Quantum Field Theory I - auxiliar lecture notes

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Abstract

These lecture notes are a synthesis of a one semester introductory QFT course. They were originated from a course given by professor Ricardo Elias Matheus in IFT/ICTP-SAIFR - Unesp. The material here contains the majority of the one-semester course and are mainly focused in the presentation of the physical ideas and derivations. I hope to improve it in the future. I also wish to acknowledge that much of the content here (specially the diagrams) were handwriting by me and transcribed in latex by AI (I took the effort to check everything though), and the first two subsections in the appendices were entirely written by AI. Except those, the work was entirely done by me, so any mistakes or imprecisions are entirely my responsability. For comments or suggestions, please contact me at `pedrocasella57@gmail.com`.

1 Introduction

The main idea of QFT is that wave functions are replaced by fields, described in all space-time.

$$\phi = \hat{\phi}(\vec{x}, t).$$

Given a lagrangian density function $\mathcal{L}$ defined on the manifold of the theory, we may define the field conjugated momentum $\hat{\pi}(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \hat{\phi}(\vec{x}, t))}$, and the **quantization procedure is to treat them as operators who respect the relations for Lie commutators as their classical counterparts obey the Poisson bracket.**

In classical field theory, we have that,

Theorem. *The transformation* $(q_1, ..., q_n, p_1, ..., p_n) \longrightarrow (Q_1, ..., Q_n, P_1, ..., P_n)$ *is canonical if, and only if,* $\Big[Q_i, Q_j\Big]_{q,p} = 0$ *;* $\Big[P_i, P_j\Big]_{q,p} = 0$ *;* $\Big[Q_i, P_j\Big]_{q,p} = \delta_{i,j}$.

In QFT, we must have the continuous quantized version,

Definition. $\Big[\hat{\pi}(x), \hat{\pi}(y)\Big] = 0$; $\Big[\hat{\phi}(x), \hat{\phi}(y)\Big] = 0$; $\Big[\hat{\phi}(x), \hat{\pi}(y)\Big] = i\delta^{(3)}(x - y)$. Where x and y have each four components (one time, three space) and the brackets denotes Lie commutator. (Those are all being taken at equal time).

We see that this is in accordance with the association

$$[,]_{Poisson} \rightarrow i[,]_{Lie}.$$

This allows us to write the definitions,

Definition. (First quantization) coordinates are transformed into operators that act on Hilbert space ($q \to \hat{q}$ and $p \to \hat{p}$).

Definition. (Second quantization) When the operators are denoted as fields (being described in every point of space-time). This allows us to treat then in a continuum of coordinates, i.e, we can associate a coordinate for each point in space-time.

Let's look at one simple example of a QFT.

2 Klein-Gordon

2.1 General properties

The Klein-Gordon equation is a relativistic equation for particles, as Schrodinger equation is for non-relativistic particles. It reads,

$$(\Box^2 + m^2)\hat{\phi}(\vec{x}) = 0.$$

We are using the metric $\eta_{uv} = diag(1, -1, -1, -1)$. Its lagrangian reads (note that we are dealing here with a complex field),

$$\mathcal{L}(\hat{\phi}, \hat{\phi}^*) = \partial_u \phi \partial^u \phi^* - m^2 \phi\phi^*.$$

So the equation is valid for both the field and its conjugate (we could also express it in terms of the real and imaginary components of the field ϕ).

Note that Klein-Gordon equation is of the same form of the harmonical oscillator,

$$(\partial_t^2 + (|p|^2 + m^2))\phi = 0.$$

This gives us an answer in terms of momentum. Fortunately, we can write in terms of coordinates by doing a Fourier transformation,

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \phi(\vec{p}, t),$$

$$\phi(\vec{p}, t) = \phi(\vec{p}) \exp(i w_p t),$$

$$w_p = \sqrt{|p|^2 + m^2}.$$

Another way to write this is,

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{-ip_u x^u} \phi(\vec{p}).$$

In analogy to how the harmonic oscillator is solved in standard quantum mechanics, we shall write the solution for the Klein-Gordon equation as,

$$\boxed{\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (\hat{a_p} e^{-ip_u x^u} + \hat{a}_p^\dagger e^{ip_u x^u}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (\hat{a_p} + \hat{a}_{-p}^\dagger) e^{-ip_u x^u}},$$

$$\boxed{\hat{\pi}(x) = \int \frac{d^3p}{(2\pi)^3} \left(-i\sqrt{\frac{w_p}{2}} \right) (\hat{a_p} - \hat{a}_{-p}^\dagger) e^{-ip_u x^u}}$$

Note that here we are associating operators $\hat{a}$ and $\hat{a}^\dagger$ to a continuum of $p's$, in accordance to second quantization. Now, by using the expression for Dirac delta,

$$\delta^{(3)}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}},$$

We may verify the following,

Theorem. *The conditions in definition 1,* $\Big[\hat{\pi}(x), \hat{\pi}(y)\Big] = 0$ *;* $\Big[\hat{\phi}(x), \hat{\phi}(y)\Big] = 0$ *;* $\Big[\hat{\phi}(x), \hat{\pi}(y)\Big] = i\delta^{(3)}(x-y)$*. Are true iff ,* $\Big[\hat{a}_p, \hat{a}_q\Big] = 0$ *;* $\Big[\hat{a}_p^\dagger, \hat{a}_q^\dagger\Big] = 0$ *;* $\Big[\hat{a}_p, \hat{a}_q^\dagger\Big] = (2\pi)^3\delta^{(3)}(p-q)$.

This is consistent to our correspondence. Note that **these expressions are being taken on equal times**.

2.2 Hamiltonian of Klein-Gordon

The hamiltonian for any field theory is the Legendre transformation of the Lagrangian, so,

$$H = \int d^3x \big(\pi(x)\dot{\phi}(x) + \pi^*(x)\dot{\phi}^*(x) - \mathcal{L}\big).$$

In this way, considering that $\dot{\phi} = \pi^*$ we find,

$$H = \int d^3x \Big(\pi^*(x)\pi(x) + \nabla\phi^*(x)\cdot\nabla\phi(x) + m^2\phi^*(x)\phi(x)\Big).$$

Writing in terms of creation and annihilation operators, we get,

$$\int d^3x \quad (\pi^*(x)\pi(x)) = \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{\sqrt{w_p w_q}}{2} (a_{-p}^\dagger - a_p)(a_q - a_{-q}^\dagger) e^{i(p+q)x},$$

$$\int d^3x \quad \nabla\phi^*\cdot\nabla\phi = \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{\vec{p}\cdot\vec{q}}{2\sqrt{w_p w_q}} (-a_{-p}^\dagger + a_p)(-a_q + a_{-q}^\dagger) e^{i(p+q)x},$$

$$\int d^3x \quad \phi^*\phi = \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{2\sqrt{w_p w_q}} (a_{-p}^\dagger + a_p)(a_q + a_{-q}^\dagger) e^{i(p+q)x}.$$

Since,

$$\delta^{(3)}(\vec{p}+\vec{q}) = \int \frac{d^3x}{(2\pi)^3} e^{i(\vec{p}+\vec{q})\vec{x}},$$

$$\int d^3x \quad (\pi^*(x)\pi(x)) = \int \frac{d^3p}{(2\pi)^3} \frac{w_p}{2} \Big[a_p^\dagger a_p - a_p^\dagger a_{-p}^\dagger - a_p a_q + a_p a_p^\dagger\Big].$$

$$\int d^3x \quad (\nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi) = \int \frac{d^3p}{(2\pi)^3} \frac{w_p}{2} \Big[a_p^\dagger a_p + a_p^\dagger a_{-p}^\dagger + a_p a_q + a_p a_p^\dagger\Big].$$

So,

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} w_p(\ a_p^\dagger a_p + a_p a_p^\dagger)}.$$

Notice that this may be restated as

$$H = \int \frac{d^3p}{(2\pi)^3} w_p (2a_p^\dagger a_p + (2\pi)^3 \delta(0)).$$

This infinite is normally dealt with by shifting the hamiltonian, so the zero energy vanishes (this will be very important later, in QFT II).

2.3 Conserved charge

The Lagrangian,

$$\mathcal{L}(\hat{\phi}, \hat{\phi}^*) = \partial_u \phi \partial^u \phi^* - m^2 \phi \phi^*,$$

is invariant under global action of $U(1)$, i.e, $\mathcal{L}(\hat{\phi}, \hat{\phi}^*) = \mathcal{L}(\hat{\phi} e^{i\varphi}, \hat{\phi}^* e^{-i\varphi})$. In terms of Noether current, this implies that, upon normal ordering

$$j^u = i\left(\partial^u \bar{\phi} \phi - \bar{\phi} \partial^u \phi\right).$$

This implies the conserved charge,

$$\hat{Q} = \int_M d^3x \frac{-i}{2}\left(\hat{\bar{\phi}} \hat{\bar{\pi}} - \hat{\pi} \hat{\phi}\right),$$

substituiting by the expressions previously found for $\hat{\phi}$ and $\hat{\pi}$,

$$\hat{Q} = \int d^3p \frac{1}{2} (\hat{a}_p^\dagger \hat{a}_p - \hat{a}_{-p}^\dagger \hat{a}_{-p}).$$

We can check that,

$$\hat{Q} \hat{a}_p^\dagger |0\rangle = \frac{1}{2} \hat{a}_p^\dagger |0\rangle,$$

$$\hat{Q} \hat{a}_{-p}^\dagger |0\rangle = -\frac{1}{2} \hat{a}_{-p}^\dagger |0\rangle.$$

This leads us to the interpretation that we have two types of particles with opposite charges (one described by the operators $\hat{a}_p$ and the other by $\hat{a}_{-p}$). In fact, we find that we can describe the particle and anti-particle state, $|p, +\rangle = a_p^\dagger |0\rangle$ and $|p, -\rangle = a_{-p}^\dagger |0\rangle$ respectively. Thus, **the existence of the anti-particle is intrinsically inside the description of the complex scalar field**.

2.4 Correlation functions

Now, we wish to compute quantities such as $D(x-y) = \langle 0| \phi^\dagger(x) \phi(y) |0\rangle$. We shall, wlog, consider a real field, $\phi(x) = \phi^\dagger(x)$. Now, computing,

$$\langle 0| \phi(x) \phi(y) |0\rangle = \frac{1}{(2\pi)^6} \int \frac{d^3p d^3q}{2\sqrt{w_p w_q}} \langle 0| \hat{a_p} \hat{a_q}^\dagger |0\rangle e^{i(q \cdot y - p \cdot x)},$$

by using $\langle 0| \hat{a_p} \hat{a_q}^\dagger |0\rangle = \langle 0| [\hat{a_p}, \hat{a_q}^\dagger] |0\rangle = (2\pi)^3 \delta^{(3)}(p-q)$, which leads us to,

$$\boxed{\langle 0| \phi(x) \phi(y) |0\rangle = D(x-y) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2w_p} e^{-ip \cdot (x-y)}}.$$

This is the two-point correlation function for scalar fields. Usually, we are worried about getting expressions that are symmetric for x and y. Thus, we will work with,

$$\langle 0| \left[\phi(x), \phi(y)\right] |0\rangle = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2w_p} [e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}].$$

We can get this answer by using contour integrals. Assuming $x^0 > y^0$, we get,

$$\langle 0| \left[\phi(x), \phi(y)\right] |0\rangle = \frac{1}{(2\pi)^3} \int d^3p \oint_{C_1} \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} (e^{-ip\cdot(x-y)}),$$

where $p^2 - m^2 = p_0^2 - (p^2 + m^2) = p_0^2 - w_p^2$, and C_1 is a path that takes both poles, as indicated in the picture below.

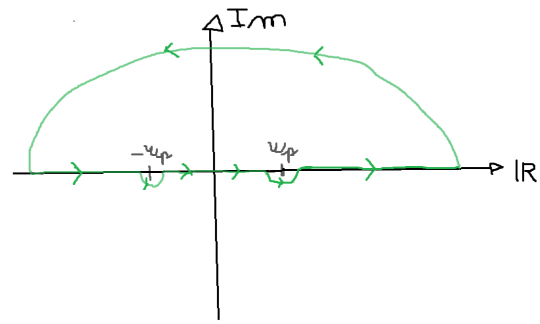


Figure : *Possible sketch of C_1.*

Since this only works for $x^0 > y^0$ (otherwise it vanishes), this is the retarded propagator,

$$\boxed{D_R(x-y) = \int_{C_1} \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)}}.$$

From this, we infer the propagator in momentum space.

$$\boxed{D_R(p) = \frac{i}{p^2 - m^2}}.$$

And we may also check that,

$$\boxed{(\Box^2 + m^2) D_R(x-y) = i\delta^{(4)}(x-y)}.$$

We can analogously define the advanced propagator. There is however a method (called Feynman´s prescription) to take into consideration both, by defining,

$$D_F(x-y) = \frac{1}{(2\pi)^3} \int d^3p \int_{C_1} \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2 + i\epsilon} (e^{-ip\cdot(x-y)}),$$

for ϵ small. This allows us to get,

$$D_F(x-y) = \theta(x^0-y^0)\langle 0|\,\phi(x)\phi(y)\,|0\rangle + \theta(y^0-x^0)\langle 0|\,\phi(y)\phi(x)\,|0\rangle = \langle 0|\,T\{\phi(x)\phi(y)\}\,|0\rangle\,,$$

where T denotes time ordering. We also get,

$$\boxed{(\Box^2+m^2)D_F(x-y) = -i\delta^4(x-y)}.$$

3 Dirac Field and algebra

3.1 Lorentz group

Lorentz transformations are, by definition, transformation that leave the line element unchanged, i.e, transformation of coordinates so as to respect,

$$g_{uv}(x'^u)(x'^v) = g_{w\sigma}x^w x^\sigma.$$

Lemma. *Lorentz transformations are linear*

Proof. Just check that $(\alpha x_1 + x_2)' = \alpha x_1' + x_2'$. (Works only in flat space). □

Thus, we have that $x'^u = \Lambda^u_v x^v$, which leads us to the relation,

$$\Lambda^u_w g_{uv}\Lambda^v_\sigma = g_{w\sigma}.$$

This leads us to the definition,

Definition. The Lorentz Group $SO(3,1) = \{\Lambda \in M_4(\mathbb{C})| \quad \Lambda\eta\Lambda^T = \eta\}$, for $\eta = diag(1,-1,-1,-1)$.

Sectors in the Lorentz group may be classified as follows (Note that they are not subgroups, except for the first one (Exercise)),

$$L_+^\uparrow := \{\Lambda \in SO(3,1) \mid \det(\Lambda) = 1 \text{ and } \Lambda^0{}_0 \geq 1\},$$
$$L_+^\downarrow := \{\Lambda \in SO(3,1) \mid \det(\Lambda) = 1 \text{ and } \Lambda^0{}_0 \leq -1\},$$
$$L_-^\uparrow := \{\Lambda \in SO(3,1) \mid \det(\Lambda) = -1 \text{ and } \Lambda^0{}_0 \geq 1\},$$
$$L_-^\downarrow := \{\Lambda \in SO(3,1) \mid \det(\Lambda) = -1 \text{ and } \Lambda^0{}_0 \leq -1\}.$$

$$SO(3,1) = L_+^\uparrow \cup L_+^\downarrow \cup L_-^\uparrow \cup L_-^\downarrow.$$

The above is a Lie group. The fields transformations under Lorentz obey,

$$\phi'^\alpha(x') = R(\Lambda)^\alpha_\beta \phi^\beta(x),$$

where $x'^\alpha = \Lambda^\alpha_\beta x^\beta$. Where $R(\Lambda)$ is a representation of such a group element, and it writes as,

$$R(A) = e^{iA^a t_a},$$

where A^a is a Lie algebra generator of Lorentz group.

3.2 Lorentz algebra

We know that,

$$Lie(SO(3,1)) = \{X \in M_4(\mathbb{C}) | \quad e^{tX} \in SO(3,1)\},$$

this implies (exercise),

$$\boxed{X = -\eta X^T \eta}.$$

This means we can write,

$$X^{\alpha\beta} = -\eta^{\alpha\omega}(X^T)^{\lambda}_{\omega}\eta^{\beta}_{\lambda},$$

which leads us to,

$$X^{\alpha\beta} = -X^{\beta\alpha}(\eta^{\alpha\alpha}\eta^{\beta\beta}).$$

This implies that the non zero elements of X obey,

$$X^{ij} = -X^{ji},$$

$$X^{0j} = X^{j0}.$$

Which means that $Lie(SO(3,1))$ has the basis of 4 by 4 matrices (the non appearing elements are implicitly zero),

$$\left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & & & \\ 0 & & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \\ 0 & & & \\ 1 & & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & & \\ 1 & & & \end{pmatrix}, \begin{pmatrix} & & & \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ & & & \end{pmatrix}, \begin{pmatrix} & & & \\ 0 & 0 & 0 & 1 \\ & & & \\ 0 & -1 & & 0 \end{pmatrix}, \begin{pmatrix} & & & \\ & & & \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \right\}.$$

Thus, we note that $\#Lie(SO(3,1)) = 6$. We can also call this basis $\{-iJ^{01}, -iJ^{02}, -iJ^{03}, -iJ^{12}, -iJ^{13}, -iJ^{23}\}$, where we are admitting $J^{uv} = -J^{vu}$. The matrices J^{uv} obey the bracket relation,

$$[J^{uv}, J^{\rho\sigma}] = i(\eta^{\sigma\mu}J^{\nu\rho} + \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\rho\mu}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho}).$$

This defines the Lie algebra of Lorentz group. One important observation is that the linear operator,

$$\boxed{J^{uv} = i(x^u\partial^v - x^v\partial^u)},$$

also obeys the Lie bracket relation. Thus, we have another representation for the Lie group.

3.3 Poincare Group

The Poincare group is the extension of Lorentz group so as to include all isometries in Minkowski space $\mathbb{R}^{3,1}$, i.e,

$$P(3,1) = \{F : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1} | \quad \eta_{uv}dx^u dx^v = \eta_{uv}(dF(x))^u(dF(x))^v\}.$$

For such transformation to obey the property above stated, we should add a translation to the coordinates x, so as to vanish when we take dx, thus, a general Poincare transformation (Minkowski space isometry) is,

$$x'^u = \Lambda^u_v x^v + a^u.$$

We thus see that,

$$\boxed{dx'^u = \Lambda^u_v dx^v},$$

as expected. Note that, $(\Lambda^u_v x^v + a^u)' = \Lambda'^u_w \Lambda^w_v x^v + \Lambda'^u_w a^w + b^w$, which is itself a transformation in $P(3,1)$, thus, the only non trivial property of group structure of $P(3,1)$ is proved.

3.4 Poincare algebra

The algebra of Poincare group is an extension of Lorentz algebra, we just have to find the generator for translation,

$$e^{tX}x^u = x^u + a^u,$$
$$tXx^u = a^u,$$

in order for this to be true, $X = \partial_u$. Taking t to be imaginary, we get the generator of translation to be,

$$\boxed{P_\rho = -i\partial_\rho},$$

which surely implies $[P_\rho, P_v] = 0$. Thus, we must have that $\#Lie(P(3,1)) = 10$, since we got 6 generators J_{uv} and 4 generators P_ρ. At last, using the representation found at the end of section 3.2, we get,

$$[J_{uv}, P_\rho] = (x_u\partial_v - x_v\partial_u)\partial_\rho - \partial_\rho(x_u\partial_v - x_v\partial_u) =,$$
$$= -\eta_{uw}\delta^w_\rho\partial_v + \eta_{vw}\delta^w_\rho\partial_u,$$
$$\Rightarrow \boxed{[J_{uv}, P_\rho] = i(\eta_{u\rho}P_v - \eta_{v\rho}Pu)}.$$

Thus, we can finally describe Poincare algebra.

Definition. The Poincare algebra is the Lie algebra of Poincare group $P(3,1)$. It is a ten dimensional complex Lie algebra that has as generators $J^{uv} = i(x^u\partial^v - x^v\partial^u)$ and $P_\rho = -i\partial_\rho$. The Lie commutation relations are,

$$1) \quad \boxed{[P_\rho, P_v] = 0}.$$
$$2) \quad \boxed{[J^{uv}, J^{\sigma\rho}] = i(\eta^{\sigma\mu}J^{\rho\nu} + \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\rho\mu}J^{\sigma\nu} - \eta^{\nu\rho}J^{\mu\sigma})}.$$
$$3) \quad \boxed{[J_{uv}, P_\rho] = i(\eta_{u\rho}P_v - \eta_{v\rho}Pu)}.$$

Important note: **Each J^{uv} is a linear operator itself** (just like P_ρ), not different matrix elements, thus, by the antisymmetric property, it has 6 independent possible outcomes.

3.5 Further details in representation theory

We first remind the definition of tensor for our purposes.

Definition. A tensor of rank (k,n) is a (k+n)-linear operator such that the following transformations apply,

$$\boxed{T'^{u_1...u_k}_{v_1,...v_n}(x') = \frac{\partial x^{u_1}}{\partial x'^{\alpha_1}}...\frac{\partial x^{u_k}}{\partial x'^{\alpha_k}}\frac{\partial x'_{v_1}}{\partial x_{\beta_1}}...\frac{\partial x'_{v_k}}{\partial x_{\beta_n}}T^{\alpha_1...\alpha_k}_{\beta_1,...\beta_n}(x)}.$$

Under this, we define a scalar field as a 0 rank tensor,

$$\phi'(x') = \phi(x),$$

and vector field as a (1,0) (contravariant) or a (0,1) (covariant) tensor,

$$\phi'^\alpha(x') = \frac{\partial x^\alpha}{\partial x'^\beta}\phi^\beta(x),$$

$$\phi'_\alpha(x') = \frac{\partial x'_\alpha}{\partial x_\beta}\phi_\beta(x).$$

Definition. Fields that transform as tensors under Poincare transformations are called Bosonic fields (or bosonic representation), i.e,

$$\boxed{B'^{u_1 \dots u_k}_{v_1, \dots v_n}(x') = \Lambda^{u_1}_{\alpha_1} \dots \Lambda^{u_k}_{\alpha_k} \Lambda^{\beta_1}_{v_1} \dots \Lambda^{\beta_n}_{v_n} B^{\alpha_1 \dots \alpha_k}_{\beta_1, \dots \beta_n}(x)}.$$

Analogously, we define fermionic fields as fields that do not transform as tensors. Examples will be given in the next section.

3.6 Dirac equation and representation

As we know it, Dirac equation (proposed exactly 98 years before I am writing this very sentence) relates the behavior of relativistic particles in order to have probability density positive defined. It is written as follows:

$$i\frac{\partial \psi}{\partial t} = -i(\vec{\alpha} \cdot \vec{\nabla}\psi) + \beta m \psi,$$

where ψ is a four component spinor (this is an example of fermionic field whose precise definition shall come later. For now, see it as a vector with four components), and the matrices are,

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The matrix form of Dirac equation is,

$$\boxed{\begin{pmatrix} i\partial\psi 1/\partial t \\ i\partial\psi 2/\partial t \\ i\partial\psi 3/\partial t \\ i\partial\psi 4/\partial t \end{pmatrix} = \begin{pmatrix} -i\partial\psi 4/\partial x - \partial\psi 4/\partial y - i\partial\psi 3/\partial z + m\psi 1 \\ -i\partial\psi 3/\partial x + \partial\psi 3/\partial y + i\partial\psi 3/\partial z + m\psi 2 \\ -i\partial\psi 2/\partial x - \partial\psi 2/\partial y - i\partial\psi 1/\partial z - m\psi 3 \\ -i\partial\psi 1/\partial x + \partial\psi 1/\partial y + i\partial\psi 2/\partial z - m\psi 2 \end{pmatrix}}.$$

We can write the Dirac equation in a shorter way by identifying $\beta = \gamma^0$ and $\gamma^i = -\beta\alpha_i = \beta\alpha^i$, we get

$$\boxed{(i\gamma^u \partial_u - m)\psi = 0}.$$

Matrices gamma $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ are the so called **Dirac matrices** .The Dirac matrices obey the so called Clifford algebra,

Definition. The Given a matrix η^{uv} and matrices γ, the so called Clifford algebra obeys ,

$$\{\gamma^u, \gamma^v\} = A\eta^{uv}.$$

Where A is a constant. In the case of Dirac matrices with the metric convention we are using $(1, -1, -1, -1)$, we get,

$$\boxed{\{\gamma^u, \gamma^v\} = 2\eta^{uv}.}$$

Where, of course,

$$\boxed{\gamma^u = \eta^{uv}\gamma_v \Longrightarrow \gamma^0 = \gamma_0,\ \gamma^i = -\gamma_i}.$$

With this, we may show a wonderful property of Dirac equation. If we take Klein-Gordon,

$$(-\partial_t^2 + \nabla^2 - m^2)\phi(x) = 0,$$

we can decompose into (exercise),

$$(i\gamma^u\partial_u - m)(i\gamma^u\partial_u + m)\phi(x) = 0.$$

We can thus see that any solution to Dirac equation is indeed a solution to Klein-Gordon (which is consistent, since Dirac equation was build in order to match it).

Now, we can give an example of a fermionic representation. Take the quantity,

$$S^{uv} = \frac{i}{4}[\gamma^u, \gamma^v].$$

We shall prove that $M = e^{-i\theta_{uv}S^{uv}}$, where θ_{uv} is antisymmetric, is an element in the Lorentz group.

Theorem. *S^{uv} obeys the Lie algebra of Lorentz group.*

Proof. From the Clifford algebra condition,

$$S^{uv} = \frac{i}{4}(\gamma^u\gamma^v - \gamma^v\gamma^u) = \frac{i}{2}(\gamma^u\gamma^v - \eta^{uv}),$$

Which implies that,

$$[S^{uv}, S^{\rho\sigma}] = \gamma^u\gamma^v\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^u\gamma^v.$$

By using Clifford algebra again, we find that we can exchange terms by,

$$\gamma^u\gamma^v\gamma^\rho\gamma^\sigma = (\gamma^\rho\gamma^v\gamma^u - \{\gamma^\rho, \gamma^u\}\gamma^v + \{\gamma^\rho, \gamma^v\}\gamma^u)\gamma^\sigma.$$

Using this property again, we get to,

$$[S^{uv}, S^{\rho\sigma}] = \frac{1}{4}(\{\gamma^\rho, \gamma^u\}\gamma^v\gamma^\sigma + \{\gamma^\sigma, \gamma^u\}\gamma^\rho\gamma^v - \{\gamma^\rho, \gamma^v\}\gamma^u\gamma^\sigma - \{\gamma^\sigma, \gamma^v\}\gamma^\rho\gamma^u).$$

By finally replacing the brackets, we arrive at,

$$\boxed{[S^{uv}, S^{\rho\sigma}] = i(\eta^{\sigma\mu}S^{\nu\rho} + \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\rho\mu}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho})},$$

which is Lorentz algebra Lie bracket. □

With this, we can finally define,

Definition. A spinor is a field $\psi(x)$ that transforms as

$$\psi'(x') = M\psi(x),$$

where $M = e^{-i\theta_{uv}S^{uv}}$ under Lorentz transformations,

$$x'^u = \Lambda^u_v x^v.$$

Also, quantities that transform like this under Lorentz are called covariant.

Theorem. $M^{-1}\gamma^u M = \Lambda^u_v \gamma^v$

Proof. We shall use first order in θ,

$$M^{-1}\gamma^u M = (1 + i\theta_{uv}S^{uv})\gamma^u(1 - i\theta_{uv}S^{uv}) = \gamma^u + i\theta_{\rho\sigma}[S^{\rho\sigma}, \gamma^u],$$

Note that $\theta_{\rho\sigma}S^{\rho\sigma} = \frac{i}{2}\theta_{\rho\sigma}\gamma^\rho\gamma^\sigma$, and, since (by the previous theorem),

$$\gamma^\rho\gamma^\sigma\gamma^u = 2(\eta^{\sigma u}\gamma^\rho - \eta^{\rho u}\gamma^\sigma) + \gamma^u\gamma^\rho\gamma^\sigma,$$

so,

$$\begin{aligned} M^{-1}\gamma^u M &= \gamma^u - \theta_{\rho\sigma}(\eta^{\sigma u}\gamma^\rho - \eta^{\rho u}\gamma^\sigma) =, \\ &= (\delta^u_\sigma + 2\theta_{\rho\sigma}\eta^{\rho u})\gamma^\sigma. \end{aligned}$$

Taking $w^u_\sigma = 2\theta_{\rho\sigma}\eta^{\rho u}$, we get,

$$M^{-1}\gamma^u M = (\delta^u_\sigma + w^u_\sigma)\gamma^\sigma = \Lambda^u_\sigma\gamma^\sigma.$$

□

With this, we shall prove the covariance of Dirac equation,

$$(i\gamma^u\partial'_u - m)\psi' = (i\gamma^u(\Lambda^{-1})^v_u\partial_v - m)M\psi =,$$

$$= M(iM^{-1}\gamma^u M(\Lambda^{-1})^v_u\partial_v - m)\psi = M[(i\gamma^v\partial_v - m)\psi].$$

Note also that, if Dirac equation is valid in one coordinate ($(i\gamma^u\partial'_u - m)\psi' = 0$), then it is valid in all coordinates under Lorentz transformation (Lorentz invariance of the validity of solution). One can also show that $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$ is invariant and $\gamma^u\partial_u\psi$ is covariant, while $\psi^\dagger\psi$ is neither. In order to construct the Lagrangian of the theory, we must take a Lorentz scalar. This is done by noting that (Exercise) $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$ is a Lorentz scalar. Thus, we find the Lagrangian,

$$\boxed{\mathcal{L} = \bar{\psi}(i\gamma^u\partial_u - m)\psi}.$$

3.7 Study of solutions of Dirac equation

This is perhaps the most exciting part of the study of Dirac equation. First, note that, having 4 components, we can write Dirac spinor as,

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

Those are so called left-handed and right-handed Weyl spinors (after German mathematician Hermann Weyl (1885-1955)). Under Dirac equation, taking the representation,

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where σ denotes Pauli matrices, we get,

$$\begin{pmatrix} -m & i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) \\ i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0.$$

For massless particles (as the case for neutrinos, approximately), we get Weyl equations,

$$i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla})\psi_R = 0,$$

$$i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla})\psi_L = 0.$$

For the free particle in Dirac theory, we have $\psi(x) = u(p)e^{-ip^u x_u} + v(p)e^{ip^u x_u}$. Substituting in the equation, we get,

$$(p^0 + \vec{\sigma} \cdot \vec{p})u_R = mu_L,$$

$$(p^0 - \vec{\sigma} \cdot \vec{p})u_L = mu_R,$$

$$(p^0 + \vec{\sigma} \cdot \vec{p})v_R = -mv_L,$$

$$(p^0 - \vec{\sigma} \cdot \vec{p})v_L = -mv_R.$$

In the rest frame, this leads to,

$$u(p) = \begin{pmatrix} \xi \\ \xi \end{pmatrix},$$

$$v(p) = \begin{pmatrix} \wp \\ -\wp \end{pmatrix}.$$

Where ξ and $\wp$ are 2 components column vector. In order to get the solution for other frames, we apply a Lorentz boost. Remember that since we are dealing with Spinor, they transform under $M = e^{-\frac{i}{2}\theta_{uv}S^{uv}}$, so, taking a Lorentz boost,

$$u'(p) = e^{-\frac{i}{2}\theta_{0j}S^{0j}} \begin{pmatrix} \xi \\ \xi \end{pmatrix} =,$$

$$= \exp\left(\frac{\theta_{0j}}{2}\begin{pmatrix}-\sigma^j & 0\\ 0 & \sigma^j\end{pmatrix}\right)\begin{pmatrix}\xi\\ \xi\end{pmatrix}.$$

Under this convention, we get (Exercise) $\theta^\rho_\sigma = \Lambda^\rho_\sigma - \delta^\rho_\sigma$. Thus, knowing that, under Lorentz boost, we get $p'^0 = m$ and $p'^j = m\theta^j_0$, we have,

$$u'(p) = \begin{pmatrix}\xi\\ \xi\end{pmatrix} + \frac{p_j}{2m}\begin{pmatrix}-\sigma^j\xi\\ \sigma^j\xi\end{pmatrix} = \begin{pmatrix}(1-\frac{p_j\sigma^j}{2m})\xi\\ (1+\frac{p_j\sigma^j}{2m})\xi\end{pmatrix},$$

this was done for an infinitesimal boost. For higher order terms, we can acknowledge a square root expansion, so, wlog, we can write,

$$u'(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi\\ \sqrt{p\cdot\bar{\sigma}}\xi\end{pmatrix},$$

where $\vec{\sigma} = (1, \sigma^i)$ and $\vec{\bar{\sigma}} = (1, -\sigma^i)$. Analogously, we find, we are actually doing the contraction of the four vector p^u with σ^u with identity metric tensor. Analogously, we find,

$$v'(p)\begin{pmatrix}\sqrt{\vec{p}\cdot\vec{\sigma}}\wp\\ -\sqrt{\vec{p}\cdot\vec{\bar{\sigma}}}\wp\end{pmatrix}.$$

Thus, **the general solution for free particle in Dirac equation is**,

$$\boxed{\psi(x) = \begin{pmatrix}\sqrt{\vec{p}\cdot\vec{\sigma}}(\xi e^{-ip^u x_u} + \wp e^{ip^u x_u})\\ \sqrt{\vec{p}\cdot\vec{\bar{\sigma}}}(\xi e^{-ip^u x_u} - \wp e^{ip^u x_u})\end{pmatrix}}.$$

We can also note that, by introducing the convention $\xi^\dagger\xi = -\wp^\dagger\wp = 1$, we find,

$$u(p)\bar{u}(p) = \not{p} + m,$$

$$v(p)\bar{v}(p) = \not{p} - m.$$

The notation $\not{p} = \gamma^u p_u$ is the so called Feynman slash notation.

3.8 Dirac current

As we know, in Weyl representation, Dirac matrices are,

$$\gamma^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad \gamma^i = \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}.$$

We introduce

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

Thus,

$$\gamma^5 = \frac{i}{4!}\,\epsilon^{\mu\nu\rho\sigma}\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma.$$

Also,

$$[\gamma^5, S^{\mu\nu}] = \frac{i}{4}\bigg[\gamma^5\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^5 - \gamma^5\gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\mu\gamma^5\bigg] = 0.$$

And,

$$\{\gamma^5, \gamma^u\} = 0.$$

Taking the Dirac Lagrangian,

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi.$$

We note a global $U(1)$ symmetry,

$$\psi \to \psi e^{i\theta}.$$

By Noether,

$$\delta_\theta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta_\theta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta_\theta(\partial_\mu\psi) + \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta_\theta\bar{\psi} = 0,$$

$$\partial_\mu\bigg(-i\bar{\psi}\gamma^\mu(-i\psi)\bigg) = 0 \qquad \Rightarrow \qquad \boxed{j^\mu = \bar{\psi}\gamma^\mu\psi},$$

the Noether current. With the advent of γ^5 we can also write another (on shell) global invariance,

$$\psi(x) \to e^{i\alpha\gamma^5}\psi(x).$$

$$\delta_\alpha\mathcal{L} = \partial_\mu\bigg(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta_\alpha\psi\bigg) = \bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi - \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi - m\bigg(-i\bar{\psi}\gamma^5\psi + i\bar{\psi}\gamma^5\psi\bigg).$$

$$\delta_\alpha\mathcal{L} = \partial_\mu\bigg(i\bar{\psi}\gamma^\mu i\gamma^5\psi\bigg) = \bar{\psi}[\gamma^5, \gamma^\mu]\partial_\mu\psi.$$

Considering the equation of motion,

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0,$$

$$i\gamma^5\gamma^\mu\partial_\mu\psi = -i\gamma^\mu\gamma^5\partial_\mu\psi = m\gamma^5\psi,$$

$$\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = 2im\bar{\psi}\gamma^5\psi.$$

$$j^{\mu5} = \bar{\psi}\gamma^\mu\gamma^5\psi \qquad \text{(Axial current)},$$

$$\partial_\mu j^{\mu5} = 2im\bar{\psi}\gamma^5\psi,$$

which is only conserved for the massless case.

3.9 Quantization of Dirac Equation and Spins

As we know, Dirac lagrangean is written as,

$$\boxed{\mathcal{L} = \bar{\psi}(i\gamma^u \partial_u - m)\psi}.$$

This implies the hamiltonian,

$$H = \int dx^3\ \psi^\dagger(-i\gamma \cdot \nabla + m\gamma^0)\psi.$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ipx}}{\sqrt{2E_p}} \sum_s (a_p^s u^s(p) + b_{-p}^s v^s(-p)).$$

Where I am summing over the spins. If we adopt a similar quantization procedure we adopted for Klein-Gordon field, namely (at equal times),

$$[\psi_a(x), \psi_b^\dagger(y)] = \delta^3(x-y)\delta_{ab},$$

we get, (straightforwardly seen, but not directly), $[a_p^r, a_q^{s\dagger}] = [b_p^r, b_q^{s\dagger}] = (2\pi)^3\delta^{(3)}(p-q)\delta^{rs}$. And the others commutators are zero. When we substitute this into the hamiltonian expression, we find,

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s (E_p(a_p^{s\dagger}a_p^s - b_p^{s\dagger}b_p^s)).$$

We find that the energy can become arbitrarily negative, because, when we act this on the vacuum, the second term can be shifted to have the creation operator at front, and the created term will coexist with delta of zero, which is not good. So, we shall use another commutation relation. Assume the following holds,

$$\boxed{\{\psi_a(x), \psi_b^\dagger(y)\} = \delta^{(3)}(x-y)\delta_{ab}}.$$

While the other relations are zero. The fields are thus written as,

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a_p^s u^s(p)e^{-ipx} + b_p^{s\dagger} v^s(p)^{ipx}).$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \Big(a_p^{\dagger\, s}\bar{u}^s(p)e^{ipx} + b^s{}_p^\dagger \bar{v}^s(p)e^{-ipx}\Big).$$

The commutation relations are $\{a_p^r, a_q^{s\dagger}\} = \{b_p^r, b_q^{s\dagger}\} = (2\pi)^3\delta^{(3)}(p-q)\delta^{rs}$. And the vacuum is defined as the state $|0\rangle$ such that $a_p^s|0\rangle = b_p^s|0\rangle = 0$. We create new states by acting the creation operators on the vacuum,

$$\boxed{|p,s\rangle_a = \sqrt{2E_p}a_p^{s\dagger}|0\rangle},$$

those states refer to **fermions**, and

$$\boxed{|p,s\rangle_b = \sqrt{2E_p}b_p^{s\dagger}|0\rangle},$$

refer to **antifermions**. (Note that this is in consistency with Dirac interpretation of the lower components of the spinor). Also, note that, for both a and b,

$$\langle p,s|\,|q,r\rangle = 2E_p\delta(p-q)\delta^{rs},$$

is Lorentz invariant. The hamiltonian of this field quantized like this is,

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s (E_p(a_p^{s\dagger}a_p^s - b_p^s b_p^{s\dagger})),$$

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s (E_p(a_p^{s\dagger}a_p^s + b_p^{s\dagger}b_p^s) - \{b_p^{s\dagger}, b_p^s\}).$$

The last term is infinite at vacuum, but we can avoid that by defyining normal ordering by,

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s (E_p(:a_p^{s\dagger}a_p^s: - :b_p^s b_p^{s\dagger}:)).$$

Where $:a_p^{s\dagger}a_p^s := a_p^{s\dagger}a_p^s$ and $:a_p^s a_p^{s\dagger} := -a_p^{s\dagger}a_p^s$ to avoid the infinite. The same is valid for operator b. We see thus that the normal ordering for fermionic operators carries a minus sign (anticommutative).

An important observation shall be made: we could, in principle, have defined

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s (a_p^s u^s(p)e^{-ipx} + b_p^s v^s(p)e^{ipx}).$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \Big(a_p^{\dagger\, s}\bar{u}^s(p)e^{ipx} + b_p^{\dagger\, s}\bar{v}^s(p)e^{-ipx}\Big).$$

This would imply in H having the same format as in the "wrong" quantization,

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s (E_p(a_p^{s\dagger}a_p^s - b_p^{s\dagger}b_p^s)).$$

Note that this does not matter, because, when we write the anticommutation relation, changing b to $b^\dagger$ does not make a difference, while it does for the Lie commutator. That´s the core of this change.

Also, since $\{b,b\} = 0$ (I am simplyfing the notation), it implies that $b^2 = 0$, so a given state cannot be filled more than once (that is, the system obeys Fermi-Dirac statistics).

4 Feynman Path Integral

4.1 methodology

We shall now introduce a new vision of what "Quantum Mechanics does" by introducing path integrals, first described by american physicist Richard Phillips Feynman (1918-1988) in 1948.

First, take $|\psi(t_f)\rangle$ and $|\psi(t_0)\rangle$ elements in Hilbert space of the theory ($\mathcal{H}$), where $|\psi(t_f)\rangle$ is the time evolution of $|\psi(t_0)\rangle$. We define the operator $U : \mathcal{H} \to \mathcal{H}$ such that $|\psi(t_f)\rangle = U(t_f,t_i)\,|\psi(t_i)\rangle$. From this, we take the following properties,

$$1) \quad U(t,t) = Id$$

$$2) \quad U(t_2,t_1)\cdot U(t_1,t_0) = \quad U(t_2,t_0),$$
$$3) \quad U^\dagger(t,\tau) = U^{-1}(t,\tau) = U(\tau,t).$$

Note that this follows easily from thinking about the non-relativistic approach (Schrodinger equation) with constant Hamiltonian, where $U(t,\tau) = e^{-iH(t-\tau)}$. With this , we define the Propagator,

$$\boxed{iG(x_f,x_i,t_f,t_i) = \langle x_f|\, U(t_f,t_i)\,|x_i\rangle}.$$

Note that $|G|^2$ denotes the probability density of transition from (x_i,t_i) to (x_f,t_f). Lets start with a free particle $H = \frac{p^2}{2m}$. Upon a Fourier transform, we get,

$$iG = \frac{1}{2\pi}\int dp \quad \langle x|\, e^{-i\frac{p^2}{2m}(t-t_0)} e^{ipx}\,|x_0\rangle = \frac{1}{2\pi}\int dp \quad e^{-i\frac{p^2}{2m}(t-t_0)} e^{ip(x-x_0)},$$

doing the integral, we get,

$$iG = \sqrt{\frac{m}{2i\pi(t-t_0)}}e^{iS_{cl}},$$

where $S_{cl} = \frac{m(x-x_0)^2}{2(t-t_0)}$ is the classical action. If we decide to take a more general Hamiltonian, $H = \frac{p^2}{2m} + V(x)$. For this, we consider infinitesimal intervals of time,

$$e^{-iHdt} = 1 - iHdt.$$

We can separate the space components as $x_0, x_1, ..., x_N = x$ and call infinitesimal time as $dt = \frac{t-t_0}{N}$. Then, we shall call,

$$iG = \lim_{N\to\infty} \langle x_N|\, \Pi_{i=0}^{N-1} U(t_{i+1},t_i)\,|x_0\rangle\,,$$

but there is a twist! We know that, if a particle goes from x_0 and gets in x, quantum mechanics state that there is no trajectory that the particle takes between those two until we measure it. That seems to invalidate our idea to partition the path. That's why **we got to sum over all possible partition points,** so we consider all possibilities! Thus, we write the function as,

$$\boxed{iG = \lim_{N\to\infty}\int ... \int dx_1...dx_{N-1}\Pi_{i=0}^{N-1}\,\langle x_{i+1}|\, U(t_{i+1},t_i)\,|x_i\rangle}.$$

Now, since we are considering infinitesimal time, we apply Fourier transform in all integration coordinates and get in each,

$$\frac{1}{2\pi}\int dp_j \quad e^{-iHdt}e^{ip_j(x_{j+1}-x_j)} = \frac{1}{2\pi}\int dp_j \quad e^{-i(H-p_j\dot{x}_j)dt}.$$

Thus, making the integral and substituting in the propagator, we get,

$$iG = \int D[x(t)]e^{iS(t,t_0)},$$

where,

$$S(t,t_0) = \int_{t_0}^{t} dt \quad \left(\frac{m\dot{x}^2}{2} - V(x,t)\right),$$

and the integration measure is,

$$D[x(t)] = \lim_{N\to\infty}\left(\sqrt{\frac{m}{2i\pi dt}}\right)^N \Pi_{k=1}^{N-1} dx_k.$$

Note that there is no guarantee that this measure will converge (it does in the free particle, as seen, and in harmonic oscillator, as we shall see). This gives us a new description of quantum mechanics, and a very interesting relation to Hamilton principle: Note that, if we were not using natural units, then,

$$iG = \int D[x(t)] e^{\frac{iS(t,t_0)}{\hbar}}.$$

This implies that δG is proportional to $\frac{\delta S}{\hbar}$. For big systems, where S is relatively big, we see that this variation is too high, where ups cancel downs and the average result is zero. Thus, the probability for the path taken by a certain system will be concentrated on the paths such that $\delta S = 0$. This explanation is surely one of the most exciting aspects of this methodology. (Note that we are working with the non-relativistic case).

4.2 Harmonic oscillator

We shall explore more the case where $V \neq 0$. For this, we take a partition of the path around the classical trajectory, $x_k = x_{cl_k} + \eta_k$. The action can be written as,

$$S_{ij}(x_{cl}) = \frac{m(x_i - x_j)^2}{2dt}(\delta_{i-1,j} + \delta_{i+1,j}) - V_{ij}(x)\delta_{i,j} dt.$$

$$V = \begin{pmatrix} V(x_{cl_1}) & & & \\ & V(x_{cl_2}) & & \\ & & ... & \\ & & & V(x_{cl_N}) \end{pmatrix}.$$

Making a Taylor expansion,

$$S(\vec{x}) = S(\vec{x}_{cl}) + S'(\vec{x}_{cl_k})\hat{\eta} + \frac{S''(\vec{x}_{cl})}{2}\hat{\eta}^2 + ...$$

Note that, by Hamilton's principle, $S'_{ij}(x_{cl_k}) = 0$, so we get,

$$e^{iS(x)} = e^{iS(x_{cl})} e^{i\frac{S''(x_{cl})}{2}\eta^2},$$

we know that $e^{i\frac{S''(x_{cl})}{2}\eta^2} = \exp\frac{i\eta^T M\eta}{2}$ and $S''_{ij}(x_{cl}) = \frac{\partial^2 S(x_{cl})}{\partial x_i \partial x_j}$, then, we find:

If $|i-j| > 1$, $S''_{ij} = 0$.

If $i = j$, $S''_{ii}(x_{cl}) = \frac{2m}{dt} - V''(x_{cl_i})dt$.

If $|i-j| = 1$, $S''_{ij} = \frac{-m}{dt}$.

Thus, we find that,

$$M = \frac{m}{dt}\begin{pmatrix} 2 & -1 & ... & & & \\ -1 & 2 & -1 & ... & & \\ & -1 & 2 & -1 & ... & \\ & & ... & & & -1 \\ & & & & -1 & 2 \end{pmatrix} - dt\begin{pmatrix} V''(x_{cl_1}) & & & \\ & V''(x_{cl_2}) & & \\ & & ... & \\ & & & V''(x_{cl_{N-1}}) \end{pmatrix}.$$

Thus, we find,

$$iG = \lim_{N\to\infty}\left(\sqrt{\frac{m}{2i\pi dt}}\right)^N e^{iS_{cl}} \int \Pi_{k=1}^{N-1} d\eta_k \exp\frac{i\eta^T M\eta}{2}.$$

We solve the integral,

$$\int \Pi_{k=1}^{N-1} d\eta_k \exp\frac{i\eta^T M\eta}{2} = \frac{\pi^{\frac{N-1}{2}}}{\sqrt{\det(\frac{iM}{2})}},$$

and arrive at,

$$\boxed{iG = \lim_{N\to\infty}\left(\sqrt{\frac{m}{2i\pi dt}}\right)^N e^{iS_{cl}} \frac{\pi^{\frac{N-1}{2}}}{\sqrt{\det(\frac{iM}{2})}}}.$$

Notice that, for $V = 0$, the integral is,

$$\left(\frac{\pi(2dt)}{m}\right)^{\frac{N-1}{2}}\sqrt{\frac{1}{N}},$$

and, substituting, we get back the expression for the free particle, as expected. Finally, if we are dealing with the harmonical oscillator, $V = \frac{m(wx)^2}{2}$, we find,

$$M = \frac{m}{dt}\begin{pmatrix} 2-(wdt)^2 & -1 & ... & & & \\ -1 & 2-(wdt)^2 & -1 & ... & & \\ & -1 & 2-(wdt)^2 & -1 & ... & \\ & & & ... & & -1 \\ & & & & -1 & 2-(wdt)^2 \end{pmatrix},$$

and then, using $\cosh(wdt) \approx \frac{2-(wdt)^2}{2}$ and $\sinh(wdt) \approx wdt$, we get,

$$\boxed{iG = \sqrt{\frac{mw}{2\pi i sin(w(t_f - t_0))}} e^{iS_{cl}}}.$$

Where $S_{cl} = \int_{t_0}^{t_f} dt\big(\frac{m\dot{x}^2}{2} - \frac{m(wx)^2}{2}\big)$. If we take $x(t) = A\cos(t) + B\sin(t)$ and $[t_0, t_f]$ to be the period of oscillation, we finally get the expression for the amplitude of quantum harmonic oscillator,

$$\boxed{iG = \sqrt{\frac{mw}{2\pi i sin(w(t_f - t_0))}} e^{\frac{imw}{2}\left[(x(t_f)^2 + x(t_0)^2)\cot(w(t_f - t_0)) - 2\frac{x(t_f)x(t_0)}{\sin(w(t_f - t_0))}\right]}}.$$

Notice that, by making $w \to 0$, we get the free particle solution, as expected. We can also treat for the quantum forced harmonical oscillator, but the method shall be more appreciated once we get deeper into application of Field Theory.

4.3 Aharanov-Bohm effect

A wonderful application of Feynman's path integral is used to describe Aharanov-Bohm effect, first described in 1961 by american physicist David Bohm (1917-1992) and israeli Yakir Aharonov (1932 -). Imagine a particle passing in the double slit experiment as the picture below.

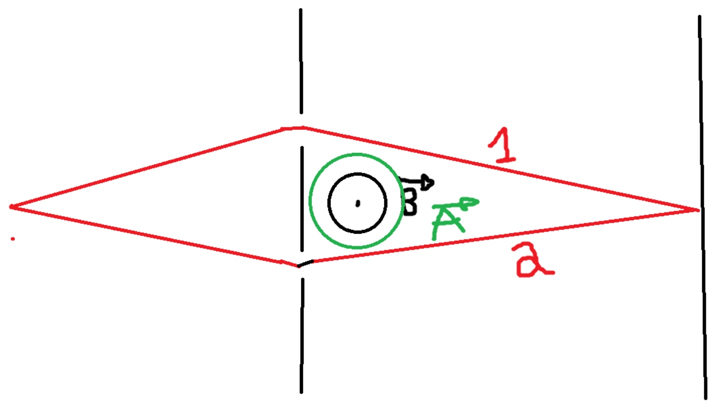


Figure : *We have a particle that is being split by the double slit. Inside it, we have a solenoid with magnetic field $\vec{B}$ in the interior, but outside is zero (the potencial $\vec{A}$ has vanishing curl outside the solenoid, but not inside it). We outline the splited paths by* 1 *and* 2*.*

Since the particle experiences no field, we know that the action is,

$$S_{EM} = -\int e\vec{A}\cdot\vec{dx}.$$

Suppose that first the magnetic field in the solenoid is shut down. There is no potential vector, and we know that the amplitude of the particle is a linear superposition of the two paths, so, by Feynman path integral, we find,

$$iG = \frac{\int_1 D[x(t)]e^{iS_1} + \int_2 D[x(t)]e^{iS_2}}{\sqrt{2\pi}} = \frac{\Phi_1 + \Phi_2}{\sqrt{2\pi}},$$

where, by symmetry, we assume $\Phi_1 = \frac{e^{i\theta_1}}{\sqrt{2}}$ and $\Phi_2 = \frac{e^{i\theta_2}}{\sqrt{2}}$. The probability density function is,

$$|iG|^2 = \frac{\cos^2(\frac{\theta_1-\theta_2}{2})}{\pi}.$$

Now, if we turn on the magnetic field, we field that the phases change by $\Phi_1 \to \Phi_1 e^{-i\int_1 e\vec{A}\cdot\vec{dx}}$ and $\Phi_2 \to \Phi_2 e^{-i\int_2 e\vec{A}\cdot\vec{dx}}$. Thus, the probability density changes to,

$$|iG|^2 = \frac{\cos^2\left(\frac{\theta_1-\theta_2}{2} - \int_{2-1} e\vec{A}\cdot\vec{dx}\right)}{\pi}.$$

By Stokes theorem,

$$\int_{2-1} e\vec{A}\cdot\vec{dx} = e\oint_{2,1} d\vec{A}\cdot\vec{\nabla}\times\vec{A} = e\Phi_B,$$

in which Φ_B is the magnetic flux present in between the paths 1 and 2. Thus, even if the particle does not interact with the field, its probability density is altered to,

$$|iG|^2 = \frac{\cos^2(\frac{\theta_1-\theta_2}{2} - e\Phi_B)}{\pi},$$

which is a measurable effect by noting how the patterns of interference are shifted when the magnetic field inside the solenoid is turned on, as the figure below indicates.

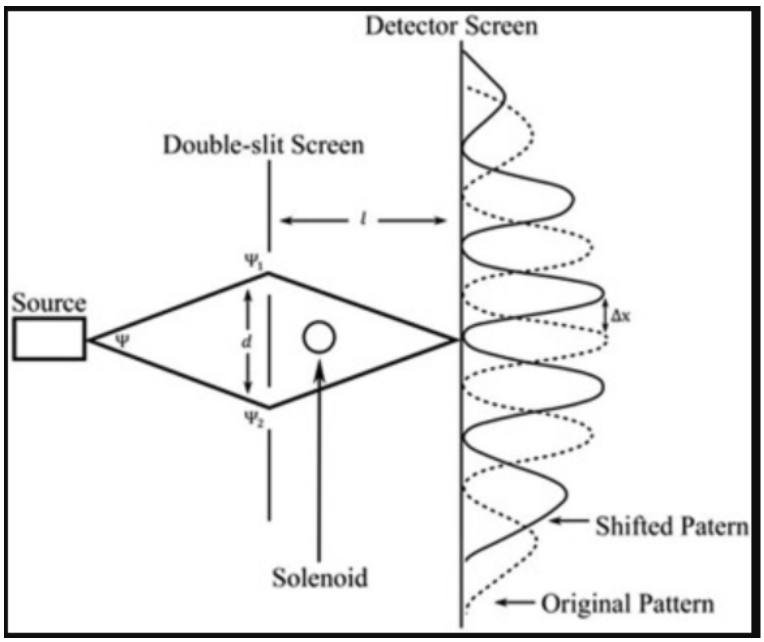


Figure : *The interference patterns are shifted. (https://physics.stackexchange.com/questions/619427/is-the-bohm-aharonov-effect-a-proof-of-real-u1-local-gauge-transformations).*

The fact that this observation is indeed seen (which is the Aharanov -Bohm effect) indicates that the vector $\vec{A}$ has physical significance instead of being a mathematical trick appearing due to the divergenceless of $\vec{B}$. Note that this is a purely quantum mechanical effect.

4.4 Dirac argument for magnetic monopole

As a final application, we shall explain about an argument formulated by Paul Dirac in 1931 to the existence of magnetic monopoles.

Suppose that there is a magnetic charge g in the universe. The expression for the magnetic field generated by this charge is presumably equivalent to the expression of an electrical field generated by a single charge,

$$\vec{B} = \frac{\rho g}{r^2}\hat{r}.$$

By consistency, excepting the origin, we must have a vector $\vec{A}$ that, written in polar coordinates, is

$$\vec{A} = \frac{\rho g(1-\cos\theta)\hat{\Phi}}{r\sin\theta}.$$

Now, imagine a particle with electrical charge q going in a loop around a line that intersects the magnetic charge, as the figure below.

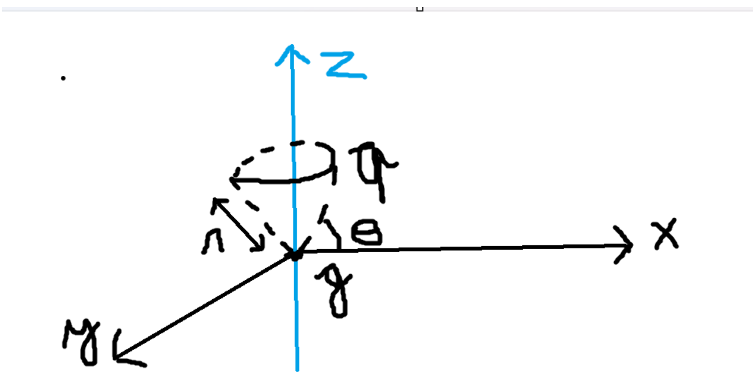


Figure : *The magnetic charge is at the origin. We consider $\theta \to \pi$ and $r \to \infty$. The line z is the so called Dirac String.*

We'll consider $\theta \to \pi$ and $r \to \infty$. By Ahranov-Bohm effect, the electrical charge would experience a change in phase going around the loop of,

$$\delta\varphi = \oint_{loop} \vec{A}\cdot\vec{dx} = \lim_{r\to\infty}\lim_{\theta\to\pi} q\int_0^{2\pi} \frac{\rho g(1-\cos\theta)\hat{\Phi}}{r\sin\theta} r\sin\theta d\Phi = 4\pi q\rho g.$$

At principle, this phase difference could be measure by doing this experiment in regions where the value ρ or the charge q changes. Since this is not observed, we infer $\delta\varphi = 2\pi n$, where n is an integer. This implies that,

$$\boxed{q = \frac{n}{2\rho g}}.$$

Thus, if a single magnetic charge exists in the universe (presumably in a very distant place from what we have reached so far), this implies the discretization of electrical charge!! This explains why particles have only quantized quantities of it, which is a long time mystery. The search for a magnetic monopole remains ongoing at the time of writing (January 7, 2026).

5 Quantization of bosonic fields

At last, we explain how Feynman's procedure is used in QFT. We shall use it to compute correlation functions. First, we´ll make a very important and beautiful connection.

5.1 Statistical Mechanics and QFT

In statistical physics, we know that the Canonical Partition function is,

$$Z(\beta) = \sum_n e^{-\beta E_n},$$

where n denotes the possible states of the system. Moreover, in the language of linear algebra that we are mostly used to in Quantum Mechanics we got,

$$Z(\beta) = \sum_n \langle n| e^{-\beta \hat{H}} |n\rangle = Tr(e^{-\beta \hat{H}}).$$

When we compute the transition amplitude from $|x, t\rangle$ to $|x', t'\rangle$, we get,

$$iG = \langle x'| e^{-i\hat{H}(t'-t)} |x\rangle .$$

Using the completeness of the base, $\sum_n |n\rangle \langle n| = \hat{1}$, we can rewrite the above expression as,

$$iG = \sum_n \langle x'| |n\rangle \langle n| e^{-i\hat{H}(t'-t)} |n\rangle \langle n| |x\rangle .$$

By calling $\psi_n(x') = \langle x'| |n\rangle$ and $\psi_n(x) = \langle x| |n\rangle$, we have,

$$iG = \sum_n \psi_n(x')\bar{\psi}_n(x) e^{-iE_n(t'-t)}.$$

Thus,

$$\int dx'dx \langle x', t'| |x, t\rangle = \sum_n \int dx'dx(\psi_n(x')\bar{\psi}_n(x))e^{-iE_n(t'-t)} = Tr(e^{-i(t'-t)\hat{H}}).$$

This implies that, under the association $\beta = i(t' - t)$, we get,

$$Z(\beta) = \int dx'dx \langle x', -i\beta| |x, 0\rangle .$$

That's an important association. We see that an integral over space of the transition amplitude considering imaginary time (a procedure named Wick rotation after Gian Carlo Wick [1909-1992]) is equivalent of a description of a statistical system of the canonical ensemble. This is not only very beautiful, but it points to a mind-blowing interpretation of what is going on (namely, that QFT is indeed a statistical system, but the interaction with microstates is replaced by interactions with vacuum). Moreover, by changing the measure of integration and considering infinitesimal time, we find,

$$Z(\beta) = \int D[x(t)] \quad e^{-i\int dtH}.$$

And we see that,

$$S = \int_0^t dt'(\dot{q}^2 - V) = \int_0^{it} -idt''(-\dot{q}^2 - V) = i\int dt'H = iS_E,$$

where S_E stands for Euclidean action. Thus, we can write this as,

$$\boxed{Z = \int D[x(t)]e^{-S_E}}.$$

We took of the time in the integral because its specific values in the limit of the integral do not matter. The above expression is called Feynman-Kac formula (after Feynman and Mark Kac[1914-1984]). We can use it to get expected values as we do in statistical physics. For the mean value of a quantity $A(x)$, we have,

$$\boxed{< A >= \frac{\int D[x(t)]A(x)e^{-S_E}}{\int D[x(t)]e^{-S_E}}}.$$

We will often normalize the denominator. Note that the correspondence of S and S_E is made by doing a Wick rotation.

Example. Take the case for an Harmonical oscillator, $S_E = \int_0^t d\tau \left[\frac{\dot{q}^2}{2} + \frac{(wq)^2}{2}\right]$. By Feynman-Kac,

$$Z = \int Dq \exp\left[-\int_0^t d\tau \left(\frac{\dot{q}^2}{2} + \frac{(wq)^2}{2}\right)\right],$$

note that, by doing a partition and decomposing the integral into a Riemann sum, we find,

$$Z = \int Dq \exp\left[-\sum_i \epsilon \left(\frac{(q_{i+1} - q_i)^2}{2\epsilon^2} + \frac{(wq_i)^2}{2}\right)\right],$$

as one can see, this is a sum of classical oscillators (as dealt with Canonical Ensemble). This allows an interpretation that, in this case, quantum oscillations may be seen as termical oscillations of a classical system.

5.2 Functional Quantization of Scalar Fields

By the last section, we find the average value of a quantity q to be,

$$< q(t) >= \int_{q_0(t_0)}^{q_1(t_1)} Dq \quad e^{iS[q]}q(t).$$

If we are to write correlation functions, we need to use time ordering

$$< T\{q(t_1), ..., q(t_n)\} >= \int Dq \quad e^{iS}T\{q(t_1), ..., q(t_n)\}.$$

We may write correlation functions for fields,

$$\boxed{< \phi(t_1)...\phi(t_n) >= \int D\phi_1...D\phi_n \quad e^{iS}T\{\phi(t_1)...\phi(t_n)\}}.$$

This leads us to define the n-point correlation function:

$$\boxed{G_n(t_1, ..., t_n) = \int D\phi_1...D\phi_n \quad e^{iS}T\{\phi(t_1)...\phi(t_n)\}}.$$

One way for us to get it is to define the generator function,

$$Z[J] = \sum_{n>0} \int dt_1 ... dt_n \quad \frac{i^n}{n!} G_n(t_1 .... t_n) J(t_1) ... J(t_n),$$

we may rewrite the above expression as,

$$\boxed{Z[J] = \int D\phi \quad e^{iS[\phi]} \sum_{n>0} \frac{\left[i \int dt J(t)\phi(t)\right]^n}{n!} = \int D\phi \quad e^{i(S[\phi] + \int dt J(t)\phi(t))}},$$

$$\boxed{\left.\frac{\delta^n Z}{i^n \delta J^n}\right|_{J=0} = G_n}.$$

(We used some facts about the derivatives of distributions of functions that were not explicit here).

Example. Take the functional $Z[J] = \exp\{\int dt J^2(t) f(t) + \lambda \int dt J^3(t) + \mu \int dt J^4(t)\}$. The three point correlation function is,

$$\left.\frac{\delta^3 Z}{\delta J^3}\right|_{J=0} = -6i\lambda.$$

While the 2-point correlation function is,

$$\left.\frac{\delta^2 Z}{\delta J^2}\right|_{J=0} = -2f(0).$$

This represents a standard pattern: only terms to the power of a number with the same parity of the degree of the correlation function being taken will appear in the final expression.

5.3 Forced Harmonic Oscillator and Fock Space

Note that, by the expression of $Z[J]$, we may find an analogous way to find the equations of motion by considering J to have a physical meaning. This may be done by applying functional derivatives to the expression and setting it to zero,

$$\boxed{\frac{\delta S(\phi(t))}{\delta \phi(t)} + J(t) = 0}.$$

Take for example the harmonic oscillator,

$$-\omega^2 q - \ddot{q} + J = 0.$$

This shows we can treat the J term as an external force. We may treat it in a more rigorous way by introducing Fock Space.

Definition. The Fock Space (after Vladimir Fock [1898-1974]), is the vector space of the form,

$$|\gamma\rangle = e^{\gamma a^\dagger} |0\rangle = \sum_n \frac{\gamma^n}{n!} (a^\dagger)^n |0\rangle .$$

For $\gamma \in \mathbb{C}$.

Proposition. *The Fock space is the Eigenspace of operator a.*

Proof.

$$a\,|\gamma\rangle = \sum_{n>0} \frac{\gamma^n}{n!}(a^\dagger a + 1)(a^\dagger)^{n-1}\,|0\rangle\,,$$

note that $a^\dagger a$ is the number operator, so, when applied to $(a^\dagger)^{n-1}\,|0\rangle$, we get $n-1$, thus getting,

$$a\,|\gamma\rangle = \gamma\sum_{n>0}\frac{\gamma^{n-1}}{(n-1)!}(a^\dagger)^{n-1}\,|0\rangle = \gamma e^{\gamma a^\dagger}\,|0\rangle = \gamma\,|\gamma\rangle\,.$$

Note that, since all the possible eigenvalues of a are there, it is also true that all Eigenstates of a can be seen as Fock space elements. □

We thus have a natural vector space to work with in the Harmonic Oscillator. We can calculate the amplitude of transition between states,

$$F(\alpha', t', \alpha, t) = \langle \alpha', t'|\,|\alpha, t\rangle = \langle \alpha'|\,e^{-iH(t'-t)}\,|\alpha\rangle\,.$$

The hamiltonian for this case is $H = \frac{p^2}{2m} + \frac{w^2q^2}{2} - qJ = w(N+\frac{1}{2}) - \frac{Ja+Ja^\dagger}{\sqrt{2w}}$. Before we shall prove an important property,

Lemma.

$$\int \frac{d\alpha d\alpha^*}{2\pi i}e^{-|\alpha|^2}\,|\alpha\rangle\,\langle\alpha^*| = \hat{1}.$$

Proof. If we take $\alpha = re^{i\theta}$ and make the substitution of variables, considering $|\alpha\rangle$ is an element in the Fock space, we find that the integral is,

$$\int_0^{2\pi}\int_0^\infty drd\theta\frac{r}{\pi}e^{-r^2}\left(\sum_n \frac{\alpha^n}{n!}(a^\dagger)^n\,|0\rangle\right)\left(\langle 0|\sum_m\frac{\bar{\alpha}^m}{m!}(a)^m\right) =,$$

$$= \int_0^{2\pi}\int_0^\infty drd\theta\frac{r}{\pi}e^{-r^2}\left(\sum_{n,m}\frac{r^{n+m}}{\sqrt{n!m!}}e^{i(n-m)\theta}\,|n\rangle\,\langle m|\right).$$

Upon doing the integration in θ and changing variables $x = r^2$, we get,

$$I = \int_0^\infty dx\ \sum_n \frac{x^n e^{-x}}{n!}\,|n\rangle\,\langle n|\,.$$

Considering those are orthonormal states, we must have that,

$$I\,|m\rangle = \int_0^\infty dx\ \frac{x^m e^{-x}}{m!}\,|m\rangle\,,$$

considering the expression for the Gamma function, we have,

$$I\,|m\rangle = |m\rangle\,.$$

Since this is valid for all states, we find that I must be the identity operator. This finishes the proof. □

With the above lemma, we can rewrite the amplitude expression to,

$$F(\alpha', t', \alpha, t) = \int \Pi_{i=1}^{n} \frac{d\alpha_i d\alpha_i^*}{2\pi i} e^{-|\alpha_i|^2} \langle \alpha_{n+1}| e^{-i\epsilon H} |\alpha_n\rangle \langle \alpha_n| e^{-i\epsilon H} |\alpha_{n-1}\rangle \ldots |\alpha_1\rangle \langle \alpha_1| e^{-i\epsilon H} |\alpha_0\rangle ,$$

and we know that,

$$\langle \alpha| e^{-i\epsilon H} |\gamma\rangle = e^{-i\epsilon H(\alpha,\gamma)} \langle \alpha| |\gamma\rangle = e^{-i\epsilon H(\alpha,\gamma)} e^{\bar{\alpha}\gamma}.$$

By substituiting,

$$F(\alpha', t', \alpha, t) = \int \left[\Pi_{i=1}^{n} \frac{d\alpha_i d\alpha_i^*}{2\pi i}\right] \exp\left(\sum_{i=0}^{n} -i\epsilon H(\alpha_{i+1}, \alpha_i) + \sum_{i=1}^{n} (\bar{\alpha}_{i+1} - \bar{\alpha}_i)\alpha_i + \bar{\alpha}_1\alpha_0\right).$$

Treating this as a Riemann sum, we find,

$$F(\alpha', t', \alpha, t) = \int D\alpha D\bar{\alpha} \exp\left(i\int_t^{t'} d\tau \ [-i\dot{\bar{\alpha}}(\tau)\alpha(\tau) - H(\tau)] + \bar{\alpha}(t)\alpha(t)\right).$$

Note that the action for this expression is

$$S(\alpha', t', \alpha, t) = \int_t^{t'} d\tau \ [-i\dot{\bar{\alpha}}(\tau)\alpha(\tau) - H(\tau)] + \bar{\alpha}(t)\alpha(t).$$

Considering the variation of α and $\bar{\alpha}$ are zero at the extremes, we find that the principal of extreme action implies,

$$\frac{\delta S}{\delta \alpha} = -i\dot{\bar{\alpha}} - \frac{\delta H}{\delta \alpha} = 0,$$
$$\frac{\delta S}{\delta \bar{\alpha}} = i\dot{\alpha} - \frac{\delta H}{\delta \bar{\alpha}} = 0.$$
$$H = \omega\bar{\alpha}\alpha - \gamma\bar{\alpha} - \bar{\gamma}\alpha.$$

Where $\gamma(t) = \frac{J(t)}{\sqrt{2w}}$. This leads to the equation of motion,

$$\dot{\bar{\beta}} - i(\omega\bar{\beta} - \gamma) = 0,$$

$$\dot{\beta} + i(\omega\beta - \bar{\gamma}) = 0.$$

This leads to the solutions,

$$\boxed{\beta(\tau) = \beta(t)e^{i\omega(t-\tau)} + i\int_t^{\tau} ds \ e^{i\omega(s-\tau)}\gamma(s)},$$

and the respective conjugated part. Note that we have the classical solution for the Harmonic oscillator ($\alpha(t)e^{i\omega(t-\tau)}$) plus another term, which refers to the forced part. We may get the partition function now,

$$Z[J] = \int Dq \ \exp\left(i\int dt \ \left[\frac{\dot{q}^2 - \omega^2 q^2}{2} + J\cdot q\right]\right).$$

Considering that the coordinates in the extreme points are fixed, we can write the action as,

$$S[q,J] = \int dt \ \frac{-1}{2} q \left(\frac{\partial^2}{\partial t^2} + \omega^2 \right) q + J \cdot q,$$

and, since the classical solution is an extreme of the action, we can write,

$$S[q,J] = S[q_{cl},J] + \frac{A}{2}(q - q_{cl})^2.$$

We can then rewrite Z expression as,

$$Z[J] = N e^{iS[q_{cl},J]},$$

where the constant N does not depend on J.

where $i\left(\frac{\partial^2}{\partial t^2} + \omega^2 \right) = \Delta^{-1}$. We can see that, upon taking the EOM on the action,

$$\Delta^{-1} q_{cl} = iJ.$$

Thus, we arrive at,

$$\boxed{q_{cl}(t,J) = q_{cl}(t,0) + i\Delta J}.$$

The form of operator Δ (up to a Feynman prescription in order to avoid singularities) is,

$$\boxed{\Delta(t,t') = i \int \frac{dp}{2\pi} \ \frac{e^{ip(t'-t)}}{p^2 - \omega^2 + i\epsilon}}.$$

At last, we may simplify the generating functional by noting,

$$\frac{\delta S}{\delta J} = q_{cl}(t,J) \Rightarrow S[q_{cl},J] = q_{cl}(t,0) \cdot J + \frac{iJ \cdot \Delta \cdot J}{2}.$$

Which leads us to,

$$\boxed{Z[J] = N \exp \left[i q_{cl}(t,0) \cdot J - \frac{J \cdot \Delta \cdot J}{2} \right]}.$$

We can, therefore, find the correlation function.

6 Interactions

No, we shall study interacting fields. This can be done by adding higher powers of the field in the Hamiltonian. Before we attack it, we ought to deal with interaction pictures.

6.1 Interaction Picture

By Heisenberg equation,

$$i\frac{d\hat{A}}{dt} = [\hat{A}, \hat{M}].$$

For states, we have Schrodinger equations,

$$i\partial_t |\psi\rangle = \hat{N} |\psi\rangle .$$

One can directly check that,

$$i\partial_t(\langle\psi| \hat{A} |\phi\rangle) = \langle\psi| [\hat{A}, \hat{M} + \hat{N}] |\phi\rangle .$$

The so called picture is precisely characterized by the choices we make for $\hat{M}$ and $\hat{N}$. For $\hat{N} = \hat{H}$ and $\hat{M} = 0$, we get Schrodinger picture. For $\hat{M} = \hat{H}$ and $\hat{N} = 0$, we get Heisenberg picture. We can write the relation,

$$i\frac{d\hat{\phi}_H}{dt} = [\hat{\phi}_S, \hat{H}] \Longrightarrow \boxed{\hat{\phi}_H(x,t) = e^{i\hat{H}t}\hat{\phi}_S(x)e^{-i\hat{H}t}}.$$

To deal with interactions, we shall consider their role in the theory via perturbations. First, we get the interacted picture by evolving ϕ_S by the free hamiltonian,

$$\boxed{\hat{\phi}_I(x,t) = e^{i\hat{H}_0 t}\hat{\phi}_S(x)e^{-i\hat{H}_0 t}},$$

$$\hat{\phi}_H(x,t) = e^{i\hat{H}t}e^{-i\hat{H}_0 t}\hat{\phi}_I(x)e^{i\hat{H}_0 t}(x)e^{-i\hat{H}t}.$$

If $\hat{U}(t,t_0) = e^{i\hat{H}_0\Delta t}e^{-i\hat{H}\Delta t}$, we get,

$$i\dot{\hat{U}}(t,t_0) = e^{i\hat{H}_0\Delta t}\hat{H}_{int}e^{-i\hat{H}_0\Delta t}\hat{U}(t,t_0),$$

$$\boxed{i\dot{\hat{U}}(t,t_0) = \hat{H}_{intI}\hat{U}(t,t_0)}.$$

To find the solution, we get that, upon integration,

$$\int_{t_0}^{t} d\tau_1 ... \int_{t_0}^{\tau_{n-1}} d\tau_n \ \hat{H}_I(\tau_1)...\hat{H}_I(\tau_n) = \frac{1}{n!}\int_{t_0}^{t} dt_1...dt_n \ T\{\hat{H}_I(t_1)...\hat{H}_I(t_n)\}.$$

Thus, we get the solution,

$$\boxed{\hat{U}(t,t_0) = \exp\bigg\{ - iT\bigg(\int_{t_0}^{t} d\tau H_I(\tau)\bigg)\bigg\}}.$$

After this, we can prove the main result of this subsection.

Theorem. *(Feynman) If one has a Hamiltonian with interaction, $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, where $|\Omega\rangle$ is the ground state for $\hat{H}$ and $|0\rangle$ is the one for $\hat{H}_0$, the two-point correlation function for the fields is,*

$$\boxed{\langle\Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0| T\bigg(\phi_I(x)\phi_I(y)\} \exp\bigg\{ - i\int_{-t}^{t} d\tau H_I(\tau)\bigg\}\bigg) |0\rangle}{\langle 0| T\bigg\{ \exp\bigg\{ - i\int_{-t}^{t} d\tau H_I(\tau)\bigg\}\bigg\} |0\rangle}}.$$

Proof. By the spectral theorem,

$$e^{-i\hat{H}t} = e^{-iE_0 t} |\Omega\rangle \langle\Omega| + \sum_{n>0} e^{-iE_n t} |n\rangle \langle n| \,.$$

We thus get,

$$|\Omega\rangle = \frac{e^{iE_0 t}}{\langle\Omega| |0\rangle} e^{-i\hat{H}t} |0\rangle \,,$$

where $\langle\Omega| |0\rangle \neq 0$ by perturbation assumption. By making $t \to \infty(1-i\epsilon)$, we get a constant in the time part wlog, also, since $\hat{H}_0 |0\rangle = 0$, we can multiply it exponentiated in front of $|0\rangle$, and get,

$$|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{e^{iE_0(t+t_0)}}{\langle\Omega| |0\rangle} e^{-i\hat{H}(t+t_0)} e^{i\hat{H}_0(t+t_0)} |0\rangle \,.$$

This leads us to,

$$|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{e^{iE_0(t+t_0)}}{\langle\Omega| |0\rangle} \hat{U}(t_0,-t) |0\rangle \,.$$

With this,

$$\langle\Omega| \phi(x)\phi(y) |\Omega\rangle = \langle\Omega| U(x_0,t_0)\phi_I(x)U^\dagger(x_0,t_0)U(y_0,t_0)\phi_I(y)U^\dagger(y_0,t_0) |\Omega\rangle \,,$$

$$\langle\Omega| \phi(x)\phi(y) |\Omega\rangle = \langle\Omega| U(x_0,t_0)\phi_I(x)U^\dagger(x_0,y_0)\phi_I(y)U^\dagger(y_0,t_0) |\Omega\rangle \,.$$

After substitutions, we get (making explicit use of the limit),

$$\langle\Omega| \phi(x)\phi(y) |\Omega\rangle = lim_{t\to\infty(1-i\epsilon)} \langle 0| \hat{U}(t,x_0)\phi_I(x)\hat{U}^\dagger(x_0,y_0)\phi_I(y)\hat{U}(y_0,-t) |0\rangle \left(\frac{e^{2iE_0 t}}{|\langle 0| |\Omega\rangle|^2}\right).$$

One can check that,

$$\left(\frac{e^{2iE_0 t}}{|\langle 0| |\Omega\rangle|^2}\right) = (\langle 0| \hat{U}(t,t_0)\hat{U}(t_0,-t) |0\rangle)^{-1}.$$

Note that the role of the imaginary part in the limit is to make the exponential of energy not vanish. Upon time ordering and making due substitutions, we finally arrive at the aimed result,

$$\boxed{\langle\Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0| T\Bigg(\phi_I(x)\phi_I(y)\exp\Big\{-i\int_{-t}^{t} d\tau H_I(\tau)\Big\}\Bigg) |0\rangle}{\langle 0| T\Bigg\{\exp\Big\{-i\int_{-t}^{t} d\tau H_I(\tau)\Big\}\Bigg\} |0\rangle}}.$$

□

This expression is exact and has a Mutatis-Mutandis generalization for when we are dealing with more fields. The approximations will come when we deal with series over the integrals. As we shall see, this will give origin to the famous Feynman Diagrams.

6.2 Wick Theorem And Feynman Diagrams

As we know, in QFT and many areas in modern mathematical sciences, we deal with operators that do not commute. There are several ways to specify their orders.

Definition. (Time Ordering) $T : L(\mathcal{H}) \to L(\mathcal{H})$, such that

$$T(\hat{\phi}(t_1)...\hat{\phi}(t_n)) = \hat{\phi}(t_{i_1})...\hat{\phi}(t_{i_n}),$$

for $t_{i_1} \geq ... \geq t_{i_n}$.

We already know this one. But, for a bosonic field, we can decompose $\phi(x) = \phi^+(x) + \phi^-(x)$, where,

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} a_p e^{-i\vec{p}\vec{x}} \Longrightarrow \phi^+(x) |0\rangle = 0,$$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger e^{i\vec{p}\vec{x}} \Longrightarrow \langle 0| \phi^-(x) = 0.$$

Definition. (Normal Ordering) $N : L(\mathcal{H}) \to L(\mathcal{H})$, such that

$$N(\hat{\phi}_1...\hat{\phi}_n) = \sum \hat{\phi}_1^{a_1}...\hat{\phi}_n^{a_n},$$

where each a is either a plus or minus and every plus is on the right of a minus.

Example.

$$N(\phi_1\phi_2) = \phi_1^+\phi_2^+ + \phi_1^-\phi_2^+ + \phi_2^-\phi_1^+ + \phi_1^-\phi_2^-.$$

Definition. (Contraction of fields) $\Delta : L(\mathcal{H}) \to L(\mathcal{H})$, such that

$$\Delta(\hat{\phi}_1...\hat{\phi}_n) = T(\hat{\phi}(t_1)...\hat{\phi}(t_n)) - N(T\{\hat{\phi}(t_1)...\hat{\phi}(t_n)\}).$$

where each a_i is either a plus or minus and every plus is on the right of a minus.

For two fields, we find,

$$\Delta(\phi(x), \phi(y)) = \theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \theta(y^0 - x^0)[\phi^+(y), \phi^-(x)].$$

We see that $\langle 0| \Delta(\phi(x), \phi(y)) |0\rangle = D_F(x-y)$, the Feynman propagator we found in section 2. This allows us to get correlation functions. We define the contraction of two fields using the notation

$$\boxed{\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) = \Delta(\phi(x), \phi(y))}.$$

Now, we shall go to one of the main results of this section.

Theorem. *(Wick)*

$$T\{\phi_1...\phi_n\} = N\{\phi_1...\phi_n + \sum_{i,j} \phi_1...\contraction{}{\phi}{_i...}{\phi}\phi_i...\phi_j...\phi_n + \sum_{i,j,k,l} \phi_1...\contraction{}{\phi}{_i...}{\phi}\phi_i...\phi_j...\contraction{}{\phi}{_k...}{\phi}\phi_k...\phi_l...\phi_n + (...)\}.$$

Proof. We shall prove it by induction. Note that the statement is trivially valid for $n = 1$. Suppose it is valid for $n - 1$,

$$T\{\phi_2 ... \phi_n\} = N\{\phi_2 ... \phi_n + \sum_{i,j} \phi_2 ... \overline{\phi_i ... \phi_j} ... \phi_n + \sum_{i,j,k,l} \phi_2 ... \overline{\phi_i ... \phi_j} ... \overline{\phi_k ... \phi_l} ... \phi_n + (...)\}.$$

Assuming wlog that $\phi_1, ..., \phi_n$ is already time ordered, we get,

$$T\{\phi_1 \phi_2 ... \phi_n\} = \phi_1 T\{\phi_2 ... \phi_n\},$$

and, by writing $\phi_1 = \phi_1^- + \phi_1^+$, we get,

$$\phi_1^- N(...) = N(\phi_1^- ...).$$

For the other part,

$$\phi_1^+ N(...) = [\phi_1^+, N(...)] + N(...)\phi_1^+,$$

since $[\phi_1^+, \phi_j^+] = 0$ due to the commutation relation of annihilators, we find,

$$N(...)\phi_1^+ = N(\phi_1^+ ...),$$

$$[\phi_1^+, N(...)] = N\left(\sum_i \phi_2 ... [\phi_1^+, \phi_i] ...\right) = N\left(\sum_i \overline{\phi_1 ... \phi_i} ...\right).$$

This last equality shows all contractions with ϕ_1 are included. Thus, we get,

$$\boxed{T\{\phi_1 ... \phi_n\} = N\{\phi_1 ... \phi_n + \sum_{i,j} \phi_1 ... \overline{\phi_i ... \phi_j} ... \phi_n + \sum_{i,j,k,l} \phi_1 ... \overline{\phi_i ... \phi_j} ... \overline{\phi_k ... \phi_l} ... \phi_n + (...)\}}.$$

By induction, the result is proven.

□

Note that, if we have an odd number of fields, there is always one field in the summation terms that remains non-contracted, which implies that

$$\langle 0| T\{\phi_1 ... \phi_n\} |0\rangle = 0.$$

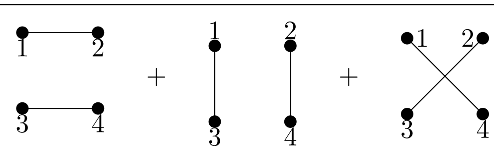
But, if we are dealing with an even number of fields, we may get a non-zero result, showing particles creation by the vacuum. Take the case for $n = 4$. We find,

$$\langle 0| T\{\phi_1 ... \phi_4\} |0\rangle = \langle 0| (\overline{\phi_1 \phi_2}\,\overline{\phi_3 \phi_4} + \phi_1 \phi_2 \phi_3 \phi_4 + \phi_1 \phi_2 \phi_3 \phi_4) |0\rangle .$$

This implies,

$$\langle 0| T\{\phi_1 ... \phi_4\} |0\rangle = D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3) D_F(x_2 - x_4) + D_F(x_1 - x_4) D_F(x_2 - x_3).$$

We can write this expression in terms of Diagrams, with each x denoting a dot. For instance,

$\langle 0| T\{\phi_1 \phi_2 \phi_3 \phi_4\} |0\rangle =$  .

Those are the so called **Feynman´s Diagrams for free fields**. They are a pictorial way to represent correlation functions, and turn out to be a crucial quantity when dealing with QFT.

6.3 Diagrams for ϕ^4 theory

Now, we get to a very interesting place. If we are dealing with an interaction hamiltonian such as: $H_{int} = \int d^3x \frac{\lambda\phi^4(x)}{4!}$.

Feynman Theorem says that the partition function for two fields interaction has as a numerator,

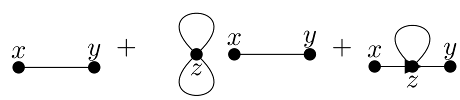
$$\langle 0| T\{\phi(x)\phi(y)\exp\{-i\int dt\ H_{int}\}\,|0\rangle = \phi(x)\phi(y) - i\frac{\lambda}{4!}(\phi(x)\phi(y)\int d^4z\phi(z)\phi(z)\phi(z)\phi(z)) + (...).$$

Keeping to the first order in λ (and that is the first approximmation we are doing!), we find,

$$\langle 0| T\{\phi(x)\phi(y)\exp\{-i\int dt\ H_{int}\}\,|0\rangle = D_F(x-y) - i\frac{\lambda}{4!}\Big[3D_F(x-y)\int d^4z D_F(z'-z'')D_F(z'''-z'''')+$$

$$+12\int d^4z D_F(x-z')D_F(y-z'')D_F(z'''-z'''')\Big].$$

The diagrams for the above expression are as follows,

$\langle 0| T\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle =$ 

.

The lines in these diagrams are the propagators, the vertices (dotted points) denote the interactions of non fixed fields (the ones that appear in H_{int}). The diagrams turn out to explicitly give the expression for the correlation function, so we are normally more interested in those. The moral of this story is essentially that,

$$\langle 0| T\{\phi_1\phi_2....\phi_n\}\,|0\rangle = \Big(Sum\ of\ diagrams\ with\ n\ external\ points\Big).$$

Notice that this is just the numerator. For the denominator of the expression in Feynman Theorem, we essentially take the disconnected diagrams (where x and y disappear), and, considering that for every disconnected diagram V_i, there are n_i copies, we can interchange them, and we get,

$$\Pi_i \exp\left[\frac{(V_i)^{n_i}}{n_i!}\right] = \exp\left[\sum_i \frac{(V_i)^{n_i}}{n_i!}\right].$$

This cancels the expression in the numerator, so we are left with (note that the arguments given here are extremely heuristical),

$$\boxed{\langle \Omega| T\{\phi_1\phi_2....\phi_n\}\,|\Omega\rangle = \Big(Sum\ of\ connected\ diagrams\ with\ n\ external\ points\Big)}.$$

There is also the notion of **symmetry factor**, which is the number of ways one could rearrange the propagators in a diagram without changing the interaction it represents. This is denoted by

$$\frac{n!}{N(appearing\quad in\quad the\quad correlation\quad function)}.$$

This turns out to be the denominator of the expression represented by the diagram in the correlation function. For example,

$$S(D_F(x-y)) = 1;$$

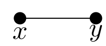
$$S\bigg(D_F(x-y)\int d^4z D_F(z'-z'')D_F(z'''-z'''')\bigg) = 8;$$

$$S\bigg(\int d^4z D_F(x-z')D_F(y-z'')D_F(z'''-z'''')\bigg) = 2.$$

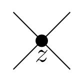
So, we are left with **Feynman Rules**.

6.4 Feynman Rules for bosonic fields

I) Each propagator is a line linking its extreme points

$$D_F(x-y) =$$ 

II) For each vertex (the result is analogous for a higher number of fields),

$$(-i\lambda)\int d^4z =$$ 

III) For each external point,

$$1 =$$ 

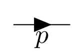
IV) Divide by symmetry factor (calculable from the diagrams).

With this, and considering the Fourier transform for the momentum space,

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon}.$$

We are also able to write **Feynman Rules for Momentum-Space**:

I) Each propagator is a line linking its extreme points

$$\frac{i}{p^2-m^2+i\epsilon} =$$ 

II) For each vertex (z), there is conservation of momentum,

$$\int d^nz\ e^{ip_1z}...e^{ip_nz} = (2\pi)^n\delta^n(p_1+...+p_n).$$

III) For each external point, the corresponding expression is e^{-ipx}.

IV) Divide by symmetry factor (calculable from the diagrams).

Example. (Example in position space and momentum space).

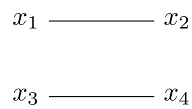
Consider the 4-point correlator in the free (Gaussian) theory. By Wick's theorem,

$$G_4(x_1, x_2, x_3, x_4) = \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) + ..., \tag{1}$$

where D_F is the Feynman propagator. For simplicity sake, we shall only consider this first term.

Diagrams (position space). We already know that the diagram for this first term is,

$$D_F(x_1 - x_2)D_F(x_3 - x_4) \; =$$



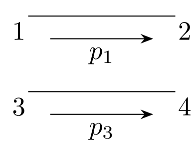
Momentum space. Defining the Fourier transform (with a convention compatible with the one used above),

$$\tilde{\phi}(p) = \int d^4x \, e^{-ip\cdot x}\phi(x), \qquad D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}, \tag{2}$$

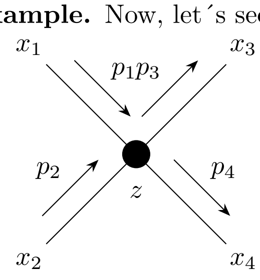
We can thus rewrite the expression in momentum space by making a choice on momentum orientation as follows,

$$\boxed{\tilde{G}_4(p_1, p_2, p_3, p_4) = (2\pi)^4\delta^{(4)}(p_1 + p_2)\,(2\pi)^4\delta^{(4)}(p_3 + p_4)\,\frac{i}{p_1^2 - m^2 + i\epsilon}\,\frac{i}{p_3^2 - m^2 + i\epsilon}}. \tag{3}$$

Diagrams (momentum space). The corresponding diagram is,



Example. Now, let´s see what happens for the case of a vertex. Take the following diagram,



$$G_4(x_1, x_2, x_3, x_4) = \int d^4z \, D_F(x_1 - z)D_F(x_2 - z)D_F(x_3 - z)D_F(x_4 - z).$$

$$G_4(x_1,x_2,x_3,x_4) = \int d^4z \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_4}{(2\pi)^4} \frac{e^{\mathrm{i}p_1\cdot(x_1-z)}}{p_1^2-m^2+i\epsilon} \cdots \frac{e^{-\mathrm{i}p_4\cdot(x_4-z)}}{p_4^2-m^2+i\epsilon}.$$

$$= \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_4}{(2\pi)^4} \tilde{G}_4(p_1,p_2,p_3,p_4)\, ie^{\mathrm{i}p_1\cdot x_1} \cdots ie^{\mathrm{i}p_4\cdot x_4}.$$

$$\boxed{\therefore \quad \tilde{G}_4(p_1,p_2,p_3,p_4) = D_F(p_1)\cdots D_F(p_4)\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4)}.$$

Where $D_F(p) = \frac{i}{p^2-m^2+i\epsilon}$. We thus see that, in order to find the diagram for momentum space, we do a Fourier transform.

6.5 Calculating diagrams via partition function

As we know, if we are dealing with scalar fields, we may get the partition function by,

$$Z = \int D\phi\; e^{-S_E[\phi]}.$$

To find the n-point correlation function,

$$G_n(x_1,...,x_n) = \left.\frac{\delta^n Z}{i^n \delta J(x_1)...J(x_n)}\right|_{J=0},$$

$$Z[J] = \int D\phi\; e^{-(S[\phi]-\int dt J(t)\phi(t))}.$$

Having perturbations, $S[\phi] = S_0[\phi] + S_I[\phi]$, we find that, by applying the definition of mean value,

$$\boxed{Z[J] = \int D\phi\; e^{-S_0[\phi]+\int dt J(t)\phi(t) - S_I[\phi]} = \langle 0|_J\, e^{-S_I[\phi]}\,|0\rangle_J\,.}$$

The above expression is usually called Dyson formula after british physicist Freeman Dyson [1923-2020]. If we are to write the interacting term as an integral, $S_I[\phi] = \int d^4x V(\phi(x))$, considering that $V(\phi)$ can be written as a sum of powers of fields, by using the identity,

$$\langle 0|_J\, \phi^{n_1}(x_1)...\phi^{n_r}(x_r)\,|0\rangle_J = \left(\frac{\delta}{\delta J(x_1)}\right)^{n_1} ... \left(\frac{\delta}{\delta J(x_r)}\right)^{n_r} \int D\phi\; e^{-S_0(\phi)+J\cdot\phi} = \left(\frac{\delta}{\delta J(x_1)}\right)^{n_1} ... \left(\frac{\delta}{\delta J(x_r)}\right)^{n_r} Z_0[J],$$

we get,

$$\boxed{\langle 0|_J\, e^{-\int d^4x V(\phi(x))}\,|0\rangle_J = e^{-\int d^4x V\left(\frac{\delta}{\delta J(x)}\right)} Z_0[J]}.$$

Where the exponential on the right side is now playing the role of an operator. Now, we write Feynman prescription and its inverse,

$$\Delta(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2-m^2+i\epsilon} e^{-ip(x-y)},$$

$$\Delta^{-1} = \Box^2 - m^2.$$

We can write the Klein-Gordon action as $S_0 = \frac{-1}{2}\int d^4x \left[\phi\Delta^{-1}\phi\right]$ (remember that we are dealing in Euclidean space, and we changed the convention used in the first section by adopting this $\frac{1}{2}$ term). By the following identity,

$$(\phi - J\Delta)\Delta^{-1}(\phi - \Delta J) = \phi\Delta^{-1}\phi - 2J\phi + J\Delta J,$$

we get,

$$Z_0[J] = \int D\phi \ \exp\left[\int d^4x \left(\frac{-1}{2}(\phi - J\Delta)\Delta^{-1}(\phi - \Delta J) + \frac{1}{2}J\Delta J\right)\right].$$

We can make the variable change from ϕ to $\phi' = \phi - J\Delta$. With this, we have,

$$Z_0[J] = e^{\frac{1}{2}J\Delta J}\int D\phi \exp\left[\int d^4x \ \frac{-1}{2}(\phi')\Delta^{-1}(\phi')\right] = e^{\frac{1}{2}J\Delta J}\left.\langle 0\right|_0 \left|0\right\rangle_0 = e^{\frac{1}{2}J\Delta J}.$$

Going back to the expression previously boxed, we get,

$$\boxed{Z[J] = e^{-\int d^4x V\left(\frac{\delta}{\delta J(x)}\right)}e^{\frac{1}{2}J\Delta J}}.$$

This is our gold nugget. With the above expression, we can calculate the correlation function until a certain order of perturbation based on the series of the potential V. Wick theorem is now trivially seen when taking derivatives of J term. Morever, when we are dealing with an odd number of derivatives, there will always be a J term left, which will make the correlation function be zero.

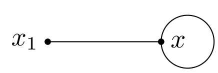
Example. Let $V(\phi) = \frac{\lambda}{3!}\phi^3$.
We expand the generating functional to first order in λ,

$$Z[J] = Z_0[J] - \int d^4y\, V\left(\frac{\delta}{\delta J(y)}\right) Z_0[J] + \mathcal{O}(\lambda^2) = Z_0[J] - \frac{\lambda}{3!}\int d^4y \left(\frac{\delta}{\delta J(y)}\right)^3 Z_0[J] + \mathcal{O}(\lambda^2).$$

Since we are interested solely in the first perturbed term, we calculate the 1-generating function,

$$G^{(1)}(x_1) = \left.\frac{\delta Z_1}{\delta J(x_1)}\right|_{J=0} = -\frac{\lambda}{2}\int d^4x \ \Delta(x_1, x)\Delta(x,x).$$

The corresponding Feynman diagram is,



Note that that, since there is a closed loop, we have a symmetry factor of 2.

Example. For $V(\phi) = \frac{\lambda\phi^4}{4!}$,

$$Z[J] = e^{-\int d^4x \frac{\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4} \exp\left[\frac{1}{2}J\cdot\Delta\cdot J\right].$$

Up to 2^{nd} order in λ,

$$\begin{cases} Z[J] = \exp\left[\frac{1}{2}J\cdot\Delta\cdot J\right] - \int d^4x \frac{\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4 e^{\frac{1}{2}J\cdot\Delta\cdot J} \\ \qquad + \frac{\lambda^2}{2(4!)^2}\int d^4x \left(\frac{\delta}{\delta J(x)}\right)^4 \int d^4y \left(\frac{\delta}{\delta J(y)}\right)^4 \exp\left[\frac{1}{2}J\cdot\Delta\cdot J\right]. \end{cases}$$

$$\begin{cases} G^{(2)}(x_1,x_2) = \Delta(x_1,x_2) - \lambda\frac{1}{4!}\int d^4x \frac{\delta^2}{\delta J_1 \delta J_2}\left(\frac{\delta}{\delta J}\right)^4 e^{\frac{1}{2}J\cdot\Delta\cdot J} \\ \qquad + \frac{\lambda^2}{2(4!)^2}\int d^4x d^4y \frac{\delta^2}{\delta J_1\delta J_2}\left(\frac{\delta}{\delta J(x)}\right)^4 \left(\frac{\delta}{\delta J(y)}\right)^4 e^{\frac{1}{2}J\cdot\Delta\cdot J}. \end{cases}$$

Calculating the derivatives for the term λ,

$$\frac{\delta^2}{\delta J_1\delta J_2}\left(\frac{\delta}{\delta J}\right)^4 e^{\frac{1}{2}J\cdot\Delta\cdot J} = \frac{\delta^2}{\delta J_1 \delta J_2}\left[6\Delta(x,x)(\Delta\cdot J)^2 + 3(\Delta(x,x))^2 + (\Delta\cdot J)^4\right]\exp\left[\frac{1}{2}J\cdot\Delta\cdot J\right]$$

$$= \frac{\delta}{\delta J_2}\left[12\Delta(x,x)\Delta(x,x_1)(\Delta\cdot J) + 12(\Delta\cdot J)^3\Delta(x,x_1) + \dots\right] =$$

$$= N_1\Delta(x,x)\Delta(x_1,x)\Delta(x,x_2)$$

(Surviving connected term for $J=0$)

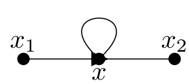
Likewise, for λ^2 terms,

$$\frac{\delta^2}{\delta J_1\delta J_2}\left(\frac{\delta}{\delta J(y)}\right)^4\left(\frac{\delta}{\delta J(x)}\right)^4 e^{\frac{1}{2}J\cdot\Delta\cdot J} =$$

$= N_2\Delta^3(x,y)\Delta(x_1,x)\Delta(x_2,y) + N_3\Delta^2(x,y)\Delta(x,x)\Delta(y,y) + N_4\Delta(x,y)\Delta(x,x)\Delta(y,y)\Delta(x,x_1)\Delta(x_2,y)$

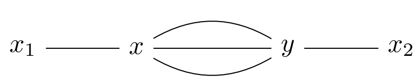
By drawing the diagram for each case,

1. Order λ: $\Delta(x,x)\Delta(x_1,x)\Delta(x,x_2)$



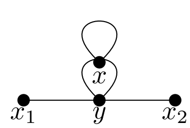
$\Rightarrow S(G) = 2$

2. Order λ^2 (Case 1 - Nut): $\Delta^3(x,y)\Delta(x_1,x)\Delta(x_2,y)$



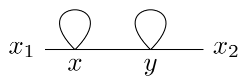
$\Rightarrow S(G) = 3! = 6$

3. Order λ^2 (Case 2 - Snowman/Cactus): $\Delta^2(x,y)\Delta(x_1,y)\Delta(x_2,y)\Delta(x,x)$



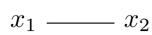
$\Rightarrow S(G) = 4$

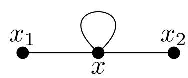
4. Order λ^2 (Case 3 - Camel) $\Delta(x,y)\Delta(x,x)\Delta(y,y)\Delta(x,x_1)\Delta(x_2,y)$

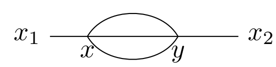


$\Rightarrow S(G) = 4$

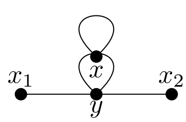
We have,

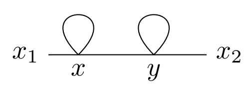
$G^{(2)}(x_1,x_2) =$ 

$+ \frac{\lambda}{2}\Bigg[$ 

$\Bigg] + \frac{\lambda^2}{6}\Bigg[$ 

$\Bigg] +$

$+ \frac{\lambda^2}{4}\Bigg[$ 

$\Bigg] + \frac{\lambda^2}{4}\Bigg[$ 

$\Bigg]$

Note that this is a way easier way to express generating functions. Moreover, we don´t even have to keep track of the multiplicities of each expression appearing. We just need to calculate the terms of each propagator multiplying to each other, and we may calculate the multiplicities by looking at the symmetry factor. That´s a wonderful tool.

The corresponding expression for momentum space is,

$$\tilde{G}^{(2)}(p_1-p_2) = (2\pi)^4 D_F(p_1)D_F(p_2)\delta^{(4)}(p_1-p_2) + \frac{\lambda}{2}\int d^4p'\,(2\pi)^4 D_F(p')\delta^{(4)}(p_1-p')\delta^{(4)}(p'-p_2)D_F(p_1)D_F(p_2)+$$

$$+\left[\tfrac{\lambda^2}{6}\int d^4p'd^4p''d^4p'''\,\tfrac{D_F(p')D_F(p'')D_F(p''')}{(2\pi)^4}\delta^{(4)}\big(p_x-p'-p''-p'''\big)\delta^{(4)}\big(p_y+p'+p''+p'''-p_a\big)D_F(p_1)D_F(p_2)\right]+$$

$$+\left[\tfrac{\lambda^2}{4}\int d^4p'd^4p''d^4p'''D_F(p')D_F(p'')D_F(p''')\delta^{(4)}\big(p_1-p'\big)\delta^{(4)}\big(p'-p''-p'''\big)\delta^{(4)}\big(p_2-p''-p'''\big)D_F(p_1)D_F(p_2)\right]+$$

$$+\Big[\tfrac{\lambda^2}{4}\int d^4p'd^4p''d^4p'''(2\pi)^4D_F(p')D_F(p'')D_F(p''')\delta^{(4)}\big(p_1-p'\big)\delta^{(4)}\big(p'-p''\big)\delta^{(4)}\big(p''-p'''\big)\delta^{(4)}\big(p'''-p_2\big)$$
$$\mathrm{D}_F(p_1)D_F(p_2)\Big].$$

6.6 S-Matrix and Optical Theorem

Let´s now find ways to calculate other quantities related to interactions. If one has an initial state $|\phi_1...\rangle_{in}$ and a final one $|\phi_1'...\rangle_{out}$, we can compute them in later times,

$$\langle\phi_1'...|_{out}\,|\phi_1...\rangle_{in} = \lim_{t\to\infty}\langle\phi_1'...|\,e^{-2iHt}\,|\phi_1...\rangle\,.$$

More generally, we can describe this intermediate operator as the **S-Matrix**, i.e,

Definition. Given a set of outgoing and incoming particles, the S-Matrix is an unitary operator such that,

$$\langle\phi_1'...|_{out}\,|\phi_1...\rangle_{in} = \langle\phi_1'...|\,S\,|\phi_1...\rangle\,.$$

We separate the identity to the interaction part,

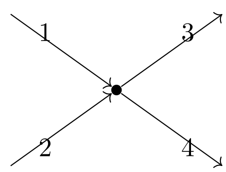
$$S = 1 + iT.$$

Thus, for conservation of momentum, we note that we ought to write,

$$\boxed{\langle p_1...|\,iT\,|p_1'...\rangle = (2\pi)^4\delta^{(4)}\left(\sum_i p_i' - \sum_i p_i\right)i\mathcal{M}(p_i \to p_i')}.$$

Where $\mathcal{M}$ is the so called *invariant matrix element.*

Example. For the ϕ^4 interaction, consider the tree-level $2\to 2$ process:



$$\langle p_1;p_2\mid iT\mid p_3;p_4\rangle = \int\frac{d^4y}{(2\pi)^4}\,(i\lambda)^1(-i)^1\,\delta^{(4)}(p_1+p_2-p_3-p_4)\,.$$

So,

$$\langle p_1;p_2\mid iT\mid p_3;p_4\rangle = i(\lambda)^1(-i)^1\,\delta^{(4)}\left(\sum p - \sum p'\right),$$

and therefore

$$\boxed{\mathcal{M} = -i\lambda}.$$

The unitarity of S allow us to write a very beautiful result,

Theorem. *(Generalized Optical theorem)*

$$\mathcal{M}(i\to f) - \mathcal{M}^*(f\to i) = i\sum_n\int d\Pi_n(2\pi)^4\delta^4\left(\sum_i p_i - \sum_n p_n\right)\mathcal{M}(i\to n)\mathcal{M}^*(f\to n)$$

Where $d\Pi_n$ is the measure for the basis completion, i.e, $\sum_n\int d\Pi_n\,|n\rangle\langle n| = \hat{1}$.

Proof. We write the S matrix as

$$S = \hat{1} + iT.$$

Since S is unitary,

$$S^\dagger S = \hat{1} \Rightarrow (\hat{1} - iT^\dagger)(\hat{1} + iT) = \hat{1},$$

which implies

$$i(T - T^\dagger) + T^\dagger T = 0.$$

Therefore,

$$T - T^\dagger = iT^\dagger T.$$

Taking matrix elements between $\langle f|$ and $|i\rangle$, and using the completeness relation for the intermediate states, we get

$$\langle f| (T - T^\dagger) |i\rangle = i \langle f| T^\dagger T |i\rangle = i \sum_m \int d\Pi_m \langle f| T^\dagger |m\rangle \langle m| T |i\rangle .$$

Now, by definition,

$$\langle f| T |i\rangle = (2\pi)^4 \delta^4\Big(\sum_i p_i - \sum_f p_f \Big) \mathcal{M}(i \to f),$$

so the left-hand side becomes

$$(2\pi)^4 \delta^4\Big(\sum_i p_i - \sum_f p_f \Big) \big(\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i)\big).$$

For the right-hand side, we have

$$i \sum_m \int d\Pi_m (2\pi)^4 \delta^4\Big(\sum_f p_f - \sum_m p_m \Big) \mathcal{M}^*(f \to m) (2\pi)^4 \delta^4\Big(\sum_i p_i - \sum_m p_m \Big) \mathcal{M}(i \to m).$$

Using the delta functions to factor out the total conservation law, this is

$$i(2\pi)^4 \delta^4\Big(\sum_i p_i - \sum_f p_f \Big) \sum_m \int d\Pi_m (2\pi)^4 \delta^4\Big(\sum_i p_i - \sum_m p_m \Big) \mathcal{M}(i \to m) \mathcal{M}^*(f \to m).$$

Finally, cancelling the common factor $(2\pi)^4 \delta^4\big(\sum_i p_i - \sum_f p_f \big)$ gives

$$\boxed{\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i) = i \sum_m \int d\Pi_m (2\pi)^4 \delta^4\Big(\sum_i p_i - \sum_m p_m \Big) \mathcal{M}(i \to m) \mathcal{M}^*(f \to m)}.$$

□

Note that we could have done with $SS^\dagger = \hat{1}$, which would imply as corollary that $\mathcal{M}(i \to m)\mathcal{M}^*(f \to m) = \mathcal{M}^*(m \to i)\mathcal{M}(m \to f)$.

As a particular case of the above theorem, if the final and initial states are the same, we get,

$$2Im(\mathcal{M}(A \to A)) = m_a \sum_n \Gamma(A \to n) = m_A \Gamma_{tot}.$$

Where $\Gamma(A \to n) = \frac{1}{2m_A} \int d\Pi_n (2\pi)^4 \delta^4 \Big(p_A - \sum_n p_n \Big) |\mathcal{M}(A \to n)|^2$ is the decay rate of a particle from state A to n. **This special case is called Optical Theorem**.

7 Quantization of fermionic fields

Now, we shall be seeing a way better way to deal with fermionic operators than what was seen in section 3. Since fermionic operators anticommutate, we must deal with these quantities.

7.1 Grassmann numbers

The Grassmann numbers (after Hermann Grassmann [1809-1877]), are variables that anticommute. One could make the case for a definition,

Definition. $(A, ., +)$ is an algebra such that, $\forall \theta, \eta \in A$,

$$\{\theta, \eta\} = \theta\eta + \eta\theta = 0,$$

which, of course, implies $\theta^2 = 0$.

The elements of such algebra are what we shall call Grassmann numbers. We can also introduce the derivative operators, $\{\frac{\partial}{\partial\theta}\}$ such that ,

$$\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta_{ij},$$

$$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0.$$

For a general field,

$$f(\{\theta_i\}) = a_0 + a_1\theta_1 + a_2\theta_1\theta_2 + ...,$$

we must specify if the derivative is acting by the right or by the left, cause, $(\frac{\partial^L f(\{\theta_i\})}{\partial\theta_1} = a_1 + a_2\theta_2 + ...)$ and $(\frac{\partial^R f(\{\theta_i\})}{\partial\theta_1} = a_1 - a_2\theta_2 + ...)$. We shall explicitate only right derivatives. We also define the Berenzian integral (Felix Berezin[1931-1980]) by,

$$\int d\theta_1 ... d\theta_n \; f(\{\theta_i\}) = \frac{\partial}{\partial\theta_1} ... \frac{\partial}{\partial\theta_n} f(...)\bigg|_{\theta_1,...,\theta_n=0}.$$

This definition leads us to,

$$\int d\theta_1 d\theta_2 \; e^{-\theta_1 b\theta_2} = b.$$

Which, for a diagonalizable matrix, this implies,

$$\int d\theta_j d\theta^j \; e^{-\theta_j A\theta^j} = \det(A).$$

This expression is important for us.

As we saw in section 3, fermionic fields anticommute, so they can be written as a sum of Grassmann numbers in a basis of usual functions,

$$\psi(x) = \sum_n \psi_i\phi_i(x).$$

So, we have

$$\int D\bar{\psi} D\psi \; e^{-\bar{\psi}A\psi} = \det(A).$$

7.2 Fermionic harmonical oscillator

We now deal with the hamiltonian,

$$H = \omega\big(bb^\dagger - \frac{1}{2}\big),$$

with $\{b, b^\dagger\} = 1$. Similarly to what we did in the bosonic case, we define the Fock space,

$$|\beta\rangle = e^{b^\dagger}\beta\,|0\rangle \Longrightarrow b\,|\beta\rangle = \beta\,|\beta\rangle\,,$$

and a completeness relation, $\int d\beta^* d\beta\,|\beta\rangle\,\langle\beta|\,e^{-\beta^*\beta} = 1$. For the forced case, we get,

$$H(b^\dagger, b, t) = \omega b^\dagger b - b^\dagger\eta(t) - \bar{\eta(t)}b,$$

thus finding the amplitude,

$$F(\beta', t', \beta, t) = \langle\beta', t'|\,|\beta, t\rangle = \int D\beta D\bar{\beta}\exp\left(i\int_t^{t'} d\tau\;[-i\dot{\bar{\beta}}(\tau)\beta(\tau) - H(b^\dagger, b, \tau)] + \bar{\beta}(t)\beta(t)\right).$$

And the equations of motion,

$$\dot{\bar{\beta}} - i(\omega\bar{\beta} - \eta) = 0,$$
$$\dot{\beta} + i(\omega\beta - \bar{\eta}) = 0.$$

This leads to the solutions,

$$\boxed{\beta(\tau) = \beta(t)e^{i\omega(t-\tau)} + i\int_t^\tau ds\; e^{i\omega(s-\tau)}\eta(s)},$$

and the respective conjugate. Notice that, once again, we have the classical solution plus a perturbed term.

7.3 Partition function

We now introduce the Green´s function for Dirac fields,

$$S_F(x, y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)},$$

note that, if $S_F(x, y)^{-1} = -i(\not{p} - m)$, we get $S_F(x, y)S_F(x, y)^{-1} = \delta^{(3)}(x - y)$ (taken at equal times). Then by the same kind of procedure we did in the bosonic case, we get,

$$Z[\eta, \bar{\eta}] = Z[0, 0]\exp\left[-\int d^4x d^4y\;\bar{\eta}(x)S_F(x, y)\eta(y)\right].$$

So, if we have a perturbative action, S_I, we can do the same kind of procedure we did for the bosonic case. For instance, if we consider a fermion interaction with a scalar bosonic field, we have,

$$\boxed{Z[\eta, \bar{\eta}, J] = e^{-S_I[-\frac{\partial}{\partial\eta}, \frac{\partial}{\partial\bar{\eta}}, \frac{\partial}{\partial J}]}Z_F[\bar{\eta}, \eta]Z_\phi[J]}.$$

The minus sign in the $\frac{\partial}{\partial\eta}$ term is due to the fact that the alteration in the action by adding η term is $\bar{\eta}\psi + \bar{\psi}\eta$, so we se that we have to change the sign due to the order of the product. Next, we prove the result due to Sidney Coleman [1937-2007].

Lemma. *(Coleman)*

$$F\left(\frac{\partial}{\partial x}\right)G(x) = G\left(\frac{\partial}{\partial y}\right)\left[F(y)e^{xy}\right]\Big|_{y=0}.$$

Proof. We assume the functions are integratable by parts and we write them in Fourier modes, $F(x) = e^{\alpha x}$, $G(x) = e^{\beta x}$. Thus,

$$F(\frac{\partial}{\partial x})G(x) = \sum_n \frac{\alpha^n}{n!}\frac{\partial^n}{\partial x^n}e^{\beta x} = \sum_n \frac{\alpha^n\beta^n}{n!}e^{\beta x} = e^{\alpha\beta}e^{\beta x}.$$

$$G(\frac{\partial}{\partial y})\left[F(y)e^{xy}\right] = \sum_n \frac{\beta^n}{n!}\frac{\partial^n}{\partial y^n}\left[e^{\alpha y}e^{xy}\right] = e^{(\beta+y)(\alpha+x)}.$$

Putting $y = 0$, we find that ,

$$F(\frac{\partial}{\partial x})G(x) = G(\frac{\partial}{\partial y})\left[F(y)e^{xy}\right]\Big|_{y=0}.$$

□

This allows us to write, from the boxed expression,

$$\boxed{Z[\eta,\bar{\eta},J] = e^{-S_I[-\frac{\partial}{\partial\eta},\frac{\partial}{\partial\bar{\eta}},\frac{\partial}{\partial J}]}Z_F[\bar{\eta},\eta]Z_\phi[J] = Z_F[-\frac{\partial}{\partial\bar{\psi}},\frac{\partial}{\partial\psi}]Z_\phi[\frac{\partial}{\partial\phi}]\left(e^{-S_I(\bar{\psi},\psi,\phi)+\bar{\psi}\eta+\bar{\eta}\psi+J\phi}\right)\Big|_{\psi=\bar{\psi}=\phi=0}}.$$

This is the fermionic version of our previous "gold nugget". When calculating correlation functions, we will use the convention $G^N_{(f,b)}$, where N denotes the degree of perturbation, f and b how many fermions and bosons we are considering.

Ex) $$G^{(0)}_{(2,0)}(x,y) = \frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\bar{\eta}(y)}Z^{(0)}[\eta,\bar{\eta}] = \frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar{\eta}}\left(e^{-\frac{\delta}{\delta\bar{\psi}}S_F\frac{\delta}{\delta\psi}+\bar{\psi}\eta+\bar{\eta}\psi}\right)\Big|_{\psi=\bar{\psi}=0} = S_F(x,y) = \underset{x}{\bullet}\!\longrightarrow\!\underset{y}{\bullet}.$$

Notice that the arrow pointing from x to y becomes important, since the order here matters!!!

7.4 Yukawa theory

Let´s now consider a simple model for interaction of scalar fields with fermions (Yukawa interaction, due to japanese physicist Hideki Yukawa [1907-1981]). We take the lagrangean,

$$\boxed{\mathcal{L}_Y = g\bar{\psi}\psi\phi}.$$

Let´s now apply it,

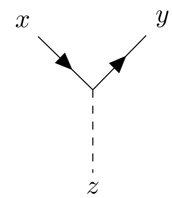
$$G^{(1)}_{(2,1)}(x,y,z) = \frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\bar{\eta}(y)}\frac{\delta}{\delta J(z)}Z^{(1)}[\eta,\bar{\eta},J]$$

$$\Rightarrow\ N\frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\bar{\eta}(y)}\frac{\delta}{\delta J(z)}\left(\frac{\delta}{\delta\bar{\psi}}S_F\frac{\delta}{\delta\psi}\right)\left(\frac{\delta}{\delta\bar{\psi}}S_F\frac{\delta}{\delta\psi}\right)(\psi\bar{\psi}\ \Delta\ Je^{\bar{\psi}\eta+\bar{\eta}\psi+J\phi})\Big|_{\bar{\eta}=\eta=J=0}$$

This leads to ,

$$G^{(1)}_{(2,1)}(x,y,z) = -Ng \int d^4w \Delta(z-w) S_F(x-w) S_F(w-y).$$

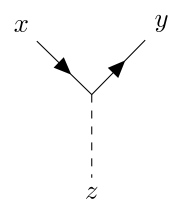
We have the diagram,



Which leads to the symmetry factor 1, so,

$$G^{(1)}_{(2,1)}(x,y,z) = -g \int d^4w \Delta(z-w) S_F(x-w) S_F(w-y).$$

This is the simplest non-trivial diagram for fermion interacting with boson. In fact, we can name the diagram after the constant g,



$= -g$

This fact, as well the the one dealed at the end of last subsection, **describes two more Feynman rules**.

7.5 Extra Feynman rules for fermion and boson propagator

From the previous discussion, we summarize the basic rules as follows:

1) Scalar (boson) propagator in momentum space:

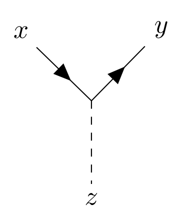
$$\langle 0| T\{\phi(x)\phi(y)\} |0\rangle \longrightarrow \frac{i}{p^2 - m^2 + i\epsilon}.$$

2) Fermion propagator in momentum space:

$$\langle 0| T\{\psi(x)\bar{\psi}(y)\} |0\rangle \longrightarrow \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}.$$

(Sign matters.)

3) Yukawa vertex factor:

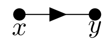


$= -g$

(In Euclidean space, sign conventions may differ.)

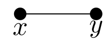
4) Coordinate-space expressions:

$$S_F(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)e^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon} =$$



,

$$\Delta(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{i\,e^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon} =$$



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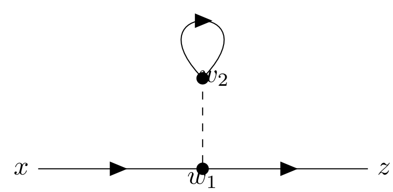
8 Some extra Feynman Diagrams

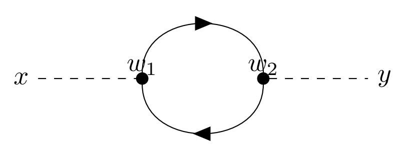
We´ve seen some quite important Feynman diagrams. In order to apply Feynman rules for fermionic and bosonic fields, we shall show some examples.

a) Fermion interacting with a buble through a boson:



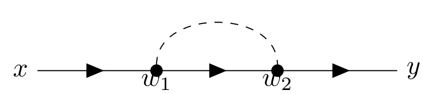
$$G^{(2)}_{(2,0)}(x,z) = -g^2 \int d^4w_1\, d^4w_2\, S_F(w_1-x)S_F(z-w_1)\Delta(w_1,w_2)S_F(w_2-w_2).$$

b) Boson with a fermion loop:



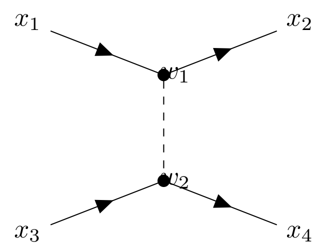
$$G^{(2)}_{(0,2)}(x,y) = g^2 \int d^4w_1\, d^4w_2\, \Delta(x,w_1)S_F(w_2-w_1)S_F(w_1-w_2)\Delta(w_2,y).$$

c) Fermion self interacting through a boson:



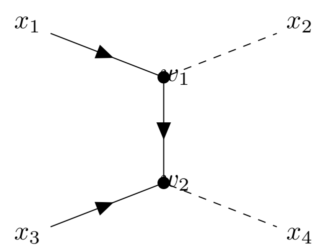
$$G^{(2)}_{(2,0)}(x,y) = -g^2 \int d^4w_1\, d^4w_2\, S_F(w_1-x)\Delta(w_1,w_2)S_F(w_2-w_1)S_F(y-w_2).$$

d) Four-fermion exchange via a boson:



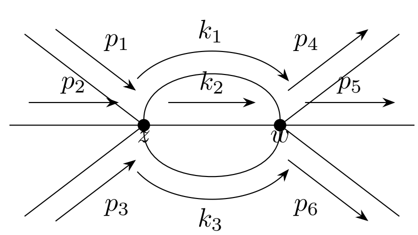
$$G^{(2)}_{(4,0)}(x_1, x_2, x_3, x_4) = g^2 \int d^4w_1\, d^4w_2\, S_F(w_1 - x_1)S_F(x_2 - w_1) \times \Delta(w_1, w_2)S_F(w_2 - x_3)S_F(x_4 - w_2).$$

e) Two-fermion, two-bosons diagram:



$$G^{(2)}_{(2,2)}(x_1, x_2, x_3, x_4) = g^2 \int d^4w_1\, d^4w_2\, S_F(w_1 - x_1)\Delta(w_1, x_2) \times S_F(w_2 - w_1)\Delta(w_2, x_4)S_F(x_3 - w_2).$$

f) Six-point diagram with three internal propagators (In momentum space):



$$\widetilde{G}_6(p_1, \ldots, p_6) = \frac{\lambda^2}{3!} \int \frac{d^4k_1\, d^4k_2}{(2\pi)^4} \left[\prod_{i=1}^{6} D_F(p_i)\right] D_F(k_1)D_F(k_2) \times D_F(p_1 + p_2 + p_3 - k_1 - k_2)\delta^{(4)}(p_1 + p_2 + p_3 - p_4 - p_5 - p_6).$$

9 QED - Quantum Electrodynamics

In this section, we finally arrive at what is until now the most precise quantitative scientific theory ever created, QED, labeled by Richard Feynman as the "Jewel of Physics" due to its incredible sucess into describing nature. Before attacking it, we shall talk a bit about one of its preliminaries, Gauge theory.

9.1 Gauge Theory and Yang-Mills

As we know, in Electrodynamics, $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$. We can make the following change in the parameters, $\delta\vec{A} = \vec{\nabla}\Lambda$ and $\delta\phi = -\frac{\partial\Lambda}{\partial t}$ and the equations for B and E remain exactly the same, and this symmetry is local, i.e, the factor of change of the fields is dependent on the position. That is the simplest non trivial example of a gauge symmetry.

In the Klein-Gordon case, we noticed that the theory had a global $U(1)$ symmetry. Suppose we take a local $U(1)$ transformation, $\phi'(x) = e^{i\varphi(x)}\phi(x)$. The lagrangean changes by,

$$\delta\mathcal{L} = i(\partial_u\varphi\phi(\partial^u\phi)^\dagger + \partial^u\phi(\partial_u\varphi\phi)^\dagger) - \partial_u\varphi\phi(\partial_u\varphi\phi)^\dagger.$$

As we see, it is not invariant. However, we can make it by altering the derivative $\partial_u \to D_u = \partial_u - iA_u$. Then, we see that the lagrangean is invariant if $(D_u\phi)'((D_u\phi)')^\dagger = (D_u\phi)((D_u\phi))^\dagger$. This is the case for $\boxed{(D_u\phi)' = e^{i\varphi}D_u\phi}$. In other words, we are adding an extra field (which is called Yang-Mills field) to the derivative such that, under this new operator, the change in the field becomes local (or the operator changes covariantly).

Thus,

$$D'_u(e^{i\varphi(x)}\phi(x)) = i\partial_u\varphi e^{i\varphi(x)}\phi(x) + e^{i\varphi(x)}\partial_u\phi(x) - iA'_u e^{i\varphi(x)}\phi(x) = e^{i\varphi(x)}(\partial_u\phi(x) - iA_u\phi(x)).$$

We can see that this implies,

$$A'_u = A_u + \partial_u\varphi(x),$$

which is the same gauge transformation for the case of electromagnetism. By the same kind of procedure, if we were to make a local $SU(N)$ transformation in a field, $\phi'(x) = w(x)\phi(x)$, then, by making use of the same covariant derivative, $D_u = \partial_u - iA_u$, and forcing the condition $\boxed{(D_u\phi)' = w(x)D_u\phi}$, we get,

$$A'_u = w(x)A_uw^{-1}(x) - i\partial_u w(x)w^{-1}(x).$$

The extension of gauge invariance to include non-Abelian group symmetry is what is called Yang-Mills theory, developed in the 1950´s by chinese Yang-Chen Ning [1922-2025] and american Robert Mills [1927-1999].

9.2 The Jewel of Physics- Lagrangean of QED

Essentially, QED is a theory on interaction between photons and fermions locally invariant under $U(1)$ (thus, it is an Abelian Yang-Mills theory). Without external sources, the lagrangean of electromagnetism is,

$$\mathcal{L}_{EM}(A) = \frac{-1}{4}F_{uv}F^{uv},$$

and it is invaraint by local change in $U(1)$ as we showed. Now, let´s take Dirac lagrangean,

$$\mathcal{L}_D = \bar{\psi}(i\gamma^u \partial_u - m)\psi.$$

Again, to make it locally $U(1)$ invariant, we change ∂_u to $D_u = \partial_u - iA_u$, and by the same procedure as before, we get

$$A'_u = A_u + \partial_u \varphi(x).$$

Thus, we can construct a theory of photons and fermions locally invariant under $U(1)$ by adding the lagrangeans, and that´s QED theory defined on a Mikowski space.

$$\boxed{\mathcal{L}_{QED}(\bar{\psi}, \psi, A_u) = \frac{-1}{4} F_{uv}F^{uv} + \bar{\psi}(i\gamma^u \partial_u - m)\psi - J^u A_u}.$$

Where $J^u = i\bar{\psi}\gamma^u\psi$. The equations of motion are,

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = 0 \ \Rightarrow \ \boxed{\partial_v F^{vu} = -\, i\,\bar{\psi}\gamma^\mu \psi}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) = 0 \Rightarrow \boxed{\gamma^\mu \partial_\mu \psi - m\psi = \ i\,\gamma^\mu A_\mu \psi}$$

Where the last Euler-lagrange equation (in relation to ψ) yields the same result as the second one. We see that we now have an equation relating the divergence of electromagnetic tensor to an expression of gamma matrices with fermionic fields and also an equation describing the generalized version od Dirac equation. These results, when confronted against experiments to measure certain quantities relating interactions of electrons and light, have been responsible for an accuracy of order 10^{-16}, an extremely successful result in describing nature.

9.3 Quantizing QED and Faddev-Popov method

Now, we shall step further and quantize QED. Consider the partition function of field A alone,

$$Z = \int DA(x) e^{-S_{EM}[A]}.$$

We may write the action as,

$$S_{EM} = \frac{1}{2} \int d^4x \ (\partial_u A_v \partial^v A^u - \partial_u A_v \partial^u A^v) = \frac{1}{2} \int d^4x [A_v(\partial^2 g^{uv} - \partial^u \partial^v)] A_u.$$

We thus note that the propagator has to obey,

$$(\partial^2 g_{uv} - \partial_u \partial_v) D^{v\rho}(x, y) = i\delta_{u\rho}\delta^4(x, y).$$

Writing in momentum sape,

$$(k^2 g_{uv} - k_u k_v) D^{v\rho} = i\delta_{u\rho}.$$

Note that the operator on the left side is singular. The way to get around this problem is to just consider physical degress of freedom, i.e, disconsider fields that differ by $\partial_u \varphi$. We do this by adopting a measure of integration called **Fadeev-Popov measure**, after russian physicists Ludwig Fadeev [1934-2017] and Victor Popov [1937-1994].

Definition. By adopting the notation $A_u^\alpha = A_u + \partial_u \alpha$,we define the Fadeev-Popov measure,

$$\Delta_G^{-1}(A_u) = \int D\alpha \; \delta(G(A_u^\alpha)),$$

where $G(A) = \partial_u A^u$ and the δ denotes a Dirac delta.

Note that $\Delta_G^{-1}(A_u^{\alpha'}) = \int D\alpha \; \delta(G(A_u^{\alpha+\alpha'}))$, and, by changing coordinates to $\alpha'' = \alpha + \alpha'$, we get,

$$\Delta_G^{-1}(A_u^{\alpha'}) = \int D\alpha'' \; \delta(G(A_u^{\alpha''})) = \Delta_G^{-1}(A_u).$$

So, this measure is invariant under gauge transformations for the vector A.

Thus, we can put this in the partition function,

$$Z = \int DA \; \Delta_G(A)\Delta_G^{-1}(A)e^{-S_{EM}[A]} = \int DA \; \Delta_G(A) \int D\alpha \; \delta(G(A_u^\alpha))e^{-S_{EM}[A]}.$$

The second integral can be rewriten by changing again the measure, $D\alpha = DG \det(\frac{\delta\alpha}{\delta G})$, so we get,

$$\Delta_G^{-1}(A_u^\alpha) = \det\left(\frac{\delta\alpha}{\delta G}\right)\Big|_{G=0}.$$

So,

$$\Delta_G(A_u^\alpha) = \det\left(\frac{\delta G}{\delta\alpha}\right)\Big|_{G=0}.$$

Thus,

$$Z = \int DAD\alpha \det\left(\frac{\delta G}{\delta\alpha}\right)\Big|_{G=0} \delta(G(A_u^\alpha))e^{-S_{EM}[A^\alpha]}.$$

Since, as we have shown, this is invariant under Gauge transformations, we find that $\delta(G(A)) = \delta(\partial_u A^u + \partial_u\partial^u\alpha)$. By denoting $w(x) = \partial_u\partial^u\alpha(x)$, we may write,

$$Z = \int Dw \exp\left[\int d^4x(2\xi)^{-1}w^2\right]\int DAD\alpha(...).$$

This mean that the action can be shifted by $\int d^4x(2\xi)^{-1}(\partial_u A^u)^2$, thus, we may call,

$$S_{EM}[A] = \frac{1}{2}\int d^4x[A_v(\partial^2 g^{uv} - \partial^u\partial^v)]A_u - (\xi)^{-1}(\partial_u A^u)^2 = \frac{1}{2}\int d^4x A^v D_{Fuv}^{-1}A^u.$$

Where, now,we have,

$$(k^2 g_{uv} - (1-\xi^{-1})k_u k_v)D^{v\rho} = i\delta_\rho^u.$$

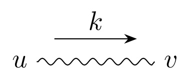
Note that this implies,

$$D_F^{uv}(k) = -\frac{4i}{k^2 + i\epsilon}\left(g^{uv} - (1-\xi)\frac{k^u k^v}{k^2}\right).$$

This is the Feynman propagator. Since all we did here is unnafected by the value we choose for the gauge ξ, we may, conveniently, choose Feynman gauge $\xi = 1$ and take this 4 out by renaming the inverse propagator. Thus, we find Feynman propagator in QED,

$$\boxed{D_F^{uv} = -i\left(\frac{g^{uv}}{k^2 + i\epsilon}\right)}.$$

Which represents the photon propagation. We represent this by the diagram,

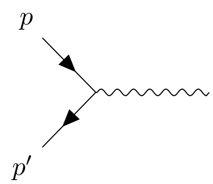
$D_F^{uv}(k) =$ 

$= -i\left(\dfrac{g^{uv}}{k^2 + i\epsilon}\right)$

We can also write a diagram representing an interaction of a fermionic field with light. For that, we use Yukawa theory,

$$H_I = \int d^3x\ e\bar{\psi}\gamma^u\psi A_u(x).$$

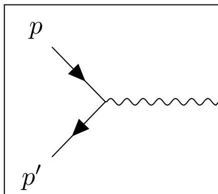
If we write the matrix element for the diagram, (we are using the fermion conventions adopted on section 3.7),



we find,

$$\langle p'|iT|p\rangle = \langle p'|T\{-i\int d^4xe\bar{\psi}\gamma^\mu\psi A_\mu\}|p\rangle\ = -ie\bar{u}(p')\gamma^\mu u(p)\int d^4xe^{-ip'x}e^{ipx}A_\mu(x)\ = -ie\bar{u}(p')\gamma^\mu u(p)A_\mu(p'-p).$$

So, we conclude,



$= -ie\gamma^u A_u(p' - p)$

With that we can write the Feynman rules for QED.

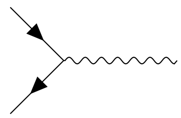
9.4 Feynman Rules for QED

The interaction Hamiltonian is given by Yukawa theory:

$$H_I = \int d^3x\, e\bar{\psi}\gamma^\mu\psi A_\mu \tag{4}$$

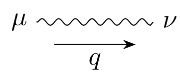
(Note: ψ represents the fermion fields.)

1. Vertex



$= -ie\gamma^{\mu} A_u$

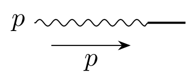
2. Photon Propagator



$$= \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$$

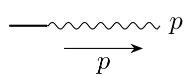
3. External Photon Lines

Incoming Photon:

$A_\mu|p\rangle =$ 

$\mu = \epsilon_\mu(p)$

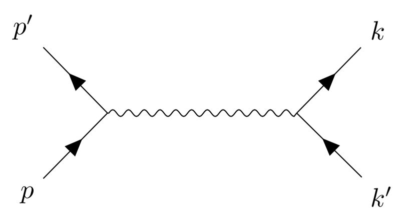
Outgoing Photon:

$\langle p|A_\mu =$ 

$\mu = \epsilon^*_\mu(p)$

Example. This is one of the most famous diagrams in QFT. The Bhabba scattering (named after Indian physicist Homi Jehangir Bhabba [1909-1966]) represents the scattering of electron-positron,

$$e^+e^- \to e^+e^-.$$



$$\Rightarrow i\mathcal{M} = (-ie)^2 \left[\bar{v}(p')\gamma^\mu u(p)\right] \frac{-ig_{\mu\nu}}{(p-p')^2} \left[\bar{v}(k')\gamma^\nu u(k)\right] \tag{5}$$

10 Effective action

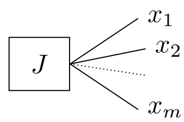
As we´ve seen in section 6, we are interested in connected diagrams. We rename the action in order

$$Z[J] = e^{-W[J]}.$$

In order for $W[J]$ to contain only connected diagrams, the correlation functions we get from it are

$$G_m(x_1, \dots, x_m) \equiv \frac{\delta^m Z[J]}{\delta J(x_1) \cdots \delta J(x_m)}.$$

Diagrammatically,

$G_m(x_1, \dots, x_m) =$  .

For the one-point function,

$$G_1(x) = -\frac{\delta W}{\delta J(x)} Z[J].$$

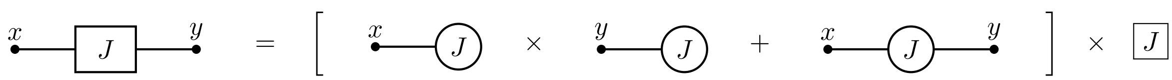
This can be represented as

$G_1(x) =$  .

For the two-point function,

$$G_2(x, y) = \left(\frac{\delta W}{\delta J(x)} \frac{\delta W}{\delta J(y)} - \frac{\delta^2 W}{\delta J(x) \delta J(y)} \right) Z[J].$$

Thus, it is the sum of a connected part and a disconnected part:

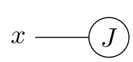
$G_2(x, y) =$ 

.

And we are only interested in the connected part (on the right of the sum operation). To get the classical field out of this formalism:

$$\phi_{\text{cl}}[J] = \frac{\langle \Omega | \hat{\phi}(x) | \Omega \rangle_J}{\langle \Omega | \Omega \rangle_J} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x)}.$$

Therefore,

$\phi_{\text{cl}}[J] = \dfrac{\delta\big(-W[J]\big)}{\delta J(x)} = G_c^1(x) =$  .

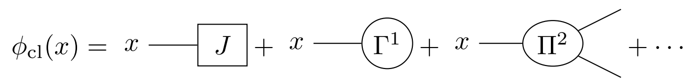
This is the connected one-leg Green function (this is where the effectiveness of the action will come from). At first, we will study how we can get the effective action of this G_c^1 for a theory with non zero potential.

10.1 Effective action for one-leg Green function with potential

For fields under a potential V:

$$W[J] = -\ln Z[J] = \int d^4x\, V\left(\frac{\delta}{\delta J}\right)\left(\frac{1}{2}J\Delta J\right).$$

The field has an unperturbed contribution and quantum corrections. So, we can write the field diagrammatically in terms of sum of irreducible diagrams:



Mathematically, we have:

$$(\Delta J)(x) = \int d^4y\, \Delta(x-y)J(y),$$

and irreducible legs insertion such as the first perturbed term,

$$\int d^4y\, \Delta(x-y)\Gamma^1(y).$$

So,

$$\boxed{\phi_{\rm cl}(x) = \int d^4y\, \Delta(x-y)\left[J(y) - \Gamma^1(y) - \int d^4z\Pi^2(y,z)\phi_{cl} + (...)\right]}.$$

So, we define the functional generator by the expression in brakets:

$$\widehat{\Gamma}[\phi_{\rm cl}] = \Gamma^1\phi_{\rm cl} + \frac{1}{2}\Pi^2\phi_{\rm cl}^2 + \frac{1}{3!}\Gamma^3\phi_{\rm cl}^3 + \cdots.$$

Which contains all the perturbations in the theory. Note:

$$\Gamma^m(x_1,\ldots,x_m) = \left.\frac{\delta^m\widehat{\Gamma}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x_1)\cdots\delta\phi_{\rm cl}(x_m)}\right|_{\phi_{\rm cl}=0}, \qquad m \neq 0.$$

Using

$$\Gamma^1(y) = \frac{\delta\widehat{\Gamma}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(y)},$$

we have

$$\phi_{\rm cl}(x) = \int d^4y\, \Delta(x-y)\left[J(y) - \frac{\delta\widehat{\Gamma}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(y)}\right].$$

Note:

$$\Delta^{-1}\phi_{\rm cl} = J - \frac{\delta\widehat{\Gamma}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}}.$$

Therefore,

$$J(x) = \frac{\delta}{\delta\phi_{\rm cl}(x)}\left[\widehat{\Gamma}[\phi_{\rm cl}] + \frac{1}{2}\phi_{\rm cl}\Delta^{-1}\phi_{\rm cl}\right].$$

Based on this, we define the effective action,

$$\Gamma_{\rm eff}[\phi_{\rm cl}] = \widehat{\Gamma}[\phi_{\rm cl}] + \frac{1}{2}\phi_{\rm cl}\Delta^{-1}\phi_{\rm cl}.$$

$$J(x) = \frac{\delta}{\delta\phi_{\rm cl}(x)}\Gamma_{\rm eff}[\phi_{\rm cl}].$$

So,

$$\Gamma^2_{\rm eff}(x,y) = \left.\frac{\delta^2\Gamma_{\rm eff}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x)\delta\phi_{\rm cl}(y)}\right|_{\phi_{\rm cl}=0} = \Pi^2(x,y) + \Delta^{-1}\delta(x-y).$$

While:

$$\Gamma^m(x_1,\ldots,x_m) = \left.\frac{\delta^m\widehat{\Gamma}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x_1)\cdots\delta\phi_{\rm cl}(x_m)}\right|_{\phi_{\rm cl}=0} = \left.\frac{\delta^m\Gamma_{\rm eff}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x_1)\cdots\delta\phi_{\rm cl}(x_m)}\right|_{\phi_{\rm cl}=0},$$

for $m \neq 2$.
So, in summary:

$$\boxed{\Gamma_{\rm effective}[\phi_{\rm cl}] = \widehat{\Gamma}[\phi_{\rm cl}] + \frac{1}{2}\phi_{\rm cl}\Delta^{-1}\phi_{\rm cl}}.$$

$$\widehat{\Gamma}[\phi_{\rm cl}] = \Gamma^1\phi_{\rm cl} + \frac{1}{2}\Pi^2\phi_{\rm cl}^2 + \frac{1}{3!}\Gamma^3\phi_{\rm cl}^3 + \cdots.$$

For $m \neq 2$,

$$\Gamma^m(x_1,\ldots,x_m) = \left.\frac{\delta^m\widehat{\Gamma}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x_1)\cdots\delta\phi_{\rm cl}(x_m)}\right|_{\phi_{\rm cl}=0} = \left.\frac{\delta^m\Gamma_{\rm eff}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x_1)\cdots\delta\phi_{\rm cl}(x_m)}\right|_{\phi_{\rm cl}=0}.$$

For $m = 2$,

$$\Gamma^2(x_1,x_2) = \left.\frac{\delta^2\Gamma_{\rm eff}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x_1)\delta\phi_{\rm cl}(x_2)}\right|_{\phi_{\rm cl}=0},$$

and

$$\Pi^2(x_1,x_2) = \left.\frac{\delta^2\widehat{\Gamma}[\phi_{\rm cl}]}{\delta\phi_{\rm cl}(x_1)\delta\phi_{\rm cl}(x_2)}\right|_{\phi_{\rm cl}=0} = \Gamma^2(x_1,x_2) - \Delta^{-1}\delta(x_1-x_2).$$

Example. Suppose we are dealing with a ϕ^3 theory. The effective action is,

$$\Gamma_{\text{eff}}[\phi_{\text{cl}}] = \frac{1}{2}\phi_{\text{cl},i}\Delta^{-1}_{ij}\phi_{\text{cl},j} + \sum_i \frac{\lambda}{3!}\phi_i^3.$$

Therefore,

$$\frac{\delta\Gamma_{\text{eff}}}{\delta\phi_{\text{cl},i}(x)} = J_i(x),$$

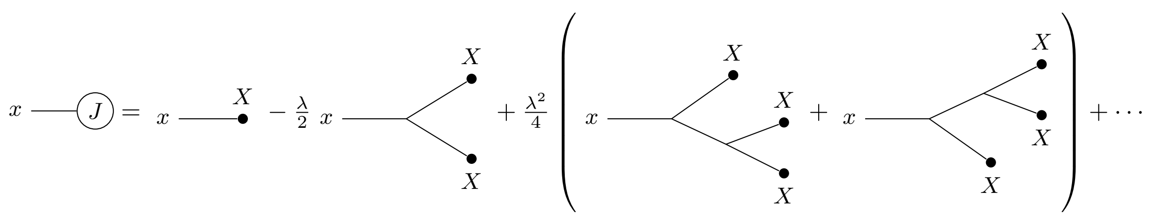
which implies

$$J_i(x) = \Delta^{-1}_{ij}\phi_{\text{cl},j} + \sum_i \frac{\lambda}{2}\phi_i^2.$$

Thus,

$$\phi_{\text{cl},j}(x) = \Delta_{ij}J_i(x) - \sum_i \frac{\lambda}{2}\Delta_{ij}\phi_i^2.$$

In diagrams terms,



This denotes a tree diagram (in which perturbations of ϕ are described in terms of the field itself).

10.2 For two legs connected diagrams

As we said before, we are interested only in the connected parts, so,

$$G_2^c(x,y) = -\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}W[J] = \frac{\delta\phi_{\text{cl}}(y)}{\delta J(x)}.$$

Since:

$$\phi_{\text{cl}}(x) = \int d^4y\,\Delta(x-y)\left[J(y) - \Gamma^1(y) - \int d^4z\Pi^2(y,z)\phi_{cl} + (...)\right],$$

we have

$$G_2^c(x,y)|_{J=0} = \Delta(y-x) - \int d^4z_1\,d^4z_2\,\Delta(y-z_1)\Pi^2(z_1,z_2)G_2^c(z_2,x)\Big|_{J=0}.$$

Since

$$G_2^c(x,y) = \int d^4z_2\,\delta(y-z_2)G_2^c(z_2,x),$$

then

$$\boxed{\int d^4 z_2 \left[\delta(y - z_2) + \int d^4 z_1 \, \Delta(y - z_1)\Pi^2(z_1, z_2)\right] G_2^c(z_2, x) = \Delta(y - x)}.$$

We may rewrite the equation above as:

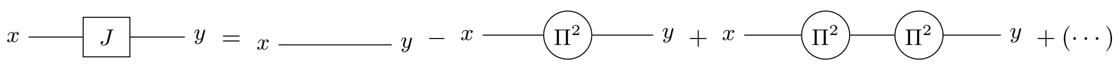
$$\left(1 + \Delta \cdot \Pi^2\right) G_2^c = \Delta,$$

$$G_2^c = \left(1 + \Delta \cdot \Pi^2\right)^{-1} \Delta.$$

Expanding:

$$G_2^c(x, y) = \Delta(x - y) - \Delta(x - z) \cdot \Pi^2(z, w)\Delta(w - y) + \Delta(x - z)\Pi^2(z, w)\Delta\Pi^2(w - k)\Delta(k - y) - (\cdots).$$

Thus, we can write:



And we can do the same procedure for higher order connected Green functions, with (much) more complicated algebra.

11 CPT symmetry and Dirac bilinears

We are now interested in finding expressions constructed from Dirac spinors and study how they tranform under Lorentz transformations. Take the expressions,

$$j = \bar{\psi}\Gamma\psi.$$

As we already know, for $\Gamma = 1$, the expression is Lorentz invariant (scalar). For $\Gamma = \gamma^u$, we find,

$$j^{u\prime} = \bar{\psi}'\gamma^u\psi' = \psi^\dagger M^\dagger \gamma^0 \gamma^u M \psi = (\psi^\dagger M^\dagger \gamma^0 M)(M^\dagger \gamma^u M \psi) = \Lambda^u_v(\bar{\psi}\gamma^v\psi) = \Lambda^u_v j^v.$$

So it is a covariant expression (vector). We essentialy wish to find expressions of the sort. To do so, we may define a basis for these matrices, $\{\Gamma^a\}_a$ such that it obbeys a similar relation as Dirac matrices, namely,

$$Tr(\Gamma_a \Gamma_b) = g_{ab},$$

for a metric g_{ab}, where we may get indices up or down. Note that, if $M = M^a\Gamma_a$, for M^a coefficients, we have,

$$M^a \Gamma_a \Gamma_b = M\Gamma_b \Rightarrow Tr(M\Gamma_b) = M^a Tr(\Gamma_a \Gamma_b) = M_b,$$

$$\boxed{M_b = Tr(M\Gamma_b)}$$

and therefore,

$$\boxed{M = Tr(M\Gamma_b)\Gamma^b}.$$

In components, this gives

$$M_{ij} = M^{kl}(\Gamma_b)_{lk}(\Gamma^b)_{ij},$$

$$\boxed{\delta_i^l \delta_j^k = (\Gamma_b)^{lk}(\Gamma^b)_{ij}}.$$

11.1 Fierz Identities

Now we show relation between elements of $\{\Gamma_a\}$ known as Fierz identities (named after swiss physicist Markus Edward Fierz [1912-2006]).

Take

$$\big(\bar{\psi}_1 A\psi_2\big)\big(\bar{\psi}_3 B\psi_4\big) = \bar{\psi}_1^i A_{ij}(\psi_2)^j(\bar{\psi}_3)^k B_{k\ell}(\psi_4)^\ell.$$

By expanding A and B in the basis $\{\Gamma_a\}$, we get

$$\big(\bar{\psi}_1 A\psi_2\big)\big(\bar{\psi}_3 B\psi_4\big) = \bar{\psi}_1^i A^a(\Gamma_a)_{ij}(\psi_2)^j(\bar{\psi}_3)^k B^b(\Gamma_b)_{k\ell}(\psi_4)^\ell,$$

so that

$$A_{ij}B_{k\ell} = A^a(\Gamma_a)_{ij}B^b(\Gamma_b)_{k\ell}.$$

We may rewrite the same expression by interchanging spinors:

$$\big(\bar{\psi}_1 M\psi_4\big)\big(\bar{\psi}_3 N\psi_2\big) = (\cdots),$$

hence

$$M_{i\ell}N_{kj} = M^c(\Gamma_c)_{i\ell}N^d(\Gamma_d)_{kj}.$$

Therefore,

$$(\Gamma_a)_{ij}(\Gamma_b)_{k\ell} = C_{ab}{}^{cd}(\Gamma_c)_{i\ell}(\Gamma_d)_{kj}.$$

Multiplying by $(\Gamma^c)^{\ell i}(\Gamma^d)^{jk}$ and summing over indices, we obtain (considering that $\mathrm{Tr}(\Gamma_a\Gamma^b) = 4\delta_a^b$),

$$16C_{ab}{}^{cd} = (\Gamma_a)_{ij}(\Gamma_b)_{k\ell}(\Gamma^c)^{\ell i}(\Gamma^d)^{jk}.$$

$$\boxed{C_{ab}{}^{cd} = \frac{1}{16}\,\mathrm{Tr}(\Gamma_a\Gamma_b\Gamma^c\Gamma^d)}.$$

Thus, we get to our **Fierz identity**,

$$\boxed{\big(\bar{\psi}_1\Gamma_a\psi_2\big)\big(\bar{\psi}_3\Gamma_b\psi_4\big) = C_{ab}{}^{cd}\big(\bar{\psi}_1\Gamma_c\psi_4\big)\big(\bar{\psi}_3\Gamma_d\psi_2\big)}.$$

11.2 Finding the bilinears

We may now specify a basis that obeys the conditions at the beggining of this section, namely,

$$\beta = \{I, \gamma^u, \gamma^{uv} = \frac{[\gamma^u, \gamma^v]}{2}, \gamma^u\gamma^5, \gamma^5\}.$$

Note that the first term includes no Dirac matrices, the second includes the multiplication of 1, the third of 2, the fourth of 3 and the last of all 4 (incubed inside γ_5).

There are 16 elements in this basis, as one might count, and it obeys the relation,

$$[A_i, A_j] = 4\delta_{ij},$$

for $A_i \in \beta$ and i going from 1 to 16. This leads to the classification of Dirac bilinears,

$$\boxed{\{\bar{\psi}\psi, \bar{\psi}\gamma^u\psi, \bar{\psi}\gamma^{uv}\psi, \bar{\psi}\gamma^u\gamma^5\psi, \bar{\psi}\gamma^5\psi\}}.$$

In a similar way that we showed that the second one is a vector, we can show that the third one is an antissymetric tensor, the fourth is a pseudo vector, and the last one is a pseudoscalar. Their linear combination describes all Dirac bilinears in four dimensions.

Example. As an example, let us show how $\bar{\psi}\gamma^\mu\gamma_5\psi$ transforms:

$$\bar{\psi}'\gamma^\mu\gamma_5\psi' = \psi^\dagger M^\dagger\gamma^0\gamma^\mu\gamma_5 M\psi = \psi^\dagger(M^{-1}\gamma^0 M)(M^{-1}\gamma^\mu M)M^{-1}\gamma_5 M\psi,$$

and therefore

$$\bar{\psi}'\gamma^\mu\gamma_5\psi' = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\gamma_5'\psi,$$

$$\gamma_5' = M^{-1}\gamma_5 M = iM^{-1}(\gamma_0\gamma_1\gamma_2\gamma_3)M = i(M^{-1}\gamma_0 M)(M^{-1}\gamma_1 M)(M^{-1}\gamma_2 M)(M^{-1}\gamma_3 M) =$$

$$= i\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\Lambda_0^\mu\Lambda_1^\nu\Lambda_2^\rho\Lambda_3^\sigma.$$

Hence,

$$\bar{\psi}'\gamma^\mu\gamma_5\psi' = \bar{\psi}\gamma^\nu\big(i\gamma_0\gamma_1\gamma_2\gamma_3\big)\psi = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\big(i\gamma_0\gamma_\rho\gamma_\sigma\gamma_\omega\big)\Lambda_1^\rho\Lambda_2^\sigma\Lambda_3^\omega\psi,$$

which shows that it is a pseudovector.

11.3 CPT symmetries.

Now we wish to see how the fields work under other types of transformation, namely CPT (charge, parity, time).

Definition. Parity change, $P : (t, \vec{x}) \to (t, -\vec{x})$.

Time reversal, $T : (t, \vec{x}) \to (-t, \vec{x})$.

Charge conjugation, $Ca_p^s C^{-1} = b_p^s$.

If we remind the definition of Lorentz group sectors, We have the relation,

$$\begin{tikzcd} L_+^{\uparrow} \arrow[r, "P"] \arrow[d, "T"'] & L_-^{\uparrow} \arrow[d, "T"] \\ L_+^{\downarrow} \arrow[r, "P"'] & L_-^{\downarrow} \end{tikzcd}$$

For the parity transformation, we take

$$Pa_p^s P^{-1} = \eta_a a_{-p}^s, \qquad Pb_p^s P^{-1} = \eta_b \, b_{-p}^s.$$

where η_a, η_b are, in principle, phases.
Writing the field expansion,

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a_p^s u^s(p) e^{-ipx} + b_p^{s\,\dagger} v^s(p) e^{ipx}).$$

We get (notice that the parity operator does not change the argument in the exponent, since both x and p are flipped).

$$P\psi(x)P^{-1} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (Pa_p^s P^{-1} u^s(p) e^{-ipx} + Pb_p^{s\,\dagger} P^{-1} v^s(p) e^{ipx}).$$

Recalling section 3, we have

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix},$$

$$v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix}.$$

By defyning $\tilde{p} = (p^0, -\vec{p})$, we have,

$$u(\tilde{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi \\ \sqrt{\tilde{p}\cdot\sigma}\,\xi \end{pmatrix} = \gamma^0 u(p),$$

$$v(\tilde{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\eta \\ -\sqrt{\tilde{p}\cdot\sigma}\,\eta \end{pmatrix} = -\gamma^0 v(p).$$

Thus,

$$P\psi(x)P^{-1} = \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2w_{\tilde{p}}}} \sum_s \Big[\eta_a a_{\tilde{p}}^s \gamma^0 u^s(\tilde{p}) e^{-ipx} - \eta_b^\dagger b_{\tilde{p}}^{s\,\dagger} \gamma^0 v^s(\tilde{p}) e^{ipx}\Big].$$

In order for the field to have a well-defined parity, we must have

$$\eta_b^* = -\eta_a.$$

Therefore,

$$P\psi(x)P^{-1} = \eta_a \gamma^0 \psi(t, -\vec{x}).$$

$$P\bar{\psi}(x)P^{-1} = \eta_a^* \bar{\psi}(t, -\vec{x})\gamma^0.$$

So,

$$\bar{\psi}\psi \to \big(P\bar{\psi}P^{-1}\big)\big(P\psi P^{-1}\big) = \eta_a^* \bar{\psi}\gamma^0 \eta_a \gamma^0 \psi = (\bar{\psi}\psi)(t, -\vec{x}),$$

which shows that $\bar{\psi}\psi$ is a scalar under parity.

For the bilinear $\bar{\psi}\gamma^5\psi$,

$$\bar{\psi}\gamma^5\psi \to P\bar{\psi}(P^{-1}P)\gamma^5\psi P^{-1} = \bar{\psi}(t,-x)\gamma^0\gamma^5\gamma^0\psi(t,-x) = -\bar{\psi}(t,-x)\gamma^5\psi(t,-x).$$

Which shows $\bar{\psi}\gamma^5\psi$ is a pseudoscalar.

Let us see about time reversal now. Reversing time reverses both momentum and spin. We know that time is antiunitary, so, if Z is an operator,

$$TZ = Z^\dagger T.$$

Which gives,

$$Ta_p^s T^{-1} = \eta_a a_{-p}^{-s},$$

$$Tb_p^s T^{-1} = \eta_b b_{-p}^{-s}.$$

Where η_a and η_b are respectivelly the phases. So, if we do the same thing we did for the parity operator,

$$T\psi(x)T^{-1} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (a_{-p}^{-s}[u^{-s}(p)]^* e^{ipx} + b_{-p}^{-s\dagger}[v^{-s}(p)]^* e^{-ipx}).$$

We may write spin up and spin down components:

$$\xi^s = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}, \qquad \xi^{-s} = \begin{pmatrix} -e^{-i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}.$$

This change is made possible by the matrix product $\gamma^1\gamma^3 = \begin{pmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & \sigma^1\sigma^3 \end{pmatrix}$. Thus, by making the same change that we did before, we get,

$$T\psi(x)T^{-1} = \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{\gamma^1\gamma^3}{\sqrt{2E_p}} \sum_s (a_{-p}^{-s} u^{-s}(\tilde{p}) e^{-i\tilde{p}x} + b_{-p}^{-s\dagger} v^{-s}(\tilde{p}) e^{i\tilde{p}x}) = \gamma^1\gamma^3\psi(-t, \vec{x}).$$

Similarly,

$$T\bar{\psi}(x)T^{-1} = \bar{\psi}(-t, \vec{x})(-\gamma^1\gamma^3).$$

So,

$$T\psi\bar(x)\psi(x)T^{-1} = \bar{\psi}(-t,\vec{x})(-\gamma^1\gamma^3)\gamma^1\gamma^3\psi(-t,\vec{x}) = \bar{\psi}\psi.$$

Which shows that $\psi(x)\psi\bar(x)$ is a scalar.

For the bilinear $\bar{\psi}\gamma^5\psi$,

$$\begin{aligned}
T\big(\bar{\psi}\gamma^5\psi\big)T^{-1} &= \big(T\bar{\psi}T^{-1}\big)\gamma^5\big(T\psi T^{-1}\big) \\
&= \bar{\psi}(-t,\vec{x})\big(-\gamma^1\gamma^3\big)\gamma^5\big(\gamma^1\gamma^3\big)\psi(-t,\vec{x}) \\
&= \bar{\psi}(-t,\vec{x})\big(-\gamma^1\gamma^3\gamma^1\gamma^3\big)\gamma^5\psi(-t,\vec{x}) \\
&= \bar{\psi}(-t,\vec{x})\gamma^5\psi(-t,\vec{x}).
\end{aligned}$$

Thus, it is also a scalar under time reversal.

Now let us show the charge operator.

The charge-conjugation operator exchanges particles and antiparticles:

$$Ca^{s\dagger}_{\vec{p}}C^{-1} = b^{s\dagger}_{\vec{p}}, \qquad Cb^{s\dagger}_{\vec{p}}C^{-1} = a^{s\dagger}_{\vec{p}}.$$

Applying charge conjugation twice gives

$$C^2 = \hat{1},$$

and therefore

$$C^{-1} = C.$$

Using the spinor identities

$$u^s(p) = -i\gamma^2\big[v^s(p)\big]^*, \qquad v^s(p) = -i\gamma^2\big[u^s(p)\big]^*,$$

we obtain the transformations of the Dirac field and its adjoint:

$$\begin{aligned}
C\psi(x)C^{-1} &= -i\gamma^2\psi^*(x) \\
&= -i\gamma^2\big(\bar{\psi}(x)\gamma^0\big)^T \\
&= -i\big(\bar{\psi}(x)\gamma^0\gamma^2\big)^T,
\end{aligned}$$

and, analogously,

$$\begin{aligned}
C\bar{\psi}(x)C^{-1} &= -i\psi^T(x)\gamma^2\gamma^0 \\
&= -i\big(\gamma^0\gamma^2\psi(x)\big)^T.
\end{aligned}$$

With these identities, the scalar bilinear transforms as

$$\begin{aligned}
C\big(\bar{\psi}\psi\big)C^{-1} &= -\big(\gamma^0\gamma^2\psi\big)^T\big(\bar{\psi}\gamma^0\gamma^2\big)^T \\
&= -\psi^T\bar{\psi}^T \\
&= \bar{\psi}\psi.
\end{aligned}$$

Thus, $\bar{\psi}\psi$ is a scalar under charge conjugation. For the pseudoscalar bilinear,

$$
\begin{aligned}
C\big(\bar\psi\gamma^5\psi\big)C^{-1} &= -\big(\gamma^0\gamma^2\psi\big)^T\gamma^5\big(\bar\psi\gamma^0\gamma^2\big)^T \\
&= -\psi^T\big(\gamma^0\gamma^2\gamma^5\gamma^0\gamma^2\big)^T\bar\psi^T \\
&= -\psi^T(\gamma^5)^T\bar\psi^T \\
&= \big(\bar\psi\gamma^5\psi\big)^T \\
&= \bar\psi\gamma^5\psi.
\end{aligned}
$$

Here the minus sign produced by exchanging the two spinors is cancelled because the fields are fermionic. Hence, $\bar\psi\gamma^5\psi$ is also even under charge conjugation.

11.4 Table of bilinears transformations

After having shown the transformation properties of each operator,
Space,

$$P\psi(x)P^{-1} = \eta_a\gamma^0\psi(t,-\vec{x}).$$

$$P\bar\psi(x)P^{-1} = \eta_a^*\bar\psi(t,-\vec{x})\gamma^0.$$

Time:

$$T\psi(x)T^{-1} = \gamma^1\gamma^3\psi(-t,\vec{x}).$$

$$T\bar\psi(x)T^{-1} = \bar\psi(-t,\vec{x})(-\gamma^1\gamma^3).$$

Charge:

$$C\psi(x)C^{-1} = -i\big(\bar\psi(x)\gamma^0\gamma^2\big)^T,$$

$$C\bar\psi(x)C^{-1} = -i\big(\gamma^0\gamma^2\psi(x)\big)^T.$$

We can prove how each bilinear transforms, just as we did show for $\bar\psi\psi$ and $\bar\psi\gamma_5\psi$. The table is the following

	$\bar\psi\psi$	$\bar\psi\gamma_5\psi$	$i\bar\psi\gamma_5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma_5\psi$	$\bar\psi\gamma^{\mu\nu}\psi$	∂_μ
C	$+1$	$+1$	$+1$	-1	1	-1	1
P	$+1$	-1	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
CPT	$+1$	-1	$+1$	-1	-1	1	-1

It is worth rewriting Dirac matrices and relations again:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\gamma^5 = \frac{i}{4!}\,\epsilon^{\mu\nu\rho\sigma}\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma.$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With the metric convention we are using $(1, -1, -1, -1)$,

$$\boxed{\{\gamma^u, \gamma^v\} = 2\eta^{uv}}.$$

$$\boxed{\{\gamma^5, \gamma^u\} = 0}.$$

$$\boxed{\gamma^u = \eta^{uv}\gamma_v \Longrightarrow \gamma^0 = \gamma_0,\ \gamma^i = -\gamma_i}.$$

We will prove for every case:

I) For $i\bar{\psi}\gamma^5\psi$:

The only transformation among C, P, and T that differs from $\bar{\psi}\gamma^5\psi$ is time reversal, because, since T is antiunitary:

$$TiT^{-1} = -i.$$

Therefore,

$$T\big(i\bar{\psi}\gamma^5\psi\big)T^{-1} = -i\bar{\psi}\gamma^5\psi,$$

and, of course,

$$CPT\big(i\bar{\psi}\gamma^5\psi\big)(CPT)^{-1} = i\bar{\psi}\gamma^5\psi.$$

II) For the vector bilinear $\bar{\psi}\gamma^\mu\psi$:

Charge conjugation gives

$$C\big(\bar{\psi}\gamma^\mu\psi\big)C^{-1} = \big(-i\gamma^0\gamma^2\psi\big)^T\gamma^\mu\big[-i\big(\bar{\psi}\gamma^0\gamma^2\big)^T\big] = -\bar{\psi}\gamma^\mu\psi.$$

Under parity,

$$P\big(\bar{\psi}\gamma^\mu\psi\big)P^{-1} = \eta_a^*\bar{\psi}\gamma^0\gamma^\mu\gamma^0\eta_a\psi = (-1)^\mu\bar{\psi}\gamma^\mu\psi.$$

Under time reversal,

$$T\big(\bar{\psi}\gamma^\mu\psi\big)T^{-1} = \bar{\psi}\gamma^3\gamma^1(\gamma^\mu)^*\gamma^1\gamma^3\psi = (-1)^\mu\bar{\psi}\gamma^\mu\psi.$$

Consequently,

$$CPT\big(\bar{\psi}\gamma^\mu\psi\big)(CPT)^{-1} = -\bar{\psi}\gamma^\mu\psi.$$

III) For the axial-vector bilinear $\bar{\psi}\gamma^\mu\gamma^5\psi$:

We obtain

$$C\big(\bar{\psi}\gamma^\mu\gamma^5\psi\big)C^{-1} = \big(-i\gamma^0\gamma^2\psi\big)^T\gamma^\mu\gamma^5\big[-i\big(\bar{\psi}\gamma^0\gamma^2\big)^T\big] = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

For parity,

$$P\big(\bar{\psi}\gamma^\mu\gamma^5\psi\big)P^{-1} = \eta_a^*\bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\eta_a\psi = -(-1)^\mu\bar{\psi}\gamma^\mu\gamma^5\psi.$$

For time reversal,

$$T\big(\bar{\psi}\gamma^\mu\gamma^5\psi\big)T^{-1} = \bar{\psi}\gamma^3\gamma^1(\gamma^\mu\gamma^5)^*\gamma^1\gamma^3\psi = (-1)^\mu\bar{\psi}\gamma^\mu\gamma^5\psi.$$

Thus,

$$CPT\big(\bar{\psi}\gamma^\mu\gamma^5\psi\big)(CPT)^{-1} = -\bar{\psi}\gamma^\mu\gamma^5\psi.$$

IV) For the tensor bilinear, $\bar{\psi}\gamma^{\mu\nu}\psi = \bar{\psi}\gamma^\mu\gamma^\nu\psi$:
Charge conjugation gives

$$C\big(\bar{\psi}\gamma^{\mu\nu}\psi\big)C^{-1} = \big(-i\gamma^0\gamma^2\psi\big)^T\gamma^\mu\gamma^\nu\big[-i\big(\bar{\psi}\gamma^0\gamma^2\big)^T\big] = -\bar{\psi}\gamma^{\mu\nu}\psi.$$

Under parity,

$$P\big(\bar{\psi}\gamma^{\mu\nu}\psi\big)P^{-1} = \eta_a^*\bar{\psi}\gamma^0\gamma^\mu\gamma^\nu\gamma^0\eta_a\psi = (-1)^\mu(-1)^\nu\bar{\psi}\gamma^{\mu\nu}\psi.$$

Under time reversal,

$$T\big(\bar{\psi}\gamma^{\mu\nu}\psi\big)T^{-1} = \bar{\psi}\gamma^3\gamma^1(\gamma^\mu\gamma^\nu)^*\gamma^1\gamma^3\psi = -(-1)^\mu(-1)^\nu\bar{\psi}\gamma^{\mu\nu}\psi.$$

Therefore,

$$CPT\big(\bar{\psi}\gamma^{\mu\nu}\psi\big)(CPT)^{-1} = \bar{\psi}\gamma^{\mu\nu}\psi.$$

V) Finally, for the derivative ∂_μ:

$$P\partial_\mu P^{-1} = \begin{cases}\partial_0, & \mu = 0,\\ -\partial_\mu, & \mu = 1,2,3,\end{cases} = (-1)^\mu\partial_\mu,$$

$$C\partial_\mu C^{-1} = \partial_\mu,$$

$$T\partial_\mu T^{-1} = \begin{cases}-\partial_0, & \mu = 0,\\ \partial_\mu, & \mu = 1,2,3,\end{cases} = -(-1)^\mu\partial_\mu,$$

and hence

$$(CPT)\partial_\mu(CPT)^{-1} = -\partial_\mu.$$

12 Extra: Conformal Field theory fundamentals

Conformal Field Theory (CFT) refers to the field theories with conformal invariance (a transformation that preserves angles in space). They turn out to be very beautiful and important for modern theoretical physics, particularly in string theory and AdS/CFT correspondence, so let us see some of its aspects.

12.1 Fundamental properties

Definition. A conformal transformation between manifold M and M' is a mapping $\varphi : U \subset M \to V \subset M'$ such that, for $x' = \varphi(x)$, there is a positive function $\Lambda(x)$ such that,

$$g'_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{u}}\frac{\partial x'^{\sigma}}{\partial x^{v}} = g_{uv}\Lambda(x).$$

Considering $M = M'$ and flat space, we get,

$$\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{u}}\frac{\partial x'^{\sigma}}{\partial x^{v}} = \eta_{uv}\Lambda(x).$$

Considering infinitesimal transformations, $x'^{\rho} = x^{\rho} + \epsilon^{\rho}$, we get,

$$\eta_{\rho\sigma}(\delta^{\rho}_{\mu} + \partial_{\mu}\epsilon^{\rho})(\delta^{\sigma}_{\nu} + \partial_{\nu}\epsilon^{\sigma}) = \eta_{\mu\nu}\Lambda(x) = \eta_{\mu\nu} + \eta_{\mu\sigma}\partial_{\nu}\epsilon^{\sigma} + \eta_{\rho\nu}\partial_{\mu}\epsilon^{\rho},$$

$$\implies \boxed{\eta_{\mu\nu}(\Lambda(x) - 1) = \partial_{\nu}\epsilon_{\mu} + \partial_{\mu}\epsilon_{\nu}}.$$

Multiplying both sides by $\eta^{\mu\nu}$, we get,

$$d(\Lambda(x) - 1) = 2\partial^{\mu}\epsilon_{\mu},$$

thus, we get to,

$$\boxed{I) \quad \frac{\eta_{\mu\nu}(2\partial\cdot\epsilon)}{d} = \partial_{(\nu}\epsilon_{\mu)}}.$$

Manipulating the above expression, we get,

$$\boxed{II) \quad (d-1)\Box^{2}(\partial\cdot\epsilon) = 0}.$$

$$\boxed{III) \quad d(\partial_{\mu}\partial_{\nu}\epsilon_{\rho}) = \eta_{[\rho\mu}\partial_{\nu]}(\partial\cdot\epsilon)}.$$

Those 3 expressions denotes conditions for an infinitesimal transformation to be canonical. Note that, from II), we get that the most general form for ϵ is,

$$\boxed{\epsilon_{\rho} = A_{\rho} + B_{\rho\nu}x^{\nu} + C_{\rho\mu\nu}x^{\mu}x^{\nu}}.$$

12.2 Conformal group and algebra

Note that the set of functions $\{f : x \to (x' - x)\}$ of conformal transformations forms a group of linear transformations (directly verifiable). Thus, we may describe its algebra by checking every element of ϵ.

We shall use the following results,

Lemma. *If G is a linear Lie group and $X \in T_eG$, then, for $\gamma(t) = e^{tX}$, $\exists\ \epsilon > 0 \quad s.t \quad \forall |t| < \epsilon, \gamma(t) \in G$.*

Theorem. *If G is a linear Lie group, then $Lie(G) = \mathfrak{g} = \{X \in GL / e^{tX} \in G\ \forall\ t \in \mathbb{R}\}$.*

Proof. Suppose first that $e^{tX} = 1 + tX + ... \in G$ for all t. Since $e^{tX}|_{t=0} = e$, the identity in the Lie group, we have that $X = \frac{d}{dt} e^{tX}|_{t=0} \in T_eG$, so $\{X \in GL/e^{tX} \in G\ \forall\ t \in \mathbb{R}\} \subset Lie(G)$.

Now, suppose $X \in T_eG$. Then, using the lemma, for any $t \in \mathbb{R}$, $\exists\ N \in \mathbb{N}$ such that $|\frac{t}{N}| < \epsilon$, thus, $e^{\frac{tX}{N}} \in G$. Since G is a group, $e^{\frac{tX}{N}} e^{\frac{tX}{N}} e^{\frac{tX}{N}} ... = e^{tX} \in G$. So $Lie(G) \subset \{X \in GL \quad /e^{tX} \in G\ \forall\ t \in \mathbb{R}\}$.

Thus, we conclude that $Lie(G) = \{X \in GL/e^{tX} \in G\ \forall\ t \in \mathbb{R}\}$. □

Thus, for the first term, considering X small,

$$e^{iX} x^\rho = x^\rho + A^\rho = x^\rho + iXx^\rho,$$

$$\implies iXx^\rho = A^\rho,$$

thus, we get that $X = -iA^\rho \partial_\rho$. So the element in the Lie algebra is $\boxed{P_\rho = -i\partial_\rho}$.

For the term B, we use expression I, and get,

$$B_{(\mu\nu)} = \frac{2}{d} \eta_{\mu\nu} (\eta^{\alpha\beta} B_{\alpha\beta}).$$

We can separate $B_{\mu\nu}$ into a symmetric and antisymmetric part, $B_{\mu\nu} = S_{\mu\nu} + A_{\mu\nu}$. Since this is infinitesimal, we can assume $S_{\mu\nu}$ is proportional to the flat metric, $S_{\mu\nu} = \alpha\eta_{\mu\nu}$, and then consider its transformations action,

$$x'^\mu = x^\mu + \alpha\eta^{\mu\nu} x_\nu,$$

thus,

$$e^{iD} x^\mu = x'^\mu \implies iDx^\mu = \alpha\eta^{\mu\nu} x_\nu \implies \boxed{D = -ix^\mu \partial_\mu}.$$

For the antissymmetric part,

$$x'^\mu = x^\mu + A^{\mu\nu} x_\nu,$$

$$iDx^\mu = A^{\mu\nu} x_\nu,$$

$$iDx^\nu = A^{\nu\mu} x_\mu = -A^{\mu\nu} x_\mu.$$

Thus, we see that for the antissymetric part $\boxed{D^\mu_\nu = i(x^\mu \partial_\nu - x^\nu \partial_\mu)}$.

For the C term, we use III,

$$C_{\rho\mu\nu} = \frac{1}{d} (\eta_{[\rho\mu} C^\sigma_{\nu]\sigma}),$$

$$C_{\rho\mu\nu} x^\mu x^\nu = \frac{1}{d} (x_\rho x^\nu C^\sigma_{\nu\sigma} - x^\nu x_\nu C^\sigma_{\rho\sigma} + x_\rho x^\mu C^\sigma_{\mu\sigma}) = \frac{1}{d} (2x_\rho x^\nu C^\sigma_{\nu\sigma} - x^\nu x_\nu C^\sigma_{\rho\sigma}).$$

Now, if,

$$e^{iK}x_\rho = x_\rho + C_{\rho\mu\nu}x^\mu x^\nu,$$

$$\implies \boxed{K^\rho = -i(2x^\rho x^\nu \partial_\nu - x^\nu x_\nu \partial^\rho)}.$$

Thus, we get that the Conformal group Lie algebra generators are,

$$1) Translation: \quad \boxed{P_\rho = -i\partial_\rho}.$$

$$2) Dilation: \quad \boxed{D = -ix^\mu \partial_\mu}.$$

$$3) Rotation: \quad \boxed{D_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)}.$$

4) Special Conformal Transformation (SCT): $\boxed{K^\rho = -i(2x^\rho x^\nu \partial_\nu - x^\nu x_\nu \partial^\rho)}$.

The dimension of such Lie algebra is $\frac{d(d-1)}{2} + 2d + 1$.

In the remaining of this project, we shall focus in $d = 2$. In this case, letting μ, ν be represented by $0, 1$, we get, by I,

$$\boxed{\partial_{(0}\epsilon_{1)} = 0},$$

$$\partial_0\epsilon_0 + \partial_1\epsilon_1 = 2\partial_0\epsilon_0 \implies \boxed{\partial_0\epsilon_0 = \partial_1\epsilon_1}.$$

Notice that those are Cauchy-Riemann equations. Thus, we state,

Theorem. *If $\epsilon(z) = \epsilon_0(x^0, x^1) + i\epsilon_1(x^0, x^1)$, the transformation $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z + \epsilon(z)$ is conformal iff it is holomorphic.*

Therefore, if one is dealing in the complex space, we can write ϵ and $\bar{\epsilon}$ as Laurent series,

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n,$$

$$\bar{\epsilon}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{n+1}\bar{\epsilon_n}.$$

Thus, in the transformation $z' = z + \epsilon(z)$ and $\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z})$, we can write the generators as,

$$l_n = z^{n+1}\partial,$$

$$\bar{l}_n = \bar{z}^{n+1}\bar{\partial}.$$

The generators obey the relations,

$$[l_n, l_m] = (n-m)l_{n+m},$$

$$[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m},$$

$$[l_n, \bar{l}_m] = 0.$$

This defines a Witt algebra, after Ernst Witt [1911 - 1991].

Definition. A Witt algebra is a Lie algebra of generators ,

$$l_n = z^{n+1}\partial,$$

$$\bar{l}_n = \bar{z}^{n+1}\bar{\partial}.$$

With the Lie bracket,

$$[l_n, l_m] = (n-m)l_{n+m},$$

$$[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m},$$

$$[l_n, \bar{l}_m] = 0.$$

There is an extension of crucial importance of this algebra, which is the so called Virasoro algebra, named after Miguel Virasoro [1940-2021].

Definition. The Virasoro Algebra is a complex Lie algebra of operators L_n, $n \in \mathbb{Z}$ with the bracket,

$$[L_n, L_m] = (n-m)L_{n+m} + \lambda(n^3 - n)\delta_{m,-n}.$$

Where $\lambda \in \mathbb{C}$.

Those definitions will have their importance shown later.

12.3 Primary Fields

We can start studying CFT in Euclidean space, with the correspondence $\mathbb{R}^2 \simeq \mathbb{C}$ by adopting $z = x^0 + ix^1$, $\bar{z} = x^0 - ix^1$. Thus, fields in our theory are described as,

$$\Phi = \Phi(z, \bar{z}).$$

Definition. Fields are Chiral and Anti-chiral if, respectively, $\Phi = \Phi(z)$ and $\Phi = \Phi(\bar{z})$.

Definition. Φ is called a primary field if, upon conformal transformation $z \to f(z)$, we get,

$$\Phi(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})).$$

Where $h, \bar{h}$ are the so called conformal dimensions.

If a field is primary, , $(\partial f)^h = 1 + h\partial\epsilon + O(\epsilon^2)$, and $\Phi(z+\epsilon, .) = \Phi(z, .) + \partial\Phi\epsilon + O(\epsilon^2)$. Considering an infinitesimal transformation, we get the transformation law,

$$\boxed{\delta_{\epsilon,\bar{\epsilon}}\Phi = (\epsilon(z)\partial + \bar{\epsilon}(\bar{z})\bar{\partial} + h\partial\epsilon + \bar{h}\bar{\partial}\bar{\epsilon})\Phi(z, \bar{z})}.$$

12.4 Stress energy tensor

In building actions of primary fields invariant under Conformal Transformations, we get the Noether current,

$$j_\mu = T_{\mu\nu}\epsilon^\nu,$$

by taking the derivative in both sides, we get,

$$\partial^\mu(T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}\partial^\mu(\epsilon^\nu) = T_{\mu\nu}\partial^\mu(\epsilon^\nu) = 0,$$

by using I and considering that $T_{\mu\nu}$ is symmetric,

$$\frac{T_{\mu\nu}\eta^{\mu\nu}(\partial\cdot\epsilon)}{d} = 0 \Longrightarrow \boxed{T^\mu_\mu = 0}.$$

Thus, we see that the stress energy tensor in a CFT in $d = 2$ is traceless.

If we wish to make a variable change,

$$T_{\mu\nu} = \frac{\partial x^\alpha}{\partial z^\mu}\frac{\partial x^\beta}{\partial z^\nu}T_{\alpha\beta},$$

where $x^\alpha \in \{x^0, x^1\}$ and $z^\mu \in \{z, \bar{z}\}$. We get,

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = 0,$$

$$T_{zz} = \frac{1}{4}(T_{00} - iT_{01}),$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + iT_{01}).$$

We also find out that the two non-vanishing components are holomorphic and anti-holomorphic,

$$\boxed{\partial_{\bar{z}}T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0.}$$

12.5 OPE

In QFT, once we have a theory with conserved charge constructed from a symmetry, we can write the variation of the field in respect to that transformation by $\delta\Phi = [Q, \Phi]$. In our case, we have that, inside a region $\mathcal{C}$,

$$\delta_{\epsilon,\bar{\epsilon}}\Phi = \oint_{\mathcal{C}} dz' \frac{[T(z')\epsilon(z'), \Phi(z,\bar{z})]}{2i\pi} + \oint_{\mathcal{C}} d\bar{z}' \frac{[\bar{T}(\bar{z}')\bar{\epsilon}(\bar{z}'), \Phi(z,\bar{z})]}{2i\pi}.$$

Comparing to our previous expression for the field variation and considering the identities,

$$\Phi(z,\bar{z})\partial\epsilon = \frac{1}{2\pi i}\oint_{\mathcal{C}} dz' \frac{\epsilon(z')}{(z-z')^2}\Phi(z,\bar{z}),$$

$$\partial\Phi\epsilon = \frac{1}{2\pi i}\oint_{\mathcal{C}} dz' \frac{\epsilon(z')}{(z-z')}\partial\Phi(z,\bar{z}),$$

we see that,

$$\oint_{\mathcal{C}} dz' \frac{[T(z')\epsilon(z'), \Phi(z,\bar{z})]}{2i\pi} = (\epsilon(z)\partial + h\partial\epsilon)\Phi(z,\bar{z}),$$

$$\implies [T(z)\epsilon(z), \Phi(w,\bar{w})] = h\frac{(\Phi(w,\bar{w})\epsilon(z))}{(z-w)^2} + \frac{\partial\Phi(w,\bar{w})\epsilon(z)}{z-w},$$

up to singular terms. There is an analog expression for the holomorphic part. Such expression allows us to define an Operator Product Expression (OPE) between the stress tensor and the fields,

Lemma. *If $\Phi = \Phi(z,\bar{z})$ is a primary, the OPE with the stress tensor in a CFT is,*

$$\boxed{T(z)\Phi(w,\bar{w}) = h\frac{(\Phi(w,\bar{w}))}{(z-w)^2} + \frac{\partial\Phi(w,\bar{w})}{z-w}.}$$

$$\boxed{\bar{T}(\bar{z})\Phi(w,\bar{w}) = \bar{h}\frac{(\Phi(w,\bar{w}))}{(z-w)^2} + \frac{\bar{\partial}\Phi(w,\bar{w})}{z-w}}.$$

The OPE can be viewed as an algebraic product of quantum fields. We may also write the Laurent modes of the Stress tensor,

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1}T(z),$$

$$[L_m, L_n] = \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i} z^{m+1}w^{n+1}[T(z), T(w)].$$

In order for this to respect Virasoro algebra, we state the OPE of stress tensors,

$$\boxed{T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}}.$$

In this case, we adopt $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{m,-n}$, where c is the so called central charge. We also note that if $c=0$, the stress energy is a primary field of conformal dimension 2. Let's now apply this formalism to QFT examples.

12.6 Free Boson

The action for free massless scalar fields is given by Polyakov action (named after Alexander Markovich Polyakov [1940-]).

$$S = \int d^2z[\partial X^\mu \bar{\partial} X_\mu].$$

Using Euler-Lagrange equations, we see that the equations of motions are

$$\partial\bar{\partial}X^\mu = 0.$$

This implies that the average value of the OPE of the above expression with any field that is not in z vanishes, i.e,

$$< \partial\bar{\partial}X^{\mu} ... >= 0.$$

Using the fact that $\partial\bar{\partial}X^{\mu} = -\frac{\delta S}{\delta X_{\mu}}$ and the above statement, we must have that

$$< \partial\bar{\partial}X^{\mu}X^{\nu}(w,\bar{w}) >= 0 \Longrightarrow \int [DX]\frac{\delta}{\delta X_{\mu}}(e^{-S}X^{\nu}(w)) = \partial\bar{\partial} < X^{\mu}(z,\bar{z})X^{\nu}(w,\bar{w}) > +\eta^{\mu\nu}\delta^{2}(z-w,\bar{z}-\bar{w}) = 0.$$

This leads to the OPE,

$$\boxed{X^{\mu}(z,\bar{z})X^{\nu}(w,\bar{w}) = -\eta^{\mu\nu}[\ln(z-w) + \ln(\bar{z}-\bar{w})]}.$$

And the energy stress tensor non-zero components are

$$T_{zz} = \frac{1}{4}(\partial X^{\mu}\partial X_{\mu}),$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}(\bar{\partial}X^{\mu}\bar{\partial}X_{\mu}).$$

With this, we see that the OPE singular terms are,

$$\boxed{T(z)X^{\mu}(w,\bar{w}) = \frac{\partial_{w}X^{\mu}}{(z-w)}},$$

thus, we see that the bosonic field has no conformal dimension(The same conclusion is taken for the antiholomorphic part).

The OPE of stress tensor leads us to,

$$T(z)T(w) = \frac{1}{16}(\partial X^{\mu}\partial X_{\mu}\partial_{w}X^{\nu}\partial_{w}X_{\nu}) = \frac{\partial X^{\mu}}{16}(\partial\partial_{w}(\eta^{\nu}_{\mu}(-\ln(z-w)))\partial_{w}X_{\nu}+\partial\partial_{w}(\eta_{\nu\mu}(-\ln(z-w)))\partial_{w}X^{\nu})+... =,$$

$$= \frac{1}{16}\left[\partial\partial_{w}(\eta^{\nu}_{\mu}(-\ln(z-w)))\partial\partial_{w}(\eta^{\mu}_{\nu}(-\ln(z-w)))+\partial\partial_{w}(\eta_{\nu\mu}(-\ln(z-w)))\partial\partial_{w}(\eta^{\nu\mu}(-\ln(z-w)))\right]+... =,$$

$$= \frac{1}{16}\left[\frac{8}{(z-w)^{4}}\right] + ... = \frac{c}{2(z-w)^{4}} + (...).$$

This implies that the bosonic free field has central charge $c = 1$. This theory is also used to describe bosonic strings in String Theory.

12.7 Free Fermion

The action for the Free fermion field $\psi(z)$ is,

$$S = \int d^{2}z[\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}].$$

Using the same procedure as the last case, we find the equations of motion,

$$\partial\bar{\psi} = \bar{\partial}\psi = 0,$$

the OPE,

$$\psi(z)\psi(w) = \frac{1}{z-w},$$

and the non-zero components of the Stress Tensor,

$$T(z) = -\frac{1}{2}(\psi(z)\partial\psi(z)),$$

$$\bar{T}(\bar{z}) = -\frac{1}{2}(\bar{\psi}(\bar{z})\bar{\partial}\bar{\psi}(\bar{z})).$$

With this, we find,

$$T(z)\psi(w) = \frac{1}{2}\frac{\psi(z)}{(z-w)^2} + (...),$$

since it has no OPE with the the conjugated part, we see that the fermionic field has conformal dimensions $(\frac{1}{2}, 0)$.

Also, since we are dealing with fermionic fields, they anticommute, so one needs to change their signs when flipping them during the OPE. Doing so, we get,

$$T(z)T(w) = \frac{1}{4}(\psi\partial\psi)(z)(\psi\partial\psi)(w) = \frac{1}{4(z-w)^4} \Longrightarrow c = \frac{1}{2}.$$

A theory with two fermions gives $c = 1$, which is related to the process of fermionization (replacing a boson for a pair of fermions).

12.8 bc Ghost System

A very similar CFT of the fermionic field is the action for the fermionic ghost fields b and c.

$$S = \int d^2z[b\bar{\partial}c + \bar{b}\partial\bar{c}].$$

The Equations of motion and OPE of the fields are

$$\bar{\partial}c = \bar{\partial}b = \partial\bar{c} = \partial\bar{b} = 0,$$

$$b(z)c(w) = \frac{1}{(z-w)},$$

$$c(w)b(z) = \frac{1}{(z-w)}.$$

The stress tensor is,

$$T_{bc}(z) = -2b(z)\partial c(z) + c(z)\partial b(z),$$

with an analog expression for the antiholomorphic part. Making the OPE with the stress tensor gives $(h_b, h_c, \bar{h}_{\bar{b}}, \bar{h}_{\bar{c}}) = (2, -1, 2, -1)$, the rest being zero. The central charge is,

$$T(z)T(w) = \frac{-13}{(z-w)^4} + ... \Longrightarrow c = -26.$$

This result has an implication that in the bosonic string, the number of dimensions is 26.

12.9 A bit about BRST

The BRST (named after Carlo Becchi[1939-2015], Alain Rouet [1942-], Raymond Stora [1930-2015] and Igor Tyutin [1940-]) is a process for quantizing fields. We are going to show an example here related to what has been previously described.

Note that, if we wish to mix bosonic fields with ghost fields, we can write the lagrangian,

$$S = \int d^2z[\partial X^\mu \bar{\partial} X_\mu + b\bar{\partial}c + \bar{b}\partial\bar{c}],$$

This avoids conformal anomalies. This action is invariant under the so called BRST transformations,

$$\delta X^\mu = \eta c \partial X^\mu,$$

$$\delta c = \eta c \partial c,$$

$$\delta b = \eta T,$$

where $T(z) = T_X + T_{bc}$ denotes the stress tensor, and $T_X(z)$ is the stress tensor for the bosonic part (shown in section 4.1).

These symmetries implies the following Noether charge,

$$Q = \oint dz(cT_X + bc\partial c),$$

which has the remarkable property of being 2-nilpotent, $Q^2 = 0$. This allows us to describe the physical states of the theory as cohomology classes of Q_B. This turns out to be a way to gauge fixing the degrees of freedom in a theory.

12.10 Virasoro Modes approach

There is an approach alternative and usually more practical to OPE, which is to define the field ϕ as a state $|\phi\rangle$ such that,

$$L_0 |\phi\rangle = h |\phi\rangle,$$

$$L_n |\phi\rangle = 0, \quad n > 0.$$

To be more precise, write $\phi(z, \bar{z})$ in Laurent modes,

$$\phi(z, \bar{z}) = \sum_{n,m \in \mathbb{Z}} z^{-n-h} \bar{z}^{-m-\bar{h}} \phi_{n,m},$$

in this sense, if one has the vacuum $|0\rangle$, we define,

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z}) |0\rangle = \phi_{-h,-\bar{h}} |0\rangle.$$

If we wish to write the hermitian conjugated version, we use Wick rotation, $x_0 \to ix_0$ and leave x_1 unchanged. Thus, we make the transformation $z \to \frac{1}{\bar{z}}$, and, for a primary field, that implies that,

$$\phi^\dagger(z, \bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \phi(z^{-1}, \bar{z}^{-1}) \Longrightarrow \boxed{\phi^\dagger_{n,m} = \phi_{-n,-m}}.$$

Using the OPE of stress tensor previously developed, we can prove that, for a chiral field,

$$[L_m, \phi_n] = ((h-1)m - n)\phi_{m+n},$$

and acting with the stress tensor is equivalent to act with $L_2 |0\rangle$. In fact, we can use the relation,

$$\langle 0| [L_2, L_{-2}] |0\rangle = \frac{c}{2}.$$

The OPE procedure is equivalent to act the field modes with Virasoro modes and use the commutation relations. In this sense, the bc system ghosts can be written as,

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2},$$

$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1}.$$

With an analog expression for the holomorphic parts. Using the OPE, we find

$$\{b_m, c_n\} = \delta_{m-m}.$$

It is in this sense that the charge Q is an operator.

13 Appendices

13.1 Linear algebra toolbox (AI written)

Throughout these notes we use basic concepts from linear algebra. Here is a compact list of conventions and facts that will be useful.

Definition. A (complex) vector space V is a set equipped with addition and scalar multiplication by $\mathbb{C}$. A linear map is a function $T : V \to V$ such that $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$.

Definition. A basis of V is a set $\{e_i\}$ such that every $v \in V$ can be written uniquely as $v = \sum_i v^i e_i$. The dimension $\dim V$ is the number of basis vectors (when finite).

In a basis, a linear map T is represented by a matrix $T^i{}_j$ defined by $T(e_j) = \sum_i T^i{}_j e_i$. Changing basis corresponds to a similarity transformation $T \mapsto S^{-1}TS$.

Definition. An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that is linear in the second entry, conjugate-linear in the first, and positive definite. In Dirac notation we write $\langle v, w\rangle = \langle v| |w\rangle$.

For an orthonormal basis $\{|i\rangle\}$ we have the completeness relation

$$\boxed{\sum_i |i\rangle \langle i| = \hat{1}}$$

(which generalizes to integrals for continuous labels).

Definition. An eigenvector of T is a nonzero v such that $Tv = \lambda v$ for some scalar $\lambda \in \mathbb{C}$ (the eigenvalue). We say that T is diagonalizable if there is a basis of eigenvectors.

In many QFT examples we deal with Hermitian (self-adjoint) operators $A = A^\dagger$, which have real eigenvalues and can be diagonalized by a unitary change of basis. Projection operators satisfy $P^2 = P$ and represent orthogonal decompositions of the space.

Finally, we frequently use the commutator and anticommutator of operators,

$$[A, B] = AB - BA, \qquad \{A, B\} = AB + BA,$$

which measure (non-)compatibility of observables and encode the algebraic structure behind quantization.

13.2 Schur decomposition (AI written)

Theorem. *(Schur decomposition) Let $A \in M_n(\mathbb{C})$. Then there exists a unitary matrix $U \in U(n)$ such that*

$$\boxed{U^\dagger A U = T}$$

where T is upper triangular. Moreover, the diagonal entries of T are the eigenvalues of A (counted with algebraic multiplicity). [Isaai Schur (1875-1941)]

Proof. We argue by induction on n. For $n = 1$ the statement is trivial.

Assume it holds for $(n-1) \times (n-1)$ matrices and let $A \in M_n(\mathbb{C})$. By the fundamental theorem of algebra, the characteristic polynomial of A has a root $\lambda \in \mathbb{C}$, hence there exists a nonzero vector v with $Av = \lambda v$.

Normalize v so that $\|v\| = 1$ and extend $\{v\}$ to an orthonormal basis $\{v, e_2, \ldots, e_n\}$ of $\mathbb{C}^n$. Let U_1 be the unitary matrix whose columns are this basis. Then the first column of $U_1^\dagger A U_1$ is $U_1^\dagger A v = U_1^\dagger(\lambda v) = \lambda e_1$, so the transformed matrix has the block form

$$U_1^\dagger A U_1 = \begin{pmatrix} \lambda & * \\ 0 & A_1 \end{pmatrix}.$$

By the induction hypothesis applied to $A_1 \in M_{n-1}(\mathbb{C})$, there exists a unitary $U_2 \in U(n-1)$ such that $U_2^\dagger A_1 U_2 = T_1$ is upper triangular. Define the block-diagonal unitary matrix

$$\widetilde{U}_2 = \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix}.$$

Then

$$\widetilde{U}_2^\dagger (U_1^\dagger A U_1) \widetilde{U}_2 = \begin{pmatrix} \lambda & *' \\ 0 & T_1 \end{pmatrix},$$

which is upper triangular. Taking $U = U_1 \widetilde{U}_2$ proves the decomposition.

Finally, since similar matrices have the same characteristic polynomial and the characteristic polynomial of an upper triangular matrix equals $\prod_i (t - T_{ii})$, the diagonal entries of T are precisely the eigenvalues of A. $\square$

13.3 Noether Theorem

Definition. A transformation $\varphi : U \to U$ defined on the domain of a Lagrangian function is called a continuous symmetry iff $\exists\ F^u : U \to \mathbb{C}$ such that $\delta_\varphi \mathcal{L} = \partial_u F^u$.

Theorem. *(Noether) A continuous symmetry implies the existence of a conserved current in the equations of motion, i.e,* $\exists\ j^u : U \to \mathbb{C}$ *such that* $\partial_u j^u = 0$.

Proof.

$$\delta_\varphi \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial \varphi} + \frac{\partial \mathcal{L}}{\partial(\partial^u \phi)}\frac{\partial(\partial^u \phi)}{\partial \varphi} = \partial_u F^u,$$

by Euler-Lagrange equations, we get,

$$\partial_u \left(\frac{\partial \mathcal{L}}{\partial \partial_u \phi}\frac{\partial \phi}{\partial \varphi} \right) = \partial_u F^u,$$

$$\boxed{j^u = \frac{\partial \mathcal{L}}{\partial(\partial_u \phi)}\frac{\partial \phi}{\partial \varphi} - F^u \Rightarrow \partial_u j^u = 0}.$$

□

Definition. If we have a Noether current, we may define the conserved charge, $\boxed{Q = \int_M d^3 x j^0}$.

Note that, by Stokes theorem, this implies $\boxed{\frac{dQ}{dt} = -\int_{\partial M} \vec{j} \cdot d\vec{S}}$.

Definition. Note that we can write $\partial_v \mathcal{L} = \partial_u(\delta^u_v \mathcal{L}) \Rightarrow \boxed{T^u_v = \frac{\partial \mathcal{L}}{\partial(\partial_u \phi)}\partial_v \phi - \delta^u_v \mathcal{L}}$. *This defines the Stress-Energy Tensor.* And we have the conserved quantities, $Q^u = \int_M d^3 x T^{u0}$. Essentially, stress tensor comes from invariance of translation of coordinates.

We can also compute the Stress Tensor based on metric perturbations of the action,

$$S = \int d^4 x \sqrt{-g}\, \mathcal{L}(g^{\mu\nu}, \Phi_i, \partial_\mu \Phi_i), \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

To first order in the perturbation $h_{\mu\nu}$, we write

$$\mathcal{L}(g, \Phi) = \mathcal{L}(\eta, \Phi) + h^{\mu\nu}\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}(\eta, \Phi),$$

and, since $\det(\eta_{\mu\nu}) = -1$,

$$\sqrt{-g} \simeq 1 - \frac{1}{2} h^\mu{}_\mu.$$

Therefore,

$$\delta S = \int d^4 x \left(h^{\mu\nu}\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2} h^{\mu\nu}\eta_{\mu\nu}\mathcal{L} \right) = \frac{1}{2}\int d^4 x \left(2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \eta_{\mu\nu}\mathcal{L} \right) h^{\mu\nu}.$$

This motivates the definition

$$\boxed{T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \eta_{\mu\nu}\mathcal{L}}.$$

By construction, this tensor is symmetric and is associated with variations of the metric rather than coordinate translations.

13.4 Quantum harmonic oscillator revisited

Quantum harmonic oscillator is here stated only in one dimension (already contains all the physics we need),

$$E\psi = -\frac{\psi''}{2m} + \frac{m(\omega x)^2}{2}\psi,$$

We can solve this differential equation by Frobenius method and get that, for n a non-negative integer,

$$E_n = \omega\left(n + \frac{1}{2}\right),$$

for eigenstates $|n\rangle$. Considering that,

$$\hat{H}\psi = \frac{1}{2m}\big(\hat{p}^2 + (m\omega\hat{x})^2\big)\psi,$$

we define,

$$\hat{a}_\pm = \frac{1}{2}\left(\sqrt{2m\omega}\,\hat{x} \pm i\sqrt{\frac{2}{m\omega}}\,\hat{p}\right).$$

And $\hat{a} = \hat{a}_+$, $\hat{a}^\dagger = \hat{a}_-$. This way, we can show that,

$$\omega\big(\hat{a}^\dagger\hat{a}\big) = \hat{H} + \frac{i\omega}{2}[\hat{x},\hat{p}] \Rightarrow \hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),$$

$$\omega\big(\hat{a}\hat{a}^\dagger\big) = \hat{H} - \frac{i\omega}{2}[\hat{x},\hat{p}] \Rightarrow \hat{H} = \omega\left(\hat{a}\hat{a}^\dagger - \frac{1}{2}\right),$$

$$[\hat{a},\hat{a}^\dagger] = \hat{1},$$

$$\hat{H}\hat{a}\,|n\rangle = (E_n - \omega)\hat{a}\,|n\rangle\,,$$

$$\hat{H}\hat{a}^\dagger\,|n\rangle = (E_n + \omega)\hat{a}^\dagger\,|n\rangle\,.$$

From the last two identities, we infer,

$$\boxed{\hat{a}\,|n\rangle = A(n)\,|n-1\rangle},$$

$$\boxed{\hat{a}^\dagger\,|n\rangle = B(n)\,|n+1\rangle}.$$

We may infer A and B by the requirement that the states are normalized, thus,

$$\langle n|n\rangle = \langle n-1|\,\frac{\hat{a}}{B(n-1)}\,\frac{\hat{a}^\dagger}{B(n-1)}\,|n-1\rangle = \frac{\omega\big(n-1+\frac{1}{2}\big)+\frac{\omega}{2}}{[B(n-1)]^2} = 1,$$

$$B(n) = \sqrt{\omega(n+1)}.$$

Similarly,

$$\langle n|n\rangle = \langle n+1|\,\frac{\hat{a}^\dagger}{A(n+1)}\,\frac{\hat{a}}{A(n+1)}\,|n+1\rangle = \frac{\omega\big(n+1-\frac{1}{2}\big)+\frac{\omega}{2}}{[A(n+1)]^2} = 1,$$

$$A(n) = \sqrt{\omega n}.$$

Thus, we can finally sum up everything we need to know about the quantum harmonic oscillator,

$$\hat{H}\left|n\right\rangle = \omega\left(n + \frac{1}{2}\right)\left|n\right\rangle,$$

$$\hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),$$

$$\hat{a}\left|n\right\rangle = \sqrt{\omega n}\left|n-1\right\rangle,$$

$$\hat{a}^\dagger\left|n\right\rangle = \sqrt{\omega(n+1)}\left|n+1\right\rangle,$$

$$[\hat{a}, \hat{a}^\dagger] = \hat{1},$$

$$\hat{x} = \frac{1}{\sqrt{2m\omega}}\big(\hat{a} + \hat{a}^\dagger\big),$$

$$\hat{p} = -i\sqrt{\frac{m\omega}{2}}\big(\hat{a} - \hat{a}^\dagger\big).$$

13.5 Klein Gordon vs Dirac probability conservation

Schrodinger equation, named after austrian physicist Erwin Schrodinger (1887-1961), express energy conservation for a non-relativistic particle with wave function ψ, mass m, momentum p and under potential energy V:

$$E\psi = \frac{(p^2)}{2m}\psi + V\psi.$$

Substituting Energy and momentum to their respective operators:

$$\hat{E} = i\hbar\frac{\partial}{\partial t},$$

$$\vec{\hat{p}} = -i\hbar\vec{\nabla},$$

We find:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + V\psi.$$

By naming $\rho = |\psi|^2$ and $\vec{j} = \frac{-i}{2m}(\vec{\nabla}\psi\psi^* - \vec{\nabla}\psi^*\psi)$, we can manipulate the above equation in order to get (Exercise),

$$\partial_t\rho + \vec{\nabla}\cdot\vec{j} = 0,$$

which is conservation of probability (note that this is consistent with Max Born interpretation that $|\psi|^2$ is the probability density).

The first attempt to apply relativity into the equation was done in 1927 by Swiss physicist Oskar Klein (1894-1977) and German physicist Walter Gordon (1893-1939). By taking Einstein energy equation and substituting for the operators shown above, the following equation was found (in natural units):

$$\left(- \frac{\partial^2}{\partial t^2} + \nabla^2 - m^2 \right) \psi = 0.$$

Which is Klein-Gordon equation. By manipulating the expression in a similar way we did with Schrodinger equation, we get,

$$\rho = -\frac{\partial \phi}{\partial t} \phi^* + \frac{\partial \phi^*}{\partial t} \phi,$$

$$\vec{j} = \vec{\nabla} \phi \phi^* - \vec{\nabla} \phi^* \phi,$$

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0.$$

As it is possible to see, the probability density can be negative defined. This troubles Born interpretation for the wave function. Dirac, intending to fix this problem, developed his equation as we have seen in section 3, and got,

$$i\psi^* \frac{\partial \psi}{\partial t} = -i\psi^* (\vec{\alpha} \cdot \vec{\nabla} \psi) + \beta m \psi^* \psi,$$

$$-i \frac{\partial \psi^*}{\partial t} \psi = i(\vec{\nabla} \psi \cdot \vec{\alpha}^\dagger \psi) + \beta m \psi^* \psi,$$

thus, taking the difference, and considering that $\alpha = \alpha^\dagger$, we get

$$\partial_t (\psi^* \psi) + \vec{\nabla} (\psi^* \vec{\alpha} \psi) = 0.$$

Thus, we get the ρ to be positive defined again, preserving Born interpretation.

13.6 Residues theorem revisited

Take $f : \mathbb{C} \to \mathbb{C}$. Here, $\mathbb{C}$ is a real vector space with dimension 2 with algebraic product defined by,

$$vw = v * w = (v_1 w_1 - v_2 w_2, v_1 w_2 + v_2 w_1).$$

This is equivalent to write $(x, y) = x + iy$, where $i^2 = -1$. Let $z = x + iy$.

Definition. We say a function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* if $f(x, y) = f(z) = f(x + iy)$ and if it is differentiable in respect to z.

Theorem. *If $f(x, y) = u(x, y) + iv(x, y)$ is holomorphic, it respects the Cauchy-Riemann equations, i.e, $u_y = -v_x$ and $u_x = v_y$.*

Proof.

$$f(x, y) = f(x + iy) \Rightarrow u_x + iv_x = \partial_r f(r),$$

$$u_y + iv_y = i \partial_r f(r),$$

By comparing the two expressions, considering both u and v are real, we get, $u_y = -v_x$ and $u_x = v_y$. $\square$

Example. The inverse of the above theorem is false for when the function f fails to be continuous everywhere. Take, for instance,

$$f(z) = e^{\frac{1}{z}},$$

for $z \neq 0$ and $f(0) = 0$. The function is not continuous at zero, but,

$$f(x, y) = e^{\frac{x-iy}{x^2+y^2}} = e^{\frac{x}{x^2+y^2}} \left(\cos\left(\frac{y}{x^2+y^2}\right) - i \sin\left(\frac{y}{x^2+y^2}\right) \right),$$

so, $u(x, y) = e^{\frac{x}{x^2+y^2}} \left(\cos\left(\frac{y}{x^2+y^2}\right)\right)$ and $v(x, y) = -e^{\frac{x}{x^2+y^2}} \left(\sin\left(\frac{y}{x^2+y^2}\right)\right)$. We can thus directly check that $u_x = v_y$ and $u_y = -v_x$, even for the limit of (x, y) going to 0.

Now, we shall talk about residues. For any holomorphic differentiable complex function, we can write its Laurent series around a point z_0,

$$f(z) = \sum_{m \in \mathbb{Z}} A_m (z - z_0)^m,$$

for $A_m \in \mathbb{C}$. If we integrate in a closed simple connected path ξ,

$$\oint_\xi f(z) dz = A_m \oint_\xi (z - z_0)^m dz,$$

$$z = re^{i\theta} + z_0 \Rightarrow dz = dre^{i\theta} + ire^{i\theta} d\theta.$$

The integral on dr vanishes, for the endpoints are equal, for the other part,

$$A_m i r^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} d\theta,$$

this is non-vanishing only for $m = -1$. Therefore, we conclude with Cauchy residues theorem,

$$\boxed{\oint_\xi f(z) dz = 2\pi i A_{-1}}.$$

Where the coefficient A_{-1} is the residue of the function in the region limited by ξ. Note that, if $f(z) = \frac{g(z)}{(z-z_0)^m}$, assuming g has no poles and for $m \geq 1$, it is true that, applying integration by parts,

$$\oint_\xi \frac{g(z)}{(z - z_0)^m} dz = \oint_\xi \frac{g^{(m-1)}(z)}{(m-1)!(z - z_0)} dz,$$

thus, we get that the residues of order m for a complex function f are,

$$\lim_{z \to z_0} \frac{d^m}{dz^m} \left(\frac{1}{(m-1)!} (z - z_0)^m f(z) \right).$$

We shall show a few examples,

Example. Let us solve Dirichlet integral,

$$I=\int_{-\infty}^{\infty}dx\frac{\sin x}{x}=Im\Big(\int_{-\infty}^{\infty}dx\frac{e^{ix}}{x}\Big).$$

We can see that this has a pole in 0, so, we write the contour,

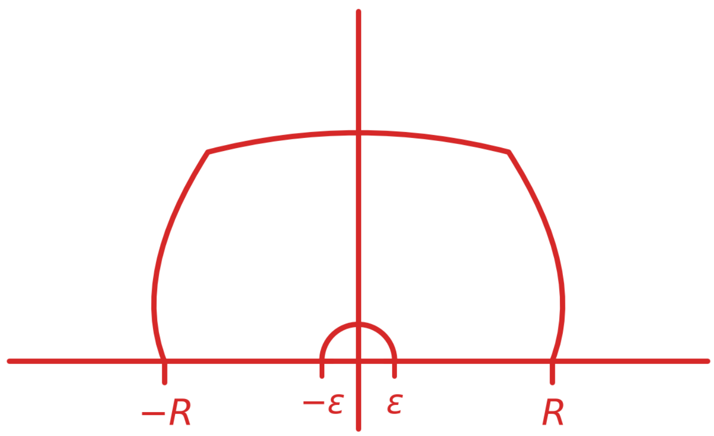


Figure : *Contour. In here, R is taken to infinity and ϵ is taken to zero.*

and, since there is no poles inside the contour, we have,

$$\int_{\epsilon}^{R}\frac{e^{ix}}{x}dx\quad+\quad\int_{0}^{\pi}id\theta(e^{i(Re^{i\theta})})\quad+\quad\int_{-R}^{-\epsilon}\frac{e^{ix}}{x}dx\quad+\quad\int_{\pi}^{0}id\theta(e^{i(\epsilon e^{i\theta})})=0.$$

Taking the limits for the values, we get,

$$\int_{-\infty}^{\infty}dx\frac{e^{ix}}{x}=i\pi,$$

thus,

$$\boxed{\int_{-\infty}^{\infty}dx\frac{\sin(x)}{x}=\pi}.$$

Example. Take the integral,

$$I=\int_{-\infty}^{\infty}dx\quad\frac{\ln(b^2+x^2)}{a^2+x^2},$$

where a,b are real. We may take $b^2=c$ and $I=I(c)$ and derivate,

$$I'(c)=\int_{-\infty}^{\infty}dx\quad\frac{1}{(a^2+x^2)(x^2+c)},$$

and now we can solve it using contour integral, considering just the upper part $a,b>0$,

$$I'(c)=2\pi i\Big(\frac{1}{2ai(c-a^2)}+\frac{1}{(-c+a^2)(2\sqrt{c}i)}\Big),$$

Integrating in respect to c, we get,

$$I(c) = \frac{2\pi}{a}\ln(a + \sqrt{c}) + C,$$

where the constant C is zero by noting the integral in the beginning of this example. Thus,

$$I = \frac{2\pi}{a}\ln(a + b).$$

13.7 Gaussian integral of matrix

Lemma. *Every symmetric real matrix A is diagonalizable.*

Proof. By Schur decomposition, $\exists\ U \in M_n(\mathbb{C})$ orthogonal such that $A = UBU^T$, with B being an upper triangular matrix. This implies $B = U^T AU$, $B^T = U^T A^T U = U^T AU = B$. Since B is symmetric and triangular, it has to be diagonal.

□

Theorem.

$$\int \Pi_{k=1}^{N-1} d\eta_k \exp\frac{i\eta^T M\eta}{2} = \frac{\pi^{\frac{N-1}{2}}}{\sqrt{\det(\frac{iM}{2})}},$$

Proof. Notice, from section 3, that M is a real diagonal matrix. This implies that it is diagonalizable by the previous lemma. So, there is a basis such that M is diagonal. Taking it, and writing eigenvalues of M as $-im_j$ (assuming none vanish), we got that the integral turns into,

$$\int d\eta_1 ... d\eta_{N-1} \exp\frac{-m_k(\eta^k)^2}{2} = \Pi_{k=1}^{N-1}\sqrt{\frac{2\pi}{m_k}} = \sqrt{\frac{\pi^{N-1}}{\det(\frac{iM}{2})}}.$$

□