1 Substitute the solution for $G^{(2)}$ (eq. 159.3 of the lecture notes) in the CS equation and show that it is actually a solution. (There are a few tips for how to do this in Peskin, page 420, starting at eq. 12.74)

2 RG according to Polchinski (NEXT PAGE ↓ ↓ ↓)

(ps: It's ok to deliver exercise 2 a bit later, the deadline for exercise 1 stays as normal)

Polchinski RG

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1 Introduction

In the early years of Quantum Field Theory, renormalization was often dismissed as a trick to get rid of infinities. Even when the understanding progressed to it being a systematic process of relating parameters in our theory to observables, understanding renormalizability becomes a rather complex discussion. It took Kenneth Wilson to modernize our comprehension of the renormalization group, and aided by his ideas, Joseph Polchinski, in his "Renormalization and Effective Lagrangians" paper developed a beatiful way to understand renormalizability. The idea of this guided problem is to develop some of the ideas of the paper, in a more digestible fashion, so that we ourselves can understand it as well.

2 Problem

We begin considering a theory which is weakly coupled at the initial cutoff Λ_0.

$$S_{\Lambda_0} = \int \mathrm{d}^4x \sum_i g_i(\Lambda_0)\Lambda^{4-\Delta_i} O_i \tag{1}$$

where the O_i are local operators in our theory, the g_i's their associated couplings, and Δ_i the coupling's mass dimension.

Note that we have many couplings g_i here which have coupled evolution under the RG. Our aim will be to "separate" as much as possible the non-renormalizable couplings from the renormalizable ones and see how they influence each other.

2.1 A Deviation from the Trajectory

To better understand the behavior of the RG flow, let us deal with renormalization group equations (RGEs) perturbatively. Consider, then, a particular solution to the RGE's $\bar{g} = \{\bar{g}_i\}$, which we often call a trajectory, and expand perturbatively in terms of a small deviation δg away from it. Using the RGE's, find the differential equation for the evolution of this deviation (remember it is small), it should look like

$$\Lambda \frac{\mathrm{d}\delta g}{\mathrm{d}\Lambda} = M\delta g \tag{2}$$

for some M_{ij} matrix which you should find.

2.2 The Projectors

Next, we define two projectors. First, one that allows us to distinguish between renormalizable and non-renormalizable couplings

$$P_{ij} = \begin{cases} \delta_{ij} & i \text{ renormalizable} \\ 0 & \text{otherwise} \end{cases}$$

Next, by defining

$$D_{ij} = \frac{\partial \bar{g}_i}{\partial \bar{g}_j^0}$$

where $\bar{g}_i^0 = \bar{g}_i(\Lambda_0)$, we can define another projector

$$\Pi = 1 - DP(PDP)^{-1}P$$

the function of which will become clear very soon. Prove that both P and Π are indeed projectors, and prove that they obey the relations

$$P\Pi = 0 \quad \text{and} \quad \Pi(1-P) = (1-P) \tag{3}$$

Bonus: If you are more mathematically inclined, are these two relations sufficient to say $\Pi = 1 - P$, if so, why? If not, why and how are they different?

These definitions allow us to define a projected coupling variation which we call ξ

$$\xi = \Pi \delta g \tag{4}$$

Which by construction respects $P\xi = 0$. Therefore, the function of Π is to isolate the deviation between non-renormalizable couplings in different trajectories. This point is worth emphasizing, we're now looking at how the difference between non-renormalizable parameters of distinct RG trajectories changes with running.

2.3 Scale Evolution

Now that we've built ξ let us see if we can learn something exclusively about the scale evolution of the non-renormalizable couplings. To do so, prove that

$$\Lambda \frac{\mathrm{d}\xi}{\mathrm{d}\Lambda} = \Pi M \xi.$$

Note here what Π has allowed us to do, we have effectively decoupled the evolution of the non-renormalizable couplings of our theory, note that implicitly however, there's still a dependency on all couplings through M and D.

2.4 Perturbation Theory

Now, the goal in this problem is to deepen our naive understanding of renormalizability in QFT. As such, since we have only developed a perturbative understanding of renormalization, it's useful to frame our discussion in this context as well. Consider then, that we're near the trivial fixed point of the renormalization group $g_i = 0$, it's to be expected that exactly on it, the theory must behave like a free theory, whose action does not, generally, depend on a cutoff. Hence, near the trivial fixed point, the cutoff dependence of the interaction terms in the action (1) should vanish as the couplings go to zero.

Use this to approximate the form of M. Using it, find an approximation for the RGE for ξ and solve it assuming we're very far from the cutoff. You should find

$$\xi = 0$$

Now write this in terms of δg, P and D, what does this tell us about the evolution of the non-renormalizable couplings in different RG trajectories? On what does it depend?

3 Quick Summary

What Polchinski elegantly illustrated in his paper, and that we've reproduced in this problem, is really the point of renormalizability. Regardless of which theory we start with (so long as it's perturbative), far from the cutoff the initial values of the non-renormalizable couplings do not matter, their values become entirely deducible from the renormalizable couplings of the theory. We can therefore set them to zero from the start.

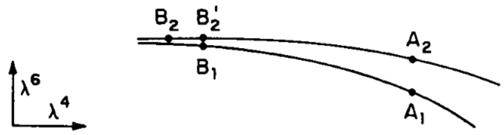

Fig. 1. Neighboring trajectories in the $\lambda_4 - \lambda_6$ plane.

Figure 1: An image from Polchinski's original paper illustrating this flow, here λ_4 is a renormalizable coupling and λ_6 is a non-renormalizable one. If two different trajectories start with the same λ_4 but different λ_6's they converge to the same region, the initial value of λ_6 does not matter in the IR: given λ_4, we can determine λ_6 to Λ^2/Λ_0^2 precision.

Note that this is a more subtle statement than saying non-renormalizable couplings go to zero far from the cutoff, they generally do not, but their intrinsic, initial value does not matter. The common jargon is that there is an attractor manifold of renormalizable theories onto which the RG flows, such that any low-energy observable may be expressed in terms of parameters on it. This is precisely what renormalizability is, all observables can be expressed as functions of a finite number of low-energy couplings, independently of the initial cutoff or bare couplings.