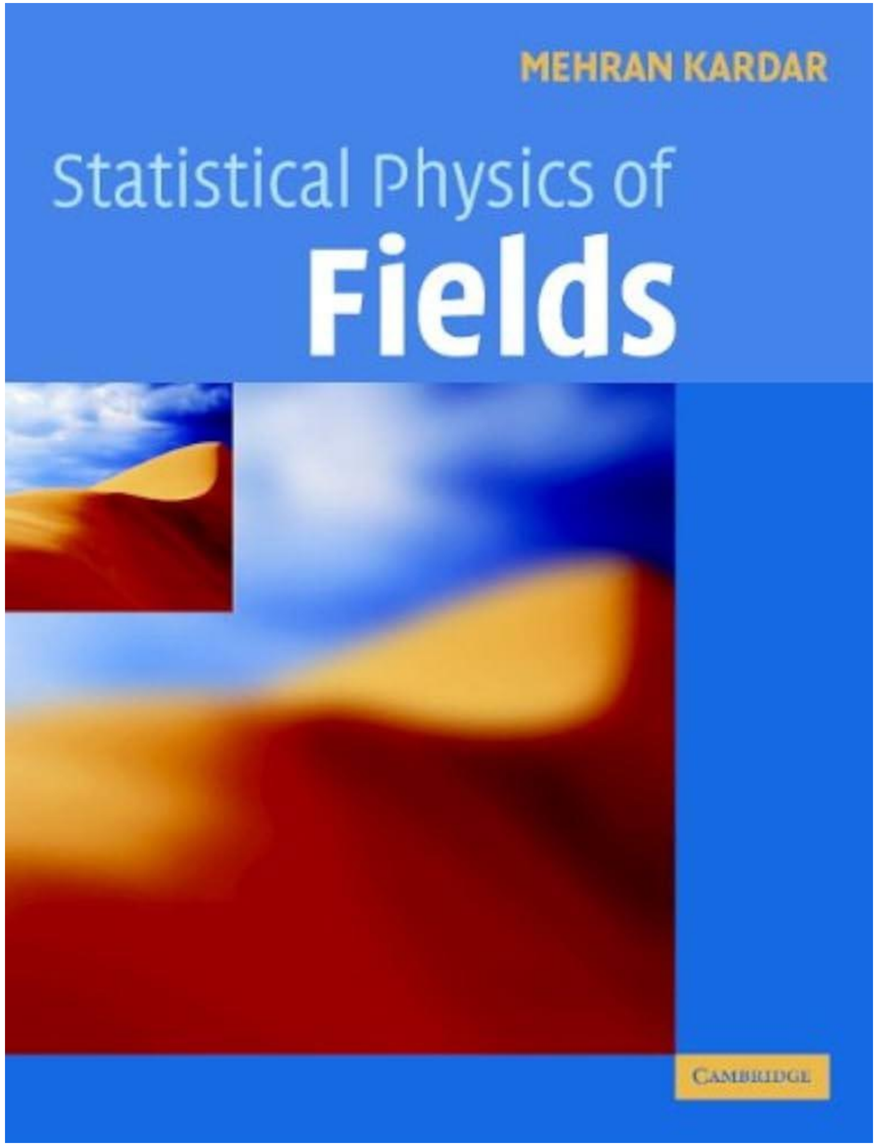
Statistical Field Theory

Jorge M. Escobar-Agudelo

Reference:



Outline

- Review Statistical Mechanics and Motivation
- Scaling and Coarse-Graining
- Statistical Fields
- Constructing the Effective Hamiltonian
- Continuous Symmetry Breaking
- Fluctuations
- Supplementary Remarks

Statistical Mechanics

Emergence of new collective properties in the *macroscopic* realm from the dynamics of the *microscopic* particles.

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$$p(\mu) = \frac{e^{-\beta\mathcal{H}(\mu)}}{\mathcal{Z}} \qquad \mathcal{Z} = \sum_{\mu} e^{-\beta\mathcal{H}(\mu)} \qquad F = -k_B T \ln \mathcal{Z}$$

Interactions

- New phases of matter
- Low energy excitations

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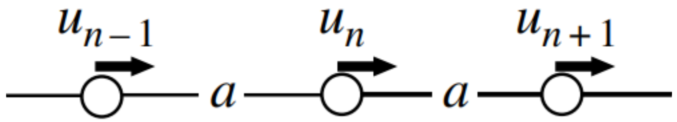
↓

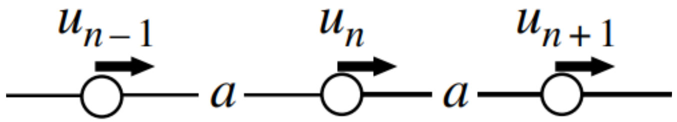
FIELDS

Scale



Scale

$$a \ll \lambda(T)$$



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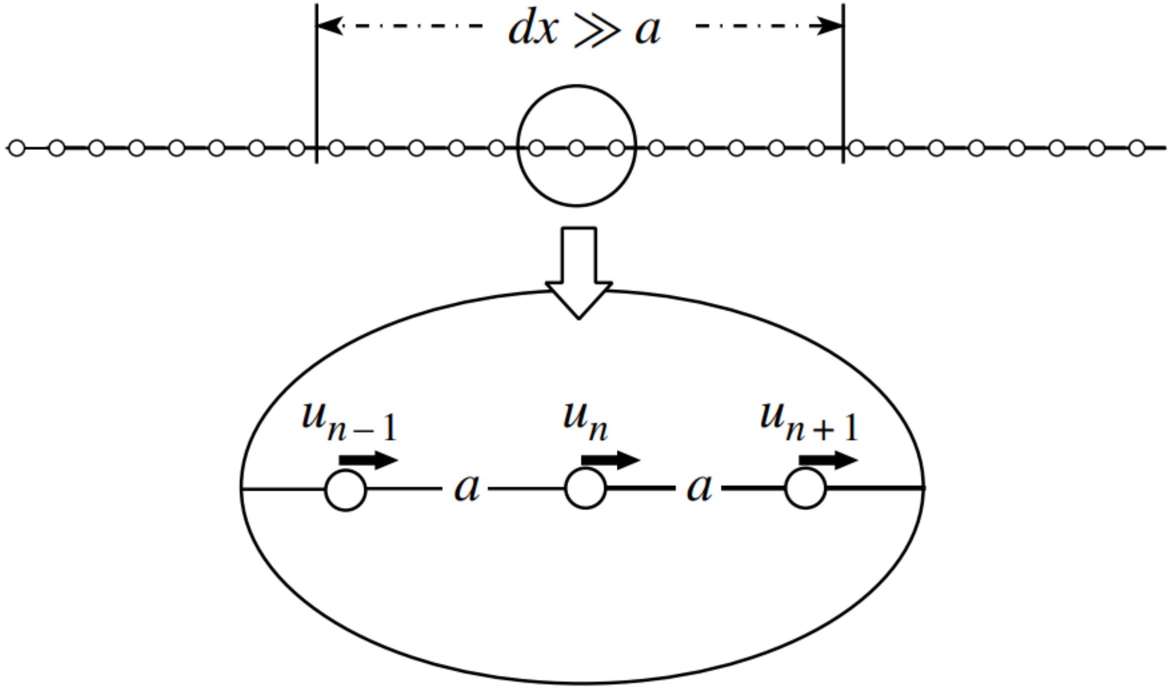

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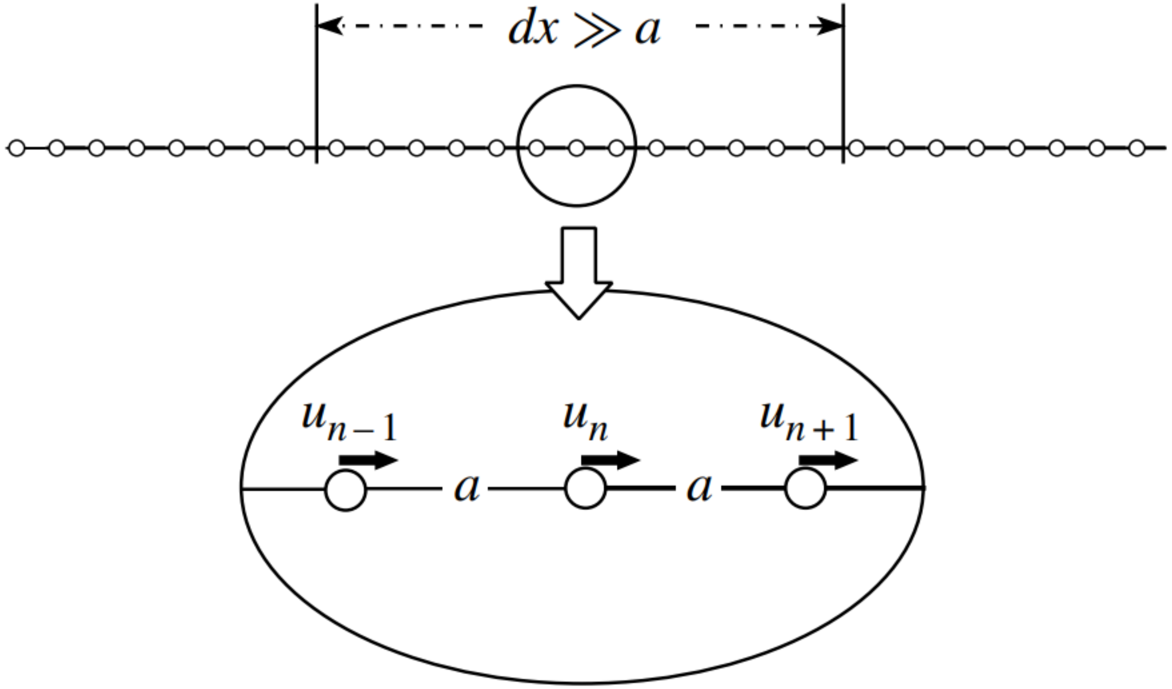

Scale

Mesoscopic scale:

Much larger than the lattice spacing but much smaller than the system size

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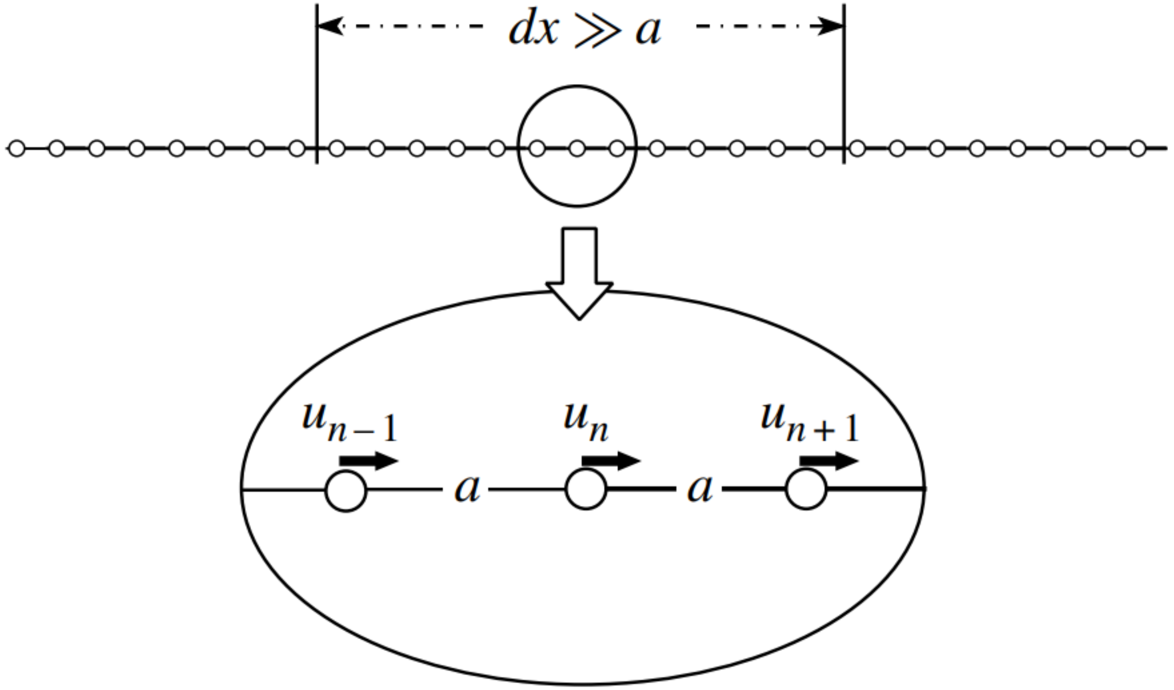

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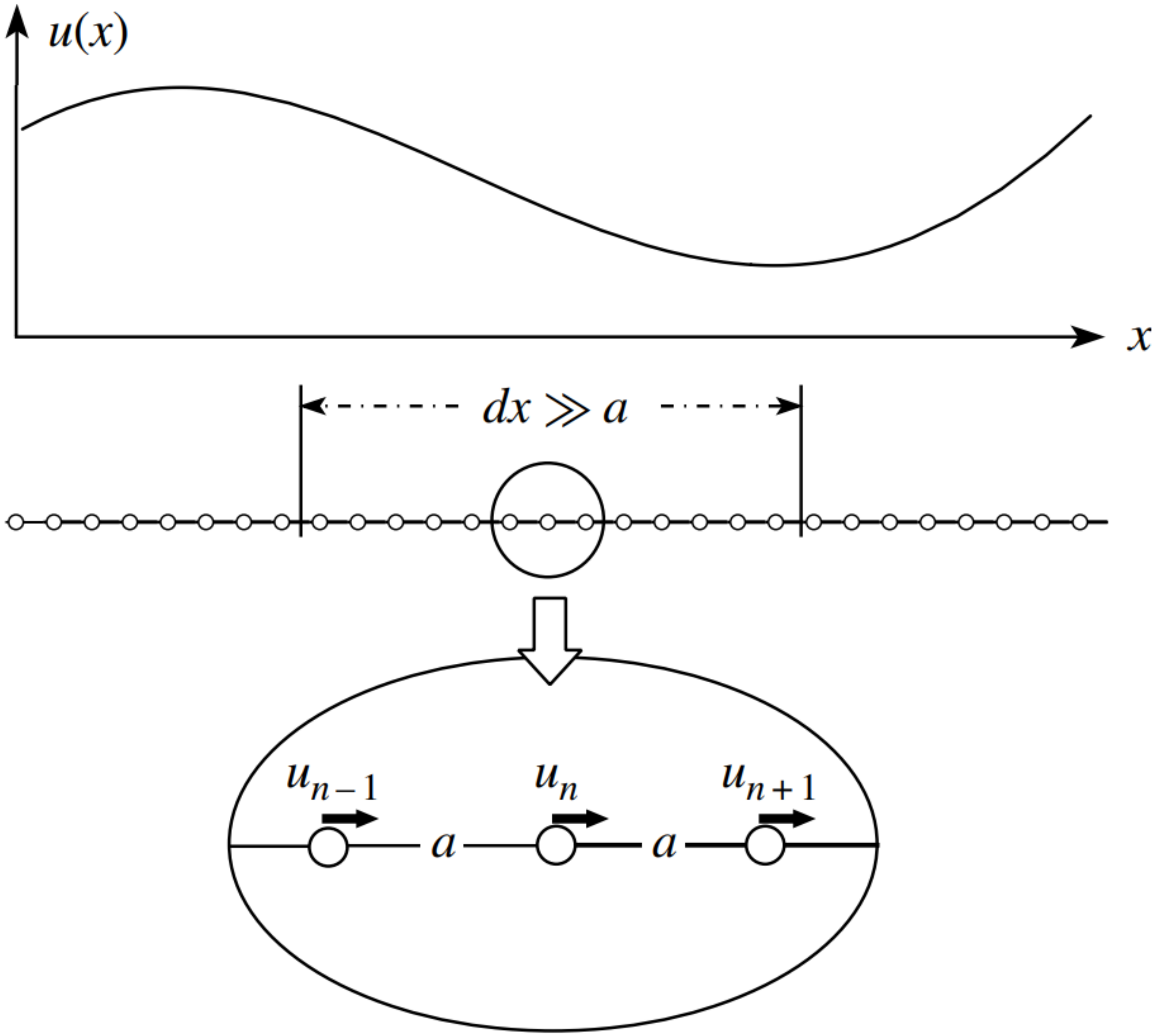

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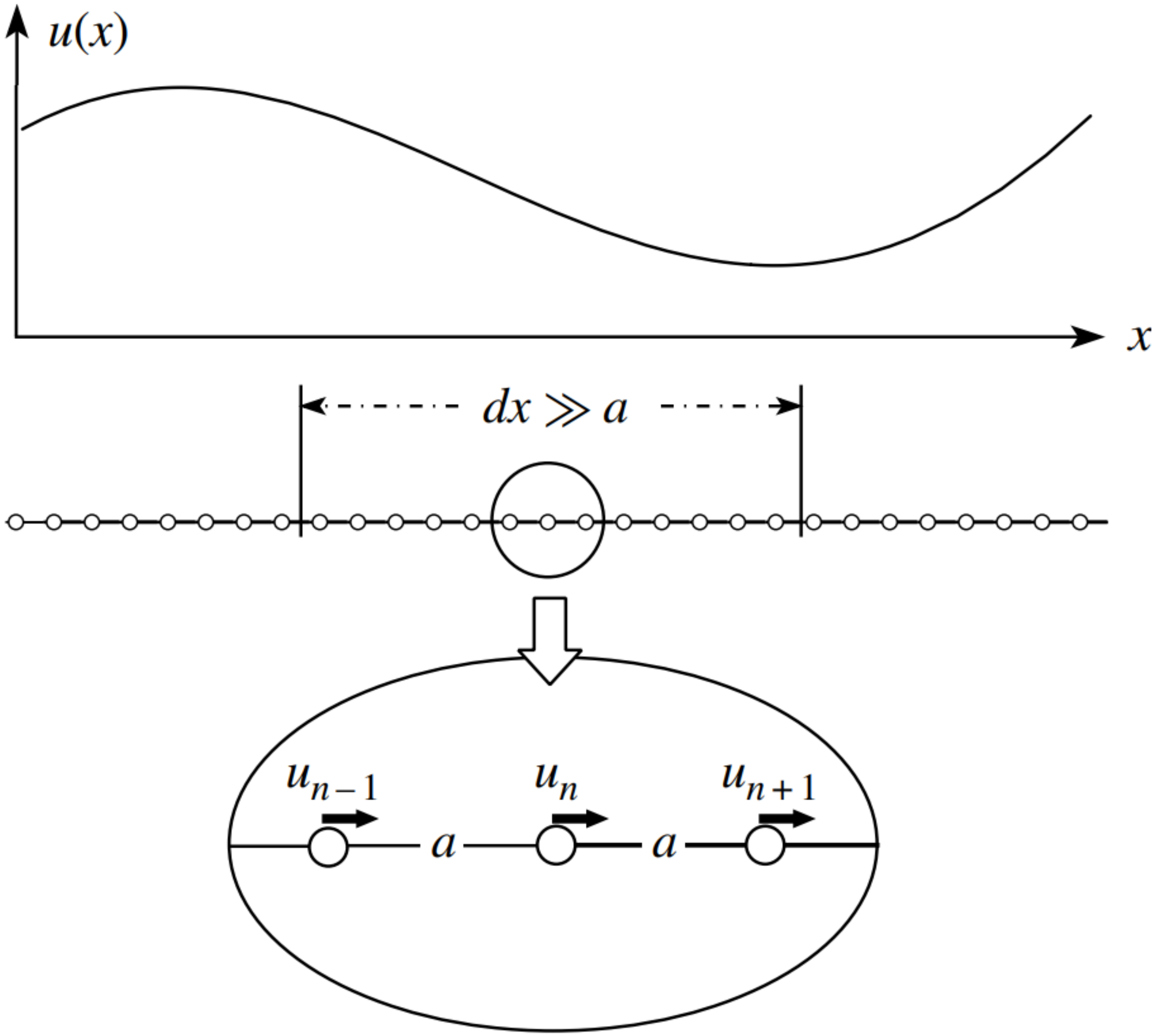

Coarse graining

Statistical fields

$$\vec{\phi}(\boldsymbol{x})$$

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$$\boldsymbol{x} \equiv (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$$ (space)

$$\vec{\phi} \equiv (\phi_1, \phi_2, \cdots, \phi_n) \in \mathbb{R}^n$$ (order parameter)

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n number of directions in which the symmetry can break:

- $n = 1$, scalar: liquid gas transitions, binary mixtures, uniaxial magnets.
- $n = 2$, 2-component vector / complex field: superfluidity, superconductivity, planar magnets.
- $n = 3$, 3-component vector: classical magnets
- $n > 3$, tensor: chirality crystal

Partition function

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Effective Hamiltonian:

$$\beta\mathcal{H}\left[\vec{\phi}(\boldsymbol{x})\right] \equiv -\ln\mathcal{P}\left[\vec{\phi}(\boldsymbol{x})\right]$$

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Disconnected:

$$\beta\mathcal{H}\left[\vec{\phi}(\boldsymbol{x})\right] = \int d^d\boldsymbol{x}\Phi\left[\vec{\phi}(\boldsymbol{x}), \boldsymbol{x}\right]$$

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Uniformity:

$$\beta\mathcal{H}\left[\vec{\phi}(\boldsymbol{x})\right] = \int d^d\boldsymbol{x}\Phi\left[\vec{\phi}(\boldsymbol{x})\right]$$

Effective Hamiltonian

Interactions:

$$\beta\mathcal{H}\left[\vec{\phi}(\boldsymbol{x})\right] = \int d^d\boldsymbol{x}\Phi\left[\vec{\phi}(\boldsymbol{x}), \nabla\vec{\phi}, \nabla^2\vec{\phi}, \cdots\right]$$

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$$\beta\mathcal{H}\left[\vec{\phi}(\boldsymbol{x})\right] = \int d^d\boldsymbol{x}\Phi\left[\underbrace{\vec{\phi}(\boldsymbol{x}), \nabla\vec{\phi}}_{\text{Locality}}, \nabla^2\vec{\phi}, \cdots\right]$$

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Analyticity:

Going from microscopic to mesoscopic scales, non-analyticities associated with microscopic degrees of freedom are washed out.

Macroscopic non-analyticities associated with phase transitions involve infinitely many degrees of freedom.

Effective Hamiltonian

Symmetries:

$$\mathcal{H}\left[R_n\vec{\phi}(\boldsymbol{x})\right] = \mathcal{H}\left[\vec{\phi}(\boldsymbol{x})\right]$$

$$\phi^2(\boldsymbol{x}) \equiv \vec{\phi}(\boldsymbol{x}) \cdot \vec{\phi}(\boldsymbol{x}) \equiv \phi_\mu(\boldsymbol{x})\phi_\mu(\boldsymbol{x})$$

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Isotropic system

$$\left(\nabla\vec{\phi}\right)^2 \equiv \partial_i\phi_\mu\partial_i\phi_\mu$$

$$\left(\nabla^2\vec{\phi}\right)^2 \equiv \partial_i\partial_i\phi_\mu\partial_j\partial_j\phi_\mu$$

$$\phi^2\left(\nabla\vec{\phi}\right)^2 \equiv \phi_\mu\phi_\mu\partial_i\phi_\nu\partial_i\phi_\nu$$

Landau-Ginzburg Hamiltonian

$$\beta\mathcal{H}\left[\vec{\phi}(\boldsymbol{x})\right] = \int d^d\boldsymbol{x}\left[\frac{t}{2}\phi^2(\boldsymbol{x}) + u\phi^4(\boldsymbol{x}) + \frac{K}{2}\left(\nabla\vec{\phi}\right)^2 + \cdots - \beta\vec{J}\cdot\vec{\phi}(\boldsymbol{x})\right]$$

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Partition Function:

$$\mathcal{Z} = \int \mathcal{D}\vec{\phi}(\boldsymbol{x})e^{-\beta\mathcal{H}\left[\vec{\phi}(\boldsymbol{x})\right]}$$

Superfluidity

$$\psi(\boldsymbol{x}) \equiv \psi_R(\boldsymbol{x}) + i\psi_I(\boldsymbol{x}) \equiv |\psi(\boldsymbol{x})|e^{i\theta(\boldsymbol{x})}$$

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$$\mathcal{V}\left[\psi(\boldsymbol{x})\right] = \frac{t}{2}\psi^2(\boldsymbol{x}) + u\psi^4(\boldsymbol{x})$$

$$\bar{\psi}^2 = -\frac{t}{4u}$$

Continuous symmetry breaking

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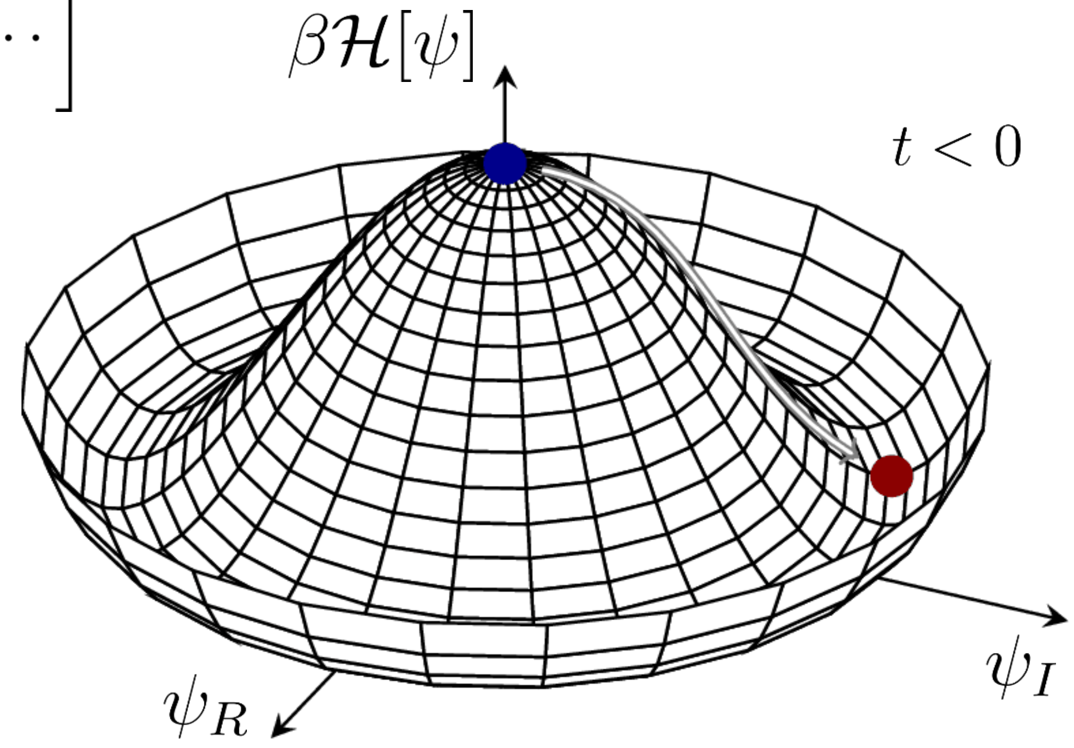

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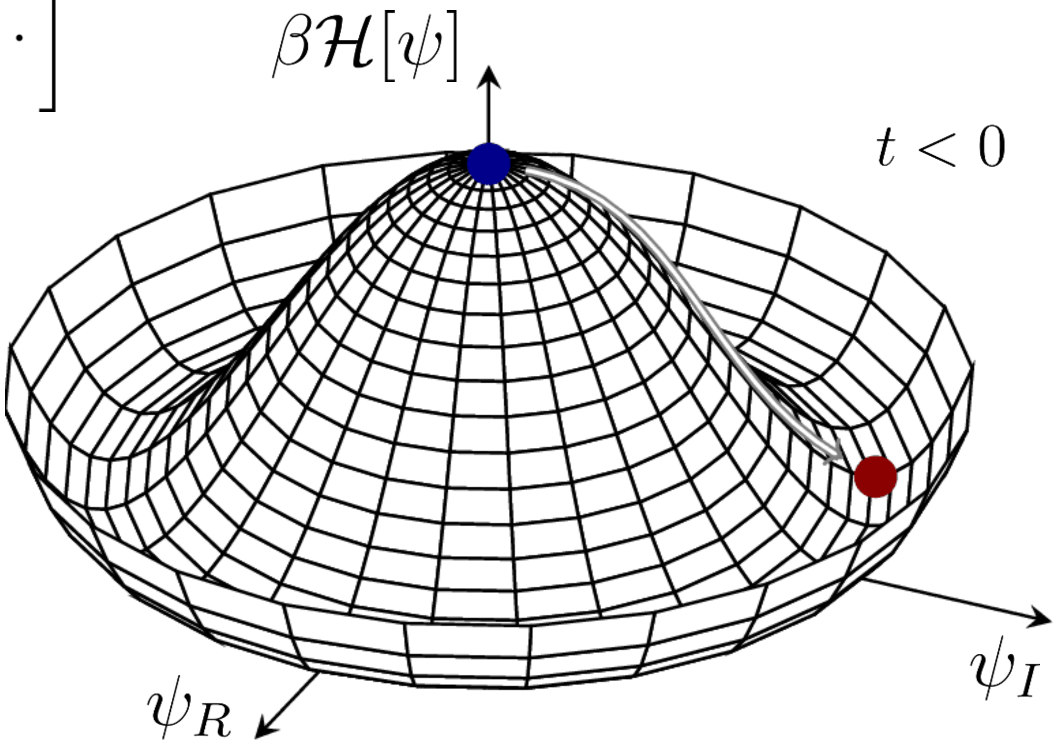
Goldstone modes: massless angular fluctuations along the manifold of minima.



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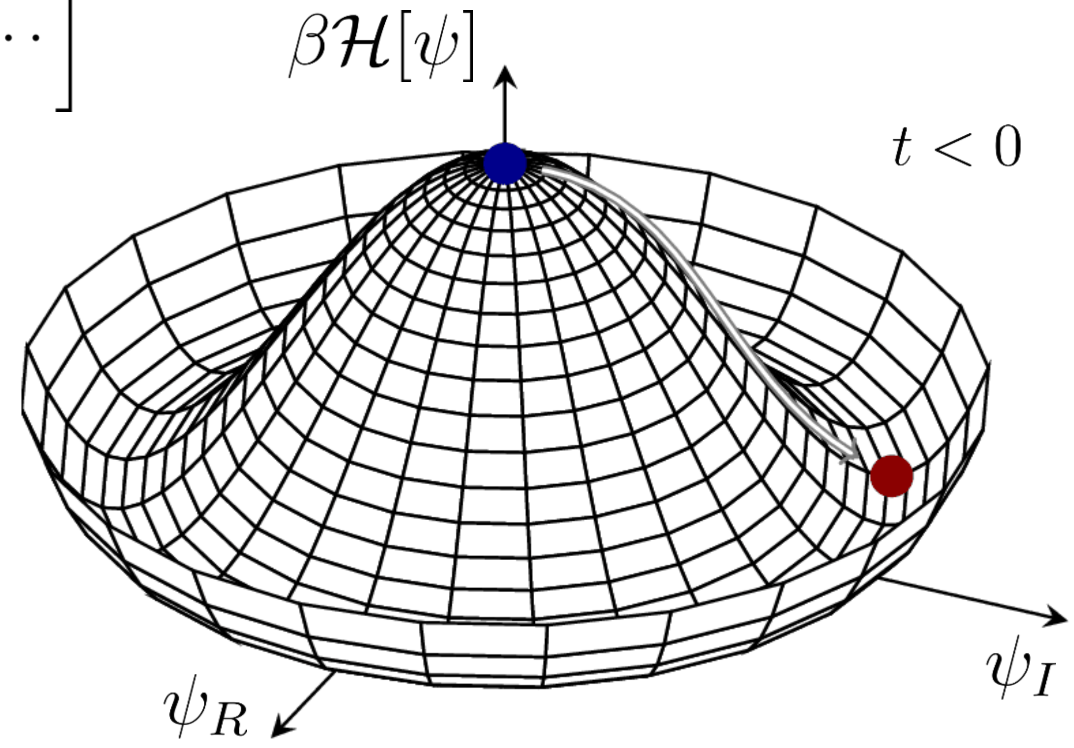

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$$\psi(\boldsymbol{x}) = \bar{\psi}e^{i\theta(\boldsymbol{x})}$$

Energy of a Goldstone mode:

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \frac{K\bar{\psi}^2}{2}\sum_{\boldsymbol{q}} q^2|\theta(\boldsymbol{q})|^2$$

Fluctuations

$$\vec{\phi}(\boldsymbol{x}) = \left[\bar{\phi} + \delta\phi_l(\boldsymbol{x})\right] \hat{e}_l + \sum_{\alpha=2}^{n} \phi_{t,\alpha}(\boldsymbol{x})\hat{e}_\alpha$$

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$$G_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}') = \langle \delta\phi_\alpha(\boldsymbol{x})\delta\phi_\beta(\boldsymbol{x}')\rangle$$

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$$\mathcal{P}\left[\vec{\phi}(\boldsymbol{x})\right] = \mathcal{N}e^{-\int d^d\boldsymbol{x}\left[\frac{K}{2}(\nabla\phi)^2+\frac{t}{2}\phi^2+u\phi^4\right]}$$

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$$\langle \delta\phi_\alpha(\boldsymbol{q})\delta\phi_\beta(\boldsymbol{q}')\rangle = (2\pi)^d \frac{\delta^{(d)}(\boldsymbol{q}+\boldsymbol{q}')\delta_{\alpha\beta}}{K(q^2+\xi_\alpha^{-2})}$$

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$$\langle \psi(\boldsymbol{x})\psi^*(\boldsymbol{0})\rangle = \bar{\psi}^2 \left\langle e^{i[\theta(\boldsymbol{x})-\theta(\boldsymbol{0})]}\right\rangle$$

$$\lim_{\boldsymbol{x}\to\infty}\langle \psi(\boldsymbol{x})\psi^*(\boldsymbol{0})\rangle = \begin{cases} \bar{\psi}^2 & \text{for } d>2 \\ 0 & \text{for } d\le 2 \end{cases}$$

Mermim-Wagner Theorem

There is no spontaneous breaking of a continuous symmetry in systems with short-range interactions in dimensions $d \le 2$.

Perturbation Theory

$$\beta\mathcal{H} = \underbrace{\int d^d\boldsymbol{x}\left[\frac{t}{2}\phi^2 + \frac{K}{2}(\nabla\phi)^2 + \cdots\right]}_{\beta\mathcal{H}_0} + \underbrace{\int d^d\boldsymbol{x}\, u\phi^4 + \cdots}_{\mathcal{U}}$$

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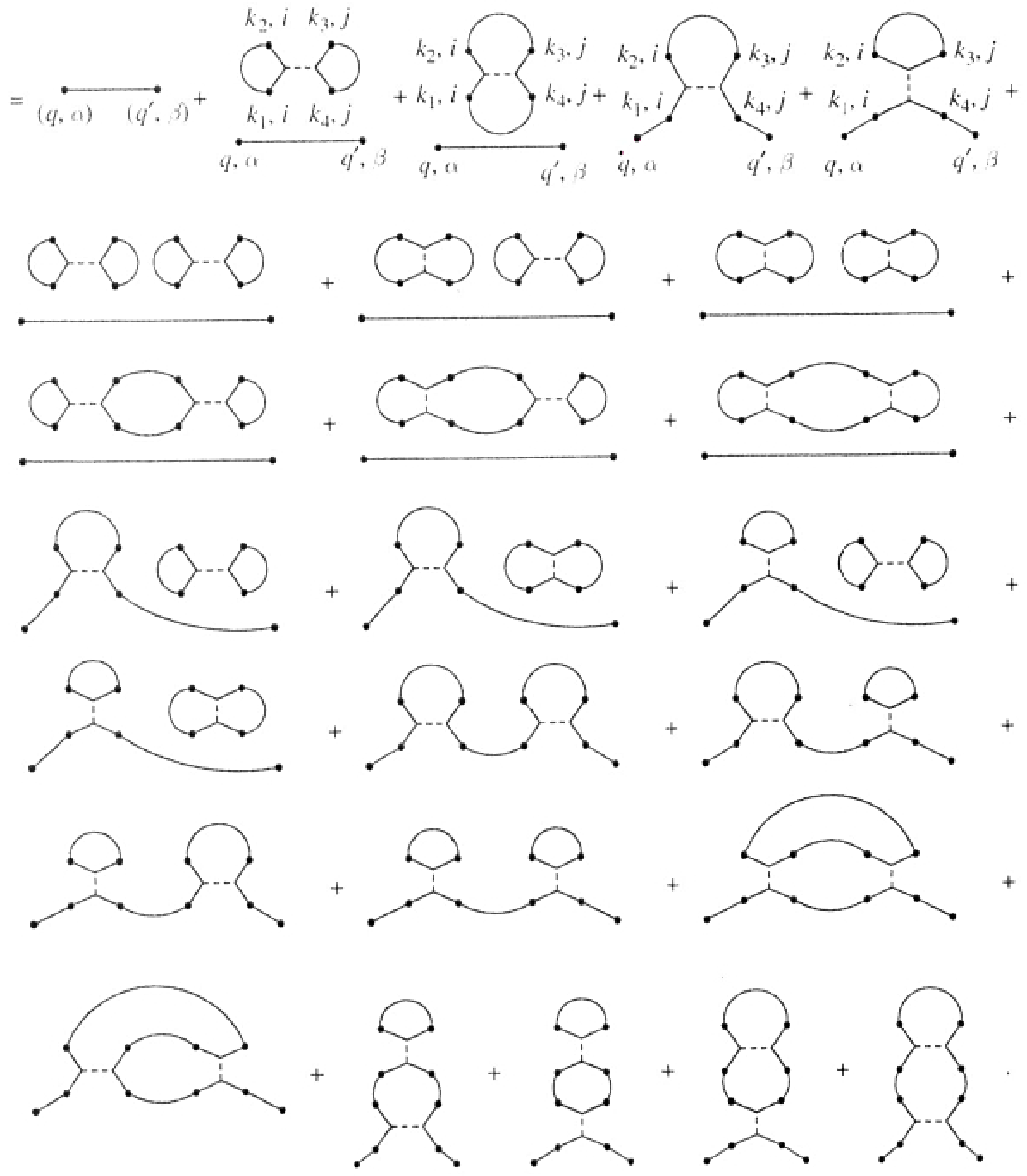
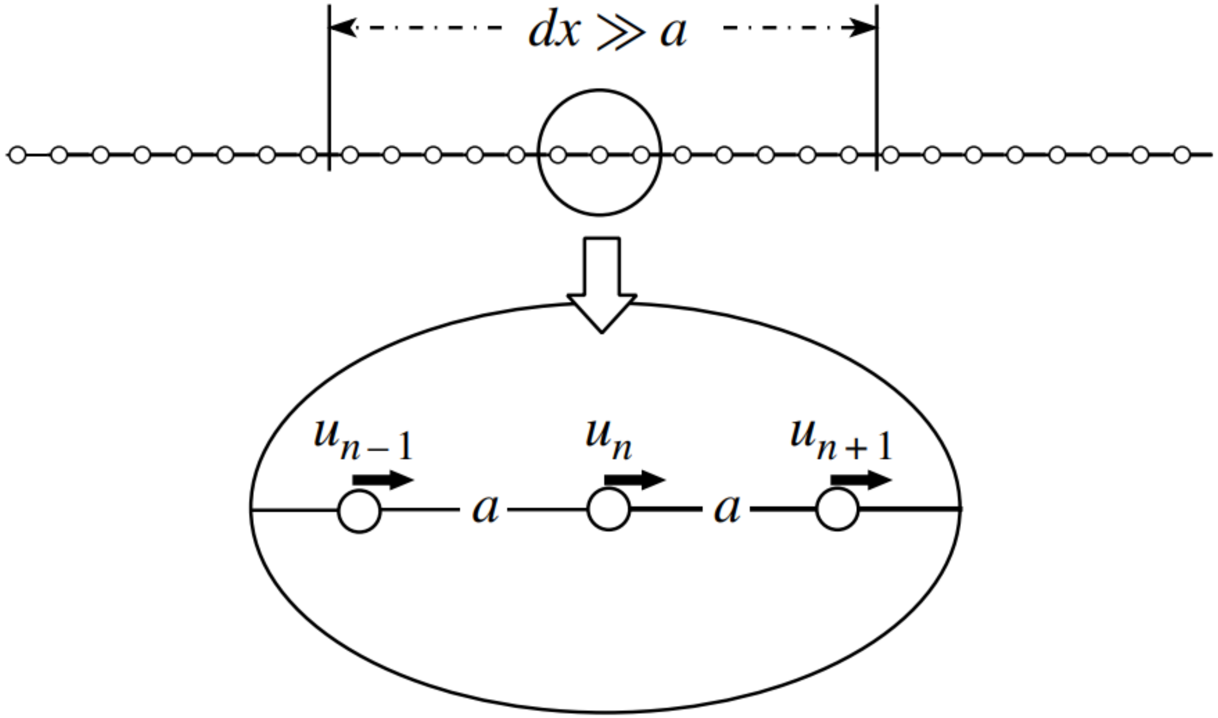
Wick's Theorem

$$\left\langle \prod_{\mu=1}^{\ell} \phi_\mu \right\rangle_0 = \begin{cases} 0 & \text{for } \ell \text{ odd} \\ \text{sum over all pairwise contractions} & \text{for } \ell \text{ even.} \end{cases}$$



=
(q, α)
(q′, β)
+
k2, i
k3, j
k1, i
k4, j
q, α
q′, β

Conclusion



$$\mathcal{Z} = \int \mathcal{D}\vec{\phi}(\boldsymbol{x}) e^{-\beta \mathcal{H}[\vec{\phi}(\boldsymbol{x})]}$$

Thanks ;)