Electron-Proton Elastic Scattering (Thomson Chapter 7)

- Scattering in a nutshell: smash a well known probe on a nucleon or nucleus in order to try to figure it's structure.

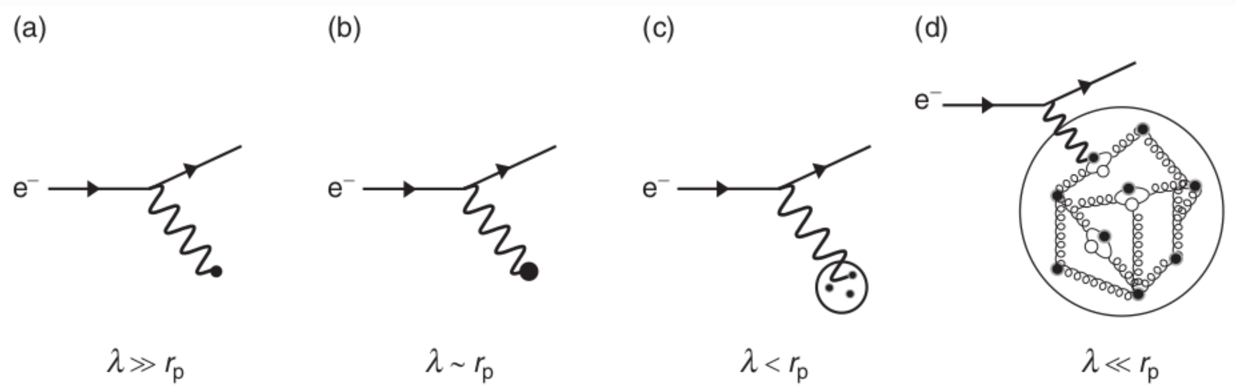
- "Photographing" an object by scattering an electron beam off is a well-proved technique in Physics.

- We're going to start with elastic $e^- p^+$ scattering in order to introduce a number of concepts that will be useful later on.

Probing the structure of the proton

- Electron-proton scattering is a useful method to investigate proton structure:
  - I) At low energies the main process is elastic scattering, where the proton stays intact. This involves a virtual photon interacting with the proton as a whole, allowing us to measure global features such as the radius.
  - II) At high energies the main process is deep inelastic scattering, where the proton breaks apart. In this case the electron scatters elastically off individual quarks, providing this way insight into the momentum distribution of the quarks.

- The nature of the scattering depends on the wavelength of the virtual photon:



a) $\lambda \gg r_p$: electrons are non relativistic and the proton Appears to be point like of the proton. The scattering is elastic and governed by static eletric potential

b) $\lambda \approx r_p$: the proton's extended charge and magnetic moment start to beco-me important

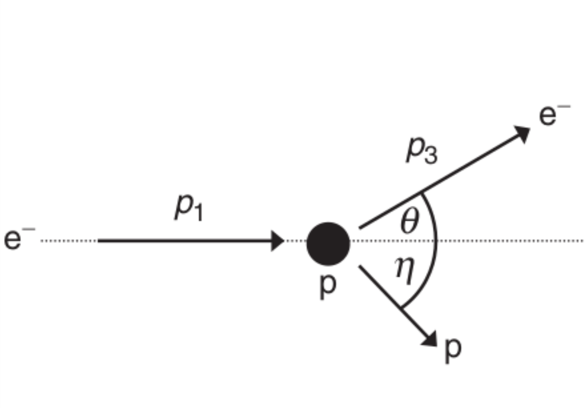
c) $\lambda < r_p$: the elastic cross section decreases. Inelastic scattering becomes dominant as the photon starts to interact with proton's constituents, and the proton subsequently breaks up.

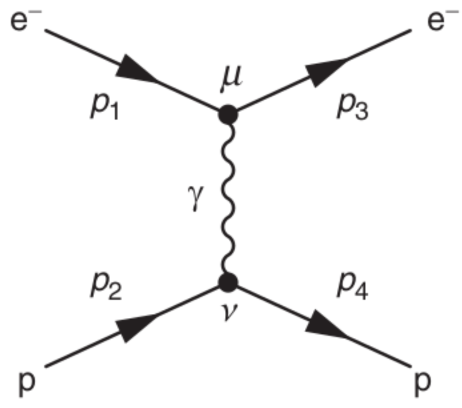
d) $\lambda \ll r_p$: the proton's structure is fully resolved, and it behaves like a sea of quarks and gluons.

the elastic scattering provides a good introduction to a number of important concepts

Rutheford and Mott Scattering

• Rutheford and Mott scattering are the two processes contributing to the e^-p elas-tic scattering. In both cases the kinetic energy of the recoiling proton is negligeble compared to its rest mass. Here we're going to consider the proton as a point-like Dirac particule, which holds if $\lambda \gg r_p$:





• If we consider the proton as just another point-like Dirac fermion, then the MATRIX element of the e^-p elastic scattering diagram is:

$$M_{fi} = \frac{Qe^2}{q^2}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] g_{\mu\nu}\left[\bar{u}(p_4)\gamma^\nu u(p_2)\right] \quad (T7.1)$$

• We can write spinors using the helicity formalism. For the electron we have:

$$u\uparrow = N_e \begin{bmatrix} \underbrace{\cos\theta/2}_{\equiv \text{"c"}} \\ \sin\theta/2\, e^{i\varphi} \\ K\cos\theta/2 \\ K\sin\theta/2\, e^{i\varphi} \end{bmatrix} \qquad u\downarrow = N_e \begin{bmatrix} -\underbrace{\sin\theta/2}_{\equiv \text{"s"}} \\ \cos\theta/2\, e^{i\varphi} \\ K\sin\theta/2 \\ -K\cos\theta/2\, e^{i\varphi} \end{bmatrix} \quad \text{(T4.65)}$$

non-relativistic: $K \ll 1$

with $N_e \equiv \sqrt{E+m_e}$ (1) $\quad K \equiv \frac{p}{E+m_e} = \frac{\beta_e\gamma_e}{\gamma_e+1}$ (2) $\quad \beta \equiv v/c \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

highly relativistic: $K \approx 1$

• If the velocity of the scattered proton is small, to a good approximation the energy of the electron does not change, and then $K_f = K_i$. If we also set $\varphi = 0$, we get then:

$$u\uparrow(p_1) = N_e(1,0,K,0) \; (3) \qquad u\uparrow(p_3) = N_e(c,s,Kc,Ks) \; (5)$$

$$u\downarrow(p_1) = N_e(0,1,0,-K) \; (4) \qquad u\downarrow(p_3) = N_e(-s,c,Ks,-Kc) \; (6)$$

• If we define objects like $[\bar{u}(p_3)\gamma^\mu u(p_1)]$ as currents J, we can rewrite (T7.1):

$$\mathcal{M}_{fi} = \frac{Qe^2}{q^2} J_e \cdot J_p \quad (7)$$

• Gladly, for any two spinors, the components of $\bar{\psi}\gamma^\mu\phi = \psi^+\gamma^0\gamma^\mu\phi$, are:

$$\bar{\psi}\gamma^0\phi = \psi^+\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \quad \text{(T6.12)}$$

$$\bar{\psi}\gamma^1\phi = \psi^+\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \quad \text{(T6.13)}$$

$$\bar{\psi}\gamma^2\phi = \psi^+\gamma^0\gamma^2\phi = -i\left[\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1\right] \quad \text{(T6.14)}$$

$$\bar{\psi}\gamma^3\phi = \psi^+\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2 \quad \text{(T6.15)}$$

so applying (3-6) on this give us:

$$J_{e\uparrow\uparrow} = \bar{u}\uparrow(p_3)\gamma^\mu u\uparrow(p_1) = (E+m_e)\left[(K^2+1)c,\, 2Ks,\, +2iKs,\, 2Kc\right] \quad \text{(T7.2)}$$

$$J_{e\downarrow\downarrow} = \bar{u}\downarrow(p_3)\gamma^\mu u\downarrow(p_1) = (E+m_e)\left[(K^2+1)c,\, 2Ks,\, -2iKs,\, 2Kc\right] \quad \text{(T7.3)}$$

$$J_{e\downarrow\uparrow} = \bar{u}\downarrow(p_3)\gamma^\mu u\uparrow(p_1) = (E+m_e)\left[(1-K^2)s,\, 0,\, 0,\, 0\right] \quad \text{(T7.4)}$$

$$J_{e\uparrow\downarrow} = \bar{u}\uparrow(p_3)\gamma^\mu u\downarrow(p_1) = (E+m_e)\left[(K^2-1)s,\, 0,\, 0,\, 0\right] \quad \text{(T7.5)}$$

• Now we turn to the proton: in the limit where the velocity of the recoiling proton is small ($\beta_P \ll 1$), then $K \approx 0$ and the 2 lower components of its spinors (the ones $\propto K$) are approximately zero. If we set $\theta_P = \eta$ and $\phi_P = \pi$, we then get

$$u_\uparrow(p_2) = \sqrt{2m_p}(1,0,0,0) \quad (8) \qquad u_\uparrow(p_4) = \sqrt{2m_p}(c_\eta, -s_\eta, 0, 0) \quad (10)$$

$$u_\downarrow(p_2) = \sqrt{2m_p}(0,1,0,0) \quad (9) \qquad u_\downarrow(p_4) = \sqrt{2m_p}(-s_\eta, -c_\eta, 0, 0) \quad (11)$$

• Again we use (T6.12–T6.15) to obtain the currents for the proton:

$$\left.\begin{aligned} J_{p\uparrow\uparrow} &= 2m_p(c_\eta, 0,0,0) = -J_{p\downarrow\downarrow} \\ J_{p\uparrow\downarrow} &= -2m_p(s_\eta, 0,0,0) = -J_{p\downarrow\uparrow} \end{aligned}\right. \quad (T7.6)$$

• Now, with (T7.2)–(T7.5) and (T7.6) in hand, we substitute them in (7), remembering to take the average over the spins:

+1 for the proton

sum over all the helicity combinations

$$M_{fi} = \frac{Qe^2}{q^2} J_e \cdot J_p \quad \Rightarrow \quad \langle |M_{fi}^2| \rangle = \frac{1}{4}\sum M_{fi}^2 = \frac{e^4}{4q^4}\sum (J_e \cdot J_p)^2 \quad (12)$$

• Let's go by parts:

electron

PROTON

i \ f	↑↑	↑↓	↓↑	↓↓
↑↑	1	2	3	4
↑↓	5	6	7	8
↓↑	9	10	11	12
↓↓	13	14	15	16

$$J_{e\uparrow\uparrow} = (E+m_e)[(K^2+1)c, 2Ks, +2iKs, 2Kc]$$

$$J_{e\downarrow\downarrow} = \bar{u}_\downarrow(p_3)\gamma^\nu u_\downarrow(p_1) = (E+m_e)[(K^2+1)c, 2Ks, -2iKs, 2Kc]$$

$$J_{e\downarrow\uparrow} = \bar{u}_\downarrow(p_3)\gamma^\nu u_\uparrow(p_1) = (E+m_e)[(1-K^2)s, 0,0,0]$$

$$J_{e\uparrow\downarrow} = \bar{u}_\uparrow(p_3)\gamma^\nu u_\downarrow(p_1) = (E+m_e)[(K^2-1)s, 0,0,0]$$

$$\left\{\begin{aligned} J_{p\uparrow\uparrow} &= 2m_p(c_\eta, 0,0,0) \\ J_{p\uparrow\downarrow} &= -2m_p(s_\eta, 0,0,0) \\ J_{p\downarrow\uparrow} &= 2m_p(s_\eta, 0,0,0) \\ J_{p\downarrow\downarrow} &= -2m_p(c_\eta, 0,0,0) \end{aligned}\right.$$

1) $\uparrow\uparrow \longrightarrow \uparrow\uparrow$: $J_{e\uparrow\uparrow} \cdot J_{p\uparrow\uparrow} = (\epsilon + m_e)\left[(\kappa^2+1)c,\ 2\kappa s,\ +2i\kappa s,\ 2\kappa c\right] \cdot 2m_p(c_\eta, 0, 0, 0)$

$$= 2m_p(\epsilon + m_e)(\kappa^2+1)c\, c_\eta$$

$$\Rightarrow \left(J_{e\uparrow\uparrow} \cdot J_{p\uparrow\uparrow}\right)^2 = 4m_p^2(\epsilon + m_e)^2(\kappa^2+1)^2c^2c_\eta^2 \quad (13)$$

2) $\uparrow\uparrow \longrightarrow \uparrow\downarrow$: $J_{e\uparrow\uparrow} \cdot J_{p\uparrow\downarrow} = (\epsilon + m_e)\left[(\kappa^2+1)c,\ 2\kappa s,\ +2i\kappa s,\ 2\kappa c\right] \cdot -2m_p(s_\eta, 0, 0, 0)$

$$= -2m_p(\epsilon + m_e)(\kappa^2+1)c\, s_\eta$$

$$\Rightarrow \left(J_{e\uparrow\uparrow} \cdot J_{p\uparrow\downarrow}\right)^2 = 4m_p^2(\epsilon + m_e)^2(\kappa^2+1)^2c^2s_\eta^2 \quad (14)$$

3) $\uparrow\uparrow \longrightarrow \downarrow\uparrow$: $J_{e\uparrow\downarrow} \cdot J_{p\uparrow\uparrow} = (\epsilon + m_e)\left[(\kappa^2-1)s, 0, 0, 0\right] \cdot 2m_p(c_\eta, 0, 0, 0)$

$$= -2m_p(\epsilon + m_e)(1-\kappa^2)\, s\, c_\eta$$

$$\Rightarrow \left(J_{e\uparrow\downarrow} \cdot J_{p\uparrow\uparrow}\right)^2 = 4m_p^2(\epsilon + m_e)^2(1-\kappa^2)^2s^2c_\eta^2 \quad (15)$$

4) $\uparrow\uparrow \longrightarrow \downarrow\downarrow$: $J_{e\uparrow\downarrow} \cdot J_{p\uparrow\downarrow} = (\epsilon + m_e)\left[(\kappa^2-1)s, 0, 0, 0\right] \cdot -2m_p(s_\eta, 0, 0, 0)$

$$= 2m_p(\epsilon + m_e)(1-\kappa^2)\, s\, s_\eta$$

$$\Rightarrow \left(J_{e\uparrow\uparrow} \cdot J_{p\uparrow\uparrow}\right)^2 = 4m_p^2(\epsilon + m_e)^2(1-\kappa^2)^2 s^2 s_\eta^2 \quad (16)$$

5) $\uparrow\downarrow \longrightarrow \uparrow\uparrow$: $J_{e\uparrow\uparrow} \cdot J_{p\downarrow\uparrow} = (\epsilon + m_e)\left[(\kappa^2+1)c,\ 2\kappa s,\ +2i\kappa s,\ 2\kappa c\right] \cdot 2m_p(s_\eta, 0, 0, 0)$

$$\Rightarrow \left(J_{e\uparrow\uparrow} \cdot J_{p\downarrow\uparrow}\right)^2 = 4m_p^2(\epsilon + m_e)^2(\kappa^2+1)^2c^2s_\eta^2 \quad (17)$$

6) $\uparrow\downarrow \longrightarrow \uparrow\downarrow$: $J_{e\uparrow\uparrow} \cdot J_{p\downarrow\downarrow} = (\epsilon + m_e)\left[(\kappa^2+1)c,\ 2\kappa s,\ +2i\kappa s,\ 2\kappa c\right] \cdot -2m_p(c_\eta, 0, 0, 0)$

$$\Rightarrow \left(J_{e\uparrow\uparrow} \cdot J_{p\downarrow\downarrow}\right)^2 = 4m_p^2(\epsilon + m_e)^2(\kappa^2+1)^2c^2c_\eta^2 \quad (18)$$

7) $\uparrow\downarrow \longrightarrow \downarrow\uparrow$: $J_{e\uparrow\downarrow} \cdot J_{p\downarrow\uparrow} = (\epsilon + m_e)\left[(\kappa^2-1)s, 0, 0, 0\right] \cdot 2m_p(s_\eta, 0, 0, 0)$

$$\Rightarrow \left(J_{e\uparrow\downarrow} \cdot J_{p\downarrow\uparrow}\right)^2 = 4m_p^2(\epsilon + m_e)^2(1-\kappa^2)^2s^2s_\eta^2 \quad (19)$$

8) $\uparrow\downarrow \longrightarrow \downarrow\downarrow$: $J_{e\uparrow\downarrow} \cdot J_{p\downarrow\downarrow} = (\epsilon + m_e)\left[(\kappa^2-1)s, 0, 0, 0\right] \cdot -2m_p(c_\eta, 0, 0, 0)$

$$\Rightarrow \left(J_{e\uparrow\downarrow} \cdot J_{p\downarrow\downarrow}\right)^2 = 4m_p^2(\epsilon + m_e)^2(1-\kappa^2)^2s^2c_\eta^2 \quad (20)$$

• Now we see that 1=6, 2=5, 3=8, 4=7, so a pattern start to show:

e^- p	e^- p	e^- p	e^- p
↑↑ ↑↑	↑↑ ↑↓	↑↑ ↓↑	↑↑ ↓↓
↑↓ ↑↑	↑↓ ↑↓	↑↓ ↓↑	↑↓ ↓↓
↓↑ ↑↑	↓↑ ↑↓	↓↑ ↓↑	↓↑ ↓↓
↓↓ ↑↑	↓↓ ↑↓	↓↓ ↓↑	↓↓ ↓↓

• We don't need to calculate all the 16 combinations, just 4! Going back to (12):

$$\langle |M_{fi}^2|\rangle = \frac{e^4}{4q^4}\sum (j_e\cdot j_p)^2 = \frac{e^4}{\cancel{4}q^4}\cancel{4}m_p^2(\overset{E=\gamma_e m_e}{E}+m_e)^2\ (\underbrace{c_\eta^2+s_\eta^2}_{1})\left[4(1+K^2)^2c^2+4(1-K^2)^2s^2\right] =$$

$$= \frac{4m_p^2m_e^2e^4(\gamma_e+1)^2}{q^4}\left[(1+K^2)^2c^2+(1-K^2)^2s^2\right] = \frac{4m_p^2m_e^2e^4(\gamma_e+1)^2}{q^4}\left[\underbrace{c^2+s^2}_{1}+2K^2(\underbrace{c^2-s^2}_{\cos\theta})+(\underbrace{c^2+s^2}_{1})K^4\right] =$$

$$= \frac{4m_p^2m_e^2e^4(\gamma_e+1)^2}{q^4}\left[1+2K^2\underbrace{\cos\theta}_{2c^2-1}+K^4\right] = \frac{4m_p^2m_e^2e^4(\gamma_e+1)^2}{q^4}\left[1+2K^2(2c^2-1)+K^4\right]$$

$$= \frac{4m_p^2m_e^2e^4(\gamma_e+1)^2}{q^4}\left[1-2K^2+K^4+4K^2c^2\right] = \frac{4m_p^2m_e^2e^4(\gamma_e+1)^2}{q^4}\left[(1-K^2)^2+4K^2c^2\right]$$

$$\Rightarrow \langle |M_{fi}^2|\rangle = \frac{4m_p^2m_e^2e^4(\gamma_e+1)^2}{q^4}\left[(1-K^2)^2+4K^2c^2\right] \quad (T\,7.7)$$

$$= \frac{16\,m_p^2m_e^2e^4}{q^4}\left[1+\beta_e^2\gamma_e^2\cos\frac{\theta}{2}\right] \quad (T\,7.8)$$ use $K=\frac{\beta_e\gamma_e}{\gamma_e+1}$ and $(1-\beta_e^2)\gamma_e^2=1$

• Now we specialize for the t-channel:

small recoil: $E_1 = E = E_2$ $\quad |\vec{p}_1| = p = |\vec{p}_3|$

$$q^2 = (p_1-p_3)^2 = p_1^2+p_3^2-2p_1\cdot p_3 = 2\underbrace{m_e^2}_{\approx 0}-2p_1\cdot p_3 = -2p_1\cdot p_3 = -2(E^2-\vec{p}_1\cdot\vec{p}_3) = -2(E^2-\vec{p}^2\cos\theta) =$$

$$= -2\left[E^2-(E^2-\underbrace{m_e^2}_{\approx 0})\cos\theta\right] = -2E^2(\underbrace{1-\cos\theta}_{2\sin^2\theta/2}) \Rightarrow q^2 = -4p^2\sin^2\theta/2 \quad (20)$$

ONLY ASSUMPTION: small recoil

$$\Rightarrow \langle |M_{fi}^2|\rangle = \frac{m_p^2m_e^2e^4}{p^4\sin^4\theta/2}\left[1+\beta_e^2\gamma_e^2\cos\frac{\theta}{2}\right] \quad (T\,7.9)$$

Rutheford Scattering

- The Rutheford scattering is the limit where the proton recoil is can be negleted (we just did that) AND the electron is non-relativistic:

$$\langle |M_{fi}^2| \rangle = \frac{m_p^2 m_e^2 e^4}{p^4 \sin^4 \theta/2}\left[1+\beta_e^2 \gamma_e^2 \cos\frac{\theta}{2}\right] \quad \xRightarrow{\beta_e \gamma_e \ll 1} \quad \langle |M_{fi}^2| \rangle_R = \frac{m_p^2 m_e^2 e^4}{p^4 \sin^4 \theta/2} \quad (T\,7.10)$$

- Lab-frame differential cross-section:

$$\text{Interaction rate} = \phi_a n_b V\sigma = (v_a+v_b) n_a n_b V\sigma \quad (T\,3.25)$$

$$\downarrow$$

IN The cm Frame

$$\sigma = \frac{1}{64\pi^2 s}\frac{p_f}{p_i}\int |M_{fi}|^2 d\Omega \quad (T\,3.31)$$

$$\downarrow$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{1}{64\pi^2}\left(\frac{1}{m_p + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \quad (T\,3.48)$$

- We can now apply (T 7.10) in (T 3.48) to get $(d\sigma/d\Omega)_{lab}$:

$$\frac{d\sigma}{d\Omega}_{lab} = \frac{1}{64\pi^2}\left(\frac{1}{m_p + \underbrace{E_1 - E_1\cos\theta}_{\sim m_e \ll m_p}}\right)^2 |M_{fi}|^2 \quad \Rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)^R_{lab} = \frac{m_e e^4}{64\pi^2 p^4 \sin^4\frac{\theta}{2}}$$

$$E_K = p^2/m_e, \quad e^2 = 4\pi\alpha \quad \Rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)^R_{lab} = \frac{\alpha^2}{16 E_K^2 \sin^4\frac{\theta}{2}} \quad (T\,7.13)$$

Mott scattering

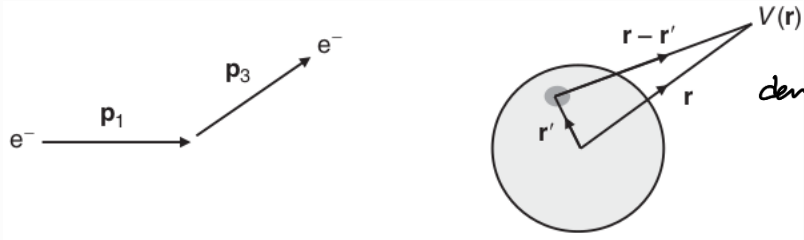
- The Mott scattering is the limit where the proton recoil is can be negleted AND the electron is relativistic:

$$\langle |M_{fi}^2| \rangle = \frac{m_p^2 m_e^2 e^4}{p^4 \sin^4 \theta/2}\left[1+\beta_e^2 \gamma_e^2 \cos\frac{\theta}{2}\right] \quad \xRightarrow{\beta_e\gamma_e > 1,\ K \approx 1,\ J_{e\uparrow\downarrow} \approx 0 \text{ and } J_{e\downarrow\uparrow} \approx 0} \quad \langle |M_{fi}^2| \rangle_M = \frac{m_p^2 e^4}{E^2 \sin^4 \theta/2}\cos^2\frac{\theta}{2} \quad (21)$$

$$\left(\frac{d\sigma}{d\Omega}\right)^M_{lab} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\cos^2\frac{\theta}{2} \quad (T\,7.14)$$

Form Factors

- let's now focus on the case were the proton is not point like. Consider the scattering of an electron by the static potential of a extended charge distribution:



density: $Q\rho(r')$

- We start by writing the charge density as $Q\rho(r')$, where Q is the total charge and $\rho(r')$ is the charge density normalized to the unity, i.e:

$$\int \rho(r')\, d^3r' = 1 \quad (22)$$

- The eletrostatic potential at a distance r from the origin is then

$$V(\vec{r}) = \int d^3\vec{r}\,' \frac{Q\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} \quad (T\ 7.15)$$

- We can then take the Born approximation where the initial and final electron states can be written as

$$\psi_i = e^{i(\vec{p}_1\cdot\vec{r} - Et)} \text{ and } \psi_f = e^{i(\vec{p}_3\cdot\vec{r} - Et)} \quad (23)$$

- Now we calculate the matrix element M_{fi}:

$$M_{fi} = \langle\psi_f|V(\vec{r})|\psi_i\rangle = \int e^{-i\vec{p}_3\cdot\vec{r}}\, V(\vec{r})\, e^{i\vec{p}_1\cdot\vec{r}}\, d^3\vec{r} = \int e^{-i\vec{p}_3\cdot\vec{r}} \int d^3\vec{r}\,' \frac{Q\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, e^{i\vec{p}_1\cdot\vec{r}}\, d^3\vec{r}$$

$\vec{q} = \vec{p}_1 - \vec{p}_3$

$$= \iint e^{i\vec{q}\cdot\vec{r}} \frac{Q\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' d^3\vec{r} = \iint e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}\, e^{i\vec{q}\cdot\vec{r}} \frac{Q\rho(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' d^3\vec{r}$$

$$= \underbrace{\int e^{i\vec{q}\cdot\vec{R}} \frac{Q}{4\pi R}\, d^3\vec{R}}_{M_{fi} \text{ for a point charge } Q \text{ at } \vec{R}} \int \rho(\vec{r}\,')\, e^{i\vec{q}\cdot\vec{r}}\, d^3\vec{r}\,' \quad (24)$$

• We call the second term on (24) of form factor $F(q^2)$, so in order to account for the extended charge distribution of the proton, the Mott scattering differential cross-section has to be modified to

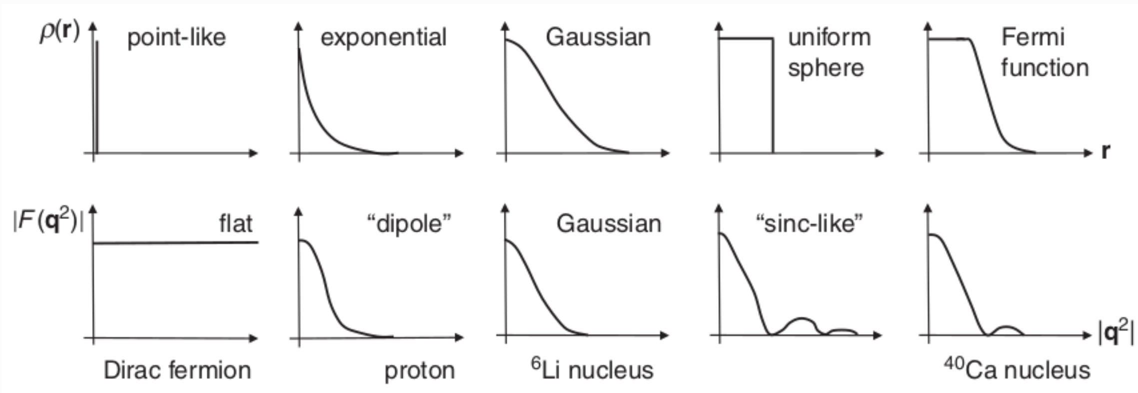
$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} |F(q^2)|^2 \quad (T7.17), (H8.1)$$

• Now we look at two cases:

i) $\lambda_r \gg$ size of the charge distribution : $\lambda \sim 1/|\vec{q}| \Rightarrow \vec{q}\cdot\vec{r} \approx 0 \Rightarrow d\sigma/d\Omega$ is the one for point-like

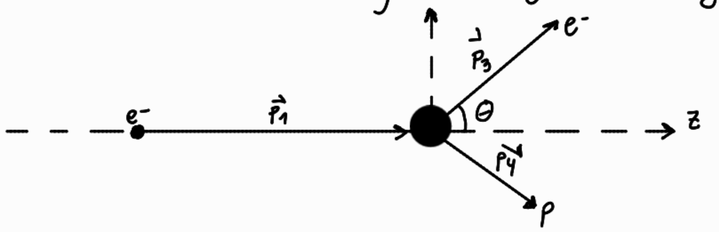
ii) $\lambda_r \ll$ size of the charge distribution : $\lambda \sim 1/|\vec{q}| \Rightarrow \vec{q}\cdot\vec{r}$ explodes

$$\left.\begin{array}{l} e^{i\vec{q}\cdot\vec{r}} \text{ oscillates rapidly over space} \\ \text{leads to a lot of cancellations} \end{array}\right\} d\sigma/d\Omega \to 0$$



The Rosenbluth formula

• So far we have assumed that the recoil of the proton could be neglected. However, for $e p$ elastic scattering at higher energies, this is not the case:



• In the general case we have:

$p_1 = (E_1, 0, 0, E_1)$ considering $m_e = 0$ (T7.18)

$p_2 = (m_p, 0, 0, 0)$ (T7.19)

$p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)$ (T7.20) $\qquad E_1 \neq E_4$

$p_4 = (E_4, \vec{p}_4)$ (T7.21)

- If we take into account both masses, $\langle |M_{fi}|^2 \rangle$ is given by:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - m_e^2(p_2\cdot p_4) - m_p^2(p_1\cdot p_3) + 2m_e^2 m_p^2\right] \quad (T\,6.67)$$

takes into account the proton recoil

- Now we take (T 6.67) and assume that the electron energy is high enough so that terms of $\mathcal{O}(m_e^2)$ can be neglected and (initially) treating the proton as a point-like Dirac particle give us:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - m_p^2(p_1\cdot p_3)\right] \quad (T\,7.22)$$

- Now our goal is to write (T7.22) in terms of more useful experimental observables: the final state energy and the scattering angle of the electron

i) using (T 7.18) – (T 7.20):

$$p_2\cdot p_3 = E_3 m_p \quad (25)$$

$$p_1\cdot p_2 = E_1 m_p \quad (26)$$

$$p_1\cdot p_3 = E_1E_3(1-\cos\theta) \quad (27)$$

ii) energy momentum conservation: $p_4 = p_1 + p_2 - p_3$

$$\Rightarrow p_3\cdot p_4 = p_3\cdot(p_1+p_2-p_3) = p_3\cdot p_1 + p_3\cdot p_2 - \underbrace{p_3\cdot p_3}_{\sim m_e^2 \approx 0} = E_1E_3(1-\cos\theta) + E_3 m_p \quad (28)$$

$$\Rightarrow p_1\cdot p_4 = p_1\cdot(p_1+p_2-p_3) = \underbrace{p_1\cdot p_1}_{\sim m_e^2 \approx 0} + p_1\cdot p_2 - p_1\cdot p_3 = E_1 m_p - E_1E_3(1-\cos\theta) \quad (29)$$

- Plugging (T 7.25) – (T 7.29) in (T 7.22) give us:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - m_p^2(p_1\cdot p_3)\right]$$

$$= \frac{8e^4}{(p_1-p_3)^4}\left[E_1 m_p\left(E_1E_3(1-\cos\theta) + E_3 m_p\right) + \left(E_1 m_p - E_1E_3(1-\cos\theta)\right)E_3 m_p - m_p^2 E_1E_3(1-\cos\theta)\right]$$

$$= \frac{8e^4}{(p_1-p_3)^4} m_p E_1E_3\left[(E_1-E_3)(1-\cos\theta) + m_p(1+\cos\theta)\right] = \frac{16 m_p E_1E_3 e^4}{(p_1-p_3)^4}\left[(E_1-E_3)\sin^2\frac{\theta}{2} + m_p\cos^2\frac{\theta}{2}\right] \quad (T\,7.23)$$

• We can also write q^2 in terms of $\mathcal{E}_3$ and θ:

$$q^2 = (p_1 - p_3)^2 = \cancel{p_1 \cdot p_1}^{\approx 0} - 2p_1 \cdot p_3 + \cancel{p_3 \cdot p_3}^{\approx 0} \approx -2\mathcal{E}_1\mathcal{E}_3(1-\cos\theta) \quad (T7.24)$$
$$= -4\mathcal{E}_1\mathcal{E}_3 \sin^2\frac{\theta}{2} \quad (T\,7.25)$$

• We define $Q^2 > 0$ in terms of q^2:

$$Q^2 \equiv -q^2 = 4\mathcal{E}_1\mathcal{E}_2 \sin^2\theta/2 \quad (T7.26)$$

• Now we've going to massage things a bit:

1) $q \cdot p_2 = (p_1 - p_3) \cdot p_2 = p_1 \cdot p_2 - p_3 \cdot p_2 = m_p(\underbrace{\mathcal{E}_1 - \mathcal{E}_3}_{\text{energy lost by the } e^-}) \quad (T7.27)$

2) $q = p_4 - p_2 \Rightarrow p_4^2 = (q + p_2)^2 \Rightarrow m_p^2 = q^2 + 2q \cdot p_2 + m_p^2 \Rightarrow q \cdot p_2 = -q^2/2 \quad (T\,7.28)$

3) Join (T7.27) and (T 7.28):

The e^- always loose energy

$$q \cdot p_2 = m_p(\mathcal{E}_1 - \mathcal{E}_3) \Rightarrow \frac{-q^2}{2} = m_p(\mathcal{E}_1 - \mathcal{E}_3) \Rightarrow \mathcal{E}_1 - \mathcal{E}_3 = \frac{-q^2}{2m_p} = \frac{Q^2}{2m_p} \quad (H\,7.29)$$

• Now back to $\langle |M_{fi}| \rangle^2$:

$$\langle |M_{fi}| \rangle^2 = \frac{16 m_p \mathcal{E}_1 \mathcal{E}_3 e^4}{(p_1 - p_3)^4}\left[(\mathcal{E}_1 - \mathcal{E}_3)\sin^2\frac{\theta}{2} + m_p \cos^2\frac{\theta}{2}\right] = \frac{\cancel{16} m_p \cancel{\mathcal{E}_1}\cancel{\mathcal{E}_3} e^4}{\cancel{16}\mathcal{E}_1^2\mathcal{E}_3^2 \sin^4\frac{\theta}{2}}\left[\frac{Q^2}{2m_p}\sin^2\frac{\theta}{2} + m_p\cos^2\frac{\theta}{2}\right]$$

$$\Rightarrow \langle |M_{fi}| \rangle^2 = \frac{m_p^2 e^4}{\mathcal{E}_1\mathcal{E}_3 \sin^4\frac{\theta}{2}}\left[\frac{Q^2}{2m_p^2}\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right] \quad (30)$$

• Now we put (30) in (T3.47):

$$\left.\frac{d\sigma}{d\Omega}\right|_{lab} = \frac{1}{64\pi^2}\left(\frac{\mathcal{E}_3}{m_p \mathcal{E}_1}\right)^2 |M_{fi}|^2 \quad (T3.47)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{\cancel{\mathcal{E}_3^2}}{\cancel{m_p^2}\mathcal{E}_1^2}\frac{\cancel{m_p^2}e^4}{\mathcal{E}_1\cancel{\mathcal{E}_3}\sin^4\frac{\theta}{2}}\left[\frac{Q^2}{2m_p^2}\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right] = \frac{\overbrace{e^4}^{\alpha^2}}{16\pi^2}\frac{1}{4\mathcal{E}_1^2\sin^4\frac{\theta}{2}}\frac{\mathcal{E}_3}{\mathcal{E}_1}\left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_p^2}\sin^2\frac{\theta}{2}\right]$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\mathcal{E}_1^2\sin^4\frac{\theta}{2}}\frac{\mathcal{E}_3}{\mathcal{E}_1}\left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_p^2}\sin^2\frac{\theta}{2}\right] \quad (T\,7.30)$$

ONLY θ is independent

• Note that

i) $\mathcal{E}_1 - \mathcal{E}_3 \overset{(T7.29)}{=} \frac{-q^2}{2m_p} \Rightarrow q^2 = -2m_p(\mathcal{E}_1 - \mathcal{E}_3) \overset{(T7.24)}{=} -2\mathcal{E}_1\mathcal{E}_3(1-\cos\theta) \Rightarrow \mathcal{E}_3 = \frac{\mathcal{E}_1 m_p}{m_p + \mathcal{E}_1(1-\cos\theta)} \quad (T7.31)$

ii) $\mathcal{E}_1 - \mathcal{E}_3 = \frac{Q^2}{2m_p} \Rightarrow Q^2 = 2m_p(\mathcal{E}_1 - \mathcal{E}_3) \Rightarrow Q^2 = \frac{2m_p\mathcal{E}_1^2(1-\cos\theta)}{m_p + \mathcal{E}_1(1-\cos\theta)} \quad (T7.32)$

Therefore, if the scattering angle of the electron is measured in the elastic scattering process, the entire kinematics of the interaction are determined. In practice, measuring the $e^- p \rightarrow e^- p$ differential cross section boils down to counting the number of electrons scattered in a particular direction for a known incident electron flux.

Furthermore, because the energy of an elastically scattered electron at a particular angle must be equal to that given by (7.31), by measuring the energy and angle of the scattered electron, it is possible to confirm that the interaction was indeed elastic and that the unobserved proton remained intact.

In the limit of $Q^2 << m^2_p$ and $E3 \approx E1$, the expression for the electron–proton differential cross section of (7.30) reduces to that for Mott scattering, demonstrating that the Mott scattering cross section formula applies when $m_e << E_1 << m_p$.

Equation (7.30) differs from the Mott scattering formula by the additional factor $E3 / E1$, which accounts for the energy lost by electron due the proton recoil, and by the new term proportional to $\sin^2 (\theta/2)$, which can be identified as being due to a purely magnetic spin–spin interaction.

• Equation (T 7.30) is the differential cross-section for elastic e-p scattering assuming a point like proton and considering its recoil. The size of the proton is accounted for by introducing two form factors, one related to the charge distribution, $G_E(Q^2)$, and the other related to the magnetic moment distribution within the proton, $G_M(Q^2)$. The most general Lorentz-invariant form is:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin\frac{\theta}{2}\right) \quad \text{(T 7.33)}$$
$$\tau \equiv Q^2/4m_p^2$$

which is called the Rosenbluth formula.

• Now we look at the behaviour of (T 7.33). We can rewrite it using a modified Mott differential cross section to account for the recoil of the proton:

$$\frac{d\sigma}{d\Omega} = \left(\frac{G_E^2+\tau G_M^2}{1+\tau} + 2\tau G_M^2\tan^2\frac{\theta}{2}\right)\frac{d\sigma}{d\Omega}^{Mott}_{recoil} \quad \text{(T 7.35)}$$

$$\frac{d\sigma}{d\Omega}^{Mott}_{recoil} = \frac{\alpha^2}{4E_1^2\sin^4\frac{\theta}{2}}\frac{E_3}{E_1}\cos^2\frac{\theta}{2} \quad \text{(T 7.36)}$$

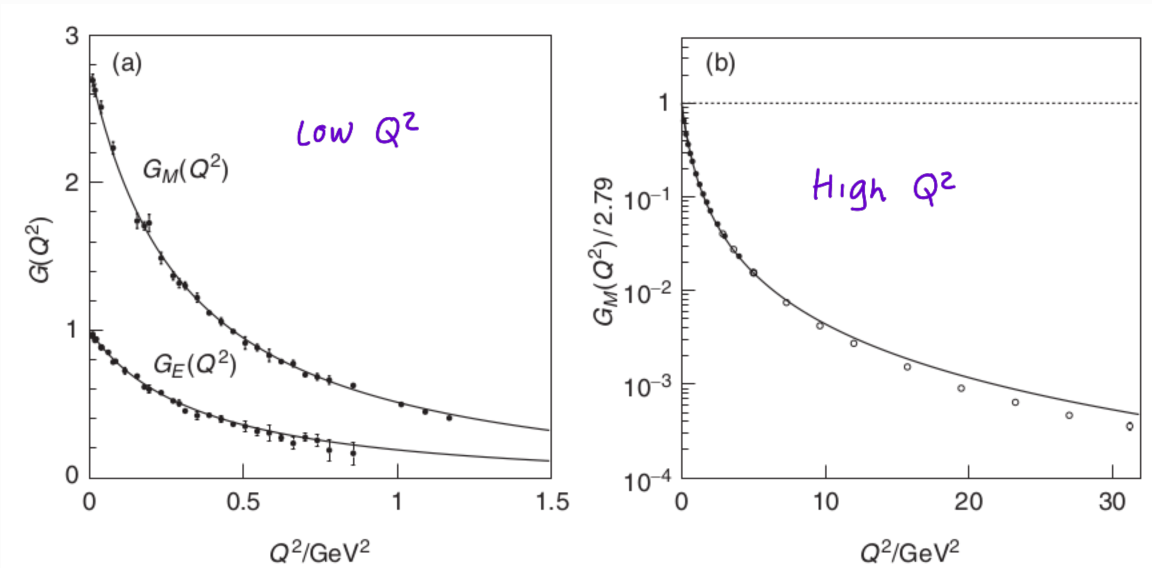
• We have 2 cases that allow us to measure the form factors:

I) Low $Q^2 \Rightarrow \tau \ll 1 \Rightarrow G_E$ dominates:

$$\frac{d\sigma}{d\Omega}\Big/\frac{d\sigma}{d\Omega}^{Mott}_{recoil} \approx G_E^2 \quad (31)$$

II) High $Q^2 \Rightarrow \tau \gg 1 \Rightarrow G_M$ dominates:

$$\frac{d\sigma}{d\Omega}\Big/\frac{d\sigma}{d\Omega}^{Mott}_{recoil} \approx \left(1+2\tau\tan^2\frac{\theta}{2}\right)G_M^2 \quad (32)$$



• From the data we can fit the form factors, which is called: "dipole function":

$$G_M(Q^2) = 2.79\, G_E^2(Q^2) \approx \frac{2.79}{(1 + Q^2/0.71\,GeV^2)^2} \quad \text{(T7.37)}$$

• One last observation is that in the high Q^2 limit (T3.35) reduces to:

$$\left(\frac{d\sigma}{d\Omega}\right)_{elastic} \sim \frac{\alpha^2}{4E_1^2 \sin^4\frac{\theta}{2}} \frac{E_3}{E_1}\left[\frac{Q^2}{2m_p^2} G_M^2 \sin^2\frac{\theta}{2}\right] \quad (33)$$

While from (T7.37), $G_M(Q^2)$ reduces to

$$G_M(Q^2) \approx Q^{-4} \quad (34)$$

So from (33) and (34) it is possible to show that:

$$\left(\frac{d\sigma}{d\Omega}\right)_{elastic} \propto \frac{1}{Q^6}\left(\frac{d\sigma}{d\Omega}\right)_{MOTT} \quad (35)$$

• Here we make a import conclusion:

Because of the finite size of the proton, both $G_E(Q^2)$ and $G_M(Q^2)$ become small at high Q^2 and the elastic scattering cross section fall rapidly with increasing Q^2: High energy e^-p scattering is dominated by inelastic processes.

Deep Inelastic Scattering

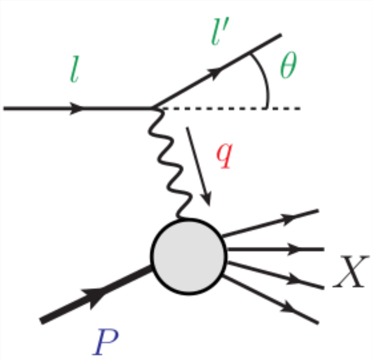
Schienbein lecture 1
Halzen chapter 8

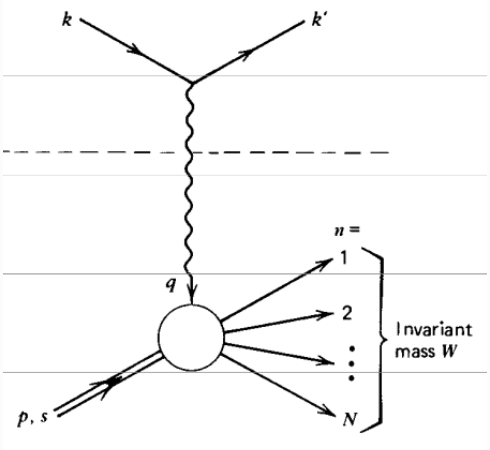
Kinematics of DIS

$\to W^2 \equiv M_X^2 > M_p^2$

Deep Inelastic Scattering

$\to Q^2 \equiv -q^2 >> M_p^2$





- let's consider:

$$e^-(\ell) + P(p) \longrightarrow e^-(\ell') + X(p_x)$$

- On shell conditions: $P^2 = m_p^2$, $\ell^2 = m_e^2 = \ell'$

- Invariants of the process:

i) $Q^2 \equiv -q^2 = -(\ell-\ell')^2 > 0$: square of the momentum transfer

ii) $\nu \equiv p\cdot q/m_p \overset{LAB}{=} E_\ell - E_{\ell'}$: ΔE_e

iii) $0 \leq x_B \equiv Q^2/(2p\cdot q) = Q^2/(2m_p\nu) \leq 1$: Bjorken scaling variable

iv) $0 \leq y \equiv p\cdot q/p\cdot \ell \overset{LAB}{=} (E_\ell - E_{\ell'})/E_\ell \leq 1$: inelasticity parameter

lab: proton rest frame = lab frame for fixed target

- We can write the invariant mass W of the hadronic final state X as:

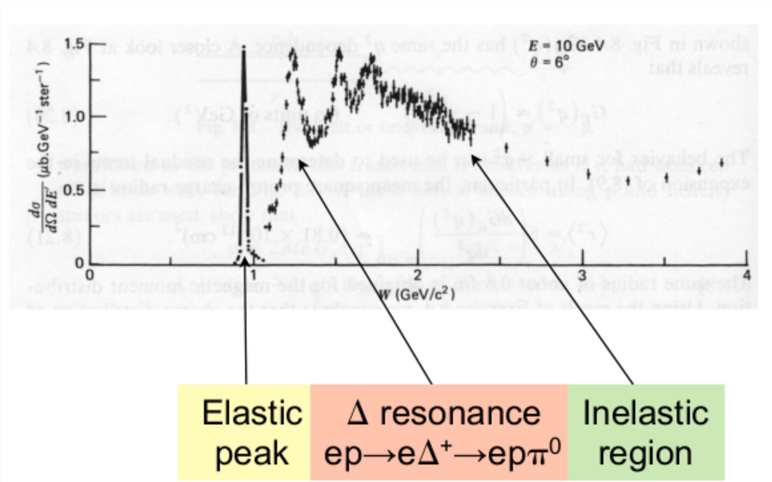
$$W^2 \equiv M_x^2 = (p+q)^2 = p^2 + 2p\cdot q + q^2 = m_p^2 + \frac{Q^2}{x_B} - Q^2 \Rightarrow W^2 = m_p^2 + \frac{Q^2}{x_B}(1-x_B) \quad (36)$$

- From (36) we see how all these variables may help us differentiate between elastic and inelastic scattering:

elastic: $W = m_p \Rightarrow x_B = 1$

inelastic: $W > m_p \quad x_B < 1$

• We can plot the $ep \to e^- X$ cross section as a function of the missing mass W:



• We can write the square of momentum transfer Q^2 as a function of the Bjorken x_B, the inelasticity parameter, E and m_p:

$$Q^2 = \frac{2p\cdot\ell}{p\cdot\ell}\,\frac{p\cdot q}{2p\cdot q}\,Q^2 = 2p\cdot\ell\,\frac{Q^2}{2p\cdot q}\,\frac{p\cdot q}{p\cdot\ell} = 2p\cdot\ell\, x_B\, y \Rightarrow Q^2 = 2m_p E x_B y \quad (37)$$

• Equation (37) allow us to plot Q^2 as a function of y and x given a electron energy E. We call this the allowed kinematic region:

1) $W^2 = (p+q)^2 =$

$= p^2 + 2p\cdot q + q^2 =$

$= m_p^2 + 2p\cdot q - Q^2$

$\Rightarrow 2p\cdot q = W^2 - m_p^2 - Q^2 \quad (38)$

2) $x_B = \dfrac{Q^2}{2p\cdot q} =$

$\Rightarrow x_B = \dfrac{Q^2}{W^2 - m_p^2 - Q^2} \quad (39)$

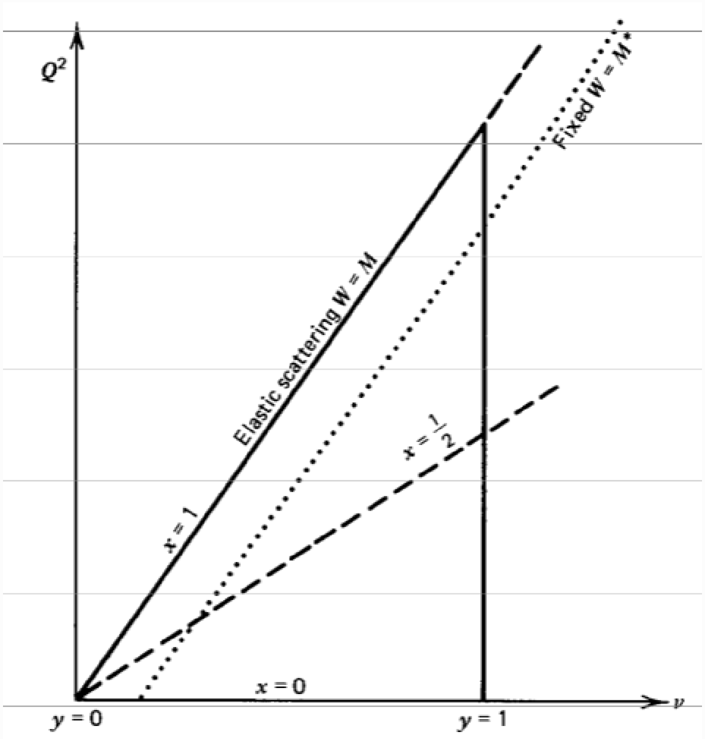


Fig. 9.3 The triangle is the allowed kinematic region for ep → eX. $\nu_{max} = E$ in the laboratory frame. W is the invariant mass of the hadronic state X, see (8.29).

3) $Q^2 = 2m_p E x_B y =$

$= 2m_p E y \left(\dfrac{Q^2}{W^2 - m_p^2 - Q^2}\right)$

$\Rightarrow Q^2(Q^2 + W^2 - m_p^2) = 2m_p E y Q^2$

$\Rightarrow (Q^2 + W^2 - m_p^2) =$
$= 2m_p E y$

$\Rightarrow Q^2 = m_p^2 - W^2 + 2m_p E y$

4) Fixed $W = m_p$:

$Q^2 = m_p^2 - m_p^2 + 2m_p E y \quad (40)$

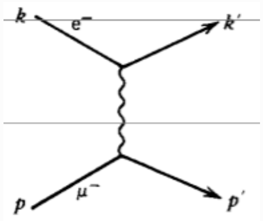
Fixed W curves are parallel to $W = m_p$

CROSS Section for INCLUSIVE DIS

↳ We don't measure the proton "debris"

- We obtain the cross section for the inclusive DIS, we can use much of the results for the $e^-\mu^- \rightarrow e^-\mu^-$ scattering. Since this topic is huge, I'm gonna use the results from Halzen. More details about $e^-\mu^- \rightarrow e^-\mu^-$ scattering can be found in sections (H6.3 ~ H6.8).

- In the case of the $e^-\mu^-$ scattering, we have that:



- The invariant amplitude follows from the Feynman rules for QED:

$$\mathcal{M}_{fi} = -e^2\, \bar{u}(k')\gamma^\mu u(k) \frac{1}{q^2} \bar{u}(p')\gamma_\mu u(p) \quad \text{(H 6.17)}$$

- To get the cross section we take the square of the modulus of $\mathcal{M}_{fi}$ and then we average over the spins of the incoming leptons and sum over the spins of the final state particles:

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4} L_e^{\mu\nu} L_{\mu\nu}^{muon} \quad \text{(H 6.18)}$$

where the L's are the leptonic tensors of the e^- and μ^-. For example:

$$L_e^{\mu\nu} \equiv \frac{1}{2}\sum_{\substack{spin\\e}} [\bar{u}(k')\gamma^\mu u(k)][\bar{u}(k')\gamma^\nu u(k)]^* \quad \text{(H 6.19)}$$

- Using the trace theorems of section H6.4 is possible to show that

$$L_e^{\mu\nu} = 2(k'^\mu k^\nu + k'^\nu k^\mu - (k\cdot k' - m_e^2)g^{\mu\nu}) \quad \text{(H6.25)}$$

So we have that

$$|\mathcal{M}_{fi}|^2 \sim L_{\mu\nu}^{e} L^{\mu\nu}_{muon} \quad (41)$$

• In the case of DIS, we generalize the expression (41) to:

$$|\mathcal{M}_l|^2 \sim L^e_{\mu\nu} W^{\mu\nu} \quad (H\,8.23)$$

where we're still using the leptonic tensor $L^e_{\mu\nu}$ since the upper part of the DIS diagram is identical to the upper part of the diagram of $e^-\mu^- \to e^-\mu^-$

• $W^{\mu\nu}$ is called the hadronic tensor. It must have only symmetric terms, since $L_e^{\mu\nu}$ is invariant under $\mu \to \nu$

↳ Antisymmetric contributions to $W^{\mu\nu}$ would not affect the cross section

• We can prove two important properties of $L^{\mu\nu}$ and $W^{\mu\nu}$:

1) $q^\mu L_{\mu\nu} = 2(q^\mu k'_\mu k_\nu + q^\mu k'_\nu k_\mu - (k \cdot k' - m_e^2) q^\mu g_{\mu\nu})$

I) $q^\mu k'_\mu k_\nu = (k^\mu - k'^\mu) k'_\mu k_\nu = (k^\mu k'_\mu - k'^\mu k'_\mu) k_\nu = (k \cdot k' - m_e^2) k_\nu$ (42)

II) $q^\mu k'_\nu k_\mu = (k^\mu - k'^\mu) k_\mu k'_\nu = -(k \cdot k' - m_e^2) k'_\nu$ (43)

III) $-(k \cdot k' - m_e^2) q^\mu g_{\mu\nu} = -(k \cdot k' - m_e^2) q_\nu$ (43)

$$\Rightarrow q^\mu L_{\mu\nu} = (k \cdot k' - m_e^2) k_\nu - (k \cdot k' - m_e^2) k'_\nu - (k \cdot k' - m_e^2) q_\nu$$

$$= (k \cdot k' - m_e^2) \left[k_{\cancel{\nu}} - k'_{\cancel{\nu}} - \cancel{q_\nu} \right] \Rightarrow q^\mu L_{\mu\nu} = 0 \quad (H\,8.25)$$

2) The hadronic tensor $W^{\mu\nu}$ is defined as

$$W^{\mu\nu} = \frac{1}{4\pi} \sum_X (2\pi)^4 \delta^4(p + q - p_X) \langle p | \hat{J}^{\mu\dagger} | X \rangle \langle X | \hat{J}^\nu | p \rangle \quad (H\,8.39)$$

• Current conservation at the hadronic vertex can be written as:

$$\partial_\mu \hat{J}^\mu = 0 \quad (44)$$

which in the momentum space can be written as $q_\mu \hat{J}^\mu = 0$. Joining this with (H 8.39) give us then:

$$q_\mu W^{\mu\nu} = 0 \quad (H\,8.26)$$

• The most general form of the tensor $W^{\mu\nu}$ must be construct out of $g^{\mu\nu}$ and the independent moment p and q:

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{m_p^2} p^\mu p^\nu + \frac{W_4}{m_p^2} q^\mu q^\nu + \frac{W_5}{m_p^2}\left(p^\mu q^\nu + q^\mu p^\nu\right) \quad (H8.24)$$

W_i = structure functions

• It can be shown that after we impose current conservation (Eq H8.26), we are left with only two independent structure functions:

$$q_\mu W^{\mu\nu} = q_\mu\left[-W_1 g^{\mu\nu} + \frac{W_2}{m_p^2} p^\mu p^\nu + \frac{W_4}{m_p^2} q^\mu q^\nu + \frac{W_5}{m_p^2}\left(p^\mu q^\nu + q^\mu p^\nu\right)\right]$$

$$= -W_1 q^\nu + \frac{W_2}{m_p^2}(q\cdot p)p^\nu + \frac{W_4}{m_p^2}(q\cdot q)q^\nu + \frac{W_5}{m_p^2}\left((q\cdot p)q^\nu + q^2 p^\nu\right)\Big]$$

$$= \left(-W_1 + \frac{W_4}{m_p^2}q^2 + \frac{W_5}{m_p^2}(p\cdot q)\right)q^\nu + \left(\frac{W_2}{m_p^2}(p\cdot q) + \frac{W_5}{m_p^2}q^2\right)p^\nu \overset{\text{must be}}{=} 0$$

$$\Rightarrow \begin{cases} i)\ \dfrac{W_2}{m_p^2}(p\cdot q) + \dfrac{W_5}{m_p^2}q^2 = 0 \ \Rightarrow\ W_5 = \dfrac{-p\cdot q}{q^2}W_2 \quad (45) \\[2ex] ii)\ -W_1 + \dfrac{W_4}{m_p^2}q^2 + \dfrac{W_5}{m_p^2}(p\cdot q) = 0 \ \Rightarrow\ W_4 = \left(\dfrac{p\cdot q}{q^2}\right)^2 W_2 + \dfrac{m_p^2}{q^2}W_1 \quad (46) \end{cases}$$

• Using (45-46) we can rewrite $W^{\mu\nu}$:

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{m_p^2} p^\mu p^\nu + \frac{W_4}{m_p^2} q^\mu q^\nu + \frac{W_5}{m_p^2}\left(p^\mu q^\nu + q^\mu p^\nu\right)$$

$$= -W_1 g^{\mu\nu} + \frac{W_2}{m_p^2} p^\mu p^\nu + \left[\left(\frac{p\cdot q}{q^2}\right)^2 W_2 + \frac{m_p^2}{q^2}W_1\right]\frac{q^\mu q^\nu}{m_p^2} + \left[\frac{-p\cdot q}{q^2}W_2\right]\frac{p^\mu q^\nu + q^\mu p^\nu}{m_p^2}$$

$$= -W_1 g^{\mu\nu} + \frac{W_2}{m_p^2} p^\mu p^\nu + \frac{W_2}{m_p^2}\left(\frac{p\cdot q}{q^2}\right)^2 q^\mu q^\nu + \frac{W_1}{q^2}q^\mu q^\nu - \frac{W_2}{m_p^2}\frac{p\cdot q}{q^2}\left(p^\mu q^\nu + q^\mu p^\nu\right)$$

$$\Rightarrow W^{\mu\nu} = W_1\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) + \frac{W_2}{m_p^2}\left(p^\mu - \frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}q^\nu\right) \quad (H8.27)$$

• Now we are ready to contract $L^e_{\mu\nu}W^{\mu\nu}$. We need to keep in mind:

$$\left.\begin{array}{l} q^\mu L_{\mu\nu} = 0 \ (H\,8.25) \\ m_e \approx 0 \end{array}\right\} \Rightarrow L^e_{\mu\nu}W^{\mu\nu} = 4W_1(k\cdot k') + \frac{2W_2}{m_p^2}\left[2(p\cdot k)(p\cdot k') - m_p^2(k\cdot k')\right] \ (H8.31)$$

LAB FRAME (fixed target) = proton rest frame

$$q^2 = -2k\cdot k' \approx -2EE'(1-\cos\theta) = -4EE'\sin^2\theta/2 \ (H6.44)$$

$$L^e_{\mu\nu}W^{\mu\nu}\Big|_{Lab} = 4EE'\left[2W_1(x,Q^2)\sin^2\frac{\theta}{2} + W_2(x,Q^2)\cos^2\frac{\theta}{2}\right] \ (H8.32)$$

• Remembering that we can write $d\sigma$ as:

$$d\sigma = \frac{|M|^2}{F} dL_{ips} \ (H4.29)$$

$\hookrightarrow$ moller flux: $F = 4((p_a\cdot p_b)^2 - m_a^2 m_b^2)^{1/2}$ (H4.32)

• Now we put everything together:

$$d\sigma \overset{(H4.29)}{=} \frac{|M|^2}{F} dL_{ips} \overset{(H4.30)(H6.18)(H8.32)}{=} \frac{1}{4[(k\cdot p)^2 - \underbrace{m_e^2}_{\approx 0} m_p^2]^{1/2}} \frac{4EE'e^4}{q^4}\left[2W_1(x,Q^2)\sin^2\frac{\theta}{2} + W_2(x,Q^2)\cos^2\frac{\theta}{2}\right]\frac{d^3k'}{2E'(2\pi)^3}$$

1) $$\frac{1}{4[(k\cdot p)^2 - \underbrace{m_e^2}_{\approx 0} m_p^2]^{1/2}} = \frac{1}{4(k\cdot p)} = \frac{1}{4Em_p} \ (47)$$

proton rest frame: $k^\mu = (E,\vec{k})$, $p^\mu = (m_p, 0)$

2) $$\frac{d^3k'}{2E'(2\pi)^3} = \frac{|\vec{k}'|^2 d|\vec{k}'| d\Omega}{(2\pi)^3 2E'} \approx \frac{E'^{\cancel{2}} dE' d\Omega}{(2\pi)^3 2\cancel{E'}} = \frac{E' dE' d\Omega}{2(2\pi)^3} \ (48)$$

So we have

$$d\sigma = \frac{1}{\cancel{4}\cancel{E}m_p}\frac{\cancel{4}\cancel{E}E'e^4}{q^4}\left[2W_1(x,Q^2)\sin^2\frac{\theta}{2} + W_2(x,Q^2)\cos^2\frac{\theta}{2}\right]\frac{E' dE' d\Omega}{2(2\pi)^3}$$

$$= \frac{E'^2 e^4}{q^4 m_p}\left[2W_1(x,Q^2)\sin^2\frac{\theta}{2} + W_2(x,Q^2)\cos^2\frac{\theta}{2}\right]\frac{dE' d\Omega}{2(2\pi)^3}$$

$$\overset{(H6.44)}{=} \frac{E'^{\cancel{2}} e^4}{16E^2 \cancel{E'^2}\sin^4\frac{\theta}{2}\, m_p}\left[2W_1(x,Q^2)\sin^2\frac{\theta}{2} + W_2(x,Q^2)\cos^2\frac{\theta}{2}\right]\frac{dE' d\Omega}{16\pi^3}$$

$$= \frac{1}{16\pi m_p\, E^2\sin^4\frac{\theta}{2}}\frac{e^4}{16\pi^2}\left[2W_1(x,Q^2)\sin^2\frac{\theta}{2} + W_2(x,Q^2)\cos^2\frac{\theta}{2}\right] dE' d\Omega \qquad \frac{e^4}{16\pi^2} \to \alpha^2$$

$$\Rightarrow \boxed{\frac{d\sigma}{dE' d\Omega} = \frac{1}{4\pi m_p}\frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left[2W_1(x,Q^2)\sin^2\frac{\theta}{2} + W_2(x,Q^2)\cos^2\frac{\theta}{2}\right]} \ (H8.34)$$

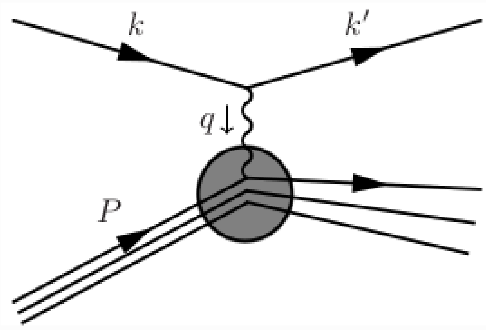
W_1 and W_2 can be determined by measuring only the energy and angular dependence of the outgoing electron

DGLAP Equations

Schwartz Chapter 32

• The defining assumption of the parton model, originally due to Feynman, is that some objects called partons within the proton are essentially free. Partons refers to not only quarks, but also gluons and antiquarks.

• To test the parton model, we need to determine what the form factors W_1 and W_2 would look like if the electron were scattering elastically off partons with mass m_q inside the proton. An elastic parton scattering diagram is:



• We call the scattered parton's initial momentum p_i^μ and its final momentum p_f^μ, so that $p_f^\mu = p_i^\mu + q^\mu$ (50). Squaring give us

$$(p_f)^2 = (p_i)^2 + 2p_i\cdot q + (q)^2 \Rightarrow m_q^2 = m_q^2 + 2p_i\cdot q - Q^2 \Rightarrow \left.\overbrace{\frac{Q^2}{2p_i\cdot q}}^{x_B} = 1 \right\} \text{elastic}$$

• However, the parton momentum is not directly measured. Let us assume that they carry a fraction ξ of the proton's momentum, i.e:

$$p_i^\mu = \xi P^\mu \Rightarrow x = \frac{\xi Q^2}{2p_i\cdot q} \quad (51)$$

Then, if the parton model were valid, by measuring x we would be measuring the fraction of the proton's momentum involved in the parton level scattering.

• Another two ingredients of the parton model:

i) We suppose that the partons are weakly interacting

ii) The probability that the Photon will hit a parton i which has a fraction ξ of the proton's momentum is given by $f_i(\xi)d\xi$, where the f_i's are called parton distribution functions PDF

• The parton model assumption is that the cross section for $e^- p^+ \to e^- X$ scattering is given by $e^- p_i \to e^- X$, summed over all partons and integrated over ξ:

$$\sigma(e^- p^+ \to e^- X) = \sum_i \int_0^1 d\xi \, f_i(\xi) \, \underbrace{\hat{\sigma}(e^- p_i \to e^- X)}_{\text{partonic quantity}} \quad (S\,32.19)$$

• Assuming the partons are free except for their QED interactions, the electron can only scatter off charged particles in the proton, which we will call quarks. For a given quark momentum p_i, the $e^- p_i \to e^- X$ partonic cross section is just a point-like scattering cross-section in QED, and will be given by the Rosenbluth formula with $F_1 = 1$ and $F_2 = 0$:

$$\left(\frac{d\sigma}{d\Omega}\right)_{LAB} = \frac{\alpha_e^2}{4E^2 \sin^4\frac{\theta}{2}} \frac{E'}{E} \left\{ \left(F_1^2 - \frac{q^2}{4m_p^2} F_2^2 \right) \cos^2\frac{\theta}{2} - \frac{q^2}{2m_p^2} (F_1 + F_2)^2 \sin^2\frac{\theta}{2} \right\} \quad (S\,32.7)$$

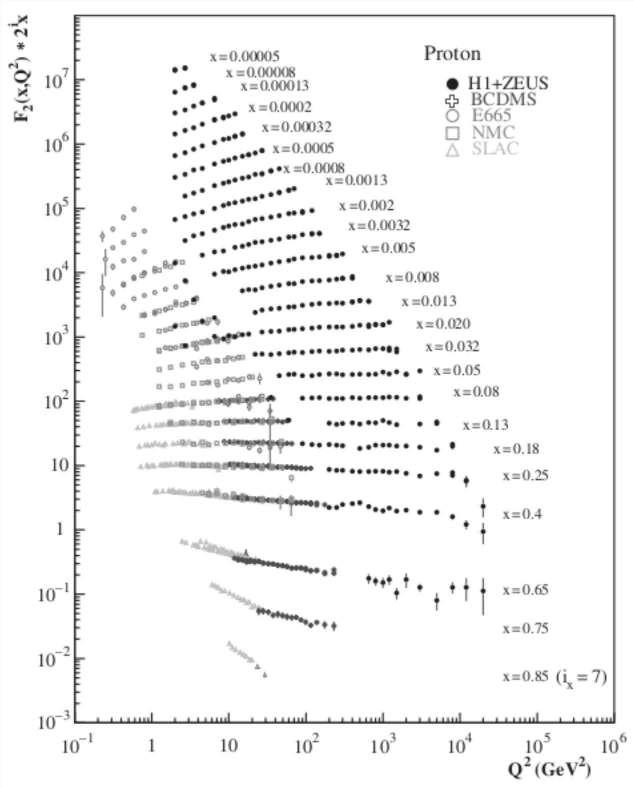
$F_1 = 1$, $F_2 = 0$

$q^2 = -4EE' \sin^2\frac{\theta}{2}\Big|_{LAB}$ (elastic)

Charge of quark i

$$\left(\frac{d\hat{\sigma}(e^- q \to e^- q)}{d\Omega \, dE'}\right)_{LAB} = \frac{\alpha_e^2 \, Q_i^2}{4E^2 \sin^4\frac{\theta}{2}} \left(\cos^2\frac{\theta}{2} - \frac{Q^2}{2m_p^2} \sin^2\frac{\theta}{2} \right) \delta\left(E - E' - \frac{Q^2}{2m_p^2} \right) \quad (S\,32.20)$$

• We call Bjorken Scalling the Appro~~x~~imate independence of the structure functions of Q^2:



In order to get the DIS cross section from (S32.20), we have to integrate over the incoming quark momentum. Since $p_i^\mu = \xi P^\mu$, and in the lab frame the proton is at rest, this implies that $m_q = \xi m_p$. We can also use that $E - E' = \nu = \frac{Q^2}{2m_p x}$, from which we get:

$$\delta\left(E - E' - \frac{Q^2}{2m_p}\right) = \delta\left(\frac{Q^2}{2m_p x} - \frac{Q^2}{2m_p \xi}\right) = \frac{2m_p}{Q^2} x^2 \delta(\xi - x) \quad (S32.21)$$

• Combining this with (S32.19), we can go from $e^- q \to e^- X$ to $e^- P \to e^- X$:

$$\left(\frac{d\sigma(e^- P \to e^- X)}{d\Omega\, dE'}\right)_{LAB} = \sum_i f_i(x) \frac{\alpha_e^2 Q_i^2}{2E^2 \sin^4\frac{\theta}{2}} \left[\frac{2m_p}{Q^2} x^2 \cos^2\frac{\theta}{2} + \frac{1}{m_p} \sin^2\frac{\theta}{2}\right] \quad (S32.22)$$

• Now we remember the cross-section for DIS calculated previously:

$$\left(\frac{d\sigma}{d\Omega\, dE'}\right)_{LAB} = \frac{\alpha_e^2}{8\pi E^2 \sin^4\frac{\theta}{2}} \left[\frac{m_p}{2} W_2(x,Q) \cos^2\frac{\theta}{2} + \frac{1}{m_p} W_1(x,Q) \sin^2\frac{\theta}{2}\right] \quad (S32.16)$$

• Comparing we can see that

$$W_1(x,Q) = 2\pi \sum_i Q_i^2 f_i(x) \quad (S32.23)$$

$$W_2(x,Q) = \frac{8\pi x^2}{Q^2} \sum_i Q_i^2 f_i(x) \quad (S32.24)$$

$$\Rightarrow W_1(x,Q) = \frac{4x^2}{Q^2} W_2(x,Q) \quad (S2)$$

Callan Gross Relation

W_1 and $Q^2 W_2$ do not depend on Q

• However, when we look at the data on the last page, we see that Bjorken scalling does not quite hold - there is some weak (logarithmic) Q^2 dependence visible in the structure function.

• We wrote the $e^- p^+ \to e^- X$ cross section in terms of the leptonic tensor $L^{\mu\nu}$ and the hadronic tensor $W^{\mu\nu}(x,Q)$ given by $|M(\gamma^* p^+ \to X)|^2$. Now let's write $\hat{W}^{\mu\nu}(z,Q)$ as the partonic version of $W^{\mu\nu}(x,Q)$ with z given by:

$$z \equiv \frac{Q^2}{2 p_i \cdot q} \quad (S32.30)$$

• Now we use the parton model assumption that the probability of finding $p_i^\nu = \xi P^\nu$ for some $0 \le \xi \le 1$ is given by the PDF $f_i(\xi)$. From the last equa-tion we see that $x = \xi z$ so we integrate over ξ:

$$W^{\mu\nu}(x,Q) = \sum_i \int_0^1 dz \int_0^1 d\xi\, f_i(\xi) \hat{W}^{\mu\nu}(z,Q)\, \delta(x - \xi z) = \sum_i \int_0^1 \frac{d\xi}{\xi} f_i(\xi) \hat{W}^{\mu\nu}\left(\frac{x}{\xi}, Q\right) \quad (S32.31)$$

• For simplicity, let us consider the form factor $W_0 \equiv -g^{\mu\nu} W_{\mu\nu}$:

$$W_0(x,Q) \equiv -g^{\mu\nu} W_{\mu\nu} = 3 W_1(x,Q) - \left(m_p^2 + \frac{Q^2}{4x^2}\right) W_2(x,Q) \quad (S32.33)$$

• Altarelli et al 1979:

$$\hat{W}_0 \text{ up to next-to-leading order} = 4\pi Q_i^2 \left\{ \left[\delta(1-z) - \frac{1}{\varepsilon} \frac{\alpha_s}{\pi} P_{qq}(z) \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon/2} \frac{\Gamma(1-\varepsilon/2)}{\Gamma(1-\varepsilon)} \right] + \right.$$

SU(3) CASIMIR: Fund. Rep

$$\left. + \frac{\alpha_s}{2\pi} C_F \left[(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ - \frac{3}{2}\left[\frac{1}{1-z}\right]_+ - \frac{1+z^2}{1-z}\ln z + 3 + 2z - \left(\frac{9}{2} + \frac{\pi^2}{3}\right)\delta(1-z) \right] \right\}$$

(S32.42)

Where:

1) $$\int_0^1 dz \frac{f(z)}{[1-z]_+} \equiv \int_0^1 dz \frac{f(z)-f(1)}{1-z}$$ (S32.39) ⇝ so that $1/[1-z]_+ = 1/1-z$ for $z \neq 1$

2) $$\int_0^1 dz \; f(z)\left[\frac{\ln^n(1-z)}{1-z}\right]_+ \equiv \int_0^1 dz \left(f(z)-f(1)\right)\frac{\ln^n(1-z)}{1-z}$$ (S32.40)

So that $\left[\frac{\ln^2(1-z)}{1-z}\right]_+ = \frac{\ln^2(1-z)}{1-z}$, for $z \neq 1$

3) C_F: SU(3) CASIMIR at the fundamental rep

4) $P_{qq}(z)$ is called the DGLAP splitting function:

$$P_{qq}(z) = C_F\left\{(1+z^2)\left[\frac{1}{1-z}\right]_+ + \frac{3}{2}\delta(1-z)\right\}$$ (S32.43)

DGLAP: Yuri Dokshitzer, Vladimir Gribov †, Lev Lipatov †, Guido Altarelli †, Giorgio Parisi

- There is a single $1/\varepsilon$ pole whose residue is proportional to $P_{qq}(z)$.

  ↳ However, what matter to us is if we can find finite differences!

- And indeed, if we consider the difference in $W_0(x,Q)$ at the same x but different scales Q and Q_0, we find:

$$W_0(x,Q) - W_0(x,Q_0) = 4\pi \sum_i q_i^2 \int_0^1 \frac{d\xi}{\xi} f_i(\xi) \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \ln Q^2/Q_0^2$$ (S32.46)

which is finite and explains exactly the violation of Bjorken scaling

- Since Q_0 is arbitrary, the independence of the cross section / W_0 of Q_0 should lead to a Renormalization GROUP Equation, so we define

$$W_0(x,Q) \equiv 4\pi \sum_i q_i^2 f_i(x, \mu = Q)$$ (S32.47)

for every scale Q. For this to be consistent with (S32.46) we need

$$f_i(x,\mu_1) = f_i(x,\mu) + \frac{\alpha_s}{2\pi}\int_0^1 \frac{d\xi}{\xi} f_i(\xi,\mu_1) \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \ln\frac{\mu_1^2}{\mu^2}$$ (S32.48)

which implies the DGLAP EVOLUTION EQUATION:

$$\boxed{\mu \frac{d}{d\mu} f_i(x,\mu) = \frac{\alpha_s}{\pi} \int_x^1 \frac{d\xi}{\xi} f_i(\xi,\mu) P_{qq}\left(\frac{x}{\xi}\right)}$$

1) $f_i(x,\mu)$: This PDF tells us the prob. of finding a parton with momentum fraction x when probed at a energy scale μ

2) $\mu \frac{d}{d\mu} f_i(x,\mu)$: the evolution of $f_i(x,\mu)$ due to changes in μ

3) $P_{qq}\left(\frac{x}{\xi}\right)$: probability of a quark with m.fraction ξ emit a gluon and retain a m.fraction x