Lattice Regularization of ϕ^4 Theory

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Outline

The Need for Regularization

- In Quantum Field Theory, calculations often lead to divergent integrals, especially in loop diagrams.
- These infinities arise from integrating over all possible momentum values, up to infinity.
- Lattice regularization make these integrals mathematically well-defined by discretizing spacetime itself, which naturally imposes a momentum cutoff.

The Lattice: A Discretized Spacetime

- We replace continuous Euclidean spacetime with a discrete grid of points.
- This grid is a hypercubical lattice, defined as:

$$\Lambda = a\mathbb{Z}^4 = \{x \mid x_\mu / a \in \mathbb{Z}\}$$

  where a is the **lattice spacing**, the smallest distance between two points.
- Continuous integrals are replaced by discrete sums:

$$\int d^4x \quad \longrightarrow \quad a^4 \sum_{x \in \Lambda}$$

- The scalar field $\phi(x)$ is now only defined at the points x on the lattice Λ.

Finite Difference Operators

- Continuous derivatives are replaced by finite differences on the lattice.
- **Forward Derivative**:

$$\Delta_\mu^f \phi(x) = \frac{1}{a}[\phi(x + a\hat{\mu}) - \phi(x)]$$

- **Backward Derivative**:

$$\Delta_\mu^b \phi(x) = \frac{1}{a}[\phi(x) - \phi(x - a\hat{\mu})]$$

where $\hat{\mu}$ is the unit vector in the μ direction.

Summation by Parts

- The lattice analogue of integration by parts is "summation by parts".
- It connects the forward and backward derivatives:

$$a^4 \sum_x (\Delta^f_\mu \phi(x))\varphi(x) = -a^4 \sum_x \phi(x)(\Delta^b_\mu \varphi(x))$$

  (Assuming boundary terms vanish, often by using periodic boundary conditions).
- This property is crucial for manipulating lattice actions in a way that mirrors the continuum.

The Lattice d'Alembert Operator

- The d'Alembert operator ($\Box = -\partial_\mu \partial^\mu$ in the continuum) is built from the difference operators.
- Applying summation by parts to the kinetic term:

$$a^4 \sum_x (\Delta_\mu^f \phi)(\Delta_\mu^f \phi) = -a^4 \sum_x \phi(\Delta_\mu^b \Delta_\mu^f \phi)$$

- This defines the lattice d'Alembertian:

$$\Box \equiv -\Delta_\mu^b \Delta_\mu^f$$

- Its explicit action on a field $\phi(x)$ is:

$$\Box\phi(x) = \sum_\mu \frac{1}{a^2}[2\phi(x) - \phi(x + a\hat{\mu}) - \phi(x - a\hat{\mu})]$$

The Free Field Action

- The continuum action for a free scalar field is:

$$S_0[\phi] = \int d^4x \frac{1}{2}\phi(x)(-\partial_\mu\partial^\mu + m^2)\phi(x)$$

- On the lattice, this becomes:

$$S_0[\phi, a] = \frac{1}{2}\sum_x a^4 \left[(\Delta^f_\mu \phi(x))(\Delta^f_\mu \phi(x)) + m^2\phi(x)^2\right]$$

- Using the definition of the lattice $\Box$ operator, this can be written more compactly as:

$$S_0[\phi, a] = \frac{1}{2}\sum_{x,y} a^8 \phi(x)(\Box + m^2)_{x,y}\phi(y)$$

The Lattice Propagator

- The propagator $G(x, y; a)$ is the inverse of the kernel of the action, $(\Box + m^2)$.
- It is the solution to the equation:

$$\sum_y a^4 (\Box + m^2)_{x,y} G(y, z; a) = a^{-4} \delta_{x,z}$$

- To solve this, we move to momentum space using a Fourier transform.

Momentum Space and the Brillouin Zone

- For a finite lattice with periodic boundary conditions, allowed momenta are also discrete. For a lattice of size L_μ in each direction:

$$p_\mu = \frac{2\pi}{aL_\mu} n_\mu, \quad n_\mu = 0, 1, ..., L_\mu - 1$$

- In the infinite volume limit ($L_\mu \to \infty$), the momentum values become continuous but are restricted to a finite range:

$$-\frac{\pi}{a} \leq p_\mu \leq \frac{\pi}{a}$$

  This range is called the **first Brillouin Zone**.

- The lattice spacing a introduces a maximum momentum, or an "ultraviolet (UV) cutoff".

The Propagator in Momentum Space

- The solution of the propagator equation is obtained through Fourier transformation:

$$G(x,y;a) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \tilde{G}(p;a),$$

- After transforming the defining equation, one finds:

$$\left\{ \sum_{\mu} \frac{2}{a^2}(1-\cos(ap_\mu)) + m^2 \right\} \tilde{G}(p;a) = 1$$

- We get the final form:

$$\tilde{G}(p;a) = \left\{ \sum_{\mu} \frac{4}{a^2} \sin^2\left(\frac{ap_\mu}{2}\right) + m^2 \right\}^{-1}$$

The Propagator in Momentum Space

- In the continuum limit ($a \to 0$), $\sin(ax)/a \approx x$, and we recover the familiar continuum propagator

$$\tilde{G}(p) = (p^2 + m^2)^{-1}$$

.

- **Simplification:** For calculations, we often set $a = 1$ and define $\hat{p}_\mu = 2\sin(p_\mu/2)$. The propagator then takes a familiar form:

$$\tilde{G}(p) \equiv \tilde{\Delta}(p) = \frac{1}{\hat{p}^2 + m^2}$$

Lattice Perturbation Theory

- Primary Role of Lattice:
  - Acts as a regularization method.
  - Mainly enables non-perturbative methods (e.g., for strongly coupled theories like QCD).
- Why Perturbation Theory on a Lattice?
  - Lattice perturbation theory provides the bridge to match non-perturbative lattice simulations with the continuum perturbation theory.
  - Estimating lattice artifacts (e.g., finite-volume effects).

The ϕ^4 Action and Perturbation Theory

- We add the simplest interaction term to the action:

$$S = \sum_x a^4 \left\{ \frac{1}{2}(\Delta^f_\mu \phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4 \right\}$$

  The subscript '0' indicates these are **bare** parameters, not the physically observed ones.

- We can now do perturbation theory using Feynman diagrams. The rules are similar to the continuum, with key differences:
  1. each line is associated with a propagator $\tilde{\Delta}(q)$,
  2. each vertex is an end point of four lines and is associated with a factor $-\lambda_0$,
  3. at inner vertices momentum conservation holds modulo 2π,
  4. loop momenta are integrated over the first Brillouin Zone.

Two-Loop Correction to the Propagator (two-Point Function)

We want to compute the two-point function, $\Gamma^{(2)}(p) = -\tilde{G}(p)^{-1}$.

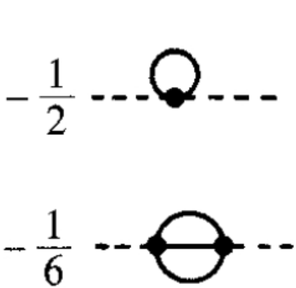

Figure: Diagrams for the two-point vertex function up to two loops.

Two-Loop Correction to the Propagator (two-Point Function)

- The calculation gives:

$$-\tilde{G}(p)^{-1} = -\left(\hat{p} + m_0^2\right) - \frac{\lambda_0}{2} J_1(m_0) + \frac{\lambda_0^2}{4} J_1(m_0) J_2(m_0)$$

$$+ \frac{\lambda_0^2}{6} I_3(m_0, p) + O\left(\lambda_0^3\right)$$

- where

$$J_n(m_0) \equiv \int_q \tilde{\Delta}(q)^n,$$

$$I_3(m_0, p) \equiv \int_{q_1} \int_{q_2} \tilde{\Delta}(q_1) \tilde{\Delta}(q_2) \tilde{\Delta}(p - q_1 - q_2)$$

One-Loop Correction to the Vertex (four-Point Function)

Next, we compute the 4-point function, $\Gamma^{(4)}$, at one-loop.

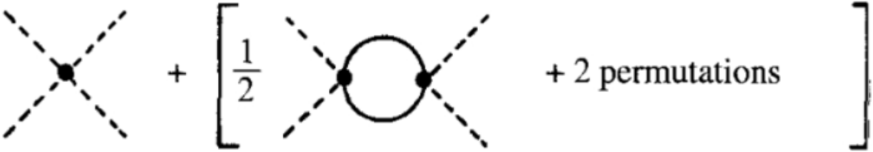

Figure: Diagrams for the four-point vertex function up to one loop contribution.

One-Loop Correction to the Vertex (four-Point Function)

- The vertex function is given by:

$$\Gamma^{(4)}(p_i) = -\lambda_0 + \frac{\lambda_0^2}{2}[I_2(m_0, p_1+p_2) + I_2(m_0, p_1+p_3) + I_2(m_0, p_1+p_4)]$$

- Where the loop integral I_2 is defined as:

$$I_2(m_0, p) = \int_q \tilde{\Delta}(q)\tilde{\Delta}(p-q)$$

- At zero external momenta, this simplifies to:

$$\Gamma^{(4)}(0,0,0,0) = -\lambda_0 + \frac{3}{2}\lambda_0^2 J_2(m_0) + O(\lambda_0^3)$$

where $J_2(m_0) = I_2(m_0, 0) = \int_q \tilde{\Delta}(q)^2$.

Defining Renormalized Quantities

- **When we take the continuum limit, $a \to 0$. We want our physical predictions to be independent of the regulator.**
- If we held the bare parameters (m_0, λ_0) fixed while sending $a \to 0$, our calculated "physical" mass and coupling would blow up. This is physically meaningless.
- We impose **renormalization conditions** on our calculated Green's functions to define the renormalized quantities.

Renormalization Conditions

- We define the renormalized field $\phi_R = Z_R^{-1/2}\phi_0$, where Z_R is the wave function renormalization factor. The renormalized Green's functions are $\Gamma_R^{(n)} = Z_R^{n/2}\Gamma^{(n)}$.
- **Condition 1 & 2 (Mass and Wave Function):** We define the renormalized mass m_R and Z_R by the behavior of the full inverse propagator at small momentum:

$$\tilde{G}(p)^{-1} \equiv \frac{1}{Z_R}(m_R^2 + p^2 + O(p^4))$$

- **Condition 3 (Coupling):** We define the renormalized coupling λ_R as the value of the 4-point function at zero external momenta:

$$\lambda_R \equiv -\Gamma_R^{(4)}(0,0,0,0)$$

Renormalization at One-Loop

- **Applying conditions to the 2-point function:**

$$\tilde{G}(p)^{-1} = (\hat{p}^2 + m_0^2) + \frac{\lambda_0}{2} J_1(m_0) \approx p^2 + \left(m_0^2 + \frac{\lambda_0}{2} J_1(m_0) \right)$$

- At this order, the momentum dependence is unchanged, so $Z_R = 1 + O(\lambda_0^2)$.
- The renormalized mass is:

$$m_R^2 = m_0^2 + \frac{\lambda_0}{2} J_1(m_0) + O(\lambda_0^2)$$

Renormalization at One-Loop

- **Applying conditions to the 4-point function:**

$$\Gamma^{(4)}(0,0,0,0) = -\lambda_0 + \frac{3}{2}\lambda_0^2 J_2(m_0)$$

With $Z_R = 1$, we have $\Gamma_R^{(4)} = \Gamma^{(4)}$, so from the condition $\lambda_R = -\Gamma_R^{(4)}(0,0,0,0)$ The renormalized coupling is:

$$\lambda_R = \lambda_0 - \frac{3}{2}\lambda_0^2 J_2(m_0) + O(\lambda_0^3)$$

Renormalization at One-Loop

- The relation between the bare parameters (m_0, λ_0) and the renormalized ones (m_R, λ_R) involves loop integrals J_1 and J_2.
- For finite lattice spacing, these integrals converge.
- To study the divergences that appear in the continuum limit $(a \to 0)$, we consider the renormalized mass m_R:

$$m_R^2 = m_0^2 + \frac{\lambda_0}{2}\frac{1}{a^2}J_1(am_0) + \ldots$$

Analysis of the Loop Integral $J_1(y)$

The behavior of $J_1(y)$ is needed for small $y = am_0$.

- At $y = 0$, the integral converges to a constant:

$$J_1(0) = \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \left(4 \sum_{\mu=1}^{4} \sin^2 \frac{q_\mu}{2} \right)^{-1} \equiv r_0 \approx 0.154933390$$

- Near this point the corrections are obtained with the help of the decomposition

$$\frac{1}{m_0^2 + \hat{q}^2} = \frac{1}{\hat{q}^2} - \frac{m_0^2}{\hat{q}^2 \left(m_0^2 + \hat{q}^2 \right)}$$

- For small y, a logarithmic singularity appears:

$$J_1(y) = r_0 + y^2 \left\{ \frac{1}{16\pi^2} \ln y^2 + r_1 + O\left(y^2\right) \right\}$$

with the constant $r_1 \approx -0.030345755$.

Divergences in Renormalized Mass

- In the limit $a \to 0$, the renormalized mass contains a quadratic and a logarithmic divergence:

$$m_R^2 = m_0^2 + \frac{\lambda_0}{2}\frac{r_0}{a^2} + \frac{\lambda_0}{32\pi^2} m_0^2 \ln\left(a^2 m_0^2\right) + \frac{\lambda_0}{2} r_1 m_0^2 + \ldots$$

- In lattice units, however, the equation

$$(am_R)^2 = (am_0)^2 + \frac{\lambda_0}{2} r_0 + \frac{\lambda_0}{32\pi^2}(am_0)^2 \ln\left(a^2 m_0^2\right) + \frac{\lambda_0}{2} r_1 (am_0)^2 + \ldots$$

  does not contain any divergent terms.

- **Tuning Condition**: In lattice units, this equation implies that for am_R to be finite in the continuum limit, the bare mass parameter must be tuned:

$$(am_0)^2 \to -\frac{\lambda_0}{2} r_0 + O\left(\lambda_0^2\right)$$

The Integral $J_2(y)$ and Renormalized Coupling

The behavior of $J_2(y)$ for small y is found via the recursion relation:

$$J_{n+1}(y) = -\frac{1}{n}\frac{d}{d\left(y^2\right)} J_n(y)$$

This gives:

$$J_2(y) = -r_1 - \frac{1}{16\pi^2}\left(1 + \ln y^2\right) + O\left(y^2\right)$$

This implies a logarithmic divergence in the renormalized coupling λ_R:

$$\lambda_R = \lambda_0 + \frac{3}{32\pi^2}\lambda_0^2 \ln\left(a^2 m_0^2\right) + \frac{3}{2}\lambda_0^2\left(\frac{1}{16\pi^2} + r_1\right) + \dots$$

Renormalized Perturbation Theory

- We now have relations between bare (m_0, λ_0) and renormalized (m_R, λ_R) parameters. We can invert them order by order:

$$m_0^2 = m_R^2 - \frac{\lambda_R}{2} J_1(m_R) + O(\lambda_R^2)$$
$$\lambda_0 = \lambda_R + \frac{3}{2}\lambda_R^2 J_2(m_R) + O(\lambda_R^3)$$

- We substitute these expressions back into our calculations for any Green's function. This procedure is called **renormalized perturbation theory**.

- For example, the 2-point function becomes:

$$\tilde{G}_R(p)^{-1} = (\hat{p}^2 + m_R^2)$$

Renormalized Perturbation Theory

- The 4-point function becomes:

$$\Gamma_R^{(4)}(p_1, p_2, p_3, p_4) = -\lambda_R + \frac{\lambda_R^2}{2}[I_2'(m_R, p_1 + p_2) + I_2'(m_R, p_1 + p_3) + I_2'(m_R, p_1 + p_4)] + O\left(\lambda_R^3\right),$$

  where

$$I_2'(m_R, p) \equiv I_2'(m_R, p) - J_2(m_R, p) = \int_q \left(\tilde{\Delta}(q)\tilde{\Delta}(p-q) - \tilde{\Delta}^2(q)\right)$$

- The result is a power series in the finite, physical coupling λ_R. The divergences as $a \to 0$ are now hidden inside the bare parameters, and the final expressions for physical observables are finite.

The RGE and the Beta Function

- **The Core Principle:** Physical reality cannot depend on our choice of regulator or the arbitrary energy scale at which we defined our renormalized parameters.

- The **Renormalization Group Equation (RGE)** tells us how our renormalized coupling constant must change with energy scale to keep physics constant.

- This change is quantified by the **beta function**:

$$\beta(\lambda_R) = m_R \frac{\partial \lambda_R}{\partial m_R}\bigg|_{\lambda_0}$$

Deriving the Beta Function

- We start with our relation: $\lambda_R = \lambda_0 - \frac{3}{2}\lambda_0^2 J_2(m_0)$.
- Differentiate with respect to m_0 at fixed λ_0:

$$\frac{\partial \lambda_R}{\partial m_0} = -\frac{3}{2}\lambda_0^2 \frac{\partial J_2(m_0)}{\partial m_0} = -\frac{3}{2}\lambda_0^2 \left(2m_0 \frac{dJ_2(m_0)}{d(m_0^2)}\right) = 6\lambda_0^2 m_0 J_3(m_0)$$

(Here we used the identity $J_{n+1}(y) = -\frac{1}{n}\frac{d}{d(y^2)}J_n(y)$).

- Using the chain rule, $\beta = m_R \frac{\partial \lambda_R}{\partial m_0}\left(\frac{\partial m_R}{\partial m_0}\right)^{-1}$, and the fact that $\frac{\partial m_R}{\partial m_0} \approx \frac{m_R}{m_0}$ at this order, we get the **lattice beta function**:

$$\beta(\lambda_R, m_R) = 6\lambda_R^2 m_R^2 J_3(m_R) + O(\lambda_R^3)$$

The Continuum Limit and Physical Meaning

- Our calculated $\beta(\lambda_R, m_R)$ explicitly depends on am_R. This is a lattice artifact known as a **scaling violation**. A physical beta function should only depend on the coupling itself.

- To get the physical result, we must take the continuum limit, $a \to 0$. We analyze the behavior of the integral J_3 in this limit.

- In the limit $y \to 0$, we have that $y^2 J_3(y) \to \frac{1}{32\pi^2}$.

- This gives the celebrated **continuum one-loop beta function for ϕ^4 theory**:

$$\beta(\lambda_R) = 6\lambda_R^2 \left(\frac{1}{32\pi^2} \right) + O(\lambda_R^3) = \frac{3}{16\pi^2}\lambda_R^2 + O(\lambda_R^3)$$

Summary and Key Takeaways

- **Lattice Regularization** provides non-perturbative definition of quantum field theory by discretizing spacetime, which introduces a natural UV cutoff.
- The framework allows for systematic perturbative calculations where divergences appear as the lattice spacing $a \to 0$.
- For ϕ^4 theory, these divergences can be absorbed by tuning the bare parameters (m_0, λ_0).This process of absorption is called **renormalization**
- The theory defined by finite, physical parameters (m_R, λ_R) has a well-behaved continuum limit, and universal quantities like the β-function can be recovered.

Summary and Key Takeaways

- **The Bridge to Non-Perturbative Physics:** The main purpose of the lattice is for non-perturbative simulations (e.g., Lattice QCD). Perturbative calculations like the one we've seen are **essential**. They provide a crucial cross-check in the weak-coupling regime, allowing us to verify that our complex numerical codes are producing correct results. This matching between perturbative and non-perturbative results is a cornerstone of confidence in lattice simulations.