcal{N}=1,10d} = \int d^{10}x \, \mathrm{Tr}\left(F_{MN}F^{MN} + i\bar{\Psi}\Gamma^{M}D_{M}\Psi\right),$$

$A \in \Omega^1 \otimes \mathfrak{su}(N)$ is a bosonic gauge field, and $F = DA = dA - ig[A, A]$
Ψ^a Majorana-Weyl spinor $(2^{\frac{10}{2}-1})$, lie algebra valued;

- Global symmetries: Lorentz $SO(1,9)$ and $\mathcal{N} = 1$ susy:

$$\delta A^a_M = \bar{\epsilon}\Gamma_M \Psi^a \tag{3}$$

$$\delta\Psi^a = \frac{i}{2}F^a_{MN}\Gamma^{MN}\epsilon, \tag{4}$$

Dimensional reduction (to 4d)

$$\mathbb{R}^{10} \longrightarrow \mathbb{R}^4 \oplus \mathbb{R}^6 \nearrow cte \quad : \qquad x^M = (x^\mu, x^i), \qquad \begin{array}{l} \mu = 1, \cdots, 4 \\ i = 1, \cdots, 6 \end{array}$$

- Dimensional reduction corresponds to fix $x^i = \text{cte}$, i.e. $\partial_i = 0$:

$$A_M = (A_\mu, \phi_i) \quad \text{and then,} \quad F_{MN} = (F_{\mu\nu}, D_\mu \phi_i, \ [\phi_i, \phi_j])$$

$\Rightarrow$ **4d gauge boson** $+$ **6 real scalars**;

- Breaks $SO(9,1) \rightarrow \underbrace{SO(3,1)}_{\text{Lorentz}} \oplus \underbrace{SO(6)}_{\text{R-Symmetry}}$

$$\downarrow$$

$SO(4,2)$: extension of the Lorentz group

- $(C_{10})_{32\times 32} = (C_4)_{4\times 4} \otimes \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix}$, such that $\Psi^T = C_{10}\bar{\Psi}$;
  Dimensional reduction: write $\Psi_\Pi = \psi_{iA}$ with $A, i = 1, \cdots, 4$;

  $$\Psi^T = C_{10}\bar{\Psi} \Rightarrow (\psi^A)^T = C_4(\bar{\psi}^A)$$

  This can be seen as **4 Weyl spinors** $\Psi^A_\alpha, \bar{\Psi}^A_{\dot{\alpha}}$, in 4d.
- The 6 scalar fields ϕ_i are in the fundamental representation of $SO(6)$, or in the anti-symmetric representation of $SU(4)$, which relation is established by the Clebsch-Gordan coefficients:

  $$\phi_{AB} = \gamma^n_{AB}\phi_n, \tag{5}$$

  with the reality condition given by $\phi^\dagger_{AB} = \epsilon^{ABCD}\phi_{CD}$.

$\mathcal{N} = 4$ Super Yang-Mills Lagrangian

The action will be:

$$S_{\mathcal{N}=4,4d} = \int d^4x \, \mathrm{tr} \Big(F_{\mu\nu}F^{\mu\nu} + D_\mu \phi_i D^\mu \phi_i + i\bar{\psi}_A \not{D} \psi^A$$

$$-g\gamma^j_{AB}\bar{\psi}^A[\phi_j, \psi^B] - g^2[\phi_i, \phi_j][\phi^i, \phi^j] \Big) \tag{6}$$

- Symmetry ($\mathcal{N} = 1 \longrightarrow \mathcal{N} = 4$):

$$\delta A_\mu = \bar{\epsilon}_A \Gamma_\mu \psi^A + \epsilon^A \Gamma_\mu \bar{\psi}_A$$
$$\delta \phi_{AB} = 2\bar{\epsilon}_{[A}\psi_{B]} + 2\epsilon_{[A}\bar{\psi}_{B]}$$
$$\delta \psi^A = -\frac{1}{2}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon^A - 2\Gamma^\mu D_\mu \phi^{AB}\epsilon_B + 2g[\phi^A, \phi^B]\epsilon_B$$

$A, B = 1, 2, 3, 4 \Rightarrow \mathcal{N} = 4$ susy's!

Content

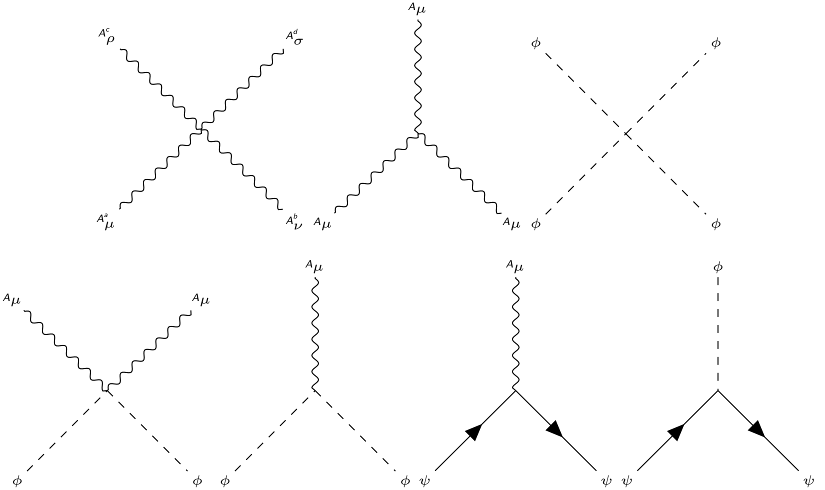
$$S_{\mathcal{N}=4,4d} = \int d^4x \;\; \frac{1}{g^2}\,\mathrm{Tr}\Big(F_{\mu\nu}F^{\mu\nu} + D_\mu\phi_i D^\mu\phi_i + i\bar{\psi}_A \not{D}\psi^A - \bar{\psi}^A[\phi_{AB},\psi^B] - [\phi_i,\phi_j][\phi^i,\phi^j]\Big) \tag{7}$$

- Field content:

$$A^a_\mu \;\; \sim\sim\sim \qquad \phi^a_i \;\; \text{------} \qquad \psi^a_A \;\; \longrightarrow \tag{8}$$

All in the adjoint representation $\Rightarrow$ $\boxed{C(r) = N}$

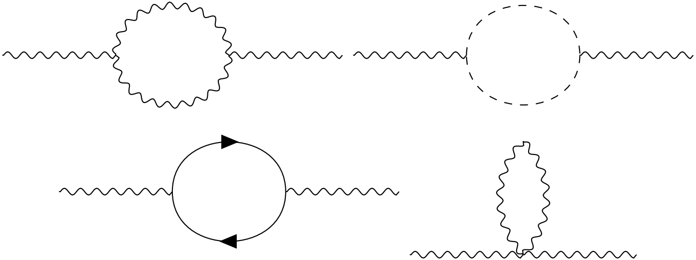
Feynman rules



(9)

$\beta-$function at 1-loop

$$(A^a_\mu)_r = Z_A^{-1/2}(A^a_\mu)_0, \quad (\phi^a_i)_r = Z_\phi^{-1/2}(\phi^a_i)_0$$



β−function at 1-loop

$$\Gamma_r^{abc}(p,q,r) = Z_1^{-1}\Gamma_0^{abc}(p,q,r) \tag{10}$$

$\beta-$function at 1-loop

$$\implies \quad \beta_1(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N - n_s\frac{1}{6}N - n_f\frac{1}{3}N\right) \tag{11}$$

n_s: number of scalars, n_f: number of fermions.

$$n_s = 6, n_f = 8 \;\Rightarrow\; \beta_1 = 0$$

Content

Symmetry of the theory

- The theory has $\mathcal{N} = 4$ super-conformal symmetry:

$$\underbrace{SO(2,4)}_{\text{Conformal}} \oplus \underbrace{SO(6)}_{\text{R-sym}} \oplus \underbrace{\mathbb{C}^{0|32}}_{\text{Super-transformations}} \tag{12}$$

$P_\mu, M_{\mu\nu}, K_\mu, D, R^I{}_J$ (bosonic generators)

$Q_{I\alpha}, \bar{Q}^I_{\dot\alpha}, S^{I\alpha}, \bar{S}^{\dot\alpha}_I$ (fermionic generators)

Conformal group

$$[D, P_\mu] = -P_\mu \ , \quad [D, M_{\mu\nu}] = 0 \ , \quad [D, K_\mu] = +K_\mu$$
$$[M_{\mu\nu}, P_\rho] = -(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \ , \quad [M_{\mu\nu}, K_\rho] = -(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) \quad (13)$$
$$[P_\mu, K_\nu] = 2(M_{\mu\nu} - \eta_{\mu\nu} D)$$

These generators acts in local operators as commutation.

Definition

A local operator $\mathcal{O}_\Delta(x)$ has a conformal dimension Δ if:

$$\mathcal{O}_\Delta(\lambda x) = \lambda^\Delta \mathcal{O}(x) \qquad \text{or} \qquad [D, \mathcal{O}_\Delta](x) = -\Delta \mathcal{O}(x) \quad (14)$$

$[K_\mu, \mathcal{O}_\Delta]$ has dimension $\Delta + 1$, $[P_\mu, \mathcal{O}_\Delta]$ has dimension $\Delta + 1$:

$$[D, [K_\mu, \mathcal{O}_\Delta]] = [[D, K_\mu], \mathcal{O}_\Delta] + [K_\mu, [D, \mathcal{O}_\Delta]] = -(\Delta - 1)[K_\mu, \mathcal{O}_\Delta]$$

$$[D, [P_\mu, \mathcal{O}_\Delta]] = [[D, P_\mu], \mathcal{O}_\Delta] + [P_\mu, [D, \mathcal{O}_\Delta]] = -(\Delta + 1)[P_\mu, \mathcal{O}_\Delta]$$

Super-conformal algebra

$$\{Q, \bar{Q}\} = 2P \ , \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0$$
$$\{S, \bar{S}\} = 2K \ , \quad \{S, S\} = \{\bar{S}, \bar{S}\} = 0 \tag{15}$$
$$\{Q, S\} = -R + \Gamma^{\mu\nu} M_{\mu\nu} - \frac{1}{2} D \ , \quad \{\bar{Q}, \bar{S}\} = -R + \bar{\Gamma}^{\mu\nu} M_{\mu\nu} - \frac{1}{2} D$$

$$[D, Q] = -\frac{1}{2} Q \ , \quad [D, \bar{Q}] = -\frac{1}{2} \bar{Q}$$
$$[D, S] = +\frac{1}{2} S \ , \quad [D, \bar{S}] = +\frac{1}{2} \bar{S} \tag{16}$$

$\Longrightarrow$ $Q, \bar{Q}$ upper the dimension by $1/2$ and $S, \bar{S}$ lowers the dimension by $1/2 \Rightarrow$ Chain of operators.

Definition

Primary Operators $\tilde{\mathcal{O}}$ are the lower bounds for a chain of operators, that is:

$$[S, \tilde{\mathcal{O}}] = [\bar{S}, \tilde{\mathcal{O}}] = 0 \tag{17}$$

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A Primary Operator $\tilde{\mathcal{O}}$ is called **chiral** operator if it is killed by some of the super-symmetries:

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- Consequences of being BPS:

$$[\{Q_{I\alpha}, S^{J\beta}\}, \mathcal{O}] = [-\epsilon_{\alpha\beta} R_I{}^J - \epsilon_{\alpha\beta}\delta_I{}^J + \delta_I{}^J \Gamma^{\mu\nu}_{\alpha\beta} M_{\mu\nu}, \mathcal{O}] \tag{19}$$

scalar

$$\therefore \boxed{[R^I_J, \tilde{\mathcal{O}}] = \Delta \delta^I_J \tilde{\mathcal{O}}} \tag{20}$$

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- Claim: The $\mathcal{N} = 4$ SYM Lagrangian is given by:

$$\mathcal{L}_{N=4} = S \circ \tilde{\mathcal{O}}$$

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- $\mathcal{L}_{N=4}$ is in the same supermultiplet as the chiral primary $\Rightarrow$ dimension protected from quantum corrections!

Quantization

- Define $\{a^\alpha, b^{\dot\alpha}, c_I\}$ with:

$$[a^\alpha, a^\dagger_\beta] = \delta^\alpha_\beta, \quad [b^{\dot\alpha}, b^\dagger_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}, \quad \{c_I, c^{\dagger J}\} = \delta^J_I \tag{21}$$

- Write the field strength as $\mathcal{F}_{\alpha\beta} = \sigma^{\mu\nu}_{\alpha\beta} F_{\mu\nu}$ and $\mathcal{F}_{\dot\alpha\dot\beta} = \sigma^{\mu\nu}_{\dot\alpha\dot\beta} F_{\mu\nu}$; Covariant derivatives as $\mathcal{D}_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha} D_\mu$

$$\begin{aligned}
\mathcal{D}_{\alpha\dot\alpha} &= a^\dagger_\alpha b^\dagger_{\dot\alpha}|0\rangle \\
\mathcal{F}_{\alpha\beta} &= a^\dagger_\alpha a^\dagger_\beta|0\rangle \\
\psi^I_\alpha &= (c^I)^\dagger a^\dagger_\alpha|0\rangle \\
\phi_{IJ} &= (c^I)^\dagger (c^J)^\dagger|0\rangle \\
\psi_{\dot\alpha I} &= \epsilon_{IJKL}(c^J)^\dagger (c^K)^\dagger (c^L)^\dagger b^\dagger_{\dot\alpha}|0\rangle \\
\mathcal{F}_{\dot\alpha\dot\beta} &= \epsilon_{IJKL}(c^I)^\dagger (c^J)^\dagger (c^K)^\dagger (c^L)^\dagger b^\dagger_{\dot\alpha} b^\dagger_{\dot\beta}|0\rangle
\end{aligned}$$

Quantization

- Write the generators in the same way

$$P_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} P_{\mu}, \quad K_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} K_{\mu}, \quad M_{\alpha\beta} = \sigma^{\mu\mu}_{\alpha\beta} M_{\mu\nu}, \quad \tilde{M}_{\dot{\alpha}\dot{\beta}} = \sigma^{\mu\mu}_{\dot{\alpha}\dot{\beta}} M_{\mu\nu}$$

such that:

$$P_{\alpha\dot{\alpha}} = a^{\dagger}_{\alpha} b^{\dagger}_{\dot{\alpha}}, \quad K^{\dot{\alpha}\alpha} = b^{\dot{\alpha}} a^{\alpha}$$

$$Q_{I\alpha} = a^{\dagger}_{\alpha} c_I, \quad \bar{Q}^{I}_{\dot{\alpha}} = b^{\dagger}_{\dot{\alpha}} c^{\dagger I}, \quad S^{I\alpha} = a^{\alpha} c^{\dagger I}, \quad \bar{S}^{\dot{\alpha}}_{I} = b^{\dot{\alpha}} c_I,$$

$$R^{I}{}_{J} = c^{\dagger I} c_J - \frac{1}{4} \delta^{I}_{J} c^{\dagger K} c_K$$

$$D = -\frac{1}{2}(a^{\dagger}_{\alpha} a^{\alpha} + b^{\dagger}_{\dot{\alpha}} b^{\dot{\alpha}} + 2)$$

$\beta = 0$: exact proof

- The following operator:

$$\mathcal{O}^{IJKL} = \mathrm{Tr}(\phi^{IJ}\phi^{KL}) - \frac{1}{4!}\epsilon^{IJKL}\epsilon_{MNOP}\,\mathrm{Tr}(\phi^{MN}\phi^{OP}) \qquad (22)$$

is Chiral $\Delta = 2 = J$!

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- The Lagrangian can be obtained from this operator:

$$\frac{1}{4!}\{Q_{\alpha I},[Q_{\beta J},\{Q_{\gamma K},[Q_{\delta L},\mathcal{O}^{IJKL}]\}]\} = \mathrm{Tr}(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta}) \tag{23}$$

$$\frac{1}{4!}\{\bar{Q}_{\dot{\alpha} I},[\bar{Q}_{\dot{\beta} J},\{\bar{Q}_{\dot{\gamma} K},[\bar{Q}_{\dot{\delta} L},\mathcal{O}^{IJKL}]\}]\} = \mathrm{Tr}(\mathcal{F}_{\dot{\alpha}\dot{\beta}}\mathcal{F}_{\dot{\gamma}\dot{\delta}}) \tag{24}$$

$$\ldots$$

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$$\cdots$$

- The Lagrangian has fixed dimension: $\Delta = 4$

$$\mathcal{L} = \frac{1}{g_{YM}^2} \mathrm{Tr}(\mathcal{F}\mathcal{F}) + \cdots \tag{25}$$

$\implies$ the coupling g_{YM} has fixed dimension

$$.: \boxed{\beta(g_{YM}) = 0} \tag{26}$$

Content

References

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