Conformal Field Theory - presentation

Pedro Ricardo Martins Casella

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Introduction

This article intends to review the basic aspects of Conformal Field Theory and relate it to techniques developed from Quantum Field Theory. Conformal Field Theory (CFT) refers to the field theories with conformal invariance (a transformation that preserves angles in space).

Fundamentals

Definition

A conformal transformation between manifold M and M' is a mapping $\varphi : U \subset M \to V \subset M'$ such that, for $x' = \varphi(x)$, there is a positive function $\Lambda(x)$ such that,

$$g'_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{u}} \frac{\partial x'^{\sigma}}{\partial x^{v}} = g_{uv} \Lambda(x).$$

Considering $M = M'$ and flat space, we get,

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{u}} \frac{\partial x'^{\sigma}}{\partial x^{v}} = \eta_{uv} \Lambda(x).$$

Considering infinitesimal transformations, $x'^{\rho} = x^{\rho} + \epsilon^{\rho}$, we get,

$$\eta_{\rho\sigma}(\delta^{\rho}_{\mu} + \partial_{\mu}\epsilon^{\rho})(\delta^{\sigma}_{\nu} + \partial_{\nu}\epsilon^{\sigma}) = \eta_{\mu\nu}\Lambda(x) = \eta_{\mu\nu} + \eta_{\mu\sigma}\partial_{\nu}\epsilon^{\sigma} + \eta_{\rho\nu}\partial_{\mu}\epsilon^{\rho},$$

$$\implies \boxed{\eta_{\mu\nu}(\Lambda(x) - 1) = \partial_{\nu}\epsilon_{\mu} + \partial_{\mu}\epsilon_{\nu}}.$$

Multiplying both sides by $\eta^{\mu\nu}$, we get,

$$d(\Lambda(x)-1)=2\partial^{\mu}\epsilon_{\mu},$$

thus, we get to,

$$\boxed{I) \quad \frac{\eta_{\mu\nu}(2\partial\cdot\epsilon)}{d}=\partial_{(\nu}\epsilon_{\mu)}}.$$

Manipulating the above expression, we get,

$$\boxed{II) \quad (d-1)\Box(\partial\cdot\epsilon)=0}.$$

$$\boxed{III) \quad d(\partial_{\mu}\partial_{\nu}\epsilon_{\rho})=\eta_{[\rho\mu}\partial_{\nu]}(\partial\cdot\epsilon)}.$$

Those 3 expressions denotes conditions for an infinitesimal transformation to be canonical. Note that, from *II*), we get that the most general form for ϵ is,

$$\boxed{\epsilon_{\rho}=A_{\rho}+B_{\rho\nu}x^{\nu}+C_{\rho\mu\nu}x^{\mu}x^{\nu}}.$$

Conformal group and algebra

Note that the set of functions $\{f : x \to (x' - x)\}$ of conformal transformations forms a group of linear transformations (directly verifiable). Thus, we may describe its algebra by checking every element of ϵ.
We shall use the following results,

Lemma

If G is a linear Lie group and $X \in T_e G$, then, for $\gamma(t) = e^{tX}$, $\exists\ \epsilon > 0 \quad s.t \quad \forall |t| < \epsilon, \gamma(t) \in G$.

Theorem

If G is a linear Lie group, then
$Lie(G) = \mathfrak{g} = \{X \in GL / e^{tX} \in G\ \forall\ t \in \mathbb{R}\}.$

Proof.

Suppose first that $e^{tX} = 1 + tX + ... \in G$ for all t. Since $e^{tX}|_{t=0} = e$, the identity in the Lie group, we have that $X = \frac{d}{dt}e^{tX}|_{t=0} \in T_eG$, so $\{X \in GL/e^{tX} \in G \ \forall \ t \in \mathbb{R}\} \subset Lie(G)$.
Now, suppose $X \in T_eG$. Then, using the lemma, for any $t \in \mathbb{R}$, $\exists \ N \in \mathbb{N}$ such that $|\frac{t}{N}| < \epsilon$, thus, $e^{\frac{tX}{N}} \in G$. Since G is a group, $e^{\frac{tX}{N}}e^{\frac{tX}{N}}e^{\frac{tX}{N}}... = e^{tX} \in G$. So $Lie(G) \subset \{X \in GL \quad /e^{tX} \in G \ \forall \ t \in \mathbb{R}\}$.
Thus, we conclude that $Lie(G) = \{X \in GL/e^{tX} \in G \ \forall \ t \in \mathbb{R}\}$. □

Thus, for the first term, considering X small,

$$e^{iX}x^{\rho} = x^{\rho} + A^{\rho} = x^{\rho} + iXx^{\rho},$$

$$\implies iXx^{\rho} = A^{\rho},$$

thus, we get that $X = -iA^{\rho}\partial_{\rho}$. So the element in the Lie algebra is $\boxed{P_{\rho} = -i\partial_{\rho}}$.

For the term B, we use expression I, and get,

$$B_{(\mu\nu)} = \frac{2}{d}\eta_{\mu\nu}(\eta^{\alpha\beta}B_{\alpha\beta}).$$

We can separate $B_{\mu\nu}$ into a symmetric and antisymmetric part, $B_{\mu\nu} = S_{\mu\nu} + A_{\mu\nu}$. Since this is infinitesimal, we can assume $S_{\mu\nu}$ is proportional to the flat metric, $S_{\mu\nu} = \alpha\eta_{\mu\nu}$, and then consider its transformations action,

$$x'^{\mu} = x^{\mu} + \alpha\eta^{\mu\nu}x_{\nu},$$

thus,

$$e^{iD}x^{\mu} = x'^{\mu} \Longrightarrow iDx^{\mu} = \alpha\eta^{\mu\nu}x_{\nu} \Longrightarrow \boxed{D = -ix^{\mu}\partial_{\mu}}.$$

For the antissymmetric part,

$$x'^{\mu} = x^{\mu} + A^{\mu\nu}x_{\nu},$$

$$iDx^{\mu} = A^{\mu\nu}x_{\nu},$$

$$iDx^{\nu} = A^{\nu\mu}x_{\mu} = -A^{\mu\nu}x_{\mu}.$$

Thus, we see that for the antissymetric part $\boxed{D^{\mu}_{\nu} = i(x^{\mu}\partial_{\nu} - x^{\nu}\partial_{\mu})}$.

For the C term, we use III,

$$C_{\rho\mu\nu} = \frac{1}{d}(\eta_{[\rho\mu} C^{\sigma}_{\nu]\sigma}),$$

$$C_{\rho\mu\nu} x^\mu x^\nu = \frac{1}{d}(x_\rho x^\nu C^{\sigma}_{\nu\sigma} - x^\nu x_\nu C^{\sigma}_{\rho\sigma} + x_\rho x^\mu C^{\sigma}_{\mu\sigma}) = \frac{1}{d}(2 x_\rho x^\nu C^{\sigma}_{\nu\sigma} - x^\nu x_\nu C^{\sigma}_{\rho\sigma}).$$

Now, if,

$$e^{iK} x_\rho = x_\rho + C_{\rho\mu\nu} x^\mu x^\nu,$$

$$\implies \boxed{K^\rho = -i(2x^\rho x^\nu \partial_\nu - x^\nu x_\nu \partial^\rho)}.$$

Conformal algebra

Thus, we get that the Conformal group Lie algebra generators are,

$$1)\,Translation: \quad \boxed{P_\rho = -i\partial_\rho}.$$

$$2)\,Dilation: \quad \boxed{D = -ix^\mu\partial_\mu}.$$

$$3)\,Rotation: \quad \boxed{D_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)}.$$

4) Special Conformal Transformation (SCT):

$$\boxed{K^\rho = -i(2x^\rho x^\nu\partial_\nu - x^\nu x_\nu\partial^\rho)}.$$

The dimension of such Lie algebra is $\frac{d(d-1)}{2} + 2d + 1$.

For $d = 2$

In the remaining of this project, we shall focus in $d = 2$. In this case, letting μ, ν be represented by $0, 1$, we get, by I,

$$\boxed{\partial_{(0}\epsilon_{1)} = 0},$$

$$\partial_0\epsilon_0 + \partial_1\epsilon_1 = 2\partial_0\epsilon_0 \Longrightarrow \boxed{\partial_0\epsilon_0 = \partial_1\epsilon_1}.$$

Notice that those are Cauchy-Riemann equations. Thus, we state,

Theorem

If $\epsilon(z) = \epsilon_0(x^0, x^1) + i\epsilon_1(x^0, x^1)$, the transformation $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z + \epsilon(z)$is conformal iff it is holomorphic.

Therefore, if one is dealing in the complex space, we can write ϵ and $\bar{\epsilon}$ as Laurent series,

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n,$$

$$\bar{\epsilon}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{n+1}\bar{\epsilon_n}.$$

Thus, in the transformation $z' = z + \epsilon(z)$ and $\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z})$, we can write the generators as,

$$l_n = z^{n+1}\partial,$$

$$\bar{l}_n = \bar{z}^{n+1}\bar{\partial}.$$

The generators obey the relations,

$$[l_n, l_m] = (n-m)l_{n+m},$$

$$[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m},$$

$$[l_n, \bar{l}_m] = 0.$$

This defines a Witt algebra, after Ernst Witt [1911 - 1991].

Witt algebra

Definition

A Witt algebra is a Lie algebra of generators ,

$$l_n = z^{n+1}\partial,$$

$$\bar{l}_n = \bar{z}^{n+1}\bar{\partial}.$$

With the Lie bracket,

$$[l_n, l_m] = (n-m)l_{n+m},$$

$$[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m},$$

$$[l_n, \bar{l}_m] = 0.$$

Virasoro algebra

There is an extension of crucial importance of this algebra, which is the so called Virasoro algebra, named after Miguel Virasoro [1940-2021].

Definition

The Virasoro Algebra is a complex Lie algebra of operators L_n, $n \in \mathbb{Z}$ with the bracket,

$$[L_n, L_m] = (n - m)L_{n+m} + \lambda(n^3 - n)\delta_{m,-n}.$$

Where $\lambda \in \mathbb{C}$.

We can start studying *CFT* in Euclidean space, with the correspondence $\mathbb{R}^2 \simeq \mathbb{C}$ by adopting $z = x^0 + ix^1$, $\bar{z} = x^0 - ix^1$. Thus, fields in our theory are described as,

$$\Phi = \Phi(z, \bar{z}).$$

Definition

Fields are Chiral and Anti-chiral if, respectively, $\Phi = \Phi(z)$ and $\Phi = \Phi(\bar{z})$.

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Definition

Φ is called a primary field if, upon conformal transformation $z \to f(z)$, we get,

$$\Phi(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})).$$

Where $h, \bar{h}$ are the so called conformal dimensions.

If a field is primary, , $(\partial f)^h = 1 + h\partial\epsilon + O(\epsilon^2)$, and $\Phi(z+\epsilon, .) = \Phi(z, .) + \partial\Phi\epsilon + O(\epsilon^2)$. Considering an infinitesimal transformation, we get the transformation law, Considering an infinitesimal transformation, we get the transformation law,

$$\boxed{\delta_{\epsilon,\bar{\epsilon}}\Phi = (\epsilon(z)\partial + \bar{\epsilon}(\bar{z})\bar{\partial} + h\partial\epsilon + \bar{h}\bar{\partial}\bar{\epsilon})\Phi(z, \bar{z})}.$$

Stress tensor

In building actions of primary fields invariant under Conformal Transformations, we get the Noether current,

$$j_\mu = T_{\mu\nu}\epsilon^\nu,$$

by taking the derivative in both sides, we get,

$$\partial^\mu(T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}\partial^\mu(\epsilon^\nu) = T_{\mu\nu}\partial^\mu(\epsilon^\nu) = 0,$$

by using I and considering that $T_{\mu\nu}$ is symmetric,

$$\frac{T_{\mu\nu}\eta^{\mu\nu}(\partial\cdot\epsilon)}{d} = 0 \Longrightarrow \boxed{T^\mu_\mu = 0}.$$

Thus, we see that the stress energy tensor in a CFT in $d = 2$ is traceless.

If we wish to make a variable change,

$$T_{\mu\nu} = \frac{\partial x^\alpha}{\partial z^\mu} \frac{\partial x^\beta}{\partial z^\nu} T_{\alpha\beta},$$

where $x^\alpha \in \{x^0, x^1\}$ and $z^\mu \in \{z, \bar{z}\}$. We get,

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = 0,$$

$$T_{zz} = \frac{1}{4}(T_{00} - iT_{01}),$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + iT_{01}).$$

We also find out that the two non-vanishing components are holomorphic and anti-holomorphic,

$$\boxed{\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0.}$$

OPE

In QFT, once we have a theory with conserved charge constructed from a symmetry, we can write the variation of the field in respect to that transformation by $\delta\Phi = [Q, \Phi]$. In our case, we have that, inside a region $\mathcal{C}$,

$$\delta_{\epsilon,\bar{\epsilon}}\Phi = \oint_{\mathcal{C}} dz' \frac{[T(z')\epsilon(z'), \Phi(z,\bar{z})]}{2i\pi} + \oint_{\mathcal{C}} d\bar{z}' \frac{[\bar{T}(\bar{z}')\bar{\epsilon}(\bar{z}'), \Phi(z,\bar{z})]}{2i\pi}.$$

Comparing to our previous expression for the field variation and considering the identities,

$$\Phi(z,\bar{z})\partial\epsilon = \frac{1}{2\pi i}\oint_{\mathcal{C}} dz' \frac{\epsilon(z')}{(z-z')^2}\Phi(z,\bar{z}),$$

$$\partial\Phi\epsilon = \frac{1}{2\pi i}\oint_{\mathcal{C}} dz' \frac{\epsilon(z')}{(z-z')}\partial\Phi(z,\bar{z}),$$

we see that,

$$\oint_{\mathcal{C}} dz' \frac{[T(z')\epsilon(z'),\Phi(z,\bar{z})]}{2i\pi} = (\epsilon(z)\partial + h\partial\epsilon)\Phi(z,\bar{z}),$$

$$\Longrightarrow [T(z)\epsilon(z),\Phi(w,\bar{w})] = h\frac{(\Phi(w,\bar{w})\epsilon(z))}{(z-w)^2} + \frac{\partial\Phi(w,\bar{w})\epsilon(z)}{z-w},$$

up to singular terms. There is an analog expression for the holomorphic part. Such expression allows us to define an Operator Product Expression (OPE) between the stress tensor and the fields,

Lemma

If $\Phi = \Phi(z, \bar{z})$ is a primary, the OPE with the stress tensor in a CFT is,

$$\boxed{T(z)\Phi(w, \bar{w}) = h\frac{(\Phi(w, \bar{w}))}{(z-w)^2} + \frac{\partial\Phi(w, \bar{w})}{z-w}.}$$

$$\boxed{\bar{T}(\bar{z})\Phi(w, \bar{w}) = \bar{h}\frac{(\Phi(w, \bar{w}))}{(z-w)^2} + \frac{\bar{\partial}\Phi(w, \bar{w})}{z-w}}.$$

The OPE can be viewed as an algebraic product of quantum fields. We may also write the Laurent modes of the Stress tensor,

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z),$$

$$[L_m, L_n] = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+1} [T(z), T(w)].$$

In order for this to respect Virasoro algebra, we state the OPE of stress tensors,

$$\boxed{T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}}.$$

In this case, we adopt $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{m,-n}$, where c is the so called central charge. We also note that if $c = 0$, the stress energy is a primary field of conformal dimension 2. Let's now apply this formalism to QFT examples.

Free Boson

The action for free massless scalar fields is given by Polyakov action (named after Alexander Markovich Polyakov [1940-]).

$$S = \int d^2 z [\partial X^\mu \bar{\partial} X_\mu].$$

Using Euler-Lagrange equations, we see that the equations of motions are

$$\partial \bar{\partial} X^\mu = 0.$$

This implies that the average value of the OPE of the above expression with any field that is not in z vanishes, i.e,

$$< \partial \bar{\partial} X^\mu ... >= 0.$$

Using the fact that $\partial\bar{\partial}X^\mu = -\frac{\delta S}{\delta X_\mu}$ and the above statement, we must have that

$$< \partial\bar{\partial}X^\mu X^\nu(w,\bar{w}) >= 0$$

$$\implies \int [DX] \frac{\delta}{\delta X_\mu}(e^{-S}X^\nu(w)) =$$

$$= \partial\bar{\partial} < X^\mu(z,\bar{z})X^\nu(w,\bar{w}) > + \eta^{\mu\nu}\delta^2(z-w,\bar{z}-\bar{w}) = 0.$$

This leads to the OPE,

$$\boxed{X^\mu(z,\bar{z})X^\nu(w,\bar{w}) = -\eta^{\mu\nu}[\ln(z-w) + \ln(\bar{z}-\bar{w})]}.$$

And the energy stress tensor non-zero components are

$$T_{zz} = \frac{1}{4}(\partial X^\mu \partial X_\mu),$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}(\bar{\partial} X^\mu \bar{\partial} X_\mu).$$

With this, we see that the OPE singular terms are,

$$\boxed{T(z)X^{\mu}(w,\bar{w}) = \frac{\partial_w X^{\mu}}{(z-w)}},$$

thus, we see that the bosonic field has no conformal dimension(The same conclusion is taken for the antiholomorphic part).
The OPE of stress tensor leads us to,

$$T(z)T(w) = \frac{1}{16}(\partial X^{\mu}\partial X_{\mu}\partial_w X^{\nu}\partial_w X_{\nu}) = \frac{\partial X^{\mu}}{16}(\partial\partial_w(\eta^{\nu}_{\mu}(-\ln(z-w)))\partial_w X_{\nu} + \partial\partial_w(\eta_{\nu\mu}(-\ln(z-w)))\partial_w X^{\nu}) + ... =$$

$$= \frac{1}{16}\left[\partial\partial_w(\eta^{\nu}_{\mu}(-\ln(z-w)))\partial\partial_w(\eta^{\mu}_{\nu}(-\ln(z-w))) + \partial\partial_w(\eta_{\nu\mu}(-\ln(z-w)))\partial\partial_w(\eta^{\nu\mu}(-\ln(z-w)))\right] + ... =,$$

$$= \frac{1}{16}\left[\frac{8}{(z-w)^4}\right] + ... = \frac{c}{2(z-w)^4} + (...).$$

This implies that the bosonic free field has central charge $c = 1$. This theory is also used to describe bosonic strings in String Theory.

Free Fermion

The action for the Free fermion field $\psi(z)$ is,

$$S = \int d^2 z[\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}].$$

Using the same procedure as the last case, we find the equations of motion,

$$\partial\bar{\psi} = \bar{\partial}\psi = 0,$$

the OPE,

$$\psi(z)\psi(w) = \frac{1}{z-w},$$

and the non-zero components of the Stress Tensor,

$$T(z) = -\frac{1}{2}(\psi(z)\partial\psi(z)),$$

$$\bar{T}(\bar{z}) = -\frac{1}{2}(\bar{\psi}(\bar{z})\bar{\partial}\bar{\psi}(\bar{z})).$$

With this, we find,

$$T(z)\psi(w) = \frac{1}{2}\frac{\psi(z)}{(z-w)^2} + (...),$$

since it has no OPE with the the conjugated part, we see that the fermionic field has conformal dimensions $(\frac{1}{2}, 0)$.
Also, since we are dealing with fermionic fields, they anticommute, so one needs to change their signs when flipping them during the OPE. Doing so, we get,

$$T(z)T(w) = \frac{1}{4}(\psi\partial\psi)(z)(\psi\partial\psi)(w) = \frac{1}{4(z-w)^4} \Longrightarrow c = \frac{1}{2}.$$

A theory with two fermions gives $c = 1$, which is related to the process of fermionization (replacing a boson for a pair of fermions).

bc Ghost system

A very similar CFT of the fermionic field is the action for the fermionic ghost fields b and c.

$$S = \int d^2z[b\bar{\partial}c + \bar{b}\partial\bar{c}].$$

The Equations of motion and OPE of the fields are

$$\bar{\partial}c = \bar{\partial}b = \partial\bar{c} = \partial\bar{b} = 0,$$

$$b(z)c(w) = \frac{1}{(z-w)},$$

$$c(w)b(z) = \frac{1}{(z-w)}.$$

The stress tensor is,

$$T_{bc}(z) = -2b(z)\partial c(z) + c(z)\partial b(z),$$

with an analog expression for the antiholomorphic part. Making the OPE with the stress tensor gives $(h_b, h_c, \bar{h}_{\bar{b}}, \bar{h}_{\bar{c}}) = (2, -1, 2, -1)$, the rest being zero. The central charge is,

$$T(z)T(w) = \frac{-13}{(z-w)^4} + ... \Longrightarrow c = -26.$$

This result has an implication that in the bosonic string, the number of dimensions is 26.

A bit about BRST

The BRST (named after Carlo Becchi[1939-2015], Alain Rouet [1942-], Raymond Stora [1930-2015] and Igor Tyutin [1940-]) is a process for quantizing fields. We are going to show an example here related to what has been previously described.
Note that, if we wish to mix bosonic fields with ghost fields, we can write the lagrangian,

$$S = \int d^2z[\partial X^{\mu}\bar{\partial}X_{\mu} + b\bar{\partial}c + \bar{b}\partial\bar{c}],$$

This action is invariant under the so called BRST transformations,

$$\delta X^\mu = \eta c \partial X^\mu,$$

$$\delta c = \eta c \partial c,$$

$$\delta b = \eta T,$$

where $T(z) = T_X + T_{bc}$ denotes the stress tensor, and $T_X(z)$ is the stress tensor for the bosonic part (shown in section 4.1). These symmetries implies the following Noether charge,

$$Q = \oint dz (c T_X + bc\partial c),$$

which has the remarkable property of being 2-nilpotent, $Q^2 = 0$. This allows us to describe the physical states of the theory as cohomology classes of Q_B. This turns out to be a way to gauge fixing the degrees of freedom in a theory.

Virasoro Modes approach

There is an approach alternative and usually more practical to OPE, which is to define the field ϕ as a state $|\phi\rangle$ such that,

$$L_0 |\phi\rangle = h |\phi\rangle ,$$

$$L_n |\phi\rangle = 0, \quad n > 0.$$

To be more precise, write $\phi(z, \bar{z})$ in Laurent modes,

$$\phi(z, \bar{z}) = \sum_{n,m \in \mathbb{Z}} z^{-n-h} \bar{z}^{-m-\bar{h}} \phi_{n,m},$$

in this sense, if one has the vacuum $|0\rangle$, we define,

$$|\phi\rangle = \lim_{z,\bar{z} \to 0} \phi(z, \bar{z}) |0\rangle = \phi_{-h,-\bar{h}} |0\rangle .$$

If we wish to write the hermitian conjugated version, we use Wick rotation, $x_0 \to i x_0$ and leave x_1 unchanged. Thus, we make the transformation $z \to \frac{1}{\bar{z}}$, and, for a primary field, that implies that,

$$\phi^\dagger(z, \bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \phi(z^{-1}, \bar{z}^{-1}) \Longrightarrow \boxed{\phi^\dagger_{n,m} = \phi_{-n,-m}}.$$

Using the OPE of stress tensor previously developed, we can prove that, for a chiral field,

$$[L_m, \phi_n] = ((h-1)m - n)\phi_{m+n},$$

and acting with the stress tensor is equivalent to act with $L_2 |0\rangle$. In fact, we can use the relation,

$$\langle 0| [L_2, L_{-2}] |0\rangle = \frac{c}{2}.$$

The OPE procedure is equivalent to act the field modes with Virasoro modes and use the commutation relations. In this sense, the bc system ghosts can be written as,

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2},$$

$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1}.$$

With an analog expression for the holomorphic parts. Using the OPE, we find

$$\{b_m, c_n\} = \delta_{m-m}.$$

It is in this sense that the charge Q is an operator.

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