Supersymmetric Field Theories

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Motivation

Supersymmetry (SUSY) is one of the core on going theories in theoretical physics. Although no major experiment has seen a direct evidence of its existence, as well as no falsification of it, it provides a possible solution to a number of problems still open today, such as "WIMP" particles in cosmology, the hierarchy problem, and it provides the necessary tools to compactify strings in string theory (where it appears naturally).

As we have seen, in QFT, non renormalizable theories can be dealt with by dealing with scales well below the cut-off,

$$g_i(n) = g_i(n_0)\left(\frac{\Lambda}{\Lambda_0}\right)^{(\alpha_i - 4)},$$

When $\alpha_i > 4$ (non-renormalizable), the coupling goes to zero for $\Lambda << \Lambda_0$. But, for high energies, this does not work. Supersymmetry is a possible solution to this problem, as it attempts to build theories that are valid for high energies, and give rise to non-supersymmetric theories in low scales by the symmetry breaking.

In 1967, Sidney Coleman (1937-2007) and Jeffrey Mandula (1941-) proved (in a theorem that now holds their names) that all Lie algebras that generate internal and space-time transformations can only combine them by trivially commuting (i.e, $[T_r, P_u] = [T_r, M_{uv}] = 0$, for T_r a internal symmetry generator). Supersymmetry evades this problem by extending the Lie algebra to a Lie super algebra, which is what we shall do now.

Supergeometry

Definition

An *I-graded vector space* V is a direct sum over a set of vector spaces V_i for $i \in I$ such that each $V_i \neq 0$, i.e,

$$V = \oplus_{i \in I} V_i,$$

An elements $v \in V$ is called homogeneous if $\exists\ i \in I$ such that $v \in V_i$.

Definition

An *I-graded algebra* A is an algebra over an I-graded vector space,

$$A = \oplus_{i \in I} A_i,$$

where each A_i is defined over V_i and the product $*$ in the algebra obeys, for $v \in V_i$ and $w \in V_j$:

$$v * w \in V_{i+j}$$

Now, we get to our central point,

Definition

The prefix super means graded by $\mathbb{Z}_2$, i.e,
$V = V_0 \oplus V_1$ is a super vector space,
$A = A_0 \oplus A_1$ is a super algebra.

Example

$$V = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Big/ A \in M_{pp}(\mathbb{C}), B \in M_{pq}(\mathbb{C}), C \in M_{qp}(\mathbb{C}), D \in M_{qq}(\mathbb{C}) \right\}$$

is a super vector space of dimension $(p^2 + q^2, 2pq)$, where the bosonic part is
$V_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \Big/ A \in M_{pp}(\mathbb{C}), D \in M_{qq}(\mathbb{C}) \right\}$ and the fermionic part is
$V_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \Big/ B \in M_{pq}(\mathbb{C}), C \in M_{qp}(\mathbb{C}) \right\}$.

Proposition

$(V, *)$, where $*$ stands for usual matrix product, is a super algebra.

Proof.

Notice,

$$V_0 * V_0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} * \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ 0 & DD' \end{pmatrix} \in V_0,$$

$$V_0 * V_1 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} * \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & AB \\ DC & 0 \end{pmatrix} \in V_1,$$

$$V_1 * V_0 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} * \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & BD \\ CA & 0 \end{pmatrix} \in V_1,$$

$$V_1 * V_1 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} * \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix} = \begin{pmatrix} BC' & 0 \\ 0 & CB' \end{pmatrix} \in V_0,$$

thus, the condition for $(V, *)$ to be a super algebra is satisfied. This is the so called $Mat(p|q)$ superalgebra. □

Super Lie Algebra

n what follows, if $x \in V$ is a homogeneous element of a super vector space, we shall adopt the notation $\bar{x} = 1$, if $x \in V_1$, and $\bar{x} = 0$ otherwise.

Definition

A super Lie algebra is a super algebra $(V, [\ ,\ ])$ such that, for all X, Y homogeneous elements, we get,

I) Bilinearity

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

II) Super anticommutativity

$$[X, Y] = -(-1)^{\bar{X}\bar{Y}}[Y, X].$$

III) Super Jacobi identity

$$(-1)^{\bar{X}\bar{Z}}[X, [Y, Z]] + (-1)^{\bar{X}\bar{Y}}[Y, [Z, X]] + (-1)^{\bar{Y}\bar{Z}}[Z, [X, Y]] = 0.$$

Note that, due to the bilinearity property, knowing how the supercommutator works for homogeneous elements show us how it works for any element.

Theorem

If $(V, *)$ is a super algebra with commutator $[x, y] = x * y - (-1)^{\bar{x}\bar{y}} y * x$, with $*$ being a distributive and associative operation, then it is a super Lie algebra.

Proof.

In what follows, we shall write the product as xy.
The bilinearity comes directly from the distributive property of the algebra product.
The super anticommutativity follows easily,

$$[x,y] = xy - (-1)^{\bar{x}\bar{y}} yx = -(-1)^{\bar{x}\bar{y}}(yx - (-1)^{\bar{x}\bar{y}} xy) = -(-1)^{\bar{x}\bar{y}}[y,x],$$

Let $I = (-1)^{\bar{X}\bar{Z}}[X,[Y,Z]]$, $II = (-1)^{\bar{X}\bar{Y}}[Y,[Z,X]]$ and $III = (-1)^{\bar{Y}\bar{Z}}[Z,[X,Y]]$. Expanding the brackets and noticing that $\overline{[x,y]} = \overline{xy} = \bar{x}\bar{y}$,

$$I = (-1)^{\bar{x}\bar{z}} xyz - (-1)^{\bar{z}(\bar{x}+\bar{y})} xzy - (-1)^{\bar{x}\bar{y}}(yzx - (-1)^{\bar{y}\bar{z}} zyx),$$

$$II = (-1)^{\bar{x}\bar{y}} yzx - (-1)^{\bar{x}(\bar{z}+\bar{y})} yxz - (-1)^{\bar{z}\bar{y}}(zxy - (-1)^{\bar{x}\bar{z}} xzy),$$

$$III = (-1)^{\bar{z}\bar{y}} zxy - (-1)^{\bar{y}(\bar{z}+\bar{x})} zyx - (-1)^{\bar{z}\bar{x}}(xyz - (-1)^{\bar{x}\bar{y}} yxz).$$

We can see that, upon summing, all terms cancel, so, super Jacobi is obeyed.

□

Super Manifolds

Definition

A supermanifold of superdimension $(m|n)$ is $(M_{rd}, A(\mathbb{R}^n))$ such that,

I) M_{rd} is manifold of dimension m

II) $A(\mathbb{R}^n)$ is a set of n Grassmann variables ($\{\theta_1, ..., \theta_n\}$ such that $\theta_i\theta_j = -\theta_j\theta_i$).

This way, a supermanifold may be parametrized by $(x_1, ..., x_m, \theta_1, ..., \theta_n)$ where x are the manifold coordinates and θ are the Grassmann variables. *Superdomains* $\mathcal{U}$ may be defined similarly, as an open set $U \subset M_{rd}$ with the sheaf of Grassmann variables. The set of functions in superdomains $\mathcal{U}$ is $C^\infty(\mathcal{U}) = C^\infty(U) \otimes A(\theta_1, ..., \theta_n)$ and, for $f \in C^\infty(\mathcal{U})$, we have,

$$f(x_1, ..., x_m, \theta_1, ..., \theta_n) = f_0(\vec{x}) + \sum_{i=1}^{n} f_i(\vec{x})\theta_i + \sum_{i<j} f_{ij}(\vec{x})\theta_i\theta_j + ...$$

We call the functions being multiplied by an even number of θ *bosons*, and for an odd number, *fermions*. The intuition behind this nomenclature can be taken from the fact that, for *n* an even number, $[\theta_{i_1}...\theta_{i_n}, \theta_{j_1}...\theta_{j_n}] = 0$, while, if *n* is odd, $[\theta_{i_1}...\theta_{i_n}, \theta_{j_1}...\theta_{j_n}] = 2\theta_{i_1}...\theta_{i_n}\theta_{j_1}...\theta_{j_n}$.
In the context of physics, Supermanifolds are usually denoted as Superspaces, and functions on it are denoted superfields, having both bosonic and fermionic part.
Formally, we give the following definition,

Definition

Given a superdomain of a supermanifold, $\mathcal{U}$, we have the super vector space $V(C^\infty(\mathcal{U})) = V_0(C^\infty(\mathcal{U})) \oplus V_1(C^\infty(\mathcal{U}))$, where,

$$V_0(C^\infty(\mathcal{U})) = \{f \in C^\infty(\mathcal{U}) \ / f \ \textit{is bosonic}\},$$

$$V_1(C^\infty(\mathcal{U})) = \{f \in C^\infty(\mathcal{U}) \ / f \ \textit{is fermionic}\}.$$

This is equivalent to describe elements of V as $f_{i_1 \dots i_m}\theta^{i_1}\dots\theta^{i_m}$ and say they belong to V_0 if m is even, and belong to V_1 otherwise. So, in this description superfields are vectors. We can induce the superalgebra $(V, *)$ with the product,

$$(f_{i_1 \dots i_m}\theta^{i_1}\dots\theta^{i_m}) * (f_{j_1 \dots j_l}\theta^{j_1}\dots\theta^{j_l}) = (-1)^{ml} f_{i_1 \dots i_m} f_{j_1 \dots j_l}\theta^{i_1}\dots\theta^{i_m}\theta^{j_1}\dots\theta^{j_l}.$$

Note that this implies that,

$$(f_{j_1 \dots j_l}\theta^{j_1}\dots\theta^{j_l}) * (f_{i_1 \dots i_m}\theta^{i_1}\dots\theta^{i_m}) = (-1)^{ml}(f_{i_1 \dots i_m}\theta^{i_1}\dots\theta^{i_m}) * (f_{j_1 \dots j_l}\theta^{j_1}\dots\theta^{j_l}).$$

So, $(V, *)$ is a super commutative superalgebra, (i.e, $x * y = (-1)^{\bar{x}\bar{y}} y * x \Rightarrow [x, y] = 0$ under the commutator of the previous theorem) which we will denote as **superfield vector space**. By the last theorem, it has a natural structure of a Super Lie Algebra.

Theorem

The number of bosonic terms (or degress of freedom) is the same of fermionic.

Proof.

Note that, when we multiply a term for k Grassmann variables, we are essentially choosing k out of n, so we have $\binom{n}{k}$ terms. Thus, we have $\sum_{k \quad even} \binom{n}{k}$ bosonic terms and $\sum_{k \quad odd} \binom{n}{k}$ fermionic. By Newton's bynomial theorem,

$$(1-1)^n = \sum_{k \quad even} \binom{n}{k} - \sum_{k \quad odd} \binom{n}{k} = 0.$$

So, they must be equal.

□

SUSY theories

Definition

A **Susy system** is a Superfield vector space (V) with an odd derivation $Q : V \to V$ such that $Q(V_0) \subset V_1$ and $Q(V_1) \subset V_0$. Also, given a constant fermionic parameter η, a **SUSY transformation** of a vector v is defined as,

$$\eta Q(v) = \delta_Q v.$$

We define a supersymmetric theory as a theory invariant under SUSY transformations.

Example

Lets take the case of $\mathbb{R}^{1|1}$ space time, i.e, one coordinate bosonic and one fermionic. In this case, we can take the superfield,

$$\Phi(t,\theta) = \phi(t) + \theta\psi(t).$$

Also, let's write the action,

$$S = \int dt\, d\theta (D\Phi\partial_t\Phi).$$

In which $D = \partial_\theta - \theta\partial_t$. Our susy generator will be,

$$Q = \partial_\theta + \theta\partial_t.$$

As one can check, this satisfies,

$$[Q, \partial_t] = 0,$$

$$\{Q, Q\} = 2\partial_t.$$

Example

Now, rewriting the action,

$$S = -\int dt\ (\psi\dot{\psi} + \dot{\phi}^2).$$

So, we may apply the susy generator, with the fermionic parameter η,

$$\eta Q(\Phi) = \eta\theta\dot{\phi} + \eta\psi = \delta_Q\phi + \theta\delta_Q\psi.$$

Which implies,

$$\delta_Q\phi = \eta\psi.$$

$$\delta_Q\psi = -\eta\dot{\phi}.$$

Those are the susy transformations. Applying this transformation into the argument of the action, we find,

$$\delta_Q(\psi\dot{\psi} + \dot{\phi}^2) = -\eta\dot{\phi}\dot{\psi} + \psi(-\eta\ddot{\phi}) + 2\dot{\phi}(\eta\dot{\psi}) = \eta\dot{\psi}\dot{\phi} + \eta\psi\ddot{\phi} = \partial_t(\eta\psi\dot{\phi}).$$

Since this is a total derivative, it vanishes when integrated, so the action is indeed supersymmetric.

SUSY algebra

We know Poincare algebra denotes all possible isometries in Mikowski space. If $P(3,1)$ is the Poincare group

$$P(3,1) = \{F : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1} | \quad \eta_{uv} dx^u dx^v = \eta_{uv} (dF(x))^u (dF(x))^v\}.$$

Definition

The Poincare algebra is the Lie algebra of Poincare group $P(3,1)$. It is a ten dimensional complex Lie algebra that has as generators $J^{uv} = i(x^u \partial^v - x^v \partial^u)$ and $P_\rho = -i\partial_\rho$. The Lie commutation relations are,

$$1) \quad \boxed{[P_\rho, P_v] = 0}.$$

$$2) \quad \boxed{[J^{uv}, J^{\sigma\rho}] = i(\eta^{\sigma\mu} J^{\rho\nu} + \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\rho\mu} J^{\sigma\nu} - \eta^{\nu\rho} J^{\mu\sigma})}.$$

$$3) \quad \boxed{[J_{uv}, P_\rho] = i(\eta_{u\rho} P_v - \eta_{v\rho} P_u)}.$$

The SUSY algebra is an extension of Poincare algebra, containing spinorial generators Q_α. They transform under,

$$Q'_\alpha = \exp\left(\frac{i}{2}\sigma_{uv}\omega^{uv}\right)Q_\alpha,$$

in which $\sigma_{uv} = \frac{1}{4}(\sigma_u\sigma_v - \sigma_v\sigma_u)$ and ω_{uv} is the infinitesimal parameter $\omega_{uv} = \Lambda_{uv} - \delta_{uv}$. For vectors transforming under Lorentz,

$$|v'\rangle = \exp(ia^u P_u - iM_{uv}\omega^{uv})\,|v\rangle = U\,|v\rangle\,.$$

his means that,

$$\begin{aligned} |v'\rangle &= U\,|v\rangle\,, \\ |b\rangle &= Q_\alpha\,|v\rangle\,. \end{aligned}$$

Therefore,

$$\begin{aligned} |b'\rangle &= Q'_\alpha\,|v'\rangle \\ \Rightarrow \quad U\,|b\rangle &= Q'_\alpha U\,|v\rangle \\ \Rightarrow \quad |b\rangle &= U^{-1}Q'_\alpha U\,|v\rangle\,. \end{aligned}$$

Thus,

$$Q_\alpha = U^{-1}Q'_\alpha U.$$

With this, we have that,

$$\begin{aligned} Q_\alpha &= \exp\left(-ia^\mu P_\mu + iM_{\mu\nu}\omega^{\mu\nu}\right) Q'_\alpha \exp\left(ia^\mu P_\mu - iM_{\mu\nu}\omega^{\mu\nu}\right) \\ &= Q'_\alpha + ia^\mu [Q'_\alpha, P_\mu] + i[M_{\mu\nu}, Q'_\alpha]\omega^{\mu\nu}. \end{aligned}$$

Therefore,

$$\delta Q_\alpha = ia^\mu [Q'_\alpha, P_\mu] + i[M_{\mu\nu}, Q'_\alpha]\omega^{\mu\nu} = -\frac{1}{2}\sigma_{\mu\nu}\omega^{\mu\nu} Q'_\alpha.$$

Thus,

$$\boxed{\begin{aligned} [Q'_\alpha, P_\mu] &= 0, \\ [M_{\mu\nu}, Q'_\alpha] &= \frac{i}{2}\sigma_{\mu\nu} Q'_\alpha \end{aligned}}.$$

Now, considering M_{uv}, P_u bosonic operators and Q_α fermionic, to make this a Super Lie algebra in accordance with theorem of section 2.2, we imply the anticommutation condition,

$$\{Q_\alpha, Q_\beta\} = Q_\alpha Q_\beta + Q_\beta Q_\alpha,$$

since Q_α is an spinor operator,

$$[Q_\alpha, Q_\beta]_{\text{super}} = \{Q_\alpha, Q_\beta\}.$$

So, for our purpose, it is a super Lie algebra. And,

$$\{Q_\alpha, Q_\beta\} = c^{\mu}_{\alpha\beta} P_\mu + \lambda^{\mu\nu}_{\alpha\beta} M_{\mu\nu}.$$

Now,

$$\begin{aligned} [P^\mu, \{Q_\alpha, Q_\beta\}] &= P^\mu Q_\alpha Q_\beta + P^\mu Q_\beta Q_\alpha - Q_\alpha Q_\beta P^\mu - Q_\beta Q_\alpha P^\mu \\ &= Q_\alpha Q_\beta P^\mu + Q_\beta Q_\alpha P^\mu - Q_\alpha Q_\beta P^\mu - Q_\beta Q_\alpha P^\mu = 0. \end{aligned}$$

Thus,

$$\lambda^{\mu\nu}_{\alpha\beta}[P^\rho, M_{\mu\nu}] = 0 \quad \Rightarrow \quad \lambda^{\mu\nu}_{\alpha\beta} = 0.$$

$$\boxed{\{Q_\alpha, Q_\beta\} = (c^{m}_{\alpha\beta}) P_m}.$$

Using Jacobi identity between generatos M_{uv}, Q_α and Q_β, we can show (with a quite hard algebra) that $c^m_{\alpha\beta} = \sigma^m_{\alpha\beta}$. So, just by adding a fermionic generator into the algebra, we find a Lie Super-algebra that non-trivially commutes susy symmetries with space-time ones, which evades the result from Coleman-Mandula theorem.
In this context, each Q_α is a susy operator, and susy transformation are $\eta^\alpha Q_\alpha$, where each η^α is a fermionic variable.

RNS superstring

For our next example, we will increase the difficult by considering a $(2, 2)$ dimensionam supermanifold. A physical interesting example of such system is RNS (Ramond-Neveu-Schwarz)superstring, which is one of the main the approaches to add fermionic degress of freedom into strings (the other two being the Grenn-Schwarz formalism and Pure Spinor formalism).

Let the RNS superstring action be

$$S = \int d\tau\, d\sigma \left(\partial_+ X^m \partial_- X_m - i\psi_-^m \partial_+ \psi_{-m} - i\psi_+^m \partial_- \psi_{+m} \right).$$

Where X^m denote the string bosonic coordinates and ψ denote the fermionic coordinates (as we will see, they Weyl spinors, which are components of the Dirac Spinor) and the string Worldsheet are written in light cone. Taking the transformations

$$\delta X^m = i\alpha^+\psi_+^m + i\alpha^-\psi_-^m,$$
$$\delta\psi_+^m = \alpha^+\partial_+ X^m,$$
$$\delta\psi_-^m = \alpha^-\partial_- X^m,$$

we get

$$\delta S = \int d\tau\, d\sigma \Big[\left(i\alpha^+ \partial_+ \psi_+^m + i\alpha^- \partial_+ \psi_-^m\right) \partial_- X_m + \partial_+ X^m \left(i\alpha^+ \partial_- \psi_{+m} + i\alpha^- \partial_- \psi_{-m}\right)$$

$$- i\alpha^- \partial_- X^m \partial_+ \psi_{-m} - i\psi_-^m \partial_+ \left(\alpha^- \partial_- X_m\right) - i\alpha^+ \partial_+ X^m \partial_- \psi_{+m} - i\psi_+^m \partial_- \left(\alpha^+ \partial_+ X_m\right) \Big]$$

$$= \int d\tau\, d\sigma \Big[i\alpha^+ \partial_+ \left(\psi_+^m \partial_- X_m\right) + i\alpha^- \partial_- \left(\psi_-^m \partial_+ X_m\right) \Big] = 0.$$

In order to find the SUSY generator, we introduce the superfield

$$\mathcal{X}^m(\tau,\sigma,\theta^+,\theta^-) = X^m(\tau,\sigma) + i\theta^+\psi_+^m(\tau,\sigma) + i\theta^-\psi_-^m(\tau,\sigma) + i\theta^+\theta^-F^m(\tau,\sigma).$$

Acting with a SUSY transformation on the superfield, we have

$$\left(\alpha^+ Q_+ + \alpha^- Q_-\right)\mathcal{X}^m = \delta X^m + i\theta^+\delta\psi_+^m + i\theta^-\delta\psi_-^m + i\theta^+\theta^-\delta F^m.$$

Therefore, disregarding the δF term, the action of the generators is

$$Q_+\mathcal{X}^m = \left(i\psi_+^m - i\theta^+\partial_+X^m\right) + A^m,$$
$$Q_-\mathcal{X}^m = \left(i\psi_-^m - i\theta^-\partial_-X^m\right) + B^m,$$

where the extra terms depend on the other field. Thus,we can state,

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} - i\theta^\pm\partial_\pm,$$

with

$$A^m = Q_+\left(i\theta^+\theta^-F^m\right), \qquad B^m = Q_-\left(i\theta^+\theta^-F^m\right).$$

In principle, this extra term is arbitrary. However, when we try to close the algebra, we get

$$[\delta_2, \delta_1]X^m = \delta_2\left(i\alpha_1^+\psi_+^m + i\alpha_1^-\psi_-^m\right) - \delta_1\left(i\alpha_2^+\psi_+^m + i\alpha_2^-\psi_-^m\right)$$
$$= \left[-2i\alpha_1^+\alpha_2^+\partial_+ - 2i\alpha_1^-\alpha_2^-\partial_-\right]X^m = \delta^P X^m.$$

Similarly,

$$[\delta_2, \delta_1]\psi_\pm^m = \left[-2i\alpha_1^\pm\alpha_2^\pm\partial_\pm\psi_\pm^m - 2i\alpha_1^\mp\alpha_2^\mp\partial_\mp\psi_\mp^m\right],$$

which does not close off-shell. We can remedy this by including auxiliary terms in the action.

$$S = \int d\tau\, d\sigma\big(\partial_+X^m\partial_-X_m - i\psi_-^m\partial_+\psi_{-m} - i\psi_+^m\partial_-\psi_{+m} + F^mF_m\big).$$

Note that, under Euler–Lagrange equations, $F^m = 0$, so this auxiliary field plays no role in the physical theory. Considering the expressions previously found for $Q_\pm$,

$$\begin{aligned}
\delta^\alpha \mathcal{X}^m &= \left(\alpha^+ Q_+ + \alpha^- Q_-\right) \mathcal{X}^m \\
&= \alpha^+ \left(i\psi_+^m - i\theta^+\partial_+ X^m + i\theta^- F^m + \theta^+\theta^-\partial_+\psi_-^m\right) \\
&\quad + \alpha^- \left(i\psi_-^m - i\theta^-\partial_- X^m - i\theta^+ F^m - \theta^+\theta^-\partial_-\psi_+^m\right) \\
&= i\alpha^+\psi_+^m + i\alpha^-\psi_-^m + i\theta^+ \left(-\alpha^+\partial_+ X^m - \alpha^- F^m\right) \\
&\quad + i\theta^- \left(-\alpha^-\partial_- X^m + \alpha^+ F^m\right) + i\theta^+\theta^- \left(-i\alpha^+\partial_+\psi_-^m + i\alpha^-\partial_-\psi_+^m\right).
\end{aligned}$$

Comparing this with

$$\delta^\alpha \mathcal{X}^m = \delta^\alpha X^m + i\theta^+\delta^\alpha\psi_+^m + i\theta^-\delta^\alpha\psi_-^m + i\theta^+\theta^-\delta^\alpha F^m,$$

we obtain the complete transformations,

$$\begin{aligned}
\delta^\alpha X^m &= i\alpha^+\psi_+^m + i\alpha^-\psi_-^m, \\
\delta^\alpha \psi_+^m &= -\alpha^+\partial_+ X^m - \alpha^- F^m, \\
\delta^\alpha \psi_-^m &= -\alpha^-\partial_- X^m + \alpha^+ F^m, \\
\delta^\alpha F^m &= -i\alpha^+\partial_+\psi_-^m + i\alpha^-\partial_-\psi_+^m.
\end{aligned}$$

ith this, the algebra closes off-shell. For ψ_+^m, the expansion is

$$\begin{aligned}
[\delta_2, \delta_1]\psi_+^m &= \delta_2\left(-\alpha_1^+\partial_+ X^m - \alpha_1^- F^m\right) - \delta_1\left(-\alpha_2^+\partial_+ X^m - \alpha_2^- F^m\right) \\
&= -\alpha_1^+\partial_+\left(i\alpha_2^+\psi_+^m + i\alpha_2^-\psi_-^m\right) - \alpha_1^-\left(-i\alpha_2^+\partial_+\psi_-^m + i\alpha_2^-\partial_-\psi_+^m\right) \\
&\quad + \alpha_2^+\partial_+\left(i\alpha_1^+\psi_+^m + i\alpha_1^-\psi_-^m\right) + \alpha_2^-\left(-i\alpha_1^+\partial_+\psi_-^m + i\alpha_1^-\partial_-\psi_+^m\right) \\
&= \left[-2i\alpha_1^+\alpha_2^+\partial_+ - 2i\alpha_1^-\alpha_2^-\partial_-\right]\psi_+^m.
\end{aligned}$$

For ψ_-^m, we similarly have

$$\begin{aligned}
[\delta_2, \delta_1]\psi_-^m &= \delta_2\left(-\alpha_1^-\partial_- X^m + \alpha_1^+ F^m\right) - \delta_1\left(-\alpha_2^-\partial_- X^m + \alpha_2^+ F^m\right) \\
&= -\alpha_1^-\partial_-\left(i\alpha_2^+\psi_+^m + i\alpha_2^-\psi_-^m\right) + \alpha_1^+\left(-i\alpha_2^+\partial_+\psi_-^m + i\alpha_2^-\partial_-\psi_+^m\right) \\
&\quad + \alpha_2^-\partial_-\left(i\alpha_1^+\psi_+^m + i\alpha_1^-\psi_-^m\right) - \alpha_2^+\left(-i\alpha_1^+\partial_+\psi_-^m + i\alpha_1^-\partial_-\psi_+^m\right) \\
&= \left[-2i\alpha_1^+\alpha_2^+\partial_+ - 2i\alpha_1^-\alpha_2^-\partial_-\right]\psi_-^m.
\end{aligned}$$

Finally, for the auxiliary field,

$$\begin{aligned}
[\delta_2, \delta_1]F^m &= \delta_2\left(-i\alpha_1^+\partial_+\psi_-^m + i\alpha_1^-\partial_-\psi_+^m\right) - \delta_1\left(-i\alpha_2^+\partial_+\psi_-^m + i\alpha_2^-\partial_-\psi_+^m\right) \\
&= -i\alpha_1^+\partial_+\left(-\alpha_2^-\partial_- X^m + \alpha_2^+ F^m\right) + i\alpha_1^-\partial_-\left(-\alpha_2^+\partial_+ X^m - \alpha_2^- F^m\right) \\
&\quad + i\alpha_2^+\partial_+\left(-\alpha_1^-\partial_- X^m + \alpha_1^+ F^m\right) - i\alpha_2^-\partial_-\left(-\alpha_1^+\partial_+ X^m - \alpha_1^- F^m\right) \\
&= \left[-2i\alpha_1^+\alpha_2^+\partial_+ - 2i\alpha_1^-\alpha_2^-\partial_-\right]F^m.
\end{aligned}$$

Moreover, from the supercharges we note that

$$\begin{aligned}
\{Q_+, Q_+\} &= 2\left(\frac{\partial}{\partial\theta^+} - i\theta^+\partial_+\right)\left(\frac{\partial}{\partial\theta^+} - i\theta^+\partial_+\right) \\
&= 2\left[-i\partial_+ + i\theta^+\partial_+\frac{\partial}{\partial\theta^+} - i\theta^+\partial_+\frac{\partial}{\partial\theta^+}\right] = -2i\partial_+, \\
\{Q_-, Q_-\} &= 2\left(\frac{\partial}{\partial\theta^-} - i\theta^-\partial_-\right)\left(\frac{\partial}{\partial\theta^-} - i\theta^-\partial_-\right) = -2i\partial_-, \\
\{Q_+, Q_-\} &= \left(\frac{\partial}{\partial\theta^+} - i\theta^+\partial_+\right)\left(\frac{\partial}{\partial\theta^-} - i\theta^-\partial_-\right) \\
&\quad + \left(\frac{\partial}{\partial\theta^-} - i\theta^-\partial_-\right)\left(\frac{\partial}{\partial\theta^+} - i\theta^+\partial_+\right) = 0.
\end{aligned}$$

Equivalently, in spinor notation,

$$\{Q_\alpha, Q_\beta\} = -2i\sigma^a_{\alpha\beta}\partial_a, \qquad [Q_\alpha, P_a] = 0.$$

Where σ^0 and σ^1 are 2 by 2 Pauli matrices. We also introduce fermionic covariant derivative generators,

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + i\theta^\pm\partial_\pm.$$

They anticommute with the SUSY generators. Explicitly,

$$
\begin{aligned}
\{D_+, Q_+\} &= \left(\frac{\partial}{\partial\theta^+} + i\theta^+\partial_+\right)\left(\frac{\partial}{\partial\theta^+} - i\theta^+\partial_+\right) \\
&\quad + \left(\frac{\partial}{\partial\theta^+} - i\theta^+\partial_+\right)\left(\frac{\partial}{\partial\theta^+} + i\theta^+\partial_+\right) = 0, \\
\{D_-, Q_-\} &= \left(\frac{\partial}{\partial\theta^-} + i\theta^-\partial_-\right)\left(\frac{\partial}{\partial\theta^-} - i\theta^-\partial_-\right) \\
&\quad + \left(\frac{\partial}{\partial\theta^-} - i\theta^-\partial_-\right)\left(\frac{\partial}{\partial\theta^-} + i\theta^-\partial_-\right) = 0, \\
\{D_+, Q_-\} &= 0, \qquad \{D_-, Q_+\} = 0.
\end{aligned}
$$

Therefore,

$$
\delta^Q(D_\alpha\phi) = D_\alpha(\delta^Q\phi).
$$

I.e, the derivative transforms supersymmetrically just like the field. Moreover,

$$D_+\mathcal{X}^m = i\psi_+^m + i\theta^+\partial_+X^m + i\theta^-F^m + \theta^+\theta^-\partial_+\psi_-^m,$$
$$D_-\mathcal{X}_m = i\psi_{-m} + i\theta^-\partial_-X_m - i\theta^+F_m - \theta^+\theta^-\partial_-\psi_{+m}.$$

Thus,

$$\begin{aligned} D_+\mathcal{X}^m D_-\mathcal{X}_m &= \left(i\psi_+^m + i\theta^+\partial_+X^m + i\theta^-F^m + \theta^+\theta^-\partial_+\psi_-^m\right) \\ &\quad \times \left(i\psi_{-m} + i\theta^-\partial_-X_m - i\theta^+F_m - \theta^+\theta^-\partial_-\psi_{+m}\right) \\ &= -\psi_+^m\psi_{-m} + \theta^+\left(-\psi_+^mF_m - \partial_+X^m\psi_{-m}\right) + \theta^-\left(\psi_+^m\partial_-X_m - F^m\psi_{-m}\right) \\ &\quad + \theta^+\theta^-\left(F^mF_m + \partial_+X^m\partial_-X_m - i\psi_+^m\partial_-\psi_{+m} - i\psi_-^m\partial_+\psi_{-m}\right). \end{aligned}$$

Taking the Grassmann integral gives

$$\int d\theta^+ \, d\theta^- \, D_+\mathcal{X}^m D_-\mathcal{X}_m = F^m F_m + \partial_+ X^m \partial_- X_m - i\left(\psi_+^m \partial_- \psi_{+m} + \psi_-^m \partial_+ \psi_{-m}\right),$$

which is the Lagrangian. Thus, the action can be written as,

$$S_{RNS} = \int d\tau \, d\sigma \, d\theta^+ \, d\theta^- \, D_+\mathcal{X}^m D_-\mathcal{X}_m.$$

Which is a nice generalization from the bosonic string. This superspace formalism allow us to deal with the theory with manifest susy. The algebra closes off-shell and we don't have to add nothing.

WZ superfields

Wess-Zumino model was the first QFT explicitly supersymmetrically to be worked on.
We first start with an action mixing bosons and fermions,

$$S = \int d^4x \Big[-\partial_m\varphi\partial^m\bar{\varphi} - i\bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^m)^{\dot{\alpha}\beta}\partial_m\psi_\beta - \frac{m}{2}\left(\psi\psi + \bar{\psi}\bar{\psi}\right)\Big].$$

As we can see, this is just a free Klein-Gordon action with a massive Dirac spinor action. The number of degrees of freedom,

$$\#\text{bosons} = \#\{\varphi, \bar{\varphi}\} = 2,$$
$$\#\text{fermions} = \#\{\psi_1, \bar{\psi}_1, \psi_2, \bar{\psi}_2\} = 4.$$

So we add two bosonic auxiliary terms, F and $\bar{F}$, and alter the action,

$$S = \int d^4x \Big[-\partial_m\varphi\partial^m\bar{\varphi} - i\bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^m)^{\dot{\alpha}\beta}\partial_m\psi_\beta + F\bar{F} + m\left(\varphi F + \bar{\varphi}\bar{F} - \frac{1}{2}\psi^\alpha\psi_\alpha - \frac{1}{2}\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}\right)\Big].$$

The SUSY transformations are

$$\delta\varphi = \sqrt{2}\lambda\psi, \qquad \delta\bar{\varphi} = \sqrt{2}\bar{\lambda}\bar{\psi},$$
$$\delta\psi_\alpha = i\sqrt{2}(\sigma^m\bar{\lambda})_\alpha\partial_m\varphi + \sqrt{2}\lambda_\alpha F, \qquad \delta\bar{\psi}^{\dot{\alpha}} = i\sqrt{2}(\bar{\sigma}^m\lambda)^{\dot{\alpha}}\partial_m\bar{\varphi} + \sqrt{2}\bar{\lambda}^{\dot{\alpha}}\bar{F},$$
$$\delta F = i\sqrt{2}\bar{\lambda}\bar{\sigma}^m\partial_m\psi, \qquad \delta\bar{F} = i\sqrt{2}\lambda\sigma^m\partial_m\bar{\psi}.$$

To make it manifest, we use superfield,

$$\boxed{\Phi(y,\theta) = \varphi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^\alpha\theta_\alpha F(y)},$$

$$\boxed{\bar{\Phi}(\bar{y},\bar{\theta}) = \bar{\varphi}(\bar{y}) + \sqrt{2}\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(\bar{y}) + \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\bar{F}(\bar{y})}.$$

And write the variation,

$$\begin{aligned}\delta\Phi &= \left(\lambda^\alpha Q_\alpha + \bar{\lambda}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)\Phi \\ &= \partial_m\varphi\,\delta y^m + \sqrt{2}\psi_\alpha\delta\theta^\alpha + \sqrt{2}\theta^\alpha\partial_m\psi_\alpha\,\delta y^m + \theta^\alpha\theta_\alpha\partial_m F\,\delta y^m + \theta^\alpha\delta\theta_\alpha F + \theta_\alpha\delta\theta^\alpha F,\end{aligned}$$

where

$$\delta\bar{y}^m = -2i\lambda\sigma^m\bar{\theta}, \qquad \delta y^m = -2i\bar{\lambda}\bar{\sigma}^m\theta, \qquad \delta\theta^\alpha = \lambda^\alpha, \qquad \delta\bar{\theta}_{\dot{\alpha}} = \lambda_{\dot{\alpha}}.$$

Under this, we find the susy generators

$$Q_\alpha(\bar{y},\theta,\bar{\theta}) = \frac{\partial}{\partial\theta^\alpha} - 2i\bar{\theta}^{\dot{\alpha}}(\sigma^m)_{\dot{\alpha}\alpha}\frac{\partial}{\partial\bar{y}^m},$$
$$\bar{Q}^{\dot{\alpha}}(y,\theta,\bar{\theta}) = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - 2i(\sigma^{m\dot{\alpha}\alpha})\theta_\alpha\frac{\partial}{\partial y^m}.$$

Note that they are not defined inder the same coordiantes. By requiring that

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad \text{(chiral field)}, \qquad D_\alpha\bar{\Phi} = 0 \quad \text{(anti-chiral field)},$$

we have the derivative operators, in $(y,\theta,\bar{\theta})$ coordinates,

$$D_\alpha = \partial_\alpha + 2i\bar{\theta}^{\dot{\alpha}}\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^m}, \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}},$$

in $(\bar{y},\theta,\bar{\theta})$ coordinates,

$$D_\alpha = \partial_\alpha, \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - 2i\theta^\beta\sigma^m_{\beta\dot{\alpha}}\frac{\partial}{\partial\bar{y}^m}.$$

Also,

$$\{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = 0, \qquad \{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0.$$

Thus, we may write a general renormalizable expression in $(4|4)$ superspace involving chiral and anti-chiral superfields,

$$S_{WZ} = \int d^4x \Big[\int d^4\theta \, \bar{\Phi}\Phi + \int d^2\theta \left(e\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3 \right) + \int d^2\bar{\theta} \left(\bar{e}\bar{\Phi} + \frac{\bar{m}}{2}\bar{\Phi}^2 + \frac{\bar{g}}{3}\bar{\Phi}^3 \right) \Big].$$

That is the Wess–Zumino action. We can also write everything in the same coordinates by writing in coordinates $(x^m, \theta, \bar{\theta})$ such that,

$$\delta x^m = -i(\lambda\sigma^m\bar{\theta} + \bar{\lambda}\bar{\sigma}^m\theta), \qquad \delta\theta^\alpha = \lambda^\alpha, \qquad \delta\bar{\theta}_{\dot{\alpha}} = \lambda_{\dot{\alpha}}.$$

In that sense, we find that $y^m = x^m + i\theta\sigma^m\bar{\theta}$ and $\bar{y}^m = x^m - i\theta\sigma^m\bar{\theta}$. Thus, since the variation is $\lambda^\alpha Q_\alpha + \bar{\lambda}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$,

$$(\lambda^\alpha Q_\alpha + \bar{\lambda}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})x^m = \delta x^m = -i(\lambda\sigma^m\bar{\theta} + \bar{\lambda}\bar{\sigma}^m\theta)$$

$$\Rightarrow Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^m\bar{\theta})_\alpha \frac{\partial}{\partial x^m},$$

$$\bar{Q}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i(\bar{\sigma}^m\theta)_{\dot{\alpha}} \frac{\partial}{\partial x^m}.$$

Also, making the change of coordinates, the derivatives are written as,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m} \quad , \quad \bar{D}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i(\bar{\sigma}^m)^{\dot{\alpha}\beta}\theta_\beta\frac{\partial}{\partial x^m}.$$

Notice that we raised the index in $\bar{D}$ term. Since everything in written in the same coordinates now, we can rewrite the action, and we have,

$$S_{WZ} = \int d^4x\Big[-\partial_m\varphi\partial^m\bar{\varphi} - i\psi\sigma^m\partial_m\bar{\psi} + F\bar{F} + \lambda F + M(\varphi F - \frac{1}{2}\psi\psi) +$$

$$+g(\varphi^2F - \psi^\alpha\psi_\alpha\varphi) + \bar{\lambda}\bar{F} + \bar{M}(\bar{\varphi}\bar{F} - \frac{1}{2}\bar{\psi}\bar{\psi}) + \bar{g}(\bar{\varphi}^2\bar{F} - \bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}\bar{\varphi})\Big].$$

Super Yang-Mills

We are still dealing with a $(4|4)$ superspace, so the operators Q and D remain form Wess-Zumino. As we know, Yang-Mills theories are constructed based on gauge symmetries that the action must obbey. In SYM, we are insterested in constructing susy versions of it. We may start by writing QED action, in the massless case,

$$S = \int d^4x \left[-\frac{F_{uv}F^{uv}}{4} - i\bar{\lambda}D\lambda \right].$$

Since we have a gauge invariance condition for the vector A,

$$A'_u = A_u + \partial_u \varphi(x),$$

we thus have 3 bosonic dof in this action and 4 fermionic dof. We apply again the superfield formalism and add a bosonic term,

$$S = \int d^4x \left[-\frac{F_{uv}F^{uv}}{4} - i\bar{\lambda}D\lambda + \frac{1}{2}d^2 \right].$$

The SUSY transformations now are,

$$\delta A_\mu = i\eta\sigma^\mu\bar{\lambda} + i\bar{\eta}\bar{\sigma}^\mu\lambda,$$
$$\delta\lambda_\alpha = F_{\mu\nu}(\sigma^{\mu\nu}\eta)_\alpha + i\eta_\alpha d,$$
$$\delta\bar{\lambda}^{\dot{\alpha}} = F_{\mu\nu}(\bar{\sigma}^{\mu\nu}\bar{\eta})^{\dot{\alpha}} - i\bar{\eta}^{\dot{\alpha}} d,$$

Note that λ is used to write the susy transformation of A. It is the superpartner of the photon, the "photino". There is another way to write the action in term of the superfield strength (a generalization of F_{uv}):

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V, \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4} D^2 \bar{D}_{\dot{\alpha}} V,$$

where $V = V^\dagger = V(x;\theta;\bar{\theta})$ is a real superfield, such that:

$$\delta V = i(\Lambda - \Lambda^\dagger) \quad \Rightarrow \quad \delta W_\alpha = \delta \bar{W}_{\dot{\alpha}} = 0,$$

for Λ chiral; $\Lambda^\dagger$ anti-chiral.
With this invariance, we may write V in the Wess-Zumino Gauge:

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta(\bar{\theta}\bar{\lambda}) - i\bar{\theta}\bar{\theta}(\theta\lambda) + \theta\theta\bar{\theta}\bar{\theta}\frac{d}{2}.$$

With this, we have:

$$S_1 = \int d^4x \left[\int d^2\theta \, W_\alpha W^\alpha + \int d^2\bar{\theta} \, \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \right].$$

Now, to properly write the sQED action, we need to write the interaction with matter.
To write the matter part, we use the superfields:

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y),$$
$$\Sigma(y,\theta) = \tau(y) + \sqrt{2}\theta\chi(y) + \theta\theta E(y),$$

where ϕ, τ are, respectively, the selectron and spositron, and ψ, χ are the electron and the positron.
The SUSY action is:

$$S_{matter} = \int d^4x \left[\int d^4\theta\, \bar{\Phi} e^{2qV} \Phi + \bar{\Sigma} e^{-2qV} \Sigma + M \int d^2\theta\, \Phi\Sigma + M \int d^2\bar{\theta}\, \bar{\Phi}\bar{\Sigma} \right].$$

With gauge transformations (for Λ denoting the same chiral field as before):

$$\delta\Phi = (e^{-2iq\Lambda} - 1)\Phi, \qquad \delta\Sigma = (e^{2iq\Lambda} - 1)\Sigma, \qquad \delta V = i(\Lambda - \bar{\Lambda}),$$
$$\delta\bar{\Phi} = (e^{2iq\bar{\Lambda}} - 1)\bar{\Phi}, \qquad \delta\bar{\Sigma} = (e^{-2iq\bar{\Lambda}} - 1)\bar{\Sigma}.$$

And susy transformations,

$$\delta\Phi = (\eta Q + \bar{\eta}\bar{Q})\Phi,$$
$$\delta\Sigma = (\eta Q + \bar{\eta}\bar{Q})\Sigma.$$

Which were already shown in the WZ model. Then, the action is,

$$S_{sQED} = S_1 + S_{matter}.$$

This is just a toy model, but it allow us an important intuition, we are extending a physical theory by making it susy, which is believed to be valid for high energy limits. By allowing this susy structure (which was already worked out in WZ model section) we extend space time symmetries non-trivially.

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