Introduction to spin-helicity formalism

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Outline

- ➢ Introduction
- ➢ Spinor representation
- ➢ Simple examples (Yukawa and QED)
- ➢ 3-particle kinematics
- ➢ BCFW recursion
- ➢ Proof of Parke-Taylor (very schematic)

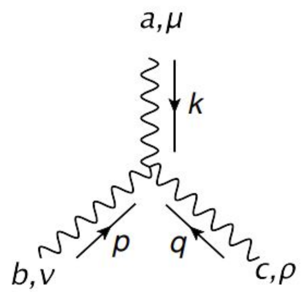
Motivation

For

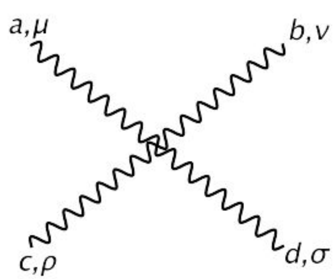
$$g + g \rightarrow g + g$$

$$S = -\frac{1}{4}\int \mathrm{d}^4x F^a_{\mu\nu}F^{a\mu\nu} \longrightarrow \text{Feynman rules} \longrightarrow \overline{|\mathcal{M}|}^2 = \frac{9g_s^4}{2}\left(3 - \frac{su}{t^2} - \frac{ut}{s^2} - \frac{st}{u^2}\right)$$

Using



$$= \quad g\, f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$$

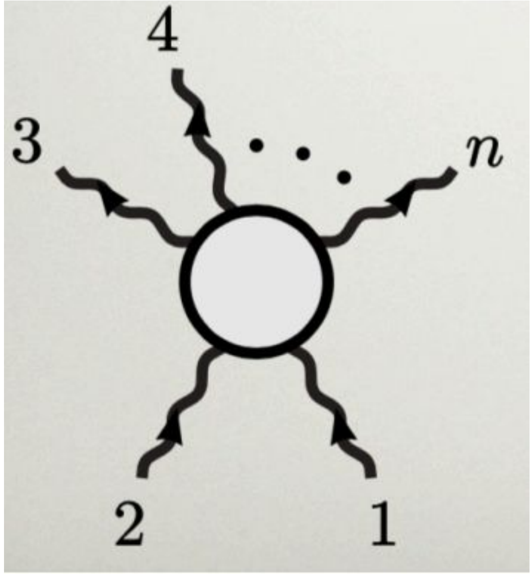


$$\begin{aligned} = \quad & -ig^2\ \left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\right. \\ & + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) \\ & \left. + f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right] \end{aligned}$$

The Parke-Taylor formula

The Maximal Helicity Violation (MHV) amplitude for cyclically ordered n-gluon is :

$$A_n[1^-2^-3^+\dots n^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$



$gg \to 2g$	4
$gg \to 3g$	25
$gg \to 4g$	220
$gg \to 5g$	2485
$gg \to 6g$	34300
$gg \to 7g$	559405
$gg \to 8g$	10525900
$gg \to 9g$	224449225
$gg \to 10g$	5348843500

Everything in terms of spinors

Recall for Dirac spinors

$$(-i\partial\!\!\!/ + m)\Psi = 0$$

with solutions

$$\Psi(x) \;\sim\; u(p)\, e^{ip.x} + v(p)\, e^{-ip.x}$$

$$(p\!\!\!/ + m)u(p) = 0 \qquad \text{and} \qquad (-p\!\!\!/ + m)v(p) = 0$$

when $m = 0$

$$p\!\!\!/\, v_\pm(p) = 0\,, \qquad \bar{u}_\pm(p)\, p\!\!\!/ = 0\,.$$

with $h = \pm 1/2$

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, \qquad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix},$$

$$\overline{u}_-(p) \;=\; \left(0\,,\, \langle p|_{\dot{a}}\right), \qquad \overline{u}_+(p) \;=\; \left([p|^a\,,\, 0\right).$$

Useful bra-ket relations

Both are anti-symmetric

$$\langle p\, q\rangle = -\langle q\, p\rangle\,, \qquad [p\, q] = -[q\, p]$$

For real momenta

$$[p\, q]^* = \langle q\, p\rangle$$

as well as

$$\langle p\, q\rangle\,[p\, q] \;=\; 2\,p\cdot q \;=\; (p+q)^2\,,$$

$$[k|\gamma^\mu|p\rangle \;=\; \langle p|\gamma^\mu|k]\,, \qquad [k|\gamma^\mu|p\rangle^* \;=\; [p|\gamma^\mu|k\rangle$$

$$\langle p|q|k] \;=\; -\langle pq\rangle[qk]\,, \qquad \langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] \;=\; 2\langle 13\rangle[24]$$

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & 0 \end{pmatrix}$$

Vectors in bi-spinor representation

For $p^\mu = (p^0, p^i) = (E, p^i)$ with $p^\mu p_\mu = -m^2$ the momentum bispinor is

$$p_{a\dot{b}} \equiv p_\mu (\sigma^\mu)_{a\dot{b}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix}, \quad \text{where} \quad \not{p} = \begin{pmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{pmatrix}$$

and similarly $p^{\dot{a}b} \equiv p_\mu (\bar{\sigma}^\mu)^{\dot{a}b}$ where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$

Again, when $m = 0$

$$p^{\dot{a}b}|p]_b = 0\,, \qquad p_{a\dot{b}}|p\rangle^{\dot{b}} = 0\,, \qquad [p|^b\, p_{b\dot{a}} = 0\,, \qquad \langle p|_{\dot{b}}\, p^{\dot{b}a} = 0\,.$$

Using the completeness equation: $-\not{p} = |p\rangle[p| + |p]\langle p|$

$$p_{a\dot{b}} = -\,|p]_a\,\langle p|_{\dot{b}}\,, \qquad p^{\dot{a}b} = -\,|p\rangle^{\dot{a}}\,[p|^b\,.$$

Feynman rules

Focusing on **outgoing** massless (anti-)fermions:

- Outgoing fermion with $h = +1/2$: $\bar{u}_+ \longleftrightarrow \left([p|^a\,,\, 0 \right)$
- Outgoing fermion with $h = -1/2$: $\bar{u}_- \longleftrightarrow \left(0\,,\, \langle p|_{\dot{a}} \right)$
- Outgoing anti-fermion with $h = +1/2$: $v_+ \longleftrightarrow \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}$
- Outgoing anti-fermion with $h = -1/2$: $v_- \longleftrightarrow \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}$

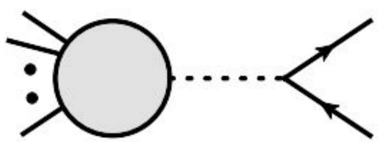
Mnemonic: *square brackets* for + and *angle brackets* for -

Example: Yukawa theory

Recall the Lagrangian

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \tfrac{1}{2}(\partial\phi)^2 + g\phi\overline{\Psi}\Psi$$

When calculation a generic amplitude



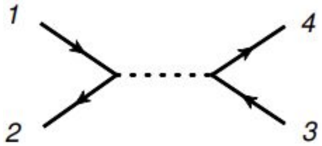
$$= \; ig\,\overline{u}_{h_1}(p_1)v_{h_2}(p_2) \times \frac{-i}{(p_1+p_2)^2} \times (\text{rest})$$

remark it survives only when helicities coincide

$$\begin{aligned}
\overline{u}_+(p_1)v_-(p_2) \;&=\; \big(\,[1|^a\,,\,0\,\big)\begin{pmatrix}0\\ |2\rangle^{\dot a}\end{pmatrix} \;=\; 0\\
\overline{u}_-(p_1)v_-(p_2) \;&=\; \big(\,0\,,\,\langle 1|_{\dot a}\big)\begin{pmatrix}0\\ |2\rangle^{\dot a}\end{pmatrix} \;=\; \langle 1|_{\dot a}|2\rangle^{\dot a} \;\equiv\; \langle 12\rangle\,.
\end{aligned}$$

Yukawa 4-fermion amplitude

The s-channel in terms of uv-products



$$= ig\,\overline{u}_4 v_3 \times \frac{-i}{(p_1+p_2)^2} \times ig\,\overline{u}_2 v_1$$

In spin-brackets

$$iA_4(\bar{f}^- f^- \bar{f}^+ f^+) = ig^2[43]\frac{1}{2p_1.p_2}\langle 21\rangle = ig^2[34]\frac{1}{\langle 12\rangle[12]}\langle 12\rangle = ig^2\frac{[34]}{[12]}$$

But using

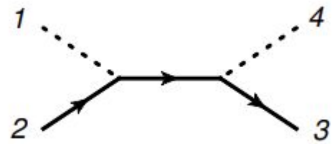
$$\langle 12\rangle[12] = 2p_1.p_2 = (p_1+p_2)^2 = (p_3+p_4)^2 = 2p_3.p_4 = \langle 34\rangle[34]$$

We get analogously

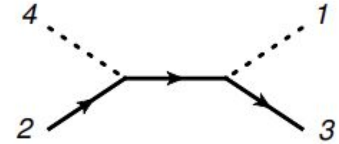
$$A_4(\bar{f}^- f^- \bar{f}^+ f^+) = g^2\frac{\langle 12\rangle}{\langle 34\rangle}$$

The φφff-vertex

$$iA_4(\phi\,\bar{f}^{h_2} f^{h_3}\phi) \;=$$



$$+$$



$$= \;(ig)^2\,\overline{u}_3\,\frac{-i(\not{p}_1+\not{p}_2)}{(p_1+p_2)^2}\,v_2 \quad + \quad (1\leftrightarrow 4)\,.$$

Remark only non-vanishing for “crossed” terms

$$\overline{u}_-(p_3)\gamma^\mu v_+(p_2) \;=\; \left(0\,,\,\langle 3|_{\dot a}\right)\begin{pmatrix} 0 & (\sigma^\mu)_{a\dot b} \\ (\bar\sigma^\mu)^{\dot a b} & 0 \end{pmatrix}\begin{pmatrix} |2]_b \\ 0 \end{pmatrix} \;\equiv\; \langle 3|\gamma^\mu|2]\,.$$

hence

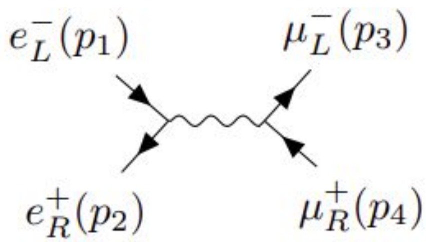
$$\begin{aligned} A_4(\phi\,\bar f^+ f^-\phi) \;&=\; -g^2\frac{\langle 3|p_1+p_2|2]}{(p_1+p_2)^2} + (1\leftrightarrow 4) \;=\; -g^2\frac{\langle 3|p_1|2]}{(p_1+p_2)^2} + (1\leftrightarrow 4) \\ &=\; -g^2\frac{-\langle 31\rangle[12]}{\langle 12\rangle[12]} + (1\leftrightarrow 4) \;=\; -g^2\frac{\langle 13\rangle}{\langle 12\rangle} + (1\leftrightarrow 4)\,, \end{aligned}$$

$$A_4(\phi\,\bar f^+ f^-\phi) \;=\; -\,g^2\left(\frac{\langle 13\rangle}{\langle 12\rangle} + \frac{\langle 34\rangle}{\langle 24\rangle}\right)$$

Example: QED (external fermions)

For the scattering $e_R^+ + e_L^- \longrightarrow \mu_R^+ + \mu_L^- \quad (\bar{f}^- f^- \bar{f}^+ f^+)$

$$i\mathcal{M} =$$



$$= \langle 2(-ie\gamma^\mu)1] \frac{-i\eta_{\mu\nu}}{s} \langle 3(-ie\gamma^\nu)4]$$

$$= \frac{ie^2}{s}\langle 2\gamma^\mu 1]\langle 3\gamma_\mu 4] = \frac{2ie^2}{s}[41]\langle 23\rangle.$$

where we used the Fierz identity (attention for signature convention)

$$\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] \quad = \quad 2\langle 13\rangle[24]$$

Polarization of massless vectors

Photon polarizations can be proven to be given by (most general form)

$$\epsilon_{-}^{\mu}(p;q) = -\frac{\langle p|\gamma^{\mu}|q]}{\sqrt{2}\,[q\,p]}, \qquad \epsilon_{+}^{\mu}(p;q) = -\frac{\langle q|\gamma^{\mu}|p]}{\sqrt{2}\,\langle q\,p\rangle},$$

when changing reference momentum $q \neq p$

$$\epsilon_{\pm}^{\mu}(p) \rightarrow \epsilon_{\pm}^{\mu}(p) + C\,p^{\mu}$$

hence conserves ward identities and it reflects gauge invariance*.

The polarization bi-spinor is a product of angle and square brackets:

$$\not{\epsilon}_{-}(p;q) = \frac{\sqrt{2}}{[qp]}\Big(|p\rangle[q| + |q]\langle p|\Big), \qquad \not{\epsilon}_{+}(p;q) = \frac{\sqrt{2}}{\langle qp\rangle}\Big(|p]\langle q| + |q\rangle[p|\Big)$$

QED 3-vertex

Let's compute $A_3(f^{h_1}\bar{f}^{h_2}\gamma^{h_3})$ for $h_1 = -1/2,\, h_2 = +1/2 \text{ and } h_3 = -1$

$$
\begin{aligned}
iA_3(f^-\bar{f}^+\gamma^-) &= \overline{u}_-(p_1) ie\gamma_\mu v_+(p_2)\, \epsilon_-^\mu(p_3;q) \\
&= -ie\langle 1|\gamma_\mu|2]\, \frac{\langle 3|\gamma^\mu|q]}{\sqrt{2}\,[3\,q]} \\
&= \sqrt{2} ie\, \frac{\langle 13\rangle[2q]}{[3\,q]}
\end{aligned}
$$

Using the conservation of momentum $p_2 = -p_1 - p_3$

$$
\langle 12\rangle[2q] = -\langle 1|p_2|q] = \langle 1|(p_1+p_3)|q] = \langle 1|3|q] = \langle 13\rangle[3q]
$$

Hence

$$
A_3(f^-\bar{f}^+\gamma^-) = \sqrt{2}e\, \frac{\langle 13\rangle^2}{\langle 12\rangle}
$$

depend only on angle brackets!!! (not coincidence)

Little group transformations

In bi-spinor representation, leaves a momentum fixed but

$$|p\rangle \to t|p\rangle\,, \qquad |p] \to t^{-1}|p] \quad \Rightarrow \quad p\rangle[p\,, p]\langle p \text{ invariant.}$$

- Scalar: does not change
- Angle and Square spinor scale as (w.r.t. helicity)

$$t^{-2h} \text{ for } h = \pm\tfrac{1}{2}$$

- Polarization vectors scale as (w.r.t. helicity)

$$t^{-2h} \text{ for } h = \pm 1$$

Therefore, an amplitude

$$\boxed{A_n\big(\{|1\rangle, |1], h_1\}, \ldots, \{t_i|i\rangle, t_i^{-1}|i], h_i\}, \ldots\big) \;=\; t_i^{-2h_i} A_n\big(\ldots\{|i\rangle, |i], h_i\}\ldots\big)\,.}$$

3-particle kinematics

For momentum conservation $p_1^\mu + p_2^\mu + p_3^\mu = 0$, thus

$$\langle 12\rangle[12] = 2p_1.p_2 = (p_1+p_2)^2 = p_3^2 = 0$$

so either $\langle 12\rangle$ or $[12]$ must vanish. Also from momentum conservation

$$1\rangle[1 + 2\rangle[2 + 3\rangle[3 = 0$$

then

$$\langle 12\rangle[2 = -\langle 13\rangle[3\,; \qquad \langle 21\rangle[1 = -\langle 23\rangle[3$$

and analogously interchanging angle and square brackets, or permuting indices.

Since bi-spinors are 2-component vectors,

$$[12] = [23] = [31] = 0,$$
$$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0$$

3-gluon amplitude

Suppose the amplitude depends on brackets only, i.e. with Ansatz

$$A_3(1^{h_1}2^{h_2}3^{h_3}) = c\langle 12\rangle^{x_{12}}\langle 13\rangle^{x_{13}}\langle 23\rangle^{x_{23}}$$

hence

$$-2h_1 = x_{12} + x_{13}\,, \qquad -2h_2 = x_{12} + x_{23}\,, \qquad -2h_3 = x_{13} + x_{23}$$

Therefore

$$A_3(1^{h_1}2^{h_2}3^{h_3}) = c\langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3}$$

For instance,

$$A_3[1^-2^-3^+] = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

fixed by dimensional analysis: $[A_n] = 4\text{-}n$ (hence cannot be square brackets).

How about if we flip helicities?

n-gluon amplitude

Schematically

$$A_n \sim \sum_{\text{diagrams}} \frac{\sum \left(\prod(\epsilon_i.\epsilon_j)\right)\left(\prod(\epsilon_i.k_j)\right)\left(\prod(k_i.k_j)\right)}{\prod P_I^2}$$

$$\epsilon_{i+}.\epsilon_{j_+} \propto \langle q_i q_j \rangle\,, \qquad \epsilon_{i-}.\epsilon_{j\_} \propto [q_i q_j]\,, \qquad \epsilon_{i-}.\epsilon_{j_+} \propto \langle i q_j \rangle [j q_i]$$

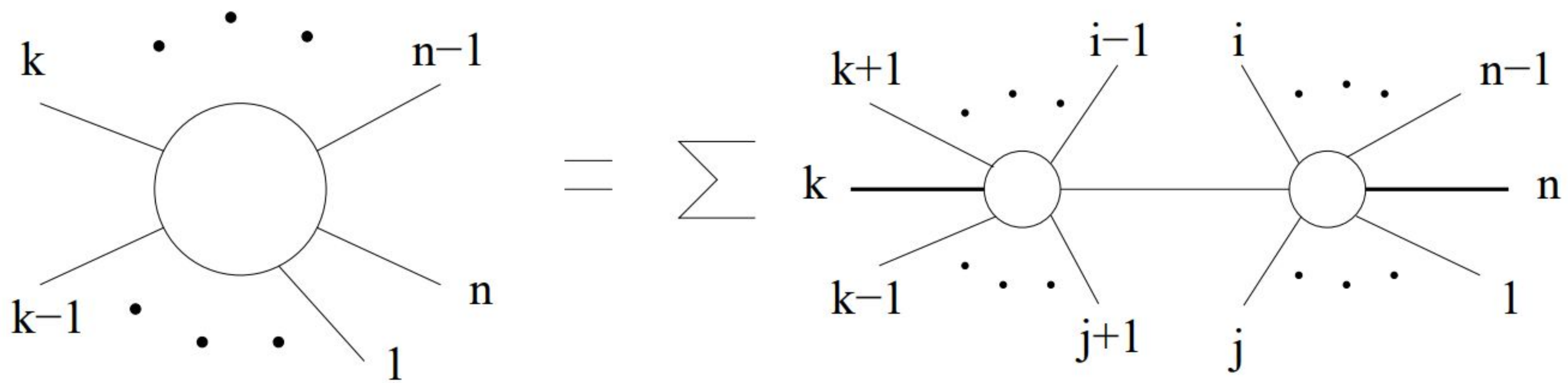
$$A_n(1^+ 2^+ \ldots n^+) = 0 \quad \text{and} \quad A_n(1^- 2^+ \ldots n^+) = 0\,.$$

Maximally Helicity Violation (MHV) amplitude is the first non-vanishing:

$$A_n(1^- 2^- 3^+ \ldots n^+)$$

BCFW recursion for n-gluons

Britto-Cachazo-Feng-Witten (2005) proved the recursion



$$A_n = \sum_r A^h_{r+1} \frac{1}{P_r^2} A^{-h}_{n-r+1}$$

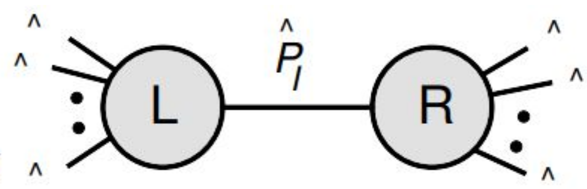
Complexified n-gluon amplitude

By introducing a shift by complex z, e.g.

$$|\hat{i}] = |i] + z\,|j]\,, \qquad |\hat{j}] = |j]\,, \qquad |\hat{i}\rangle = |i\rangle\,, \qquad |\hat{j}\rangle = |j\rangle - z|i\rangle$$

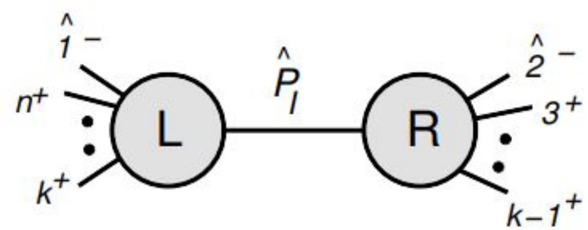
It can be shown (generally)

$$A_n \;=\; \sum_{\text{diagrams } I} \hat{A}_{\rm L}(z_I)\,\frac{1}{P_I^2}\,\hat{A}_{\rm R}(z_I) \;=\; \sum_{\text{diagrams } I}$$



and in particular (internal momentum must be complex)

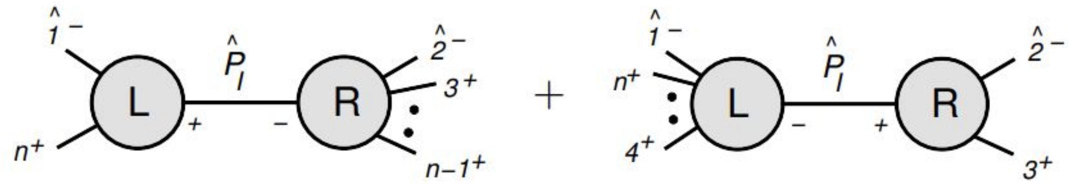
$$A_n[1^-2^-3^+\ldots n^+] \;=\; \sum_{k=4}^{n}$$



Complexified n-gluon amplitude

Since one-minus amplitudes vanish except for 3-amplitudes:

$$A_n[1^- 2^- 3^+ \ldots n^+] \;=$$



$$= \hat{A}_3\big[\hat{1}^-, -\hat{P}^+_{1n}, n^+\big]\,\frac{1}{P^2_{1n}}\,\hat{A}_{n-1}\big[\hat{P}^-_{1n}, \hat{2}^-, 3^+ \ldots (n-1)^+\big]$$

$$+\; \hat{A}_{n-1}\big[\hat{1}^-, \hat{P}^-_{23}, 4^+ \ldots, n^+\big]\,\frac{1}{P^2_{23}}\,\hat{A}_3\big[-\hat{P}^+_{23}, \hat{2}^-, 3^+\big]\,.$$

But $\quad \hat{A}_3\big[\hat{1}^-, -\hat{P}^+_{1n}, n^+\big] \;=\; \dfrac{[\hat{P}_{1n}\, n]^3}{[n\hat{1}][\hat{1}\hat{P}_{1n}]} = 0.$

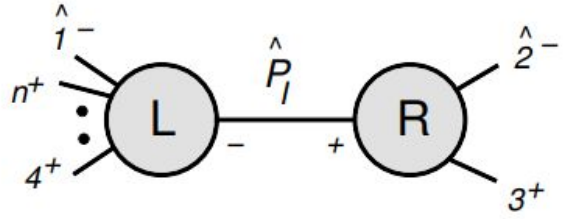
Since $\;[\hat{1}n] = 0\;$ from on-shell condition

$$0 = \hat{P}^2_{1n} = 2\hat{p}_1 \cdot p_n = \langle \hat{1}n\rangle[\hat{1}n] = \langle 1n\rangle[\hat{1}n]$$

$$|\hat{P}_{1n}\rangle[\hat{P}_{1n}\, n] = -\hat{P}_{1n}|n] \;=\; -(\hat{p}_1 + p_n)|n] \;=\; |1\rangle[\hat{1}n] \;=\; 0$$

Final step by induction

$$A_n[1^-2^-3^+\dots n^+] \;=$$



$$= \hat{A}_{n-1}[\hat{1}^-, \hat{P}_{23}^-, 4^+, \dots, n^+]\,\frac{1}{P_{23}^2}\,\hat{A}_3[-\hat{P}_{23}^+, \hat{2}^-, 3^+]$$

By induction:

$$A_n[1^-2^-3^+\dots n^+] = \frac{\langle \hat{1}\hat{P}_{23}\rangle^4}{\langle \hat{1}\hat{P}_{23}\rangle\langle \hat{P}_{23}\,4\rangle\langle 45\rangle\cdots\langle n\hat{1}\rangle} \times \frac{1}{\langle 23\rangle[23]} \times \frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\,\hat{2}][\hat{2}3]}$$

$$= -\frac{\langle 12\rangle^3[23]^3}{(-\langle 34\rangle[23])\;\langle 45\rangle\cdots\langle n1\rangle\;\langle 23\rangle[23]\;[23]} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\cdots\langle n1\rangle}\,.$$

using

$$\langle \hat{1}\hat{P}_{23}\rangle[3\hat{P}_{23}] = -\langle \hat{1}\hat{P}_{23}\rangle[\hat{P}_{23}\,3] = \langle \hat{1}|\hat{P}_{23}|3] =$$

$$= \langle \hat{1}|(\hat{p}_2 + p_3)|3] = \langle \hat{1}|\hat{p}_2|3] = -\langle \hat{1}\hat{2}\rangle[\hat{2}3] = -\langle 12\rangle[23]$$

$$\langle \hat{P}_{23}\,4\rangle[\hat{P}_{23}\,\hat{2}] = \langle 4|\hat{P}_{23}|\hat{2}] = \langle 4|3|2] = -\langle 43\rangle[32] = -\langle 34\rangle[23]$$

THANK YOU!

Bonus: there is a "magic" recursion

For cyclic-ordered n-gluon MHV amplitude, e.g.

$$
\begin{aligned}
A(1^+,2^+,3^+,4^-,5^+,6^+,7^-) &= \frac{\langle 72\rangle}{\langle 71\rangle\langle 12\rangle}A(2^+,3^+,4^-,5^+,6^+,7^-) = \\
&= \frac{\langle 72\rangle}{\langle 71\rangle\langle 12\rangle}\frac{\langle 73\rangle}{\langle 72\rangle\langle 23\rangle}A(3^+,4^-,5^+,6^+,7^-) = \\
&= \frac{\langle 72\rangle}{\langle 71\rangle\langle 12\rangle}\frac{\langle 73\rangle}{\langle 72\rangle\langle 23\rangle}\frac{\langle 74\rangle}{\langle 73\rangle\langle 34\rangle}A(4^-,5^+,6^+,7^-) = \\
&= \frac{\langle 74\rangle}{\langle 71\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle}A(5^+,6^+,7^-,4^-) = \\
&= \frac{\langle 74\rangle}{\langle 71\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle}\frac{\langle 46\rangle}{\langle 45\rangle\langle 56\rangle}A(6^+,7^-,4^-) = \\
&= \frac{\langle 74\rangle}{\langle 71\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle}\frac{\langle 46\rangle}{\langle 45\rangle\langle 56\rangle}\frac{\langle 74\rangle^3}{\langle 46\rangle\langle 67\rangle}
\end{aligned}
$$