Global Symmetry, Local Symmetry, and the Lattice

QFT II Seminar

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Global Symmetry and spontaneous symmetry breaking

through the Ising model of ferromagnetism

Ising model

Hamiltonian:

$$H = -J \sum_{x} \sum_{\mu=1}^{D} s(x)s(x+\hat{\mu}) \quad (1)$$

Probability of any given spin configuration:

$$\text{Prob}[\{s(x)\}] = \frac{1}{Z} e^{-H/(kT)} \quad (2)$$

with

$$Z = \sum_{\{s(x)\}} e^{-H/(kT)} \quad (3)$$

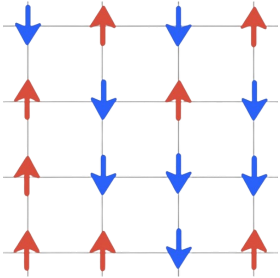


Figure: Spins arranged in a two dimensional quadratic lattice.

The magnetization (average spin)

$$\langle s \rangle = \sum_{\{s(x)\}} \frac{1}{N_{spins}} \left(\sum_{x} s(x) \right) \text{Prob}[\{s(x)\}] \quad (4)$$

Ising model - Global Symmetry

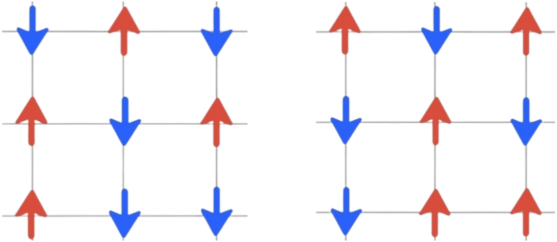
Hamiltonian:

$$H = -J \sum_{x} \sum_{\mu=1}^{D} s(x)s(x+\hat{\mu}) \tag{5}$$

is invariant under the **global transformation**

$$s(x) \longrightarrow s'(x) = zs(x) \tag{6}$$

where $z = \pm 1$



Ising model - Global Symmetry

Hamiltonian:

$$H' = -J \sum_{x} \sum_{\mu=1}^{D} s'(x) s'(x+\hat{\mu}) \tag{5}$$

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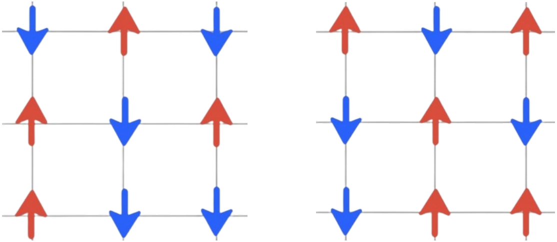

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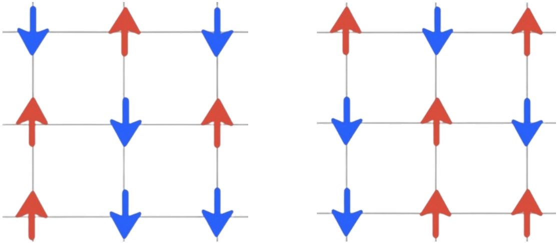

Ising model - Global Symmetry

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$$H' = -J \sum_{x} \sum_{\mu=1}^{D} z^2 s(x) s(x+\hat{\mu}) = H \tag{5}$$

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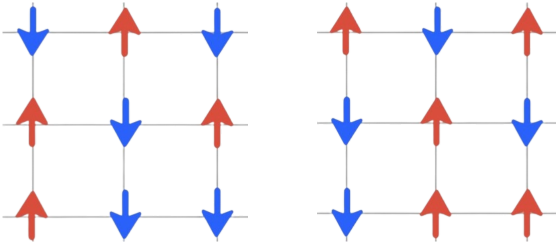
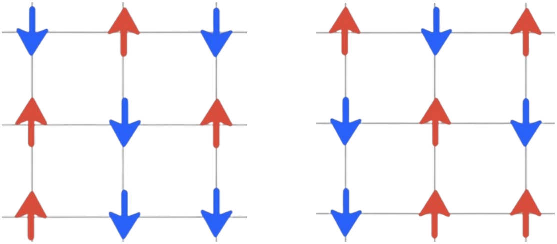
where $z = \pm 1 \longrightarrow Z_2$ group



Ising model - Global Symmetry



As a consequence

$$\langle s \rangle = \sum_{\{s(x)\}} \frac{1}{N_{spins}} \left(\sum_x s(x) \right) \text{Prob}[\{s(x)\}] = 0 \tag{5}$$

So, if we look at the average spin, **Permanent Magnets are IMPOSSIBLE!**

Ising model - Practical purpose

Hamiltonian:

$$H_h = -J \sum_x \sum_{\mu=1}^{D} s(x)s(x+\hat{\mu}) - h \sum_x s(x) \tag{6}$$

$$Z_h = \sum_{\{s(x)\}} \exp\left[-H_h/(kT)\right] \tag{7}$$

Ising model - Practical purpose

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Now, $\langle s \rangle \neq 0$ at any temperature. We consider what happens in the following limits

$$m = \lim_{h\to 0} \lim_{N\to\infty} \langle s \rangle \tag{8}$$

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- **High T**, thermal fluctuations, disordered phase: $\langle s(x) \rangle = 0$.
- **Low T** $(D > 1)$, ordered phase, most spins aligned and $\langle s(x) \rangle \neq 0$.

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So, Z_2 global symmetry is **spontaneously broken**.

Ising model - Correlation Function

To characterize the phases more precisely, one can compute the spin-spin correlation function:

$$G(R) = \langle s(0)s(R) \rangle \tag{9}$$

$$= \frac{1}{Z} \sum_{\{s(x)\}} s(0)s(R)e^{-H/kT} \tag{10}$$

- In the **ordered phase**, spins are correlated over long distances, so:

$$G(R) \to m^2 \quad \text{as} \quad R \to \infty.$$

- In the **disordered phase**, spins are uncorrelated at large distances, and:

$$G(R) \sim e^{-R/\xi},$$

where ξ is the correlation length.

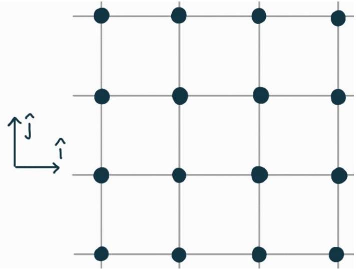
Gauge Invariance

The Unbreakable Symmetry

Lattice Theory

Discrete #D.o.F., ultralocal interactions, underlating lattice, built-in cut-off

- **sites:** points in lattice
- **links:** lines joining neighboring sites
- **plaquettes:** squares formed by 4 adjacent links



Gauge transformation: an be chosen independently at each site.

We associate dynamical degrees of freedom with the links in the lattice, and the gauge transformation is specified at each site.

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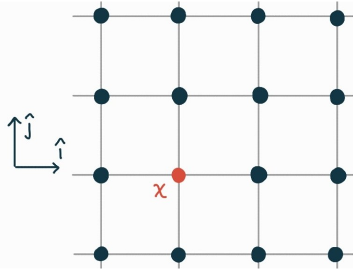

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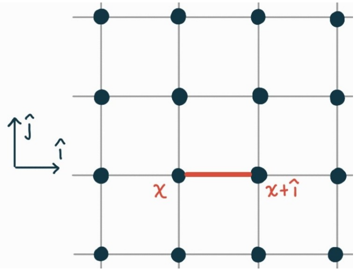

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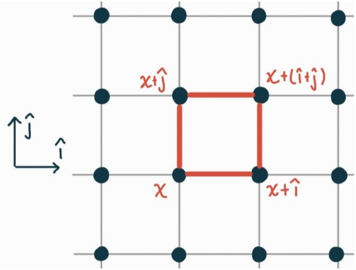

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Elitzur's Theorem

Hamiltonian of gauge invariant Ising model (Z_2 lattice gauge theory):

$$H = -J \sum_{x} \sum_{\mu=1}^{D-1} \sum_{\nu>\mu}^{D} s_\mu(x) s_\nu(x+\hat{\mu}) s_\mu(x+\hat{\nu}) s_\nu(x) \qquad (11)$$

is invariant under

$$s_\mu \longrightarrow s'_\mu = z(x) s_\mu(x) z(x+\hat{\mu}) \qquad (12)$$

Local Z_2 gauge symmetry is vastly larger than global Z_2 symmetry.

A gauge symmetry cannot break spontaneously.

There is no analog to magnetization, $\langle s(x) \rangle = 0$ even iw we introduce h and then apply the same limits as before.

Gauge invariant observables can be constructed from **Wilson loops**.

Wilson loops

Wilson line:

$$W(x,y) = P\exp\left[ig\int_x^y A_\mu(z)dz^\mu\right] \tag{13}$$

Link:

$$U_\mu = W(x, x+a\hat{\mu}) = P\exp\left[igaA_\mu(z)\right] \tag{14}$$

Wilson loop:

$$W(C) = \left\langle \prod_{(x,\mu)\in C} U_\mu(x) \right\rangle \tag{15}$$

We can generalize the Z_2 construction to any symmetry group G.

- $SU(3)$: strong interaction
- $SU(2)\times U(1)$: electroweak interaction
- $U(1)$: compact QED

Compact QED

Link variables

$$U_\mu(x) = \exp[iaeA_\mu(x)], \qquad A_\mu(x) \in \left[-\frac{\pi}{ae}, \frac{\pi}{ae}\right] \tag{16}$$

where a is the lattice spacing and e is the electric charge. The probability distribution,

$$\text{Prob}[\{U_\mu(x)\}] = \frac{1}{Z} e^{-S[U]} \tag{17}$$

where the Euclidean action

$$S[U] = -\frac{\beta}{2} \sum_{x,\mu<\nu} U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^*(x+\hat{\nu}) U_\nu^*(x) \ + \ \text{c.c.} \tag{18}$$

Local gauge transformation

$$U_\mu(x) \longrightarrow e^{i\theta(x)} U_\mu(x) e^{-i\theta(x+\hat{\mu})} \tag{19}$$

Compact QED

$$U_\mu(x) = 1 + iaeA_\mu(x) - \frac{1}{2}a^2e^2A_\mu^2(x) + \dots \tag{20}$$

So, the action

$$S = \frac{\beta}{2}\sum_x a^4 e^2 \sum_{\mu<\nu}\left[\frac{A_\nu(x+\mu) - A_\nu(x)}{a} - \frac{A_\mu(x+\nu) - A_\mu(x)}{a}\right]^2 \tag{21}$$

where $\beta = 1/e^2$. Then becomes

$$S = \int d^4x \frac{1}{4} F_{\mu\nu}F_{\mu\nu} \tag{22}$$

and the gauge transformation

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x) \tag{23}$$

Non-abelian groups: SU(N)

Link variable

$$U(x) = e^{iagA_\mu(x)} \tag{24}$$

where

$$A_\mu(x) = A_\mu^a(x)\frac{\lambda_a}{2} \tag{25}$$

Euclidean action,

$$S = -\frac{\beta}{2N}\sum_{x,\mu<\nu}\left\{\mathrm{Tr}\left[U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)\right] \ + \ \mathrm{c.c.}\right\} \tag{26}$$

then

$$S = \int d^4x\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right] \tag{27}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \tag{28}$$

$$A_\mu(x) \to G(x)A_\mu(x)G^\dagger(x) - \frac{i}{g}G(x)\partial_\mu G^\dagger(x) \tag{29}$$

The Monte Carlo Method

Stochastic Sampling in Lattice Field Theories

The Monte Carlo Method

The connection between the Euclidean path integral and the Minkowski space

$$\langle Q^\dagger_{t_2} Q_{t_1} \rangle = \frac{1}{Z} \int DA \; Q^\dagger_{t_2} Q_{t_1} e^{-S} \tag{30}$$

$$= \langle \Psi_0 | Q^\dagger_{t_2} e^{-H(t_2 - t_1)} Q_{t_1} | \Psi_0 \rangle \tag{31}$$

in lattice for Monte Carlo,

$$\langle Q \rangle = \int DU \; Q[U] \frac{1}{Z} e^{-S[U]} \tag{32}$$

$$\approx \frac{1}{N_{conf}} \sum_{n=1}^{N_{conf}} Q \left[U^{(n)} \right] \tag{33}$$

Metropolis Algorithm

1. **Initial state**: choose any convenient configuration $\{U_\mu(x)\}$.
2. **Propose update**: for each link choose a random element in G as a possible replacement: $U'_\mu(x)$.
3. **Compute** $\Delta S = S_E[U'] - S_E[U]$.
4. **Acceptance rule**

   $$P_{\text{acc}} = \begin{cases} 1, & \Delta S \leq 0, \\ e^{-\Delta S}, & \Delta S > 0. \end{cases}$$

   random number generator: χ in $[0,1]$ with uniform weight.
   Then:
   - $P_{acc} > \chi$: Make the change
   - $P_{acc} < \chi$: Don't
5. **Iterate**: sweep over all links repeatedly to build an ensemble.

Guarantees: Detailed balance and (with a suitable proposal) ergodicity $\Rightarrow$ configurations distributed according to the Boltzmann weight e^{-S_E}.

Static Quark Potential

Action,

$$S_{matter} = -\sum_{x,\mu}\left(\phi^\dagger(x)U_\mu(x)\phi(x+\hat{\mu}) + \text{c.c.}\right) + \sum_x \left(m^2 + 2D\right)\phi^\dagger(x)\phi(x)$$

we consider the observable

$$Q_t = \phi^\dagger(0,t)U_i(0,t)U_i(\hat{e},t)U_i(2\hat{e},t)\dots U_i((R-1)\hat{e},t)\phi(R\hat{e},t) \quad (34)$$

and $m >> 1$. We can get to

$$\langle Q_T^\dagger Q_0\rangle = \text{const.} \times \langle \text{Tr}[UU\dots UU]_C\rangle \quad (35)$$

$$= \frac{\sum_{nm}\langle 0|Q^\dagger|n\rangle\langle n|e^{-HT}|m\rangle\langle m|Q|0\rangle}{\langle 0|e^{-HT}|0\rangle} \quad (36)$$

$$= \sum_n |c_n|^2 e^{-\Delta E_n T} \quad (37)$$

$$\sim e^{-\Delta E_{min} T} \quad \text{as } T \to \infty \quad (38)$$

Wilson loop

$$W(R,T) \sim e^{-V(R)T} \ (\text{as } T \to \infty) \quad (39)$$