Gaussian Fluctuations In Atomic Bose-Bose Gases: A Droplet Phase

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Presented to:

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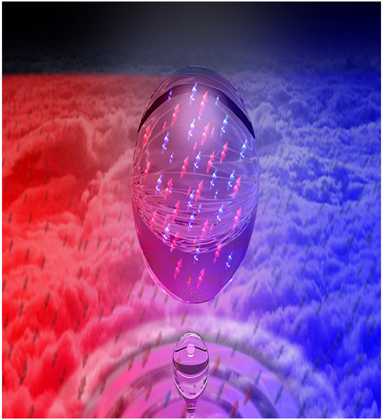
Universidade Estadual Paulista - UNESP
São Paulo

June 30, 2025

Outline



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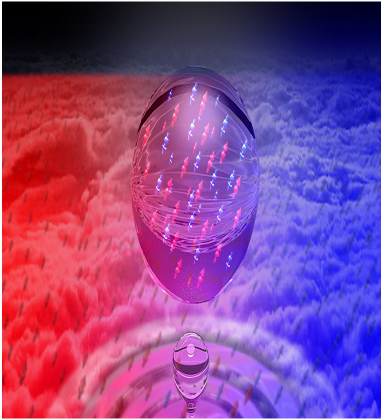
Outline

1. Model and Preliminaries
2. The mean-field grand potential
3. Gaussian Fluctuations
   - Zero-temperature grand potential
   - Two-dimensional Bose-Bose gases
4. The droplet phase
5. Finite-temperature grand potential
6. Summary and perspectives



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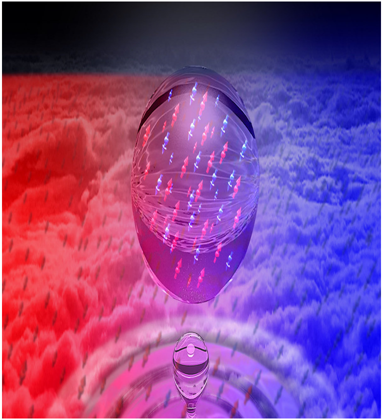

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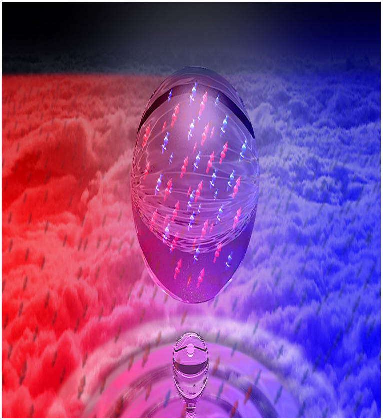

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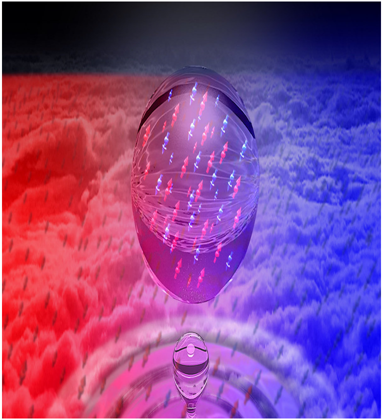

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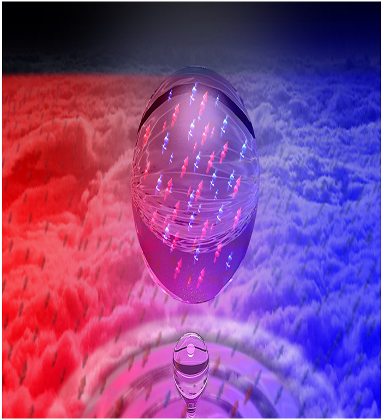

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Model and Preliminaries

At leading order (PRA **63**, 063609 (2001))

$$\mathcal{L}_{\mathsf{EFF}} = \psi^*\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi + c_1(\psi^*\psi)^2 + c_2[\nabla(\psi^*\psi)]^2 + c_3(\psi^*\psi)^3 + \cdots \quad (1)$$

- Galilean invariance
- Parity
- Time reversal
- Z_2
- $U(1)$

$c_1, c_2,$ and c_3 are related to the the low-momentum expansions for the scattering amplitudes of atoms.

- c_1 Two-body scattering (universal) (RMP **76**, 599 (2004))
- c_2 Nonuniversal corrections to the two-body scattering
- c_3 Three-body scattering

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Bose-Bose gases

We consider two interacting and equal-mass bosonic species with hyperfine states ($\alpha = 1, 2$), in a d-dimensional box of volume L^d ($d = 3, 2, 1$), governed by the Euclidean action

$$S[\Psi, \Psi^*] = \int_0^{\hbar\beta} d\tau \int_{L^d} d^d r \, \mathcal{L}[\Psi, \Psi^*] \qquad (2)$$

where $\beta^{-1} = k_B T$, k_B is the Boltzmann constant, $\Psi = (\psi_\alpha, \psi_\sigma)^T$ (with periodic conditions),

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \qquad (3)$$

with

$$\mathcal{L}_0 = \sum_{\alpha=1,2} \psi_\alpha^* \big(\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \nabla^2 - \mu_\alpha \big) \psi_\alpha, \qquad (4)$$

and

$$\mathcal{L}_{\text{int}} = \frac{1}{2} \sum_{\sigma=1,2} g_{\alpha\sigma} |\psi_\alpha|^2 |\psi_\sigma|^2, \qquad \boldsymbol{g_{\alpha\sigma}, g_{\alpha\alpha} > 0 (< 0)} \qquad (5)$$

In order to obtain the ground state of the mixture we calculate $\Omega = -\beta^{-1} \ln \mathcal{Z}$,

$$\mathcal{Z} = \int \mathcal{D}[\psi, \psi^*] \text{exp}\Big[-\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \int_{L^d} d^d r \mathcal{L}(\psi, \psi^*) \Big]. \qquad (6)$$

So we can set (*Ultracold Quantum Fields* (2009))

$$\boldsymbol{\psi_\alpha(\mathbf{r}, \tau) = \phi_\alpha + \eta_\alpha(\mathbf{r}, \tau)}, \qquad (7)$$

with $\phi_\alpha(\mathbf{r}) \equiv \langle \psi_\alpha(\mathbf{r}, \tau) \rangle$, and $n_\alpha \equiv |\phi_\alpha|^2$ is the macroscopic condensate or quasicondensate density in the mean-field approximation.

The fluctuations around ψ_α are given by $\eta_\alpha(\mathbf{r}, \tau)$.

Now, by expanding the action up to the second order (Gaussian) in $\eta_\alpha(\mathbf{r}, \tau)$ and $\eta_\alpha^*(\mathbf{r}, \tau)$, the action becomes (PRA **91**, 043641 (2015))

$$S = S_0 + \sum_{\sigma=1,2} S_{1\sigma} + S_{\text{GF}} + \text{higher orders} \qquad (8)$$

where the zeroth-order contribution is

$$S_0 = \hbar\beta L^d\Big(- \mu_1\phi_1^2 + \frac{1}{2}g\phi_1^4 - \mu_2\phi_2^2 + \frac{1}{2}g\phi_2^4 + g_{12}\phi_1^2\phi_2^2\Big), \qquad (9)$$

the term linear in fluctuations reads ($\alpha \neq \sigma$)

$$S_{1\alpha} = \int \Big[\eta_\alpha^*\big(- \mu_\alpha + g\phi_\alpha^2 + g_{12}\phi_\sigma^2\big)\phi_\alpha + \text{c.c.}\Big], \qquad (10)$$

where $\int \equiv \int_0^{\hbar\beta} d\tau \int_{L^d} d^d r$, and the quadratic term in fluctuations is ($\alpha \neq \sigma$)

$$\begin{aligned} S_{\mathsf{GF}} &= \sum\!\!\!\!\!\!\int \Big[\eta_\alpha^*\Big(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu_\alpha + 2g\phi_\alpha^2 + g_{12}\phi_\sigma^2\Big)\eta_\alpha \\ &+ \frac{1}{2}g\phi_\alpha^2(\eta_\alpha^*\eta_\alpha^* + \eta_\alpha\eta_\alpha)\Big] \\ &+ g_{12}\phi_1\phi_2 \int (\eta_1\eta_2 + \eta_1^*\eta_2 + \text{c.c.}), \end{aligned} \qquad (11)$$

where $\sum\!\!\!\!\!\!\int \equiv \int_0^{\hbar\beta} d\tau \int_{L^d} d^d r \sum_\alpha$.

The mean-field grand potential

In terms of the grand-potential we get

$$\Omega = \Omega_0 + \Omega_{\mathsf{GF}}. \tag{12}$$

Where Ω_0 is the mean-field contribution, while Ω_{GF} takes into account both zero- and finite-temperature fluctuations.

$$\frac{\Omega_0}{L^d} = \sum_{\alpha=1,2} \Big(-\mu_\alpha \phi_\alpha^2 + \frac{1}{2} \sum_{\sigma=1,2} g_{\alpha\sigma} \phi_\alpha^2 \phi_\sigma^2 \Big), \tag{13}$$

The ϕ_α's are determined by means of the saddle-point approximation, $\boldsymbol{\partial\Omega_0/\partial\phi_\alpha = 0}$, from which

$$\mu_\alpha = g_{\alpha\alpha}\phi_\alpha^2 + g_{\alpha\lambda}\phi_\lambda^2 \qquad \alpha, \lambda = 1, 2 \ \text{ and } \ \alpha \neq \lambda. \tag{14}$$

Hence the mean-field grand potential becomes (PRA **97**, 063605 (2018))

$$\boxed{\frac{\Omega_0}{L^d} = -\frac{1}{2} \sum_{\alpha} \frac{g_{\sigma\sigma}\mu_\alpha^2 - g_{\alpha\sigma}\mu_\alpha\mu_\sigma}{g_{\alpha\alpha}g_{\sigma\sigma} - g_{\alpha\sigma}^2}} \tag{15}$$

where $\alpha, \sigma = 1, 2$, and, $\alpha \neq \sigma$.

Gaussian Fluctuations

In such a case the partition function is (Phys. Rep. **640**, 1 (2016))

$$\mathcal{Z}_{\mathsf{GF}} = \int \mathcal{D}[\eta,\eta^*] \exp\Big[-\frac{1}{2}\sum_{\substack{k\neq 0\\ n=-\infty}}^{\infty} \eta^*(k,\omega_n)\mathbb{G}^{-1}(k,\omega_n)\eta(k,\omega_n)\Big] \quad (16)$$

with the bosonic Matsubara's frequencies $\omega_n = 2\pi n/(\hbar\beta)$, and, the 4×4 inverse propagator $\mathbb{G}^{-1}$ given by (PRA **97**, 063605 (2018))

$$\mathbb{G}^{-1} = \begin{pmatrix} \mathbf{G}_{11}^{-1} & \mathbf{G}_{12}^{-1} \\ \mathbf{G}_{12}^{-1} & \mathbf{G}_{22}^{-1} \end{pmatrix}, \quad (17)$$

with the symmetric 2×2 matrices

$$\mathbf{G}_{11}^{-1} = \begin{pmatrix} -\mathrm{i}\hbar\omega_n + \epsilon_k + h_{11} & g_{11}\phi_1^2 \\ g_{11}\phi_1^2 & \mathrm{i}\hbar\omega_n + \epsilon_k + h_{11} \end{pmatrix}, \quad (18)$$

with the free-particle energy $\epsilon_k = \hbar^2 k^2/2m$, $h_{11} = 2g_{11}\phi_1^2 + g_{12}\phi_2^2 - \mu_1$, $\mathbf{G}_{22}^{-1} = \mathbf{G}_{11}^{-1}(1 \leftrightarrow 2)$, and

$$\mathbf{G}_{12}^{-1} = g_{12}\phi_1\phi_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (19)$$

$$\mathcal{Z}_{\mathsf{GF}} = \prod_{\substack{k\neq 0\\ \omega_n\neq 0}} [\det \mathbb{G}(k,\omega_n)]^{1/2} = \exp\left[\frac{1}{2}\sum_{k,\omega_n\neq 0} \ln\det\mathbb{G}(k,\omega_n)\right], \quad (20)$$

So

$$\Omega_{\mathsf{GF}} = -\frac{1}{2\beta}\sum_{k\neq 0,\omega_n} \ln\det\mathbb{G} = \frac{1}{2\beta}\sum_{k\neq 0,\omega_n} \ln\left(\frac{1}{\det\mathbb{G}}\right). \quad (21)$$

Therefore, the grand potential of the Gaussian fluctuations is *(Le Bellac, Thermal Field Theory, 1996)*

$$\boldsymbol{\Omega_{\mathsf{GF}} = \frac{1}{2\beta}\sum_{\substack{k>0\\ n=-\infty}}^{+\infty} \ln\det[\mathbb{G}^{-1}(k,\omega_n)].} \quad (22)$$

By solving the determinant of the propagator, we get

$$\boxed{\Omega_{\mathsf{GF}} = -\frac{1}{2\beta}\sum_{\substack{k>0\\ n=-\infty}}^{+\infty} \ln\left[(\hbar^2\omega_n^2 + E_+^2)(\hbar^2\omega_n^2 + E_-^2)\right]}, \quad (23)$$

with the Bogoliubov spectra (dispersion relation)

$$\boldsymbol{E_\pm(k, \mu_1, \mu_2) = [\epsilon_k^2 + 2\epsilon_k f_\pm^2(\mu_1, \mu_2)]^{1/2},} \tag{24}$$

and

$$f_\pm^2 = \frac{\xi}{2}\left[A + B \pm \sqrt{(A-B)^2 + 4\Delta AB}\right], \tag{25}$$

where

$$\xi = [g_{11}g_{22}(1-\Delta)]^{-1}, \qquad \Delta = g_{12}^2/g_{11}g_{22}, \tag{26}$$

$$A = g_{11}(g_{22}\mu_1 - g_{12}\mu_2), \qquad B = g_{22}(g_{11}\mu_2 - g_{12}\mu_1). \tag{27}$$

The sum over the bosonic Matsubara's frequencies can be read as

$$\begin{aligned} \Omega_{\mathsf{GF}} &= \frac{1}{2}\sum_{k,\pm} E_\pm + \frac{1}{\beta}\sum_{k,\pm} \ln\left(1 - e^{-\beta E_\pm}\right) \\ &= \Omega_{\mathsf{GF}}^{(0)} + \Omega_{\mathsf{GF}}^{(T)} \end{aligned} \tag{28}$$

Zero-temperature grand potential

In the continuum limit, $\sum_k \to L^d \int d^dk/(2\pi)^d$, we have

$$\frac{\Omega^{(0)}_{\mathsf{GF}}}{L^d} = \frac{1}{2}\frac{S_d}{(2\pi)^d}\sum_{\pm}\int_0^{\infty} dk\, k^{d-1} E_{\pm}, \tag{29}$$

with $S_d = 2\pi^{d/2}/\Gamma(d/2)$ the solid angle in d dimensions and $\Gamma(x)$ the Euler gamma function.

By means of

$$x_{\pm}^2 = \frac{\hbar^2 k^2}{4 m f_{\pm}^2} \quad \to \quad x_{\pm}^2 + 1 = y_{\pm} \quad \to \quad y_{\pm} - 1 = t_{\pm}, \tag{30}$$

and considering the beta-Euler function

$$\mathsf{B}(a,b) = \int_0^{\infty} dt \frac{t^{a-1}}{(1+t)^{a+b}} \qquad \mathcal{R}\{a\}, \mathcal{R}\{b\} > 0, \tag{31}$$

the gaussian corrections to the zero-temperature grand-potential are (PRA **97**, 063605 (2018))

$$\boldsymbol{\Omega^{(0)}_{\mathsf{GF}} = \frac{L^d}{\Gamma(d/2)}\Big(\frac{m}{\pi\hbar}\Big)^{d/2}\mathrm{B}\Big(\frac{1+d}{2}, -\frac{2+d}{2}\Big)\sum_{\pm} f_{\pm}^{d+2}}. \quad (32)$$

For $d = 3$

$$\frac{\Omega^{(0)}_{\mathsf{GF}}}{L^3} = \frac{8}{15\pi^2}\Big(\frac{m}{\hbar^2}\Big)^{3/2}\sum_{\pm} f_{\pm}^{5} \quad (33)$$

For $d = 1$

$$\frac{\Omega^{(0)}_{\mathsf{GF}}}{L} = -\frac{2}{3\pi}\Big(\frac{m}{\hbar^2}\Big)^{1/2}\sum_{\pm} f_{\pm}^{3} \quad (34)$$

For $\boldsymbol{d = 2}$

$$\boxed{\frac{\Omega^{(0)}_{\mathsf{GF}}}{L^2} \sim \mathrm{B}\Big(\frac{3}{2}, \boldsymbol{-2}\Big)} \quad (35)$$

To simplify we consider that the intra-species are equal $\boldsymbol{g_{11} = g_{22} \equiv g}$, and it is also natural to take $\boldsymbol{n_1 = n_2 \equiv n/2}$, so $\boldsymbol{\mu_1 = \mu_2 \equiv \mu}$. Therefore,

For $d = 3$

$$\frac{\Omega_{\mathsf{GF}}}{L^3} = \frac{8}{15\pi^2}\left(\frac{m}{\hbar^2}\right)^{3/2}[1 + \lambda^{5/2}(g, g_{12})]\mu^{5/2}. \tag{36}$$

For $d = 1$

$$\frac{\Omega_{\mathsf{GF}}}{L} = -\frac{2}{3\pi}\left(\frac{m}{\hbar^2}\right)^{1/2}[1 + \lambda^{3/2}(g, g_{12})]\mu^{3/2}, \tag{37}$$

with

$$\lambda(g, g_{12}) = \frac{g - g_{12}}{g + g_{12}} \tag{38}$$

and what happens for $d = 2$?

Two-dimensional Bose-Bose gases

$$\frac{\Omega_{\mathsf{GF}}}{L^2} = \frac{1}{2}\frac{S_2}{(2\pi)^2}\int_0^\infty dk\, k^2\left[\sqrt{\epsilon_k^2 + 2\epsilon_k f_+} + (f_+ \to f_-)\right] \equiv \frac{\Omega^+_{\mathsf{GF}}}{L^2} + \frac{\Omega^-_{\mathsf{GF}}}{L^2} \qquad (39)$$

with $S_2 = 2\pi/\Gamma(1)$, and

$$f_+ = \frac{1}{2}\mu, \qquad f_- = \frac{1}{2}\lambda\mu. \qquad (40)$$

Considering dimensional regularization, we have

$$2\bar{\epsilon} \equiv 2 - d, \qquad dk^2 \to \boldsymbol{\bar{\kappa}^{2\bar{\epsilon}}} dk^d, \qquad \boldsymbol{\bar{\kappa} \equiv \frac{e^{\gamma/2} L}{2\sqrt{\pi}}\kappa} \qquad (41)$$

$$\boxed{\frac{\Omega^+_{\mathsf{GF}}}{L^2} = -\frac{\mu\,\bar{\kappa}^{2\bar{\epsilon}}}{2\sqrt{\pi}}\left(\frac{m\mu L^2}{\pi\hbar^2}\right)^{1-\bar{\epsilon}}\frac{\Gamma(\frac{3}{2}-\bar{\epsilon})\Gamma(-2+\bar{\epsilon})}{\Gamma(1-\bar{\epsilon})}} \qquad (42)$$

$$\Gamma\left(\frac{3}{2}-\bar{\epsilon}\right)=\Gamma\left(\frac{3}{2}\right)\left[1-\bar{\epsilon}\psi\left(\frac{3}{2}\right)+\frac{1}{2}\bar{\epsilon}^2\left[\psi'\left(\frac{3}{2}\right)+\psi^2\left(\frac{3}{2}\right)\right]\right]+\mathcal{O}(\bar{\epsilon}^3)$$

$$\Gamma(-2+\bar{\epsilon})=\frac{1}{2}\left[\frac{1}{\bar{\epsilon}}+\psi(3)+\frac{1}{2}\bar{\epsilon}\left[\frac{\pi^2}{3}+\psi^2(3)-\psi'(3)\right]\right]+\mathcal{O}(\bar{\epsilon}^2)$$

$$\Gamma(1-\bar{\epsilon})=\Gamma(1)\left[1-\bar{\epsilon}\psi(1)+\frac{1}{2}\bar{\epsilon}^2\left[\psi'(1)+\psi^2(1)\right]\right]+\mathcal{O}(\bar{\epsilon}^3) \qquad (43)$$

with the Euler digamma function $\psi(z)=\frac{d}{dz}\ln\Gamma(z)$. So,

$$\frac{\Gamma(\frac{3}{2}-\bar{\epsilon})\Gamma(-2+\bar{\epsilon})}{\Gamma(1-\bar{\epsilon})}=\frac{1}{2}\Gamma\left(\frac{3}{2}\right)\left[\frac{1}{\bar{\epsilon}}+\ln 4-\frac{1}{2}-\gamma\right]+\mathcal{O}(\bar{\epsilon}), \qquad (44)$$

and

$$\frac{\Omega^{+}_{\mathsf{GF}}}{L^2}=-\frac{m\mu^2}{8\pi\hbar^2}\left[\frac{1}{\bar{\epsilon}}+\ln\left(\frac{\kappa^2\hbar^2}{e^{1/2}m\mu}\right)\right] \qquad (45)$$

Therefore in the $\overline{\mathrm{MS}}$-scheme

$$\boxed{\frac{\Omega_{\mathsf{GF}}}{L^2}=-\frac{m\mu^2}{8\pi\hbar^2}\ln\left(\frac{\hbar^2\boldsymbol{\kappa^2}}{e^{1/2}m\mu}\right)-\frac{m\lambda^2\mu^2}{8\pi\hbar^2}\ln\left(\frac{\hbar^2\boldsymbol{\kappa^2}}{e^{1/2}m\lambda\mu}\right)} \qquad (46)$$

The coupling constants

The matrix element $T_{\mathbf{k}\mathbf{k}'} = \langle \mathbf{k}|\hat{T}|\mathbf{k}'\rangle$ of the transition operator $\hat{T}$ of scattering theory satisfies the T-matrix equation (*Introduction to the Quantum Theory of Scattering, 1970. Sakurai and Napolitano, Modern Quantum Mechanics 2017.*)

$$T_{\mathbf{k}\mathbf{k}'} = V_{\mathbf{k}\mathbf{k}'} + \int d^d k'' \frac{V_{\mathbf{k}\mathbf{k}''}}{\frac{\hbar^2 k^2}{2m_r} - \frac{\hbar^2 (k'')^2}{2m_r} + i\epsilon} T_{\mathbf{k}''\mathbf{k}'} \tag{47}$$

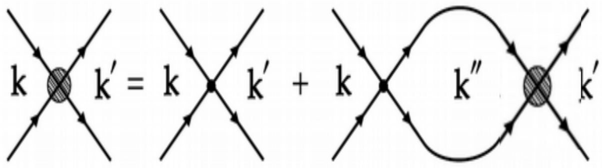

Figure 1: Pictoral representation of the scattering process in the center-of-mass reference frame in the construction of Eq (47), (adapted from PRA **107**, 033325 (2023)).

For $k \simeq k''$ we have $T_{l=0}(k'', k) \simeq T_{l=0}(k, k) = T_{l=0}(k)$ and $V_{l=0}(k, k'') \simeq V_{l=0}(k, k) = V_{l=0}(k)$. So Eq. (47) becomes

$$T_{l=0}(k) = V_{l=0}(k) + V_{l=0}(k) C(k) T_{l=0}(k), \tag{48}$$

where in the limit $\epsilon \to 0$, we have

$$\begin{aligned} C(k) &= -\frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \frac{dk''}{(2\pi)^d} \frac{(k'')^{d-1}}{\frac{\hbar^2 k^2}{m} - \frac{\hbar^2 (k'')^2}{m}} \\ &= -\frac{m}{\hbar^2} (-ik)^{d-2} \frac{\mathrm{B}(d/2, \mathbf{1 - d/2})}{(4\pi)^{d/2} \Gamma(d/2)}, \end{aligned} \tag{49}$$

where (PRA **107**, 033325 (2023))

$$C_{d=3}(k) = -i \frac{m}{4\pi\hbar^2} k \qquad C_{d=1}(k) = -i \frac{m}{2\hbar^2} \frac{1}{k} \tag{50}$$

For $\boldsymbol{d = 2}$ in the $\overline{\mathrm{MS}}$-scheme we get (with the low-energy cutoff $\epsilon_c = \hbar^2 \bar{\kappa}^2 / m$).

$$C_{d=2}(k) = \frac{m}{2\pi\hbar^2} \ln\left(\frac{k}{\sqrt{\boldsymbol{\epsilon_c}}} \right) - \frac{m}{4\hbar^2} \mathrm{i}, \tag{51}$$

Therefore,

- $d = 3$

$$V_{l=0}(k) = -\frac{4\pi\hbar^2}{m}\frac{\tan[\delta_{l=0}(k)]}{k}, \quad k\cot[\delta_{l=0}(k)] = -\frac{1}{a} + \cdots \quad (52)$$

- $d = 2$

$$V_{l=0}(k) = -\frac{4\pi\hbar^2}{m}\frac{1}{\cot[\delta_{l=0}(k)] - \frac{2}{\pi}\ln\left(\frac{k}{\sqrt{\epsilon_c}}\right)},$$

$$\cot[\delta_{l=0}(k)] = \frac{2}{\pi}\ln\left(\frac{k}{2}ae^{\gamma}\right) + \cdots \quad (53)$$

- $d = 1$

$$V_{l=0}(k) = -\frac{2\hbar^2}{m}k\tan[\delta_{l=0}(k)], \quad k\tan[\delta_{l=0}(k)] = \frac{1}{a} + \cdots, \quad (54)$$

and at low momentum of $V_{l=0}(k)$ where

$$V_{l=0}(k) = g + \mathcal{O}(k^2) + \cdots, \quad (55)$$

we obtain the intra-species coupling constants

$$g^{(3d)} = \frac{4\pi\hbar^2}{m}a, \qquad g^{(2d)} = \frac{4\pi\hbar^2}{m}\frac{1}{\ln\left(\frac{4e^{-2\gamma}}{a^2\epsilon_c}\right)},$$
$$g^{(1d)} = -\frac{2\hbar^2}{ma}, \tag{56}$$

with a the s-wave scattering length **(experimentally controllable parameter)**.

In a similar way the inter-species coupling constants are

$$g_{12}^{(3d)} = \frac{4\pi\hbar^2}{m}a_{12}, \qquad g_{12}^{(2d)} = \frac{4\pi\hbar^2}{m}\frac{1}{\ln\left(\frac{4e^{-2\gamma}}{a_{12}^2\epsilon_c}\right)},$$
$$g_{12}^{(1d)} = -\frac{2\hbar^2}{ma_{12}}, \tag{57}$$

d-dimensional grand potential for Bose-Bose gases

For $d = 3$

$$\frac{\Omega}{L^3} = -\frac{\mu^2}{g+g_{12}} + \frac{8}{15\pi^2}\left(\frac{m}{\hbar^2}\right)^{3/2}[1+\lambda^{5/2}(g,g_{12})]\mu^{5/2}. \qquad (58)$$

For $d = 2$

$$\frac{\Omega}{L^2} = -\frac{\mu^2}{g+g_{12}} - \frac{m\mu^2}{8\pi\hbar^2}\ln\left(\frac{\hbar^2\boldsymbol{\kappa^2}}{e^{1/2}m\mu}\right) - \frac{m\lambda^2\mu^2}{8\pi\hbar^2}\ln\left(\frac{\hbar^2\boldsymbol{\kappa^2}}{e^{1/2}m\lambda\mu}\right). \qquad (59)$$

For $d = 1$

$$\frac{\Omega}{L} = -\frac{\mu^2}{g+g_{12}} - \frac{2}{3\pi}\left(\frac{m}{\hbar^2}\right)^{1/2}[1+\lambda^{3/2}(g,g_{12})]\mu^{3/2}. \qquad (60)$$

$$\lambda(g,g_{12}) = \frac{g-g_{12}}{g+g_{12}} \qquad (61)$$

The droplet phase

From thermodynamics $\Omega = E - \mu N - TS$. So

$$\frac{E_{3d}}{N} = (a + a_{12})\frac{\pi\hbar^2}{m}n + \frac{32\sqrt{2\pi}}{15}\frac{\hbar^2}{m}[(a+a_{12})^{5/2} + (a-a_{12})^{5/2}]n^{3/2}. \quad (62)$$

In a local density approximation (LDA)

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = \Big[-\frac{\hbar^2}{2m}\nabla^2 + \frac{2\pi\hbar^2}{m}(a_{12}+a)|\phi|^2\Big]\phi, \quad (63)$$

and we have a mean-field instability by collapse if $-|a_{12}| \gtrsim |a|$.

3d Droplet, PRL **115**, 155302 (2015) (using quantum mechanics)

$$\boxed{\frac{E_D}{N} = -|a-a_{12}|\frac{\pi\hbar^2}{m}n + \frac{256\sqrt{\pi}}{15}\frac{\hbar^2}{m}a^{5/2}n^{3/2}}. \quad (64)$$

Experimental evidence: Science **359**, 301 (2018); PRL **120**, 235301 (2018); PRL **120**, 135301 (2018) $+\cdots$

In what region would the droplets exist?

$$E_+ = \sqrt{\epsilon_k^2 - \epsilon_k |g - g_{12}| n}\,, \tag{65}$$

then, the threshold where the energy E_+ becomes imaginary is

$$k_c < \sqrt{\frac{2m}{\hbar^2} |g - g_{12}| n} \tag{66}$$

In terms of the healing length ξ we have a region where the contribution of E_+ can be neglected (for the $3d$ case)

$$\xi \ll (8\pi |a - a_{12}| n)^{-1/2}, \tag{67}$$

if $|a - a_{12}| = |1 - 1.1| a = 0.1 \times 100 \times 10^{-10}$m, $n \sim 10^{19}$ atoms/m^3, $\xi \ll 2\mu$m.

$$\int_{k_c}^{\infty} dk k^{d-1} \left[\sqrt{\epsilon_k^2 + \epsilon_k |g + g_{12}| n} + \sqrt{\epsilon_k^2 - \epsilon_k |g - g_{12}| n} \right]$$
$$\sim \quad \int_{0}^{\infty} dk\, k^{d-1} \sqrt{\epsilon_k^2 + 2\epsilon_k |g| n} \tag{68}$$

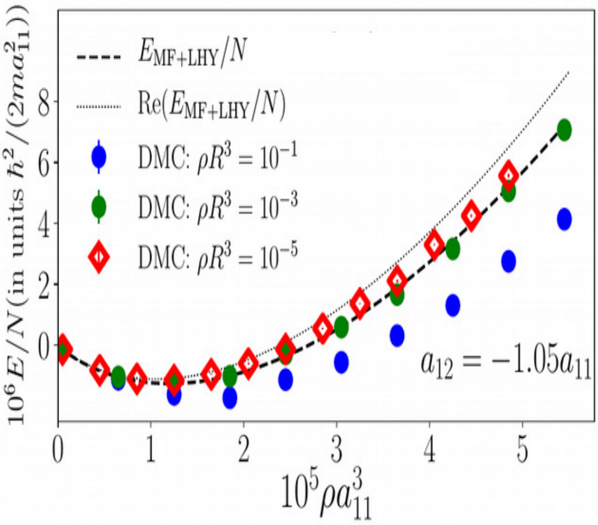

Figure 2: $3d$ energy per particle of Bose-Bose gases as function of the density ((PRA **99**, 023618 (2019))). The droplet energy per particle in Eq. (64) (dashed black line), the real part of Eq. (62) (dotted line), and the diffusion Monte Carlo (DMC) results (colors).

$1d$-dimensional droplet

PRL **117**, 100401 (2016)

$$\frac{E_D}{N} = \frac{1}{4}(g+g_{12})n - \frac{1}{3\sqrt{2}\pi}\left(\frac{m}{\hbar^2}\right)^{1/2}[(g+g_{12})^{3/2}+(g-g_{12})^{3/2}]n^{1/2} \quad (69)$$

$$\boxed{\frac{E_D}{N} = \frac{1}{4}|g-g_{12}|\,n - \frac{2}{3\pi}\left(\frac{m}{\hbar^2}\right)^{1/2} g^{3/2}n^{1/2}} \quad (70)$$

In a LDA we have

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}|g-g_{12}||\phi|^2 - \frac{1}{\pi}\left(\frac{m}{\hbar^2}\right)^{1/2} g^{3/2}|\phi|\right]\phi. \quad (71)$$

$2d$-dimensional droplet

For simplicity $\hbar = m = 1$, and (PRL **117**, 100401 (2016))

$$\epsilon_c = \kappa^2 = \frac{4e^{-2\gamma}}{aa_{12}}, \tag{72}$$

$$g = \frac{4\pi}{\ln(a_{12}/a)}, \qquad g_{12} = \frac{4\pi}{\ln(a/a_{12})}, \tag{73}$$

where $g = -g_{12}$

$$\boxed{\frac{E_D}{N} = \frac{2\pi n}{\ln^2(a_{12}/a)} \ln\left(\frac{n}{en_0}\right)}, \tag{74}$$

with n_0 the equilibrium density

$$n_0 = \frac{1}{\pi} e^{-2\gamma-3/2} \frac{\ln(a_{12}/a)}{aa_{12}}. \tag{75}$$

Finally, in a LDA we obtain

$$\mathrm{i}\frac{\partial\phi}{\partial t} = \left[-\frac{1}{2}\nabla^2 + \frac{4\pi}{\ln^2(a_{12}/a)}|\phi|^2 \ln\left(\frac{|\phi|^2}{\sqrt{e}n_0}\right)\right]\phi, \tag{76}$$

Finite-temperature grand potential

$$\Omega_{\mathsf{GF}}^{(T)} = \frac{1}{\beta}\sum_{k,\pm}\ln\left(1 - e^{-\beta E_{\pm}}\right), \tag{77}$$

$$\frac{\Omega_{\mathsf{GF}}^{(T)}}{L^d} = A_d \sum_{\pm}\int_0^{+\infty} dE_{\pm}(k)\,\frac{k_{\pm}^d}{e^{\beta E_{\pm}(k)} - 1}, \tag{78}$$

where $A_{d=3} = -(6\pi^2)^{-1}$, $A_{d=2} = -(4\pi)^{-1}$ and $A_{d=1} = -\pi^{-1}$. The **Gaussian** contribution to the grand-potential of d-dimensional Bose-Bose gases at low temperature is

$$\begin{aligned}\frac{\Omega_{\mathsf{GF}}^{(T)}}{L^d} &= A_d (k_B T)^{d+1}\sum_{\pm}\left(\frac{m}{\hbar^2 f_{\pm}^2}\right)^{d/2}\Big[\Gamma(d+1)\zeta(d+1) \\ &- \frac{d}{8}\left(\frac{k_B T}{f_{\pm}^2}\right)^2 \Gamma(d+3)\zeta(d+3) + \cdots\Big],\end{aligned} \tag{79}$$

with the Euler's gamma function $\Gamma(x)$ and the Riemann's zeta function $\zeta(x)$.

Summary and perspectives

- Leading quantum corrections to the equation of state in atomic Bose-Bose gases are obtained employing elements of QFT II.
- The presence of the Gaussian fluctuations allow the droplet existence in Bose-Bose gases.

Under construction...

1. Dipole-dipole interaction

$$U_{dd}(\mathbf{r}) = \frac{\mu_0 m^2}{4\pi} \frac{1 - 3\cos^2\theta}{r^3},$$

   $m =$ permanent dipole moment

2. RG in $2d$, critical exponents?
3. Two-loops corrections, $\sim (\psi^* \psi)^3$ (EPJ **B11**, 143 (1999))