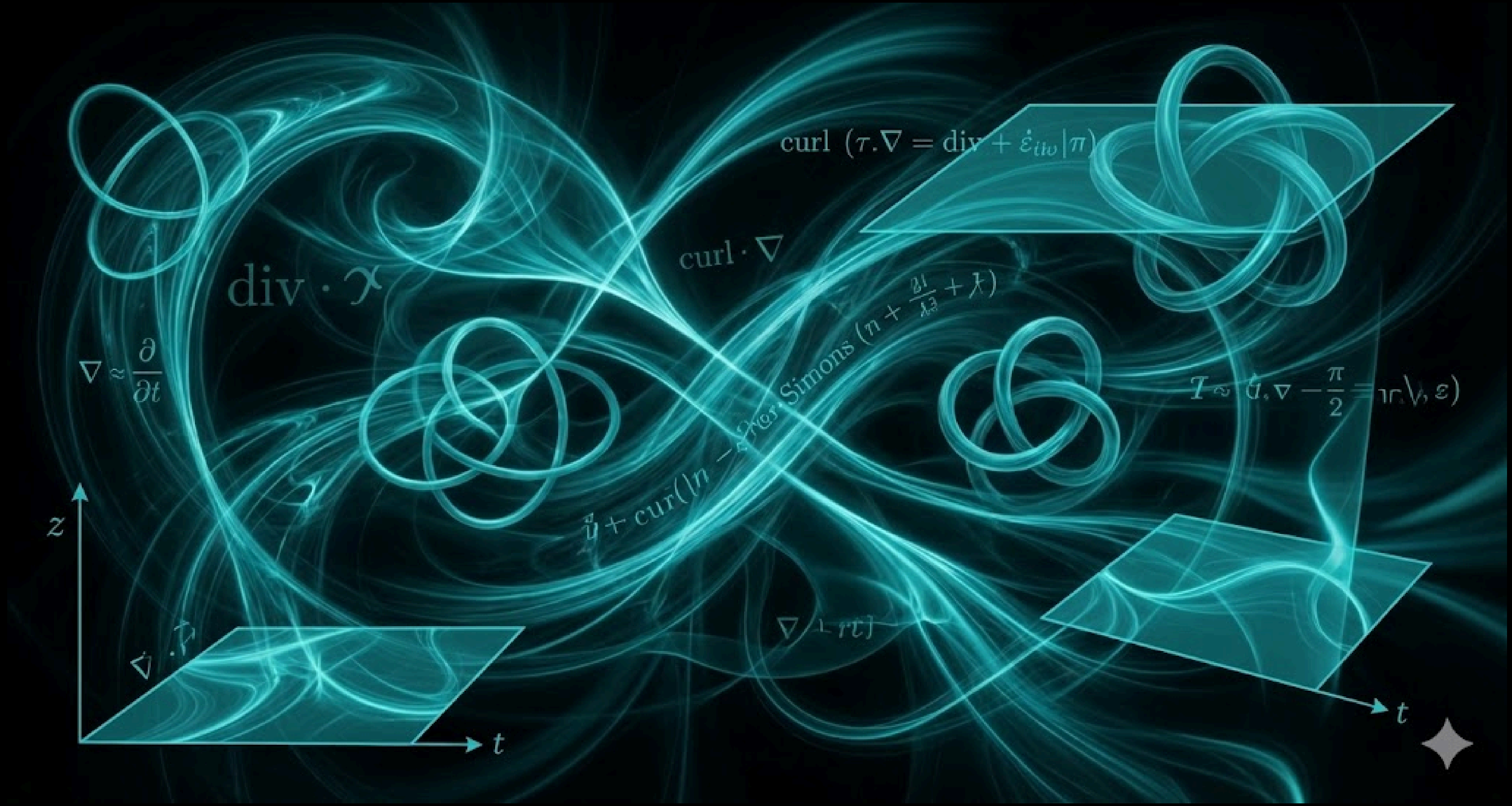
QED in 2+1 dimentions & Chern-Simons Theory



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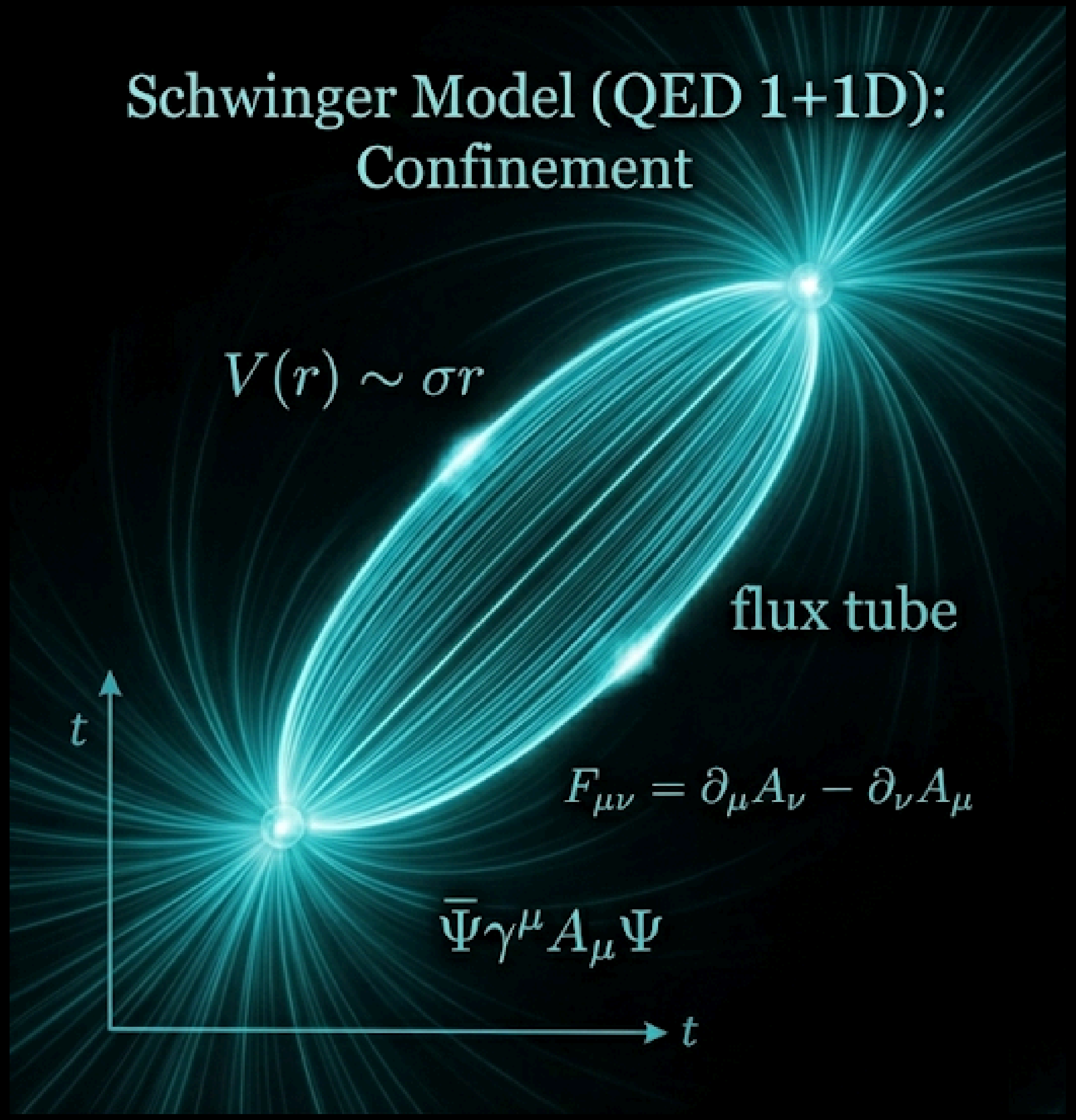
Introduction and Motivation

Theoretical modification for low-dimensional field theories

Schwinger Model: QED in 1+1 dimentions

- Exhibits confinements of fermions;
- Is a toy model for QCD;
- This model can be solved exactly;



Models like these, serve as toy models for gaining a better understanding os some phenomena in field theory

Furthermore, 2+1 dimensional models find applications in real physical systems, such as in condensed matter on surfaces.

Maxwell Theory in 2+1 dimensions

The Lagrangian itself appears the same, only some considerations regarding the dimensions of the fields will be necessary.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\gamma^{\mu}\partial_{\mu})\psi + e\overline{\psi}\gamma^{\mu}\psi A_{\mu} - m\overline{\psi}\psi$$

The action must be dimensionless: $[S] = M^0$ given that $S = \int d^3x\mathcal{L} \Rightarrow [\mathcal{L}] = M^3$

$\mathcal{L}_{\text{Dirac}} = \overline{\psi}(\mathrm{i}\gamma^{\mu}\partial_{\mu})\psi \Rightarrow [\mathcal{L}] = [\psi]^2[\partial]$ since $[\partial] = M^1 \Rightarrow M^3 = M^1[\psi]^2$ so $[\psi] = M^1$

$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ dimensionally $[\mathcal{L}] = [F]^2$ as $[F] = [\partial][A] \to M^3 = M^2[A] \Rightarrow [A] = M^{\frac{1}{2}}$

Looking at the interaction term $\mathcal{L} \supset e\overline{\psi}\gamma^{\mu}\psi A_{\mu} \Rightarrow [e] = M^{\frac{1}{2}}$

We will see later the consequences of the coupling constant having a positive mass dimension.

Chenr -Simons Term

$$\frac{\mu}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_{\alpha}$$

Maxwell-Chern-Simons

In addition to the conventional terms in the QED Lagrangian, in 2+1 dimensions there is an extra term.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\gamma^{\mu}\partial_{\mu})\psi + e\overline{\psi}\gamma^{\mu}\psi A_{\mu} - m\overline{\psi}\psi + \underbrace{\frac{\mu}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_{\alpha}}_{\text{Chern-Simons Term}}$$

Invariance of the Chern-Simons term under gauge U(1):

Under gauge U(1) the photom field transforms like: $A_{\mu} \to A_{\mu} + \partial_{\mu}\Lambda$

$$\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_{\alpha} \to \epsilon^{\mu\nu\alpha}[\partial_{\mu}(A_{\nu} - \partial_{\nu}\Lambda) - \partial_{\nu}(A_{\mu} - \partial_{\mu}\Lambda)](A_{\alpha} + \partial_{\alpha}\Lambda) = \epsilon^{\mu\nu\alpha}F_{\mu\nu}(A_{\alpha} + \partial_{\alpha}\Lambda)$$

The variantion is $\epsilon^{\mu\nu\alpha}F_{\mu\nu}\partial_{\alpha}\Lambda = \partial_{\alpha}\left(\epsilon^{\mu\nu\alpha}F_{\mu\nu}\Lambda\right)$

The dimension of the coupling constant is: $[\mu] = M^{1}$

Maxwell-Chern-Simons: The Mass of the photon

Considering only the Lagrangian for the photon field $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\mu}{4} \epsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha$

The equation of motion will be: $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial A_\nu} \Rightarrow \partial_\mu F^{\mu\nu} + \frac{\mu}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0$

By differentiating, we obtain Bianchi's identity. $\partial_\nu \left(\partial_\mu F^{\mu\nu} + \frac{\mu}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} \right) = 0 \Rightarrow \epsilon^{\nu\alpha\beta} \partial_\nu F_{\alpha\beta} = 0$

In this theory, the electric field is a vector. $\vec{E} = (E_x, E_y)$ And the magnetic field is a pseudoscalar B

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1) \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & -B \\ E_y & B & 0 \end{pmatrix} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & -B \\ -E_y & B & 0 \end{pmatrix}$$

Maxwell-Chern-Simons: The Mass of the photon

Defining the Dual field: $\tilde{F}^{\mu} = \frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \leftrightarrow F^{\mu\nu} = -\epsilon^{\mu\nu\rho}\tilde{F}_{\rho}$

$\tilde{F}^{0} = \frac{1}{2}(\epsilon^{012}F_{12} + \epsilon^{021}F_{21}) = F_{12} = -B$ and $\tilde{F}^{i} = \frac{1}{2}(\epsilon^{i0j}F_{0j} + \epsilon^{ij0}F_{j0}) = E_j$

Writing the equation of motion in terms of the dual field.

$$\partial_{\nu}(-\epsilon^{\mu\nu\rho}\tilde{F}_{\rho}) + \mu\tilde{F}^{\mu} = 0 \Rightarrow \epsilon^{\mu\nu\rho}\partial_{\nu}\tilde{F}_{\rho} = -\mu\tilde{F}^{\mu}$$

$$\epsilon_{\alpha\beta\mu}\partial^{\beta}(\epsilon^{\mu\nu\rho}\partial_{\nu}\tilde{F}_{\rho}) = -\mu\epsilon_{\alpha\beta\mu}\partial^{\beta}\tilde{F}^{\mu}$$

using the relation: $\epsilon_{\alpha\beta\mu}\epsilon^{\mu\nu\rho} = -(\delta^{\nu}_{\alpha}\delta^{\rho}_{\beta} - \delta^{\rho}_{\alpha}\delta^{\nu}_{\beta})$

$$\epsilon_{\alpha\beta\mu}\epsilon^{\mu\nu\rho}\partial^{\beta}\partial_{\nu}\tilde{F}_{\rho} = -(\delta^{\nu}_{\alpha}\delta^{\rho}_{\beta} - \delta^{\rho}_{\alpha}\delta^{\nu}_{\beta})\partial^{\beta}\partial_{\nu}\tilde{F}_{\rho} = -\partial^{\rho}\partial_{\alpha}\tilde{F}_{\rho} + \partial^{\nu}\partial_{\nu}\tilde{F}_{\alpha}$$

From: $\partial^{\rho}\tilde{F}_{\rho} = 0 \Rightarrow \epsilon_{\alpha\beta\mu}\epsilon^{\mu\nu\rho}\partial^{\beta}\partial_{\nu}\tilde{F}_{\rho} = \Box\tilde{F}_{\alpha}$

Maxwell-Chern-Simons: The Mass of the photon

Therefore $\epsilon_{\alpha\beta\mu}\partial^{\beta}(\epsilon^{\mu\nu\rho}\partial_{\nu}\tilde{F}_{\rho}) = -\mu\epsilon_{\alpha\beta\mu}\partial^{\beta}\tilde{F}^{\mu} \Rightarrow \Box\tilde{F}_{\alpha} = -\mu\epsilon_{\alpha\beta\mu}\partial^{\beta}\tilde{F}^{\mu}$

From the equation of motion $\epsilon_{\mu\nu\rho}\partial^{\nu}\tilde{F}^{\rho} = -\mu\tilde{F}_{\mu}$

$$\Box\tilde{F}_{\alpha} = -\mu\epsilon_{\alpha\beta\mu}\partial^{\beta}\tilde{F}^{\mu} \Rightarrow \Box\tilde{F}_{\alpha} = -\mu(-\mu\tilde{F}_{\alpha})$$

The equation of motion for the electric and magnetic fields in this case becomes:

$$(\Box - \mu^2)\tilde{F}_{\alpha} = 0 \qquad \tilde{F}_{\alpha} = (-B, E_y, E_x)^T$$

The photon acquires mass without breaking the U(1) symmetry.

Maxwell-Chern-Simons: The Topological origin of the photon Mass

The Chern-Simons term is said to be topological because it does not depend on the local geometry of spacetime (metric), but only on the global structure.

$$S_{\text{Maxwell}} = \int d^3x \sqrt{-g} \left(-\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} F_{\mu\nu} \right)$$

This term depends on the local geometry of the space; if the space is deformed, this term changes accordingly.

$$S_{CS} = \int d^3x \frac{\mu}{4} \epsilon^{\mu\nu\alpha} A_\mu F_{\nu\alpha}$$

This term, however, is invariant under these distortions; in a curved spacetime, it is invariant. This term is a measure of topological aspects of space. It would be as if common terms measured areas, while this term measures something like knots and holes.

Maxwell-Chern-Simons: Why only in 2+1 D

Why can we only write this term in 3 dimensions?

$$\epsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha \xrightarrow{d=4} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} A_\alpha$$

This term is not a Lorentz invariant.

Actually, it is possible to write down a "Chern-Simons theory" in any odd space-time dimension.

For example, in 4+1 dimensions it is possible to write the term

$$\mathcal{L} \supset \epsilon^{\mu\nu\alpha\beta\rho} F_{\mu\nu} F_{\alpha\beta} A_\rho$$

Can a photon have mass?

- The existence of this term was postulated because it is invariant.
- We follow Gell-Mann totalian principle “Everything not forbidden is compulsory.”
- But the question remains, if I hadn't explicitly included this term, could it have been dynamically induced in some other way?

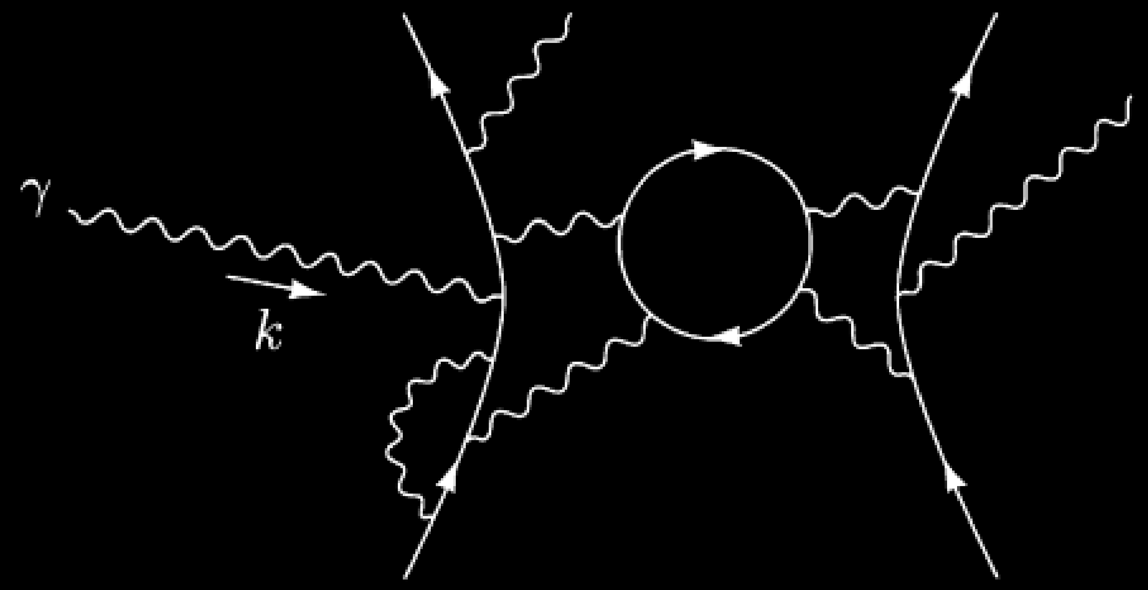
Ward-Takahashi Identity: The 2+1 dimensional scenario

On page 189 of the lecture notes, it ends with the following statement:

$$k_\mu \Pi_2^{\mu\nu} = 0$$

"This implies that the photon must remain massless even under radiative corrections."

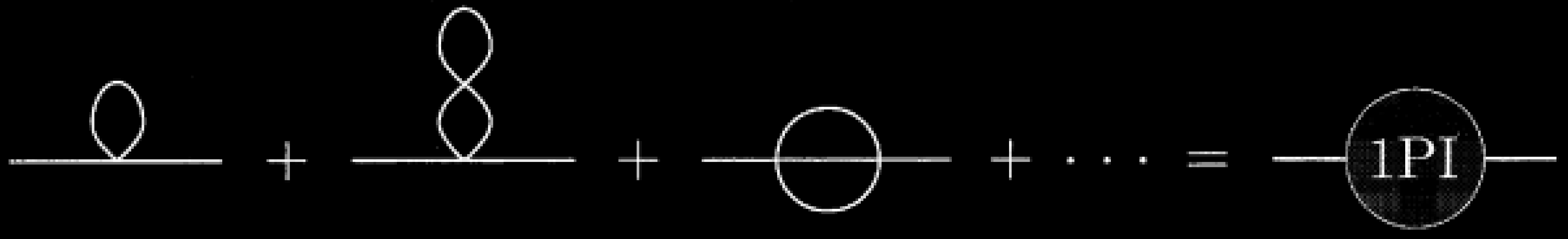
We'll see that the scenario here is considerably different. The identity remains preserved, but the interpretation regarding the generation of radioactive mass will no longer be valid.



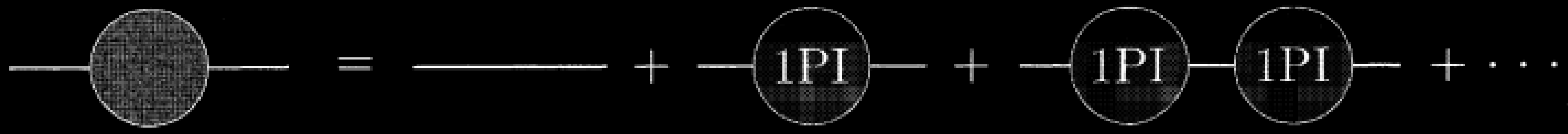
But first, let's review some definitions.

$\Pi_2^{\mu\nu}$

$\Pi_2^{\mu\nu}$ is the 1PI function from two points for the photon.



So that the complete propagator is given by



Great! But how can we get the expressions for these guys?

Effective Action

Let's define the effective action. First, we look for a functional that manages only the connected Green functions.

$$Z[J] = e^{-W[J]} \Rightarrow W[J] = -\ln\{Z[J]\}$$

$$\Phi_{cl}[J] = \frac{\delta W[J]}{\delta J(x)}$$

This classical field is defined as being "not very relativistic," meaning no particles are created, but all quantum corrections are included.

$$\hat{\Gamma}[\Phi_{cl}] \equiv \Gamma^1 . \Phi_{cl} + \frac{1}{2}\Pi^2 . \Phi_{cl} . \Phi_{cl} + \frac{1}{3!}\Gamma^3 . \Phi_{cl} . \Phi_{cl} . \Phi_{cl} + \dots$$

$$\Gamma^n(x_1, \dots, x_n) = \frac{\delta^n}{\delta\Phi_{cl} \dots \delta\Phi_{cl}} \hat{\Gamma}[\Phi_{cl}]\Big|_{\Phi_{cl}=0}$$

for $n \neq 2$

$$\Pi^2(x_1, x_2) = \frac{\delta^2}{\delta\Phi_{cl}(x_1)\delta\Phi_{cl}(x_2)} \hat{\Gamma}[\Phi_{cl}]\Big|_{\Phi_{cl}=0}$$

Finally, we can define the effective action

Effective Action

Therefore, effective action is defined as follows: $$\Gamma[\Phi_{cl}] = \hat{\Gamma}[\Phi_{cl}] + \frac{1}{2}\Phi_{cl}.\, D_0^{-1}.\, \Phi_{cl}$$

It satisfies a relationship analogous to that of classical action: $$\frac{\delta}{\delta\Phi_{cl}}\Gamma[\Phi_{cl}] = J(x)$$

But the effective action is the action that actually dictates the behavior of this field, including quantum fluctuations.

The complete propagator, considering all quantum corrections, can be obtained from:

$$\frac{\delta^2}{\delta\Phi_{cl}(x)\delta\Phi_{cl}(y)}\Gamma[\Phi_{cl}] = D_C^{-1}(x,y) = \Pi_2(x,y) + D_0^{-1}(x-y)$$

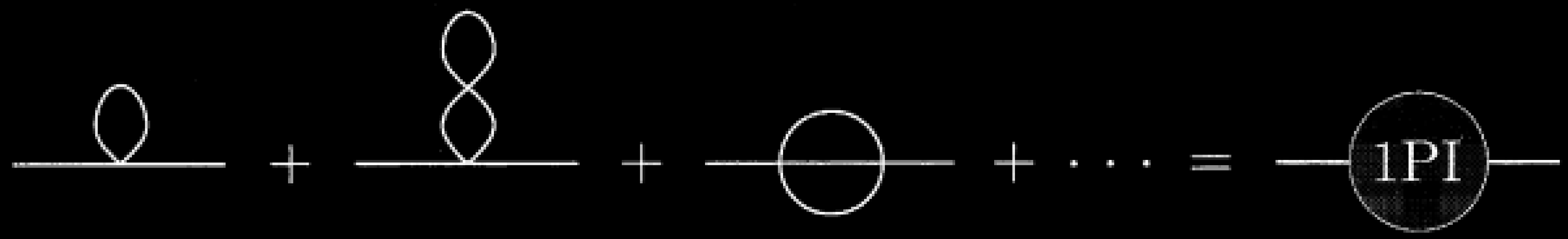
$$k_\mu \Pi_2^{\mu\nu} = 0$$

We want to verify whether the conclusion we reached regarding Ward's identity in the lecture notes holds true within a theory that classically allows for a massive photon.

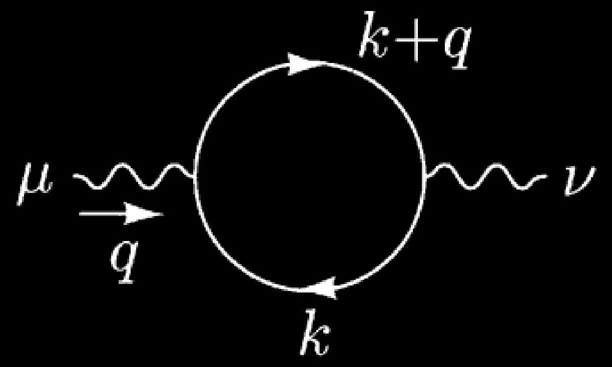
In the case of QED, who is this $\Pi_2^{\mu\nu}$?

$$\equiv i\Pi_2^{\mu\nu}$$

This is very complicated, because



Let's focus on a single contribution, a fermion loop:



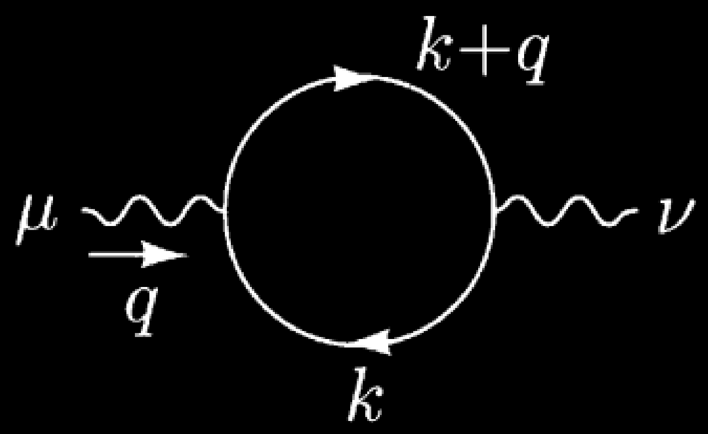
Who's afraid of infinity?

Quantum Effects

Throughout the course we avoided calculating diagrams with loops, but in QED3, the problem becomes a little easier. Furthermore, the modification of the Dirac matrices allows for a whole new phenomenology. To show this, consider pure QED, without the Chern-Simons term but in 2+1 dimensions.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\gamma^\mu\partial_\mu)\psi + e\overline{\psi}\gamma^\mu\psi A_\mu - m\overline{\psi}\psi$$

Feynman's rules will be the same as those of QED in 3+1 dimensions.



$$= (-\mathrm{i}e)^2(-1)\int \frac{d^3k}{(2\pi)^3}\mathrm{Tr}\left[\gamma^\mu \frac{i}{\gamma^\alpha k_\alpha - m}\gamma^\nu \frac{i}{\gamma^\beta (k+q)_\beta - m}\right]$$

$$\equiv i\Pi_2^{\mu\nu}(q)$$

This diagram is discussed extensively in section 7.5 of Peskin's book, where it accounts for the renormalization of electric charge. In what follows, I will highlight the similarities and differences.

Traces of Dirac matrices in 2+1 Dimensions

Dirac algebra in 2+1 dimensions and Trace identities

In 2+1 dimensions we will have 3 DIrac matrices with the following representation:

$$\gamma^0 = \sigma^3, \quad \gamma^1 = \mathrm{i}\sigma^1, \quad \gamma^2 = \mathrm{i}\sigma^2$$

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1)$$

$$\gamma^\mu \gamma^\nu = -g^{\mu\nu} + \mathrm{i}\epsilon^{\mu\nu\alpha}\gamma_\alpha$$

Unlike the case in 3+1 dimensions, where: $\mathrm{Tr}[\underbrace{\gamma^\mu\gamma^\nu\gamma^\alpha \dots \gamma^\rho}_{n}] = 0$ for odd n. Here

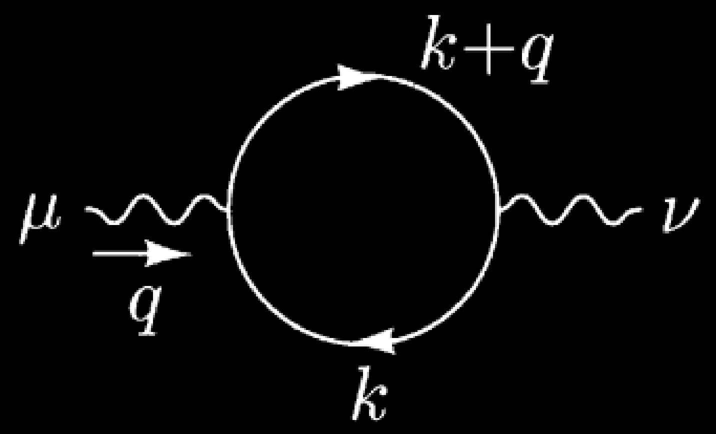
$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha] = \mathrm{Tr}[(-g^{\mu\nu} + \mathrm{i}\epsilon^{\mu\nu\rho}\gamma_\rho)\gamma^\alpha] = -g^{\mu\nu}\underbrace{\mathrm{Tr}[\gamma^\alpha]}_{=0} + \mathrm{i}\epsilon^{\mu\nu\rho}\mathrm{Tr}[\gamma_\rho\gamma^\alpha] = \mathrm{i}\epsilon^{\mu\nu\rho}\mathrm{Tr}[\gamma_\rho\gamma^\alpha]$$

$$\mathrm{Tr}[\gamma_\rho\gamma^\alpha] = g_{\beta\rho}\mathrm{Tr}[\gamma^\beta\gamma^\alpha] = g_{\beta\rho}\mathrm{Tr}[-g^{\beta\alpha} + \mathrm{i}\epsilon^{\beta\alpha\delta}\gamma_\delta] = -g_{\beta\rho}g^{\beta\alpha}\mathrm{Tr}[I_{2\times 2}] = -2\delta^\alpha_\rho$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha] = -2\mathrm{i}\epsilon^{\mu\nu\alpha}$$ Which is completly different from 3+1

Back to the loop

Quantum Effects : Loop calculation



$$= (-\mathrm{i}e)^2(-1)\int \frac{d^3k}{(2\pi)^3}\mathrm{Tr}\left[\gamma^\mu \frac{i}{\gamma^\alpha k_\alpha - m}\gamma^\nu \frac{i}{\gamma^\beta (k+q)_\beta - m}\right]$$

$$\equiv i\Pi_2^{\mu\nu}(q)$$

Let's move all the matrices to the numerator.

$$\mathrm{i}\Pi_2^{\mu\nu}(q) = -(-\mathrm{i}e)^2\int \frac{d^3k}{(2\pi)^3}\mathrm{Tr}\left[\gamma^\mu \frac{i(\gamma^\alpha k_\alpha + m)}{k^2 - m^2}\gamma^\nu \frac{i(\gamma^\beta (k+q)_\beta + m)}{(k+q)^2 - m^2}\right]$$

Let's focus on matrices.

$$\mathrm{Tr}\left[\gamma^\mu \frac{i(\gamma^\alpha k_\alpha + m)}{k^2 - m^2}\gamma^\nu \frac{i(\gamma^\beta (k+q)_\beta + m)}{(k+q)^2 - m^2}\right] = \frac{\mathrm{i}^2}{(k^2-m^2)((k+q)^2-m^2)}\mathrm{Tr}\left[\gamma^\mu(\gamma^\alpha k_\alpha + m)\gamma^\nu(\gamma^\beta(k+q)_\beta + m)\right]$$

Quantum Effects : Loop calculation

Because of the new result that the trace of three Dirac matrices will not result in zero, we have a completely new term.

$$\mathrm{Tr}\left[\gamma^\mu(\gamma^\alpha k_\alpha + m)\gamma^\nu(\gamma^\beta(k+q)_\beta + m)\right] = k_\alpha(K+q)_\beta \mathrm{Tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta] + m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] + \underbrace{mk_\alpha\mathrm{Tr}[\gamma^\mu\gamma^\alpha\gamma^\nu] + m(k+q)_\beta\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\beta]}_{\neq 0 \quad \text{because we are in 2+1 D}}$$

For this new term we have $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha] = -2\mathrm{i}\epsilon^{\mu\nu\alpha}$

Which results in $mk_\alpha\mathrm{Tr}[\gamma^\mu\gamma^\alpha\gamma^\nu] + m(k+q)_\beta\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\beta] = m(-2\mathrm{i}\epsilon^{\mu\nu\alpha})k_\alpha + m(-2\mathrm{i}\epsilon^{\mu\beta\nu})(k+q)_\beta = 2\mathrm{i}m\epsilon^{\mu\nu\alpha}q_\alpha$

This part is new!

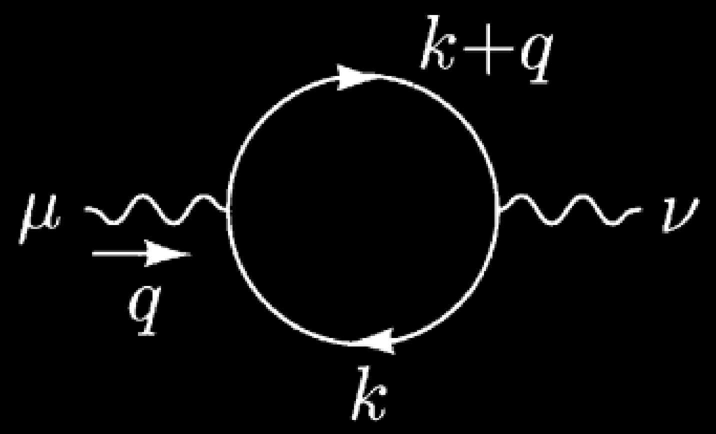
The other term is what is found in peskin and which generates the shielding effect on the electrical charge.

It is also the time that can be written as: $\Pi^{\mu\nu}_{old}(q) = (q^2g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$

Which leads to the conclusion I mentioned earlier in my lecture notes on page 189.

Quantum Effects : Loop calculation

Generally speaking, we have



$$\equiv i\Pi_2^{\mu\nu}(q) = i\Pi_{\text{old}}^{\mu\nu}(q) + i\Pi_{\text{new}}^{\mu\nu}(q)$$

The new term is only possible here due to the result for the traces of the Dirac matrices.

$$\Pi_{old}^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^{\mu}q^{\nu})\Pi(q^2)$$

term responsible for cargo renormalization (section 7.5 of Peskin's book and page 189 of lecture notes)

$$\Pi_{\text{new}}^{\mu\nu}(q) = e^2 \int \frac{d^3k}{(2\pi)^3} \frac{2i\epsilon^{\mu\nu\alpha}}{(k^2 - m^2)((k+q)^2 - m^2)}$$

A new term whose effect we will try to discover.

Quantum Effects of the new term

Let's calculate this term:

$$\Pi^{\mu\nu}_{\text{new}}(q) = e^2 \int \frac{d^3k}{(2\pi)^3} \frac{2im\epsilon^{\mu\nu\alpha}q_\alpha}{(k^2 - m^2)((k+q)^2 - m^2)}$$

We can simplify this integral using a method known as the Feynman parameter.

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}$$

The "trick" consists of merging the two distinct terms of the denominator into one. By doing so, we transform the problem into an integral that depends on the square of the displaced momentum, which is easy to solve using spherical coordinates. The price of this simplification is that we exchange a difficult momentum integral for additional integrals in the new Feynman parameter.

Quantum Effects of the new term

In our case, we can use the exact same substitution that Peskin uses in calculating the renormalization of electric charge.

$$\frac{1}{(k^2 - m^2)((k+q)^2 - m^2)} = \int_0^1 dx \frac{1}{(k^2 + 2xkq + xq^2 - m^2)^2} = \int_0^1 dx \frac{1}{(\ell^2 + x(1-x)q^2 - m^2)^2}$$

where: $\ell = k + xq$

we can also define: $\Delta = m^2 - x(1-x)q^2$ such that

$$\frac{1}{(k^2 - m^2)((k+q)^2 - m^2)} = \int_0^1 dx \frac{1}{(\ell^2 - \Delta)^2}$$

Quantum Effects of the new term

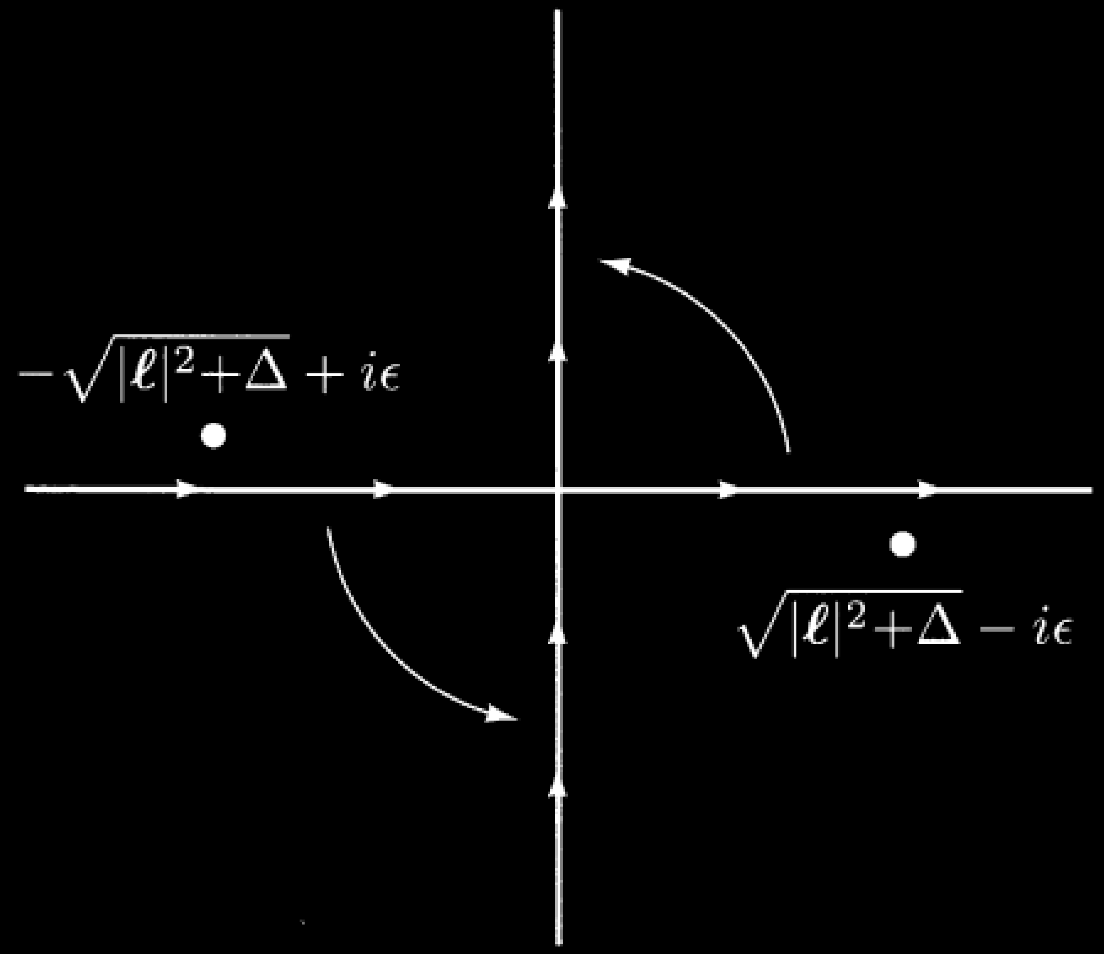
We can rewrite the term we want to calculate as follows.

$$\Pi^{\mu\nu}_{\text{new}}(q) = e^2 2\mathrm{i} m \epsilon^{\mu\nu\alpha} q_\alpha \int_0^1 dx \int \frac{d^3\ell}{(2\pi)^3} \frac{1}{(\ell^2 - \Delta)^2}$$

If we want to integrate into a sphere, we need to perform a Wick rotation.

$$\ell^0 \to \mathrm{i}\ell^0_E, \quad d^3\ell \to \mathrm{i}d^3\ell_E, \quad \ell^2 \to -\ell_E$$

$$\int \frac{d^3\ell}{(2\pi)^3} \frac{1}{(\ell^2 - \Delta)^2} = \mathrm{i} \int \frac{d^3\ell_E}{(2\pi)^3} \frac{1}{(\ell_E^2 + \Delta)^2}$$



Comparing dimensions

This is when being in 2+1 d saves us

2+1

$$\int \frac{d^3\ell_E}{(2\pi)^3} \frac{1}{(\ell_E^2 + \Delta)^2}$$

Solving the Euclidean integral in 3D using spherical coordinates

$$d^3\ell_E = 4\pi\ell_E^2 d\ell_E$$

$$\int \frac{d^3\ell_E}{(2\pi)^3} \frac{1}{(\ell_E^2 + \Delta)^2} = \frac{1}{2\pi^2} \int_0^\infty d\ell_E \frac{\ell_E^2}{(\ell_E^2 + \Delta)^2}$$

3+1

$$\int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{(\ell_E^2 + \Delta)^2}$$

Solving the Euclidean integral in 4D using spherical coordinates

$$d^4\ell_E = 2\pi^2\ell_E^3 d\ell_E$$

$$\int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{(\ell_E^2 + \Delta)^2} = \frac{1}{8\pi^2} \int d\ell_E \frac{\ell_E^3}{(\ell_E^2 + \Delta)^2}$$

In[1]:= `Integrate[x^2/(x^2 + δ)^2, {x, 0, Infinity}]`

integra infinito

Out[1]= $\frac{\pi}{4\sqrt{\delta}}$ if $\mathrm{Re}[\sqrt{\delta}] > 0$

`Integrate[x^3/(x^2 + δ)^2, {x, 0, Infinity}]`

integra infinito

Integrate: Integral of $\frac{x^3}{(x^2+\delta)^2}$ does not converge on $\{0, \infty\}$.

Out[2]= $\int_0^\infty \frac{x^3}{(x^2+\delta)^2}\,dx$

Quantum Effects of the new term

The dimensionality of the problem saved us from divergent integrals. We can proceed with no fear of infinities.

$$\int \frac{d^3\ell_E}{(2\pi)^3}\frac{1}{(\ell_E^2+\Delta)^2} = \frac{1}{2\pi^2}\int_0^\infty d\ell_E \frac{\ell_E^2}{(\ell_E^2+\Delta)^2} = \frac{1}{2\pi^2}\frac{\pi}{4\sqrt{\Delta}}$$

Is that really so?

$$\Pi^{\mu\nu}_{\text{new}}(q) = e^2 2\mathrm{i} m \epsilon^{\mu\nu\alpha} q_\alpha \int_0^1 dx \frac{\mathrm{i}}{2\pi^2}\frac{\pi}{4\sqrt{\Delta}} = -2e^2 m \epsilon^{\mu\nu\alpha} q_\alpha \frac{1}{8\pi}\int_0^1 \frac{1}{\sqrt{\Delta}}$$

Quantum Effects of the new term

This integral also has a finite result.

$$\int_0^1 dx \frac{1}{\sqrt{m^2 - x(1-x)q^2}} = \frac{1}{q}\ln\left(\frac{1+q/2|m|}{1-q/2|m|}\right) \xrightarrow{q\to 0} \frac{q}{q|m|}$$

We are interested in low-energy physics.

We found the finite result for this term.

$$\mathrm{i}\Pi_2^{\mu\nu}(q) = -\frac{e^2}{4\pi}\frac{m}{|m|}\epsilon^{\mu\nu\alpha}q_\alpha$$

We calculated (partially) the loop diagram without problems with infinity!

We calculated this term for two reasons.

- **First, we want to know if this new term maintains Ward's identity intact;**
- **second, we want to know what effect this guy has.**

Ward-Takahashi Identity

Returning to this question if $q_\mu \Pi_2^{\mu\nu} = 0$ still holds

$$\Pi_2^{\mu\nu}(q) \propto \epsilon^{\mu\nu\alpha} q_\alpha \Rightarrow q_\mu \Pi_2^{\mu\nu}(q) \propto \epsilon^{\mu\nu\alpha} q_\mu q_\alpha = -\epsilon^{\mu\nu\alpha} q_\mu q_\alpha = 0$$

The product of two tensors, one symmetric and the other antisymmetric, will be zero.

We verified that this new term does not break this identity, therefore it does not violate gauge invariance.

But what is the effect of this term?

Effective Action

Let's get back to our effective action.

$$\frac{\delta^2}{\delta\Phi_{cl}(x)\delta\Phi_{cl}(y)}\Gamma[\Phi_{cl}] = \Pi_2(x,y) + D_0^{-1}(x-y) \Rightarrow \Gamma[\Phi_{cl}] \supset \Phi_{cl}.\,\Pi_2(x,y).\,\Phi_{cl}$$

Let's focus on the part induced by the two-point function. $\mathrm{i}\Pi_2^{\mu\nu}(q) = -\frac{e^2}{4\pi}\mathrm{sgn}(m)\epsilon^{\mu\nu\alpha}q_\alpha$

In the space of moments $\Gamma^{(2)} = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3} A_\mu(-q)\Pi_2^{\mu\nu}(q)A_\nu(q)$

$$\Gamma^{(2)} = -\mathrm{sgn}(m)\frac{e^2}{8\pi}\int \frac{d^3q}{(2\pi)^3} A_\mu(-q)\epsilon^{\mu\nu\alpha}q_\alpha A_\nu(q)$$

Performing a Fourier transform for the position space: $\mathrm{i}q_\alpha A_\nu(q) \rightarrow \partial_\alpha A_\nu(x)$

Effective Action

Finally, we have

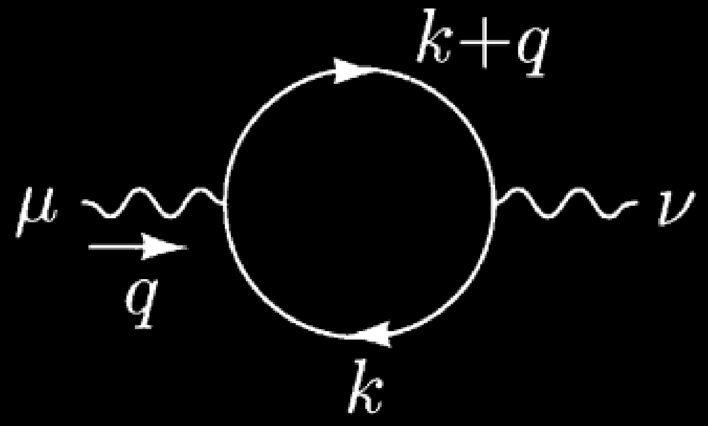
$$\Gamma^{(2)} = \frac{e^2}{8\pi}\frac{m}{|m|}\int d^3x \epsilon^{\mu\nu\alpha} A_\mu(x)\partial_\alpha A_\nu(x)$$

$$\Gamma^{(2)} = \frac{e^2}{4\pi}\frac{m}{|m|}\int d^3x \epsilon^{\mu\nu\alpha} A_\mu(x)F_{\alpha\nu}(x)$$

The effect that the extra term generates is a Chern-Simons term. We see that here, this term is not only allowed, but it is introduced by the quantum corrections of the theory!

Photon propagator Corrections

Total contribution of the loop



$$\equiv i\Pi_2^{\mu\nu}(q)$$

$$i\Pi_2^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2) + \frac{e^2}{4\pi}\mathrm{sgn}(m)\epsilon^{\mu\nu\alpha}q_\alpha$$

$$D_C^{-1}(x,y) = \Pi_2(x,y) + D_0^{-1}(x-y)$$

Complete propagator

Let's see how this term alters the photon propagation. If everything is consistent, we should see the mass term appear in the corrected propagator.

Let's focus exclusively on the corrections that the new term introduces, without worrying about the other part (Peskin7.5)

$$D^{-1}_{\mu\nu} = \underbrace{-\mathrm{i}q^2\left(g_{\mu\nu} - \frac{q^{\mu\nu}}{q^2}\right)}_{\text{Free}} - \underbrace{(-\mathrm{i})(\mathrm{i}\mu\epsilon_{\mu\nu\alpha}q^\alpha)}_{\text{Chern-Simons}}$$

where $\mu = \frac{e^2}{4\pi}$

And we define $P_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$ and $S_{\mu\nu} = \frac{\mathrm{i}\epsilon_{\mu\nu\alpha}q^\alpha}{\sqrt{q^2}}$

Complete propagator

With these operators, we can write $D^{-1}_{\mu\nu} = -\mathrm{i}\left[q^2 P_{\mu\nu} + \mu\sqrt{q^2} S_{\mu\nu}\right]$

These tensors satisfy the relations

$$(P^2)^{\nu}_{\mu} = P_{\mu\lambda} P^{\lambda\nu} \qquad \Rightarrow (P^2)^{\nu}_{\mu} = \left(g_{\mu\lambda} - \frac{q_\mu q_\lambda}{q^2}\right)\left(g^{\lambda\nu} - \frac{q^\lambda q^\nu}{q^2}\right) = \delta^{\nu}_{\mu} - \frac{q_\mu q^\nu}{q^2} = P^{\nu}_{\mu}$$

We can represent this concisely as follows: $P.P = P$

$$(S^2)^{\nu}_{\mu} = S_{\mu\lambda} S^{\lambda\nu} \Rightarrow (S^2)^{\nu}_{\mu} = -\frac{\epsilon_{\mu\lambda\alpha}\epsilon^{\lambda\nu\beta} q^\alpha q_\beta}{q^2} \qquad \epsilon_{\mu\lambda\alpha}\epsilon^{\lambda\nu\beta} = -(\delta^{\nu}_{\mu}\delta^{\beta}_{\alpha} - \delta^{\beta}_{\mu}\delta^{\nu}_{\alpha})$$

$$(S^2)^{\nu}_{\mu} = \frac{1}{q^2} q^\alpha q_\beta (\delta^{\nu}_{\mu}\delta^{\beta}_{\alpha} - \delta^{\beta}_{\mu}\delta^{\nu}_{\alpha}) = \delta^{\nu}_{\mu} - \frac{q^\mu q^\nu}{q^2} = P^{\nu}_{\mu} \qquad S.S = P$$

Complete propagator

The products of the two still need to be determined:

$$P_{\mu\lambda}S^{\lambda\nu} = \mathrm{i}\left(g_{\mu\lambda} - \frac{q_\mu q_\lambda}{q^2}\right)\frac{\epsilon^{\lambda\nu\alpha}q_\alpha}{\sqrt{q^2}} = \mathrm{i}\frac{\epsilon_\mu^{\ \nu\alpha}q_\alpha}{\sqrt{q^2}} = S_\mu^\nu$$

Similarly $S.P = P.S = S$

We want to find an operator such that: $D^{-1}_{\mu\lambda}D^{\lambda\nu} = P_\mu^\nu$

The ansatz $D_{\mu\nu} = \mathrm{i}\left(AP_{\mu\nu} + BS_{\mu\nu}\right)$

$$D^{-1}_{\mu\lambda}D^{\lambda\nu} = \left(q^2 P_{\mu\lambda} + \mu\sqrt{q^2}S_{\mu\lambda}\right)\left(AP^{\lambda\nu} + BS^{\lambda\nu}\right)$$

Complete propagator

Expanding the product

$$D^{-1}_{\mu\lambda} D^{\lambda\nu} = (q^2 A + \mu\sqrt{q^2} B) P^{\nu}_{\mu} + (q^2 B + \mu\sqrt{q^2} A) S^{\nu}_{\mu} = 1.P^{\nu}_{\mu} + 0.S^{\nu}_{\mu}$$

Solving the linear system

$$q^2 A + \mu\sqrt{q^2} B = 1$$
$$q^2 B + \mu\sqrt{q^2} A = 0$$

$$A = \frac{1}{q^2 - \mu^2} \qquad B = -\frac{\mu}{\sqrt{q^2}} \frac{1}{q^2 - \mu^2}$$

The photon propagator after this correction will be

$$D_{\mu\nu}(q) = \frac{\mathrm{i}}{q^2 - \mu^2} \left[\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \mathrm{i} \frac{\mu}{q^2} \epsilon_{\mu\nu\alpha} q^\alpha \right]$$

Note that the pole no longer occurs at zero. We have the propagator of a massive field!

Conclusions

- We show that in 2+1 dimensions a completely new term can be written in the Lagrangian of a gauge-invariant theory U(1).
- So we start from a quantum theory without this new term.
- We show how Dirac algebra in 3D changes the Trace property, allowing us to write a new term.
- This term is precisely the Chern-Simons mass term, which in this dimension preserves the Ward identity and allows for the radiative generation of mass for the photon.
- Finally, we show that the propagator pole is no longer at zero, but rather at the mass of this term.
- Throughout the text we calculate the diagram for a loop, obtaining a finite result.

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