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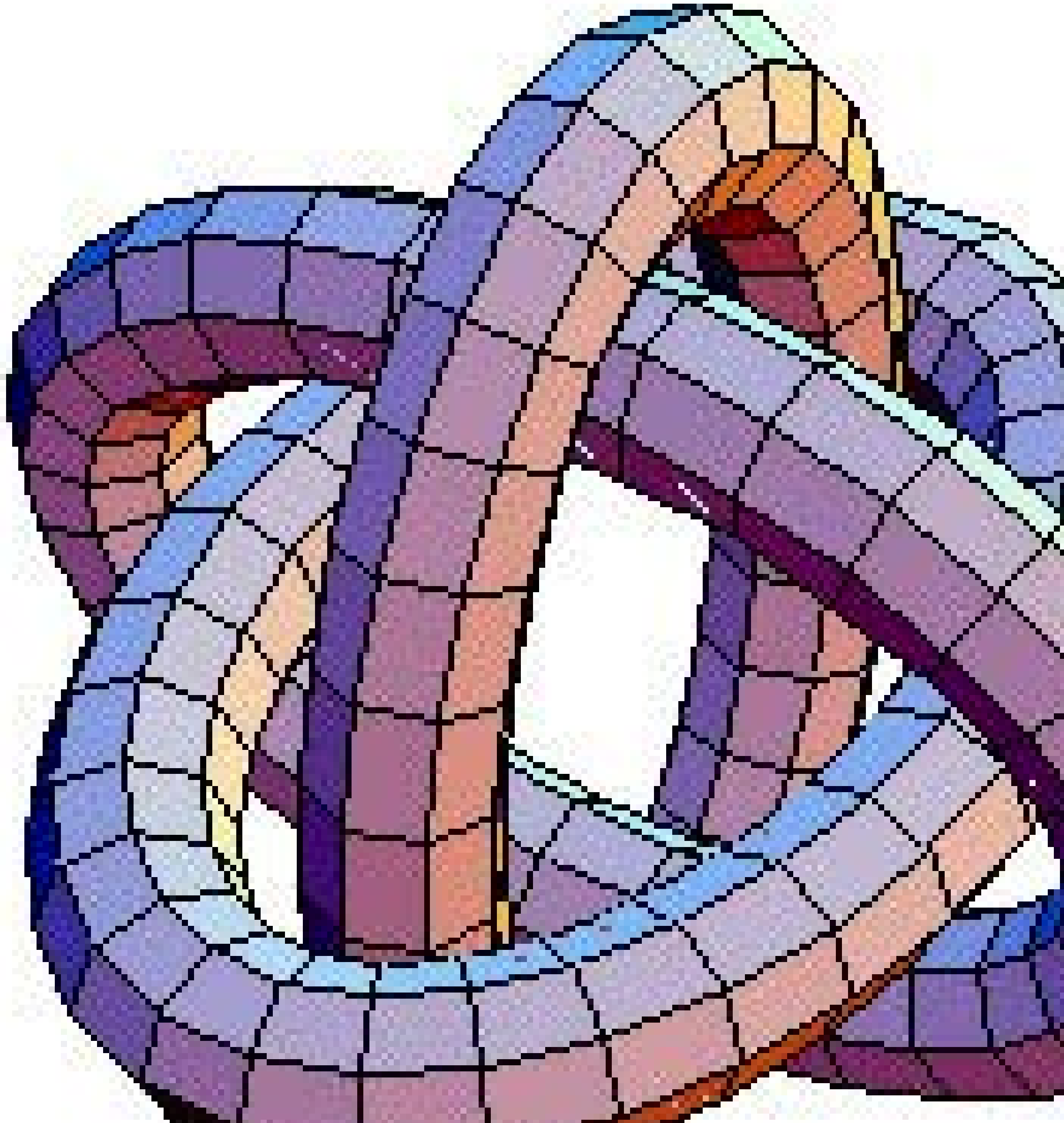
Topological Quantum Field Theories



june - 2026

Rafael Ottoni

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- Introduction
- Gauge theories and topology
  - Holonomy and the Aharonov-Bohm effect
- Definition of TQFTs
  - Overview of BRST
  - Witten-type theories
    - Instantons and Donaldson polynomials
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- Chern-Simons
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Introduction

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British mathematician Michael Atiyah proposed to quantum field theorists two challenges: finding physical interpretations of **Donaldson polynomials** and **knot theory**

Michael Attiyah

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Mathematicians **Simon Donaldson** and **Andreas Floer** developed the study of **topological invariants** of 3- and 4-dimensional spacetime using ideas from Yang-Mills and classical field theory

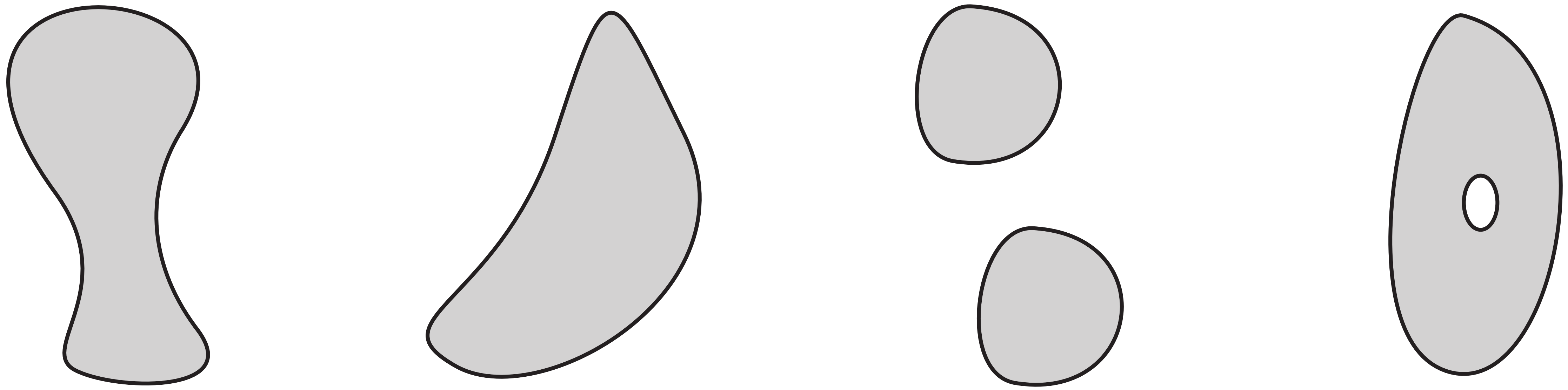
There was still no clear connection to quantum theory

Simon Donaldson

Andreas Floer

Introduction

Topological invariants are properties of spaces that do not change when you stretch, twist and deform it in many ways (that don't involve cutting it) → For instance, a space is **simply connected** when every point can be connected to each other and every loop can be smoothly deformed to a point



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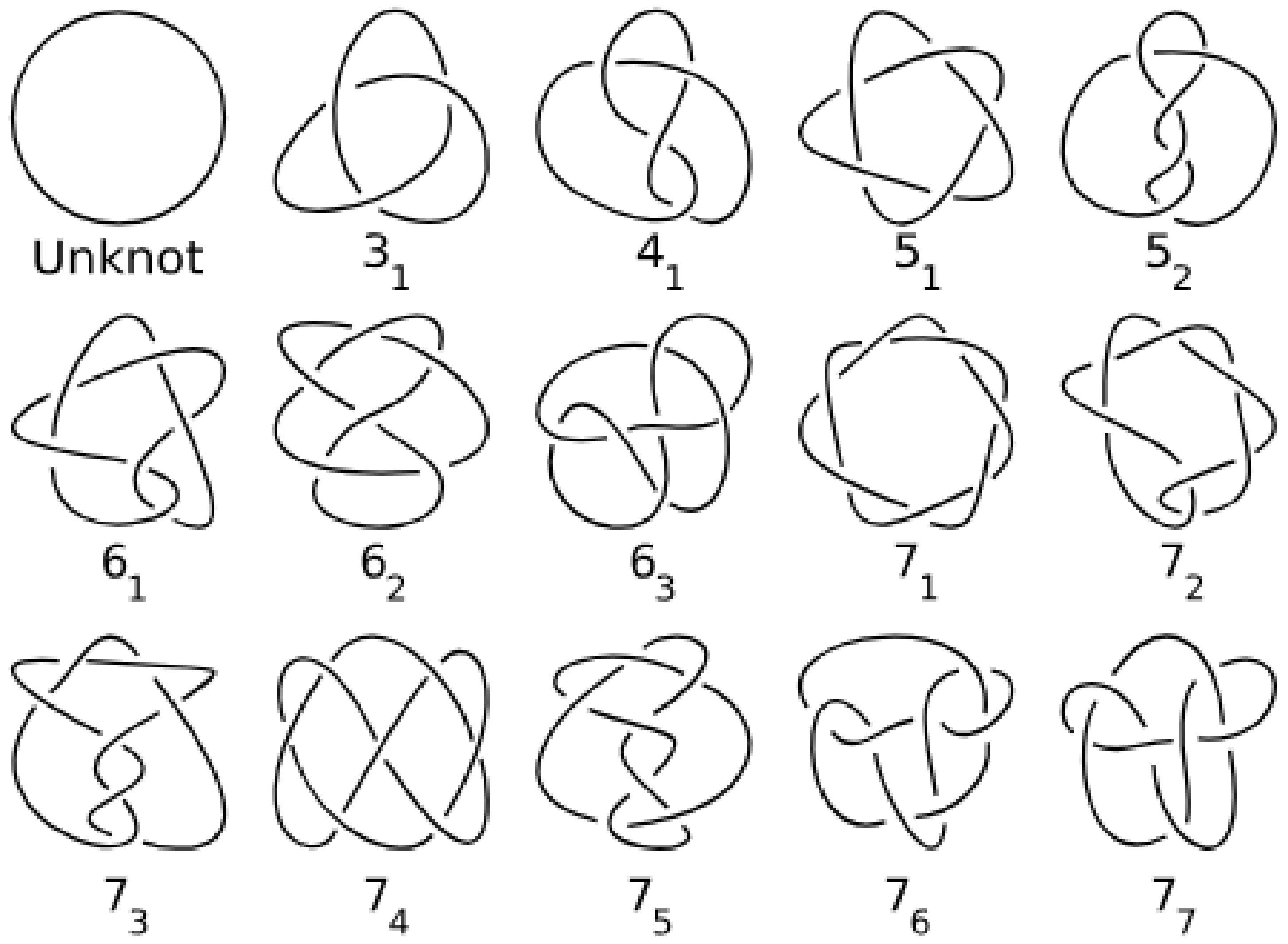
A **knot** is an **embedding of a circle** in 3d Euclidean space → two knots are equivalent if they can be transformed into one another via smooth deformations



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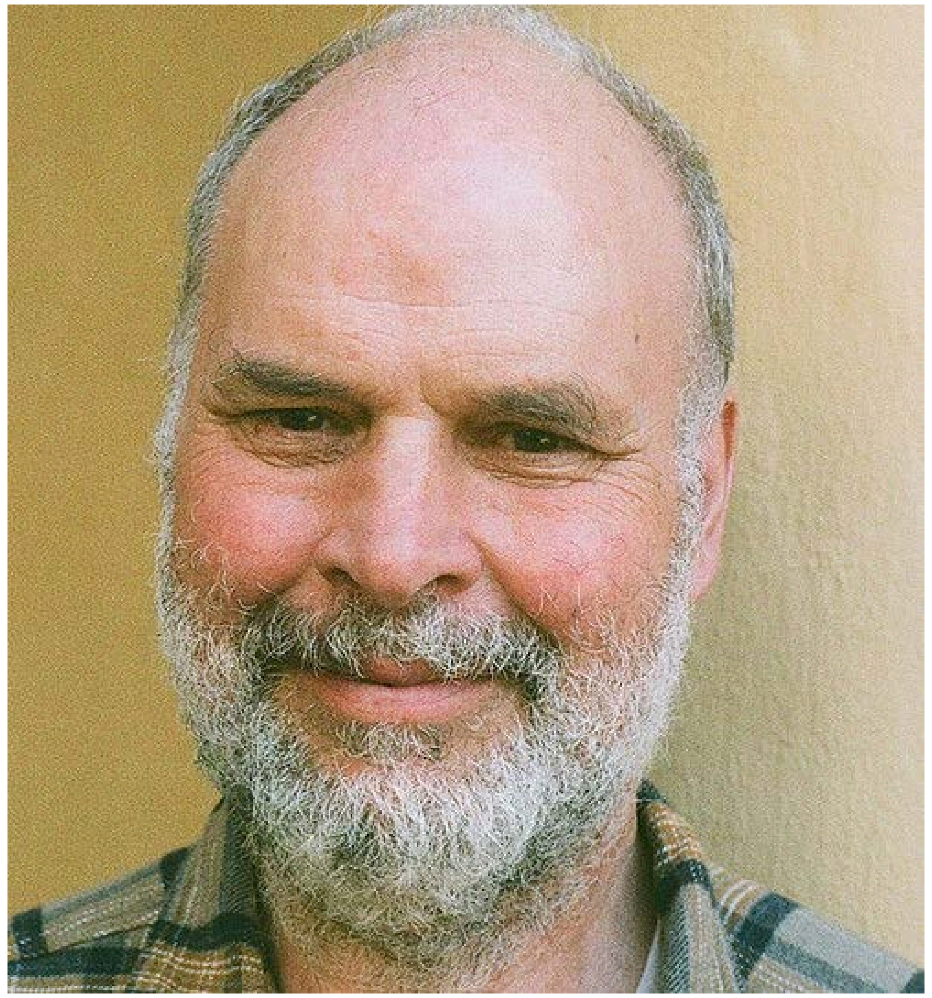
In 1984, **Vaughan Jones** made one of the most important recent discoveries in knot theory, of a topological invariant that assigns a polynomial to each knot

The coefficients of this polynomial encodes topological properties of a given knot



Vaughan Jones

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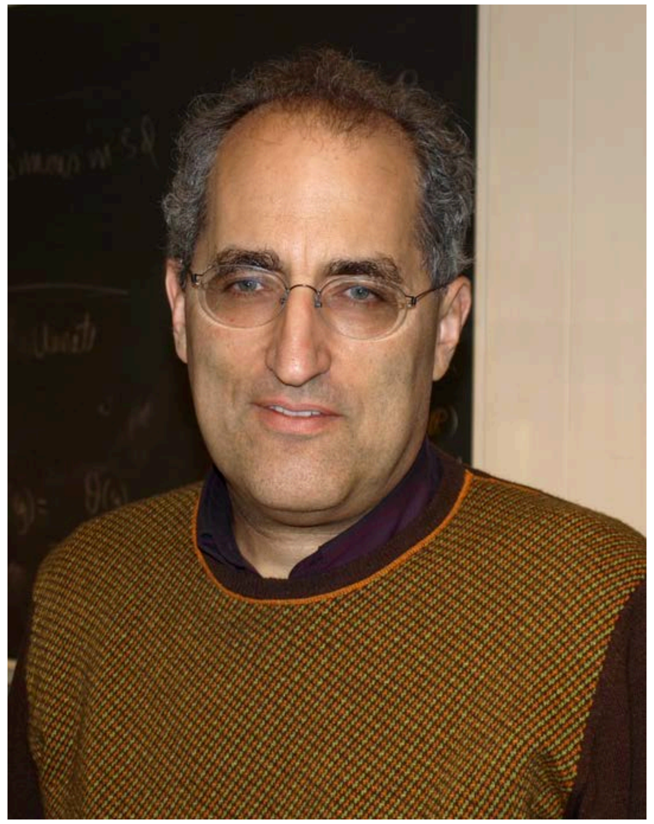
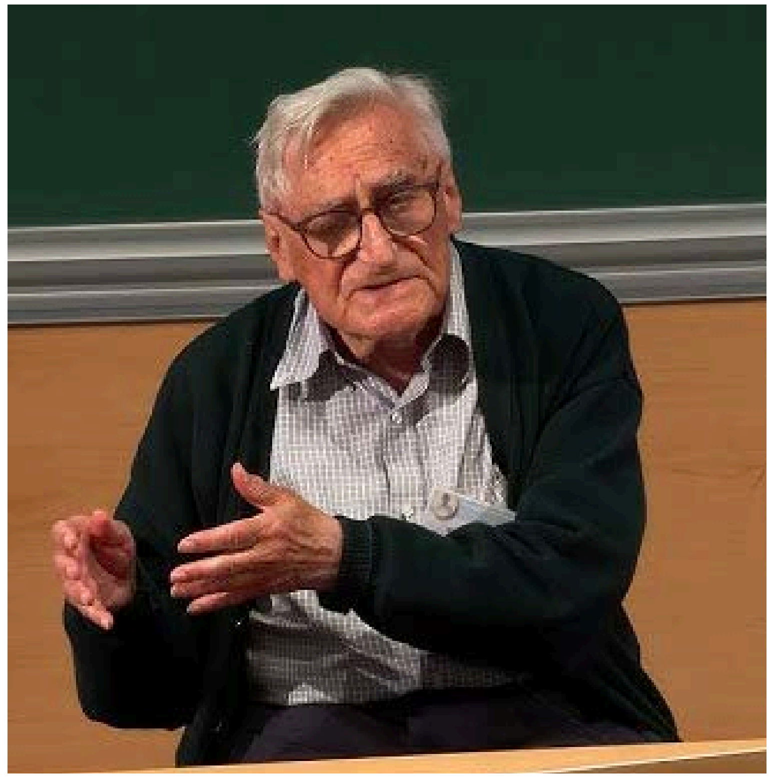


Edward Witten



Albert Schwarz

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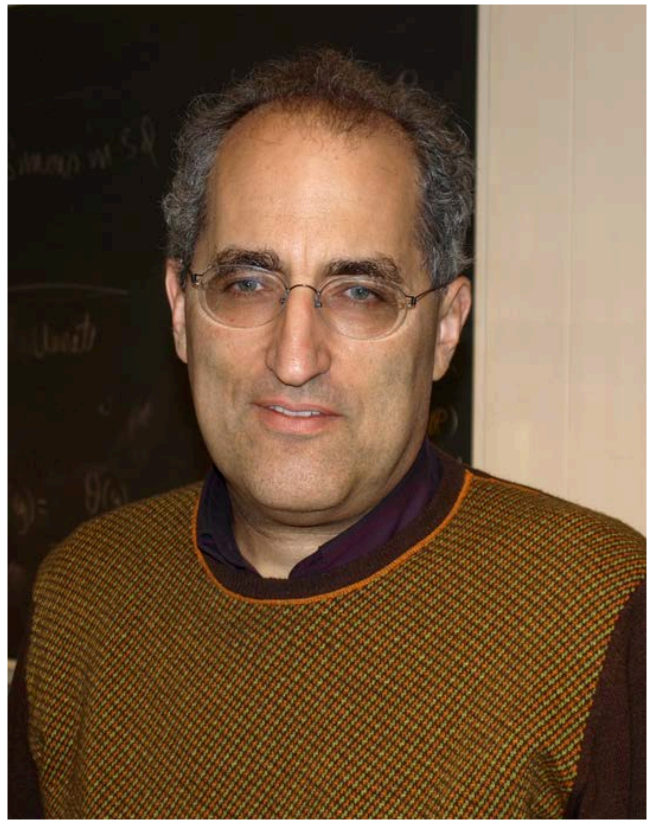
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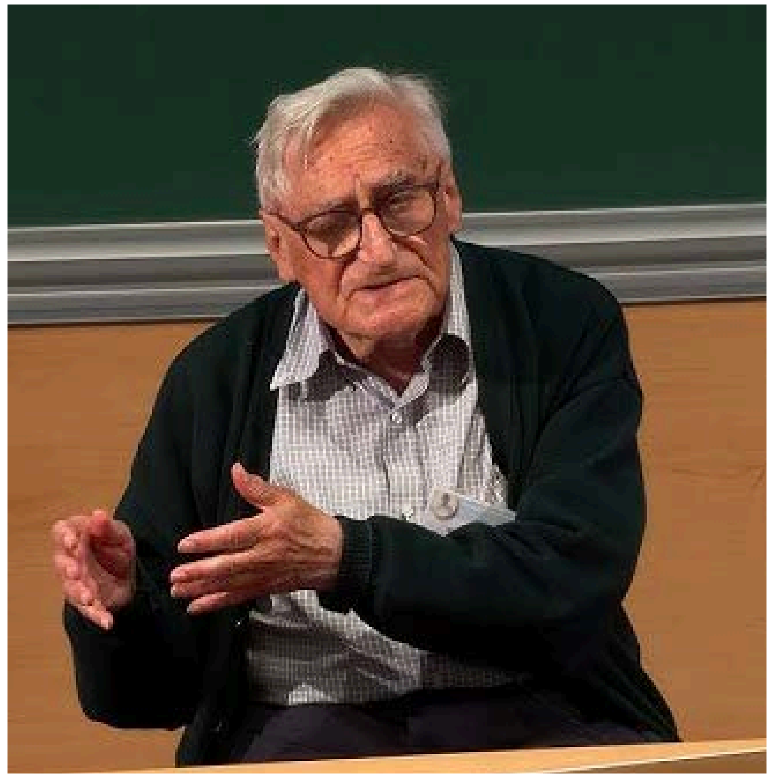
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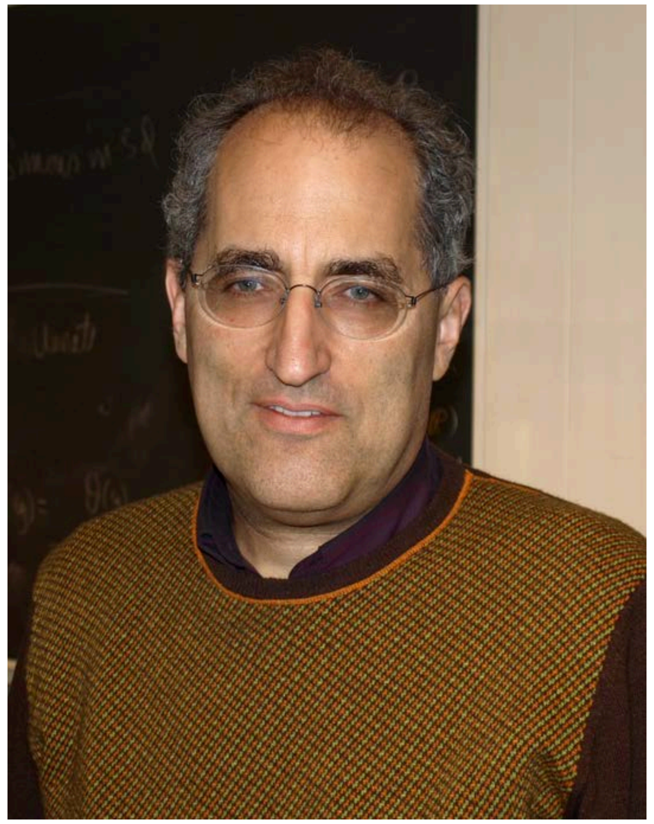
In fact, Witten-type theories are those closely related with Donaldson-Floer theory and Schwarz-type are related to knot theory and Jones polynomials



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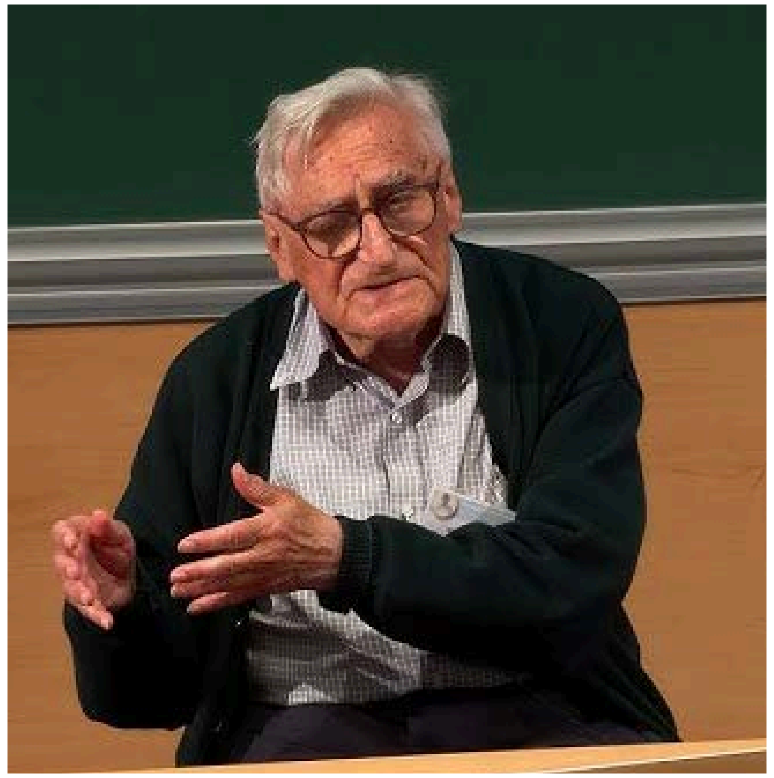

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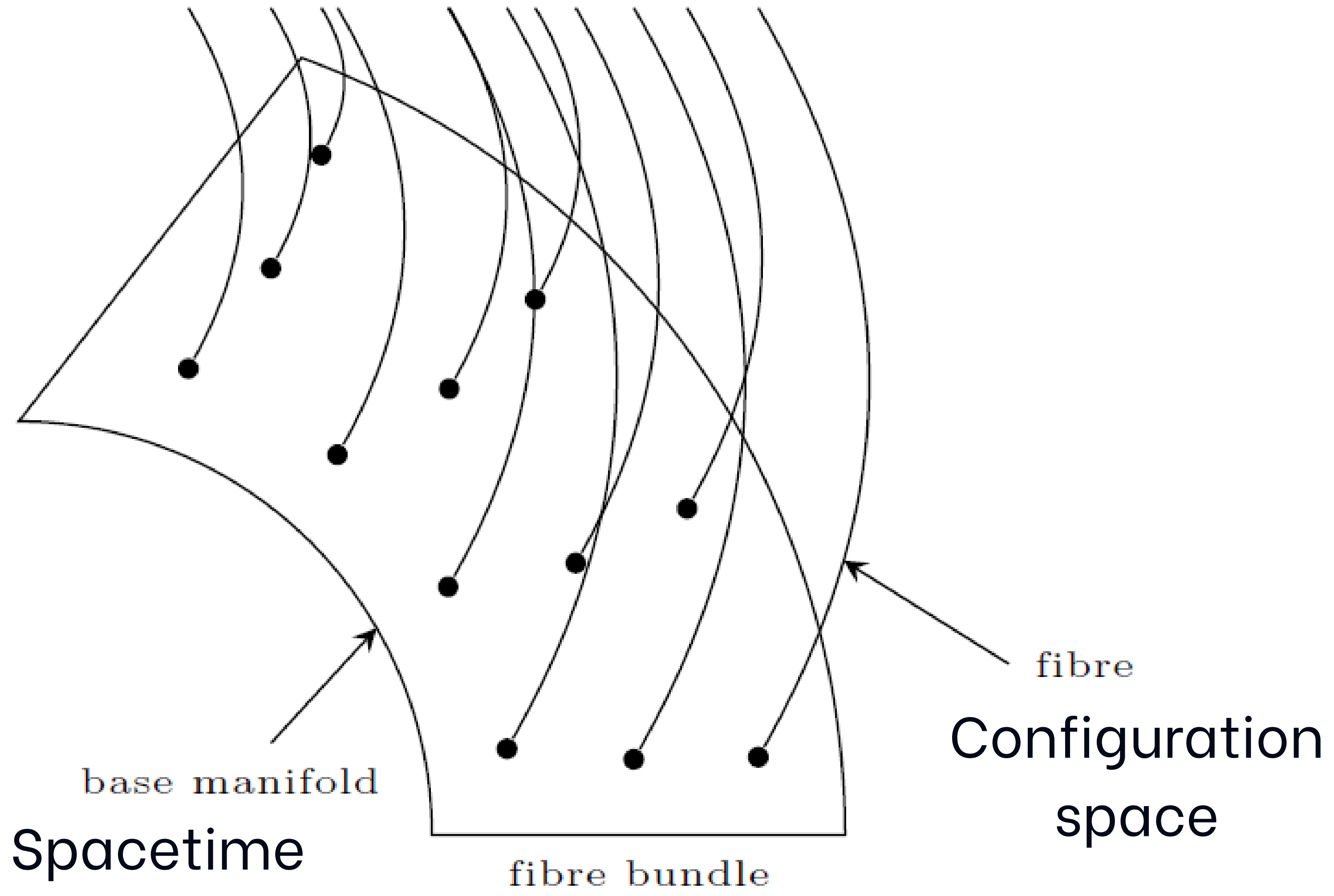
Chern-Simons, however, is a gauge theory. Therefore, before talking about TQFTs, we will need to talk about the topology of gauge theories and how to obtain topological invariants using them

Gauge theories and topology

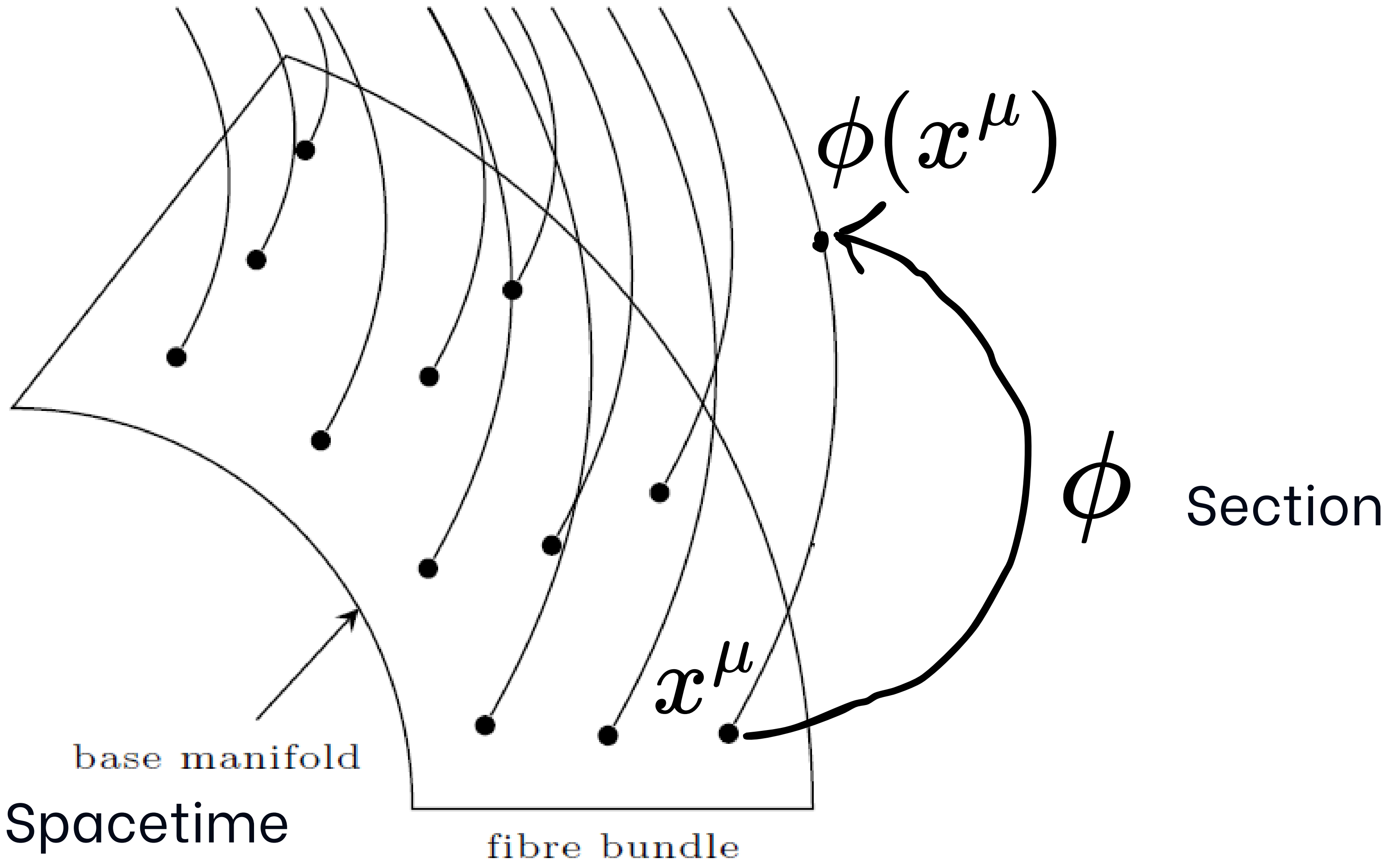
A field theory can be understood as a **fibre bundle**, which is a composite manifold whose **base** is the spacetime and the **fibres** are the configuration spaces

This can be understood as “glueing” a copy of the configuration space in each point of spacetime



Gauge theories and topology

The field histories in this formalism are **sections** of this bundle, which is a map that gives a field configuration for every point of spacetime



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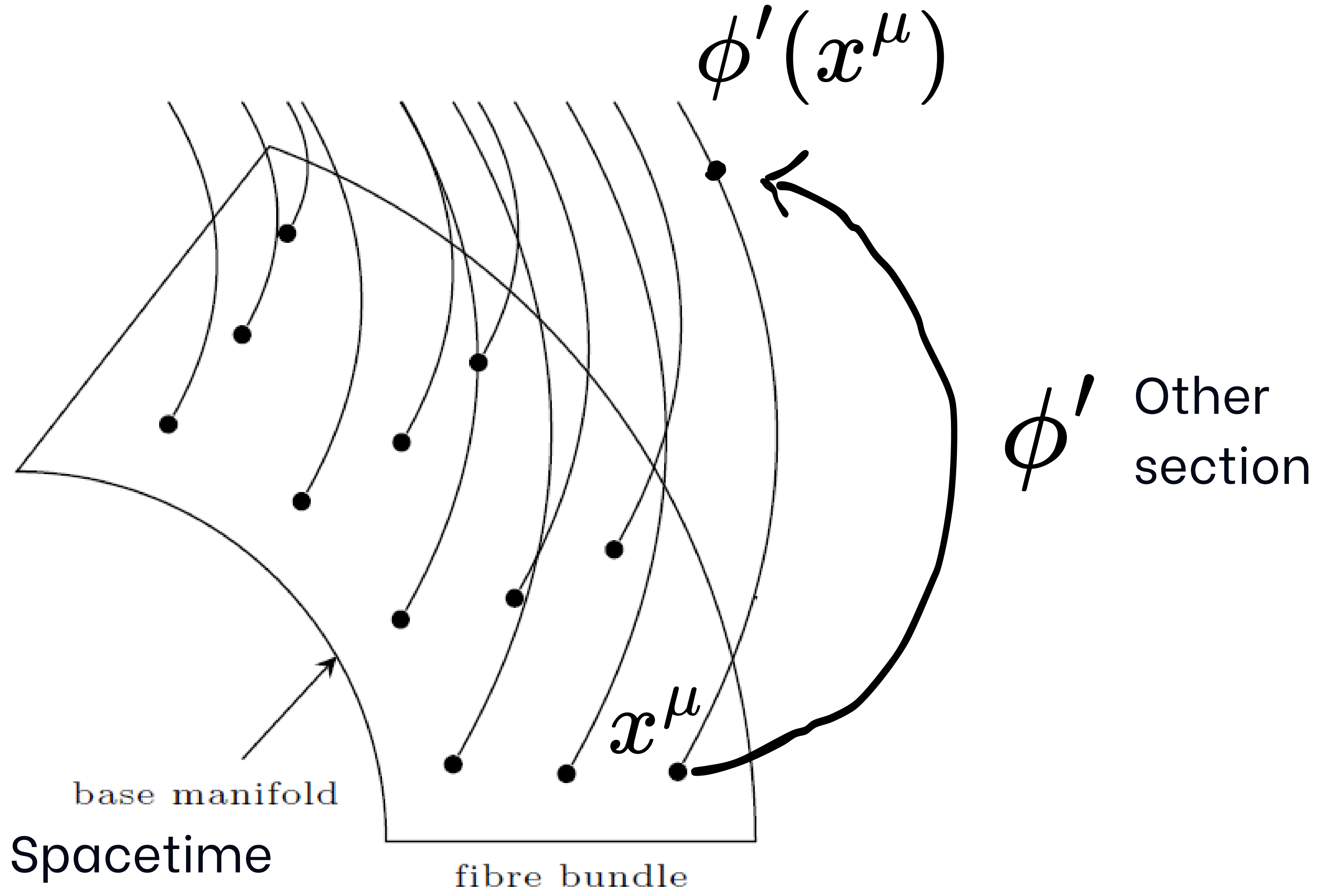

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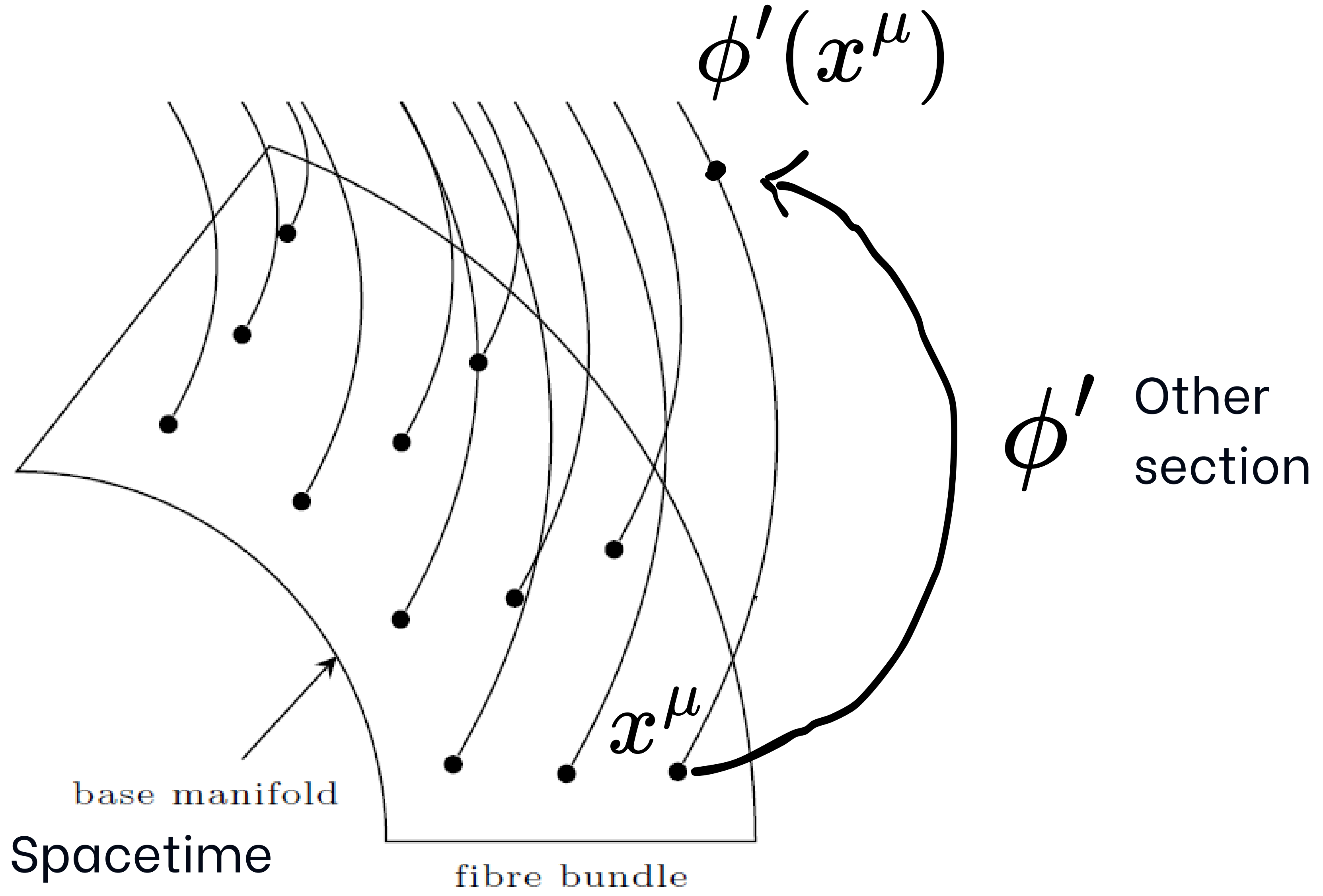
The space of all physical histories/states is, therefore, the space of all sections over a bundle



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Gauge theories are defined in **associated bundles**, which are constructed from representations R of a given Lie group G

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Gauge symmetries are understood classically as redundancies of the physical states related by a gauge transformation, so we take

$$\phi' \sim \phi \qquad \text{if} \qquad \phi' = R(g)\phi,\ g \in G$$

and consider them as the same state. Then we redefine our space of physical states as only those that are **physically inequivalent** → this is called the **moduli space** of solutions

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Our equations of motion need to be written in this space using **only gauge-invariant quantities**

Gauge theories and topology

As we have seen, the ordinary derivative ∂_μ is **not** a gauge invariant quantity, so we need to define a **covariant derivative**

$$D_\mu \equiv \partial_\mu + igA_\mu$$

where $A_\mu = A_\mu^a t^a$ is a matrix in the algebra of the Lie group and t^a are the generators of the algebra in a given representation

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where $A_\mu = A_\mu^a t^a$ is a matrix in the algebra of the Lie group and t^a are the generators of the algebra in a given representation. The Lagrangian of the theory is thus written in this space of physically inequivalent states as

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) + (D^\mu\phi)^*(D_\mu\phi) + V(|\phi|^2) \qquad \text{for bosons}$$

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) + (i\gamma^\mu D_\mu - m)\psi + V(|\psi|^2) \qquad \text{for fermions}$$

with $F_{\mu\nu} = F_{\mu\nu}^a t^a$ the matrix in the algebra of G representing the field strength

Gauge theories and topology

We can get plenty of relevant information about the topology of our spacetime by studying the topology of the space of states. The most important object for this is the **Wilson loop**, that we will construct now:

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The covariant derivative defined before serves a purpose very much like the one in general relativity. It can be used to **parallel transport** physical states trough the space of states

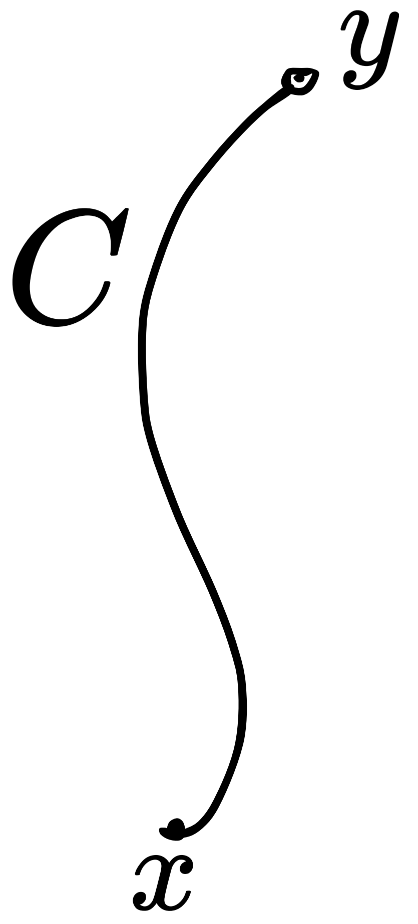
When a charged particle moves through some space with electric and magnetic fields, its wavefunction changes → so it also moves in the space of states. This movement is determined by the parallel transport, which is the solution of the geodesic equation

$$D_\mu \phi(x) = 0 \qquad \therefore \partial_\mu \phi(x) = igA_\mu(x)\phi(x)$$

Gauge theories and topology

Consider a curve $C(x, y)$ on spacetime that starts at point x and ends at point y. The solution of the parallel transport from $\phi(x)$ to $\phi(y)$ can be obtained by integrating both sides of the equation along this path

$$\phi(y) - \phi(x) = ig \int_{C(x,y)} dz^{\mu} A_{\mu}(z)\phi(z) \implies \phi(y) = \phi(x) + ig \int_{C(x,y)} dz^{\mu} A_{\mu}(z)\phi(z)$$



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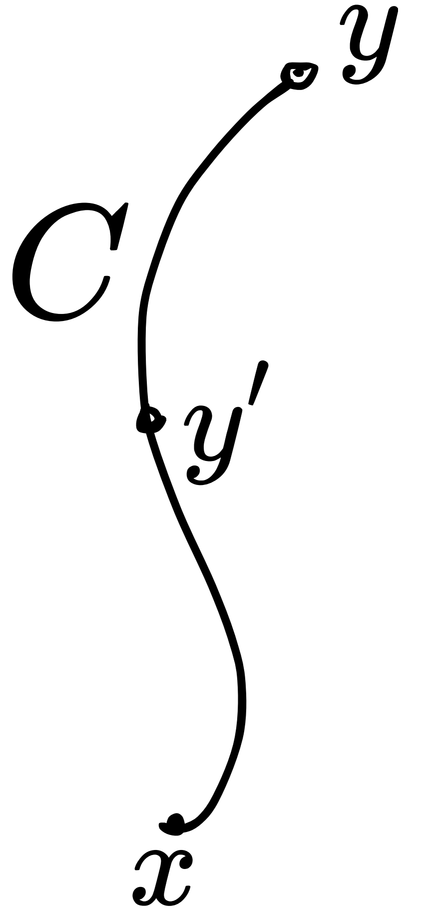


and then reinsterting this solution onto itself

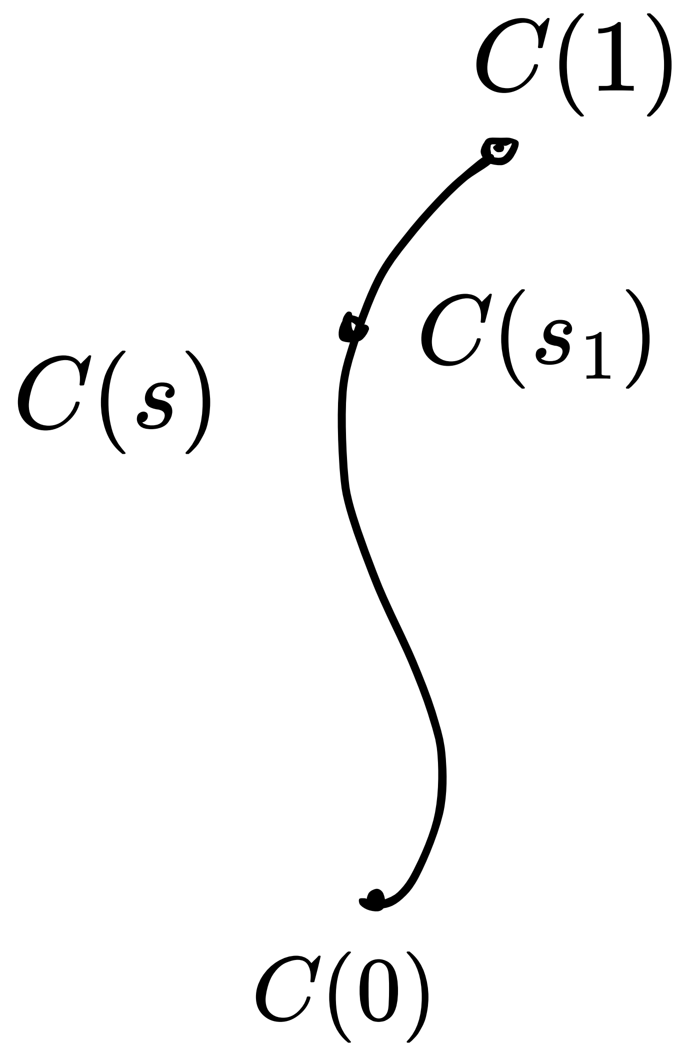
$$\phi(y) = \phi(x) + ig \int_{C(x,y')} dz^\mu A_\mu(z)\left[\phi(x) + ig \int_{C(y',y)} dz'^\mu A_\mu(z')\phi(z')\right]$$

$$= \phi(x) + ig \int_{C(x,y')} dz^\mu A_\mu(z)\phi(x) + (ig)^2 \int_{C(x,y')} dz^\mu \int_{C(y',y)} dz'^\mu A_\mu(z)A_\mu(z')\phi(z')$$

Gauge theories and topology

We repeat this procedure iteratively to get the final solution. Notice, however, that if we parametrize the curve by s, the first iteration becomes

$$\phi(y) = \phi(x) + ig \int_0^1 ds \frac{dz^\mu}{ds} A^\mu(z(s))\phi(z(s))$$



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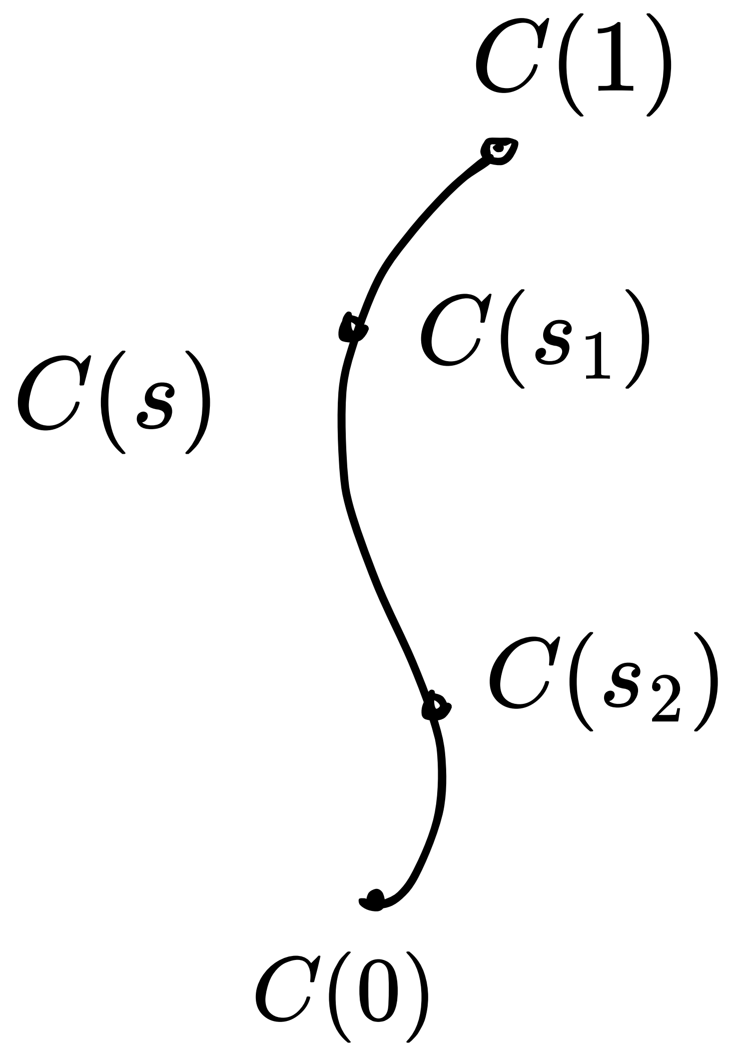
$$+(ig)^2\int_0^1 ds_1\int_0^{s_1} ds_2\frac{dz^{\mu_1}}{ds_1}\frac{dz^{\mu_2}}{ds}A_{\mu_1}(z(s_1))A_{\mu}\mu 1(z(s_2))\phi(z)$$



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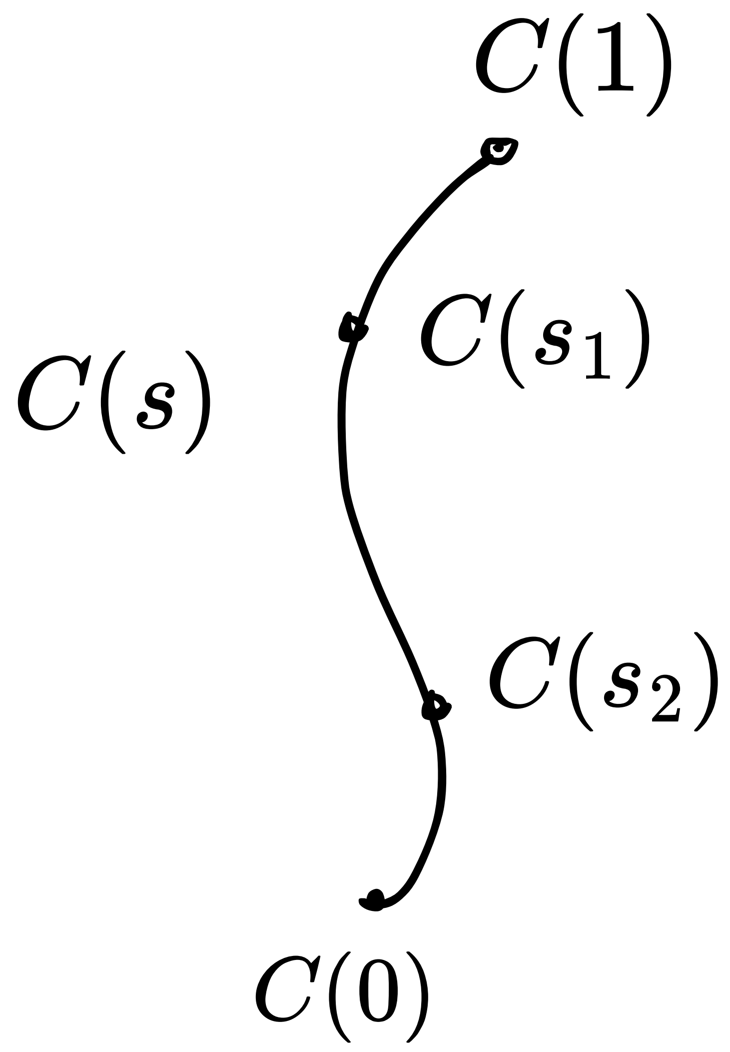

We repeat this procedure infinitely, getting to infinitesimal divisions of our path

Gauge theories and topology

The resulting solution becomes exactly the **Dyson series** that we previously used to define the path integral

$$\phi(y) = \sum_n \frac{(ig)^n}{n!} \int_0^1 ds_1 \int_0^{s_2} ds_2 \ldots \int_0^{s_n} ds_n \prod_j \left(\frac{dz^{\mu_j}(s_j)}{ds_j} A_{\mu_j}(z(s_j)) \right) \phi(x)$$

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which allows us to define the **path-ordered exponential** (just like the time-ordered one)

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This object is an operator containing information about how a state changes when we move it along a curve $C(x,y)$ in spacetime → It has **non-local** information about the theory

Gauge theories and topology

We call this object the **Wilson line**

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To get physical information about our theory, however, we need gauge invariant objects. How does the Wilson line transform under such a transformation?

Gauge theories and topology

Let us consider the Abelian theory first: Suppose $A_\mu \to A_\mu - \partial_\mu \alpha$. This yields

$$U(x,y) \to \exp\left(\int_C A_\mu dz^\mu + \int_C \partial_\mu \alpha \, dz^\mu\right)$$

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If $x = y$, however, this object becomes invariant. We define, therefore, the **Wilson loop** as a Wilson line around a closed path (a loop)

$$W[C] \equiv U(x,x) = \exp \oint_C dz^\mu A_\mu$$

Gauge theories and topology

For non-Abelian theories, we have $A_\mu \to A_\mu - D_\mu\alpha$, and this calculation generalizes to

$$U(x,y) \to P\exp\left(\int_C A_\mu dz^\mu + \int_C D_\mu\alpha\, dz^\mu\right) = e^{\alpha(y)}U(x,y)e^{-\alpha(x)},$$

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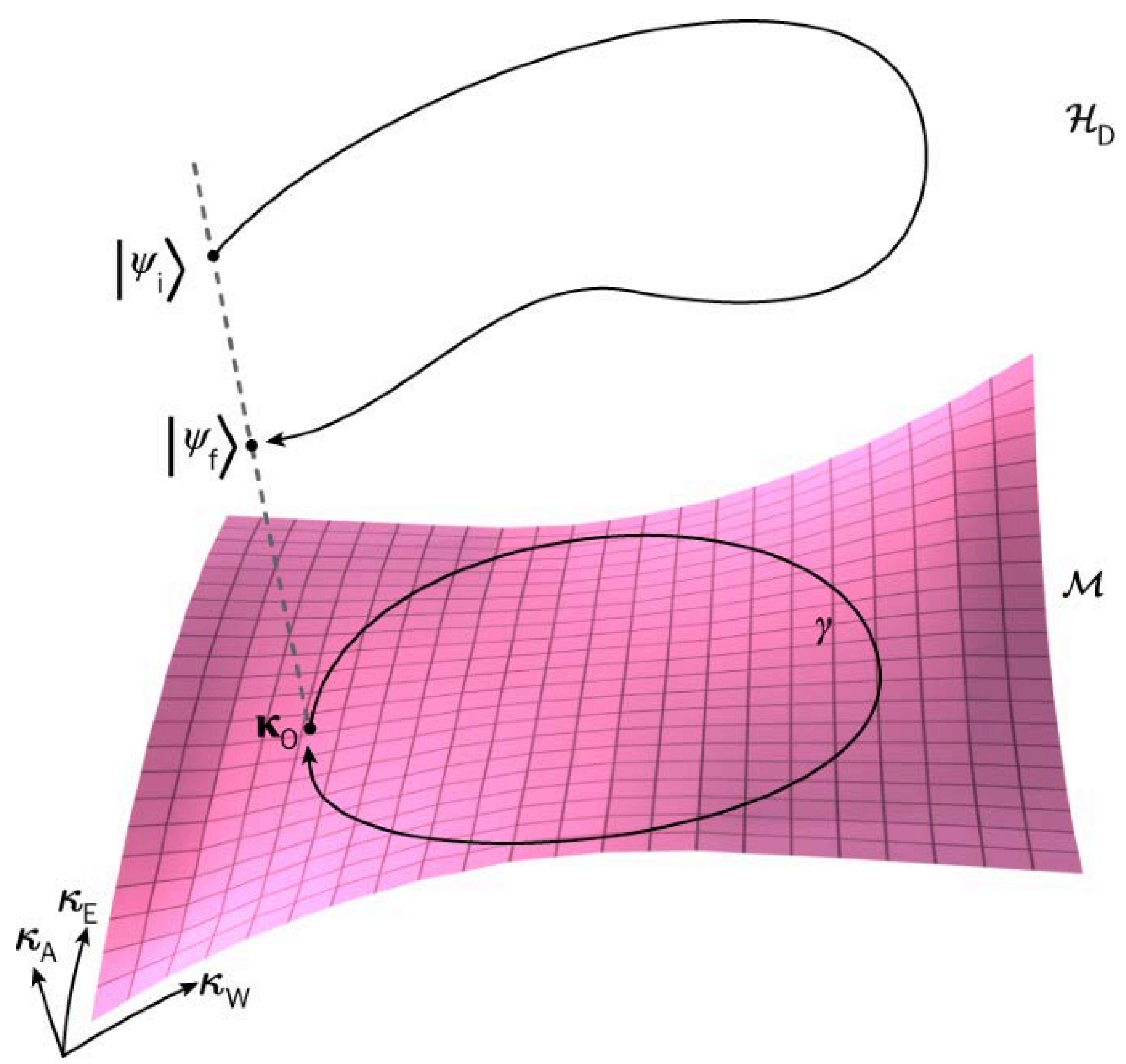
where these 3 terms are now matrices that do not commute with each other. The gauge invariant object is then defined as the **trace** of the Wilson loop, since, due to its cyclic property,

$$\mathrm{tr}\, W[C] = \mathrm{tr}\left(P\exp \oint_C dx^\mu A_\mu\right) \to \mathrm{tr}\left(e^{\alpha(x)} W[C] e^{-\alpha(x)}\right) \mathrm{tr}\left(e^{-\alpha(x)} e^{\alpha(x)} W[C]\right) = \mathrm{tr}\, W[C]$$

Gauge theories and topology

Geometrically, this gauge invariant object is known as the **holonomy**. This is the object that contains information about how a quantum state changes when we go around a loop on spacetime



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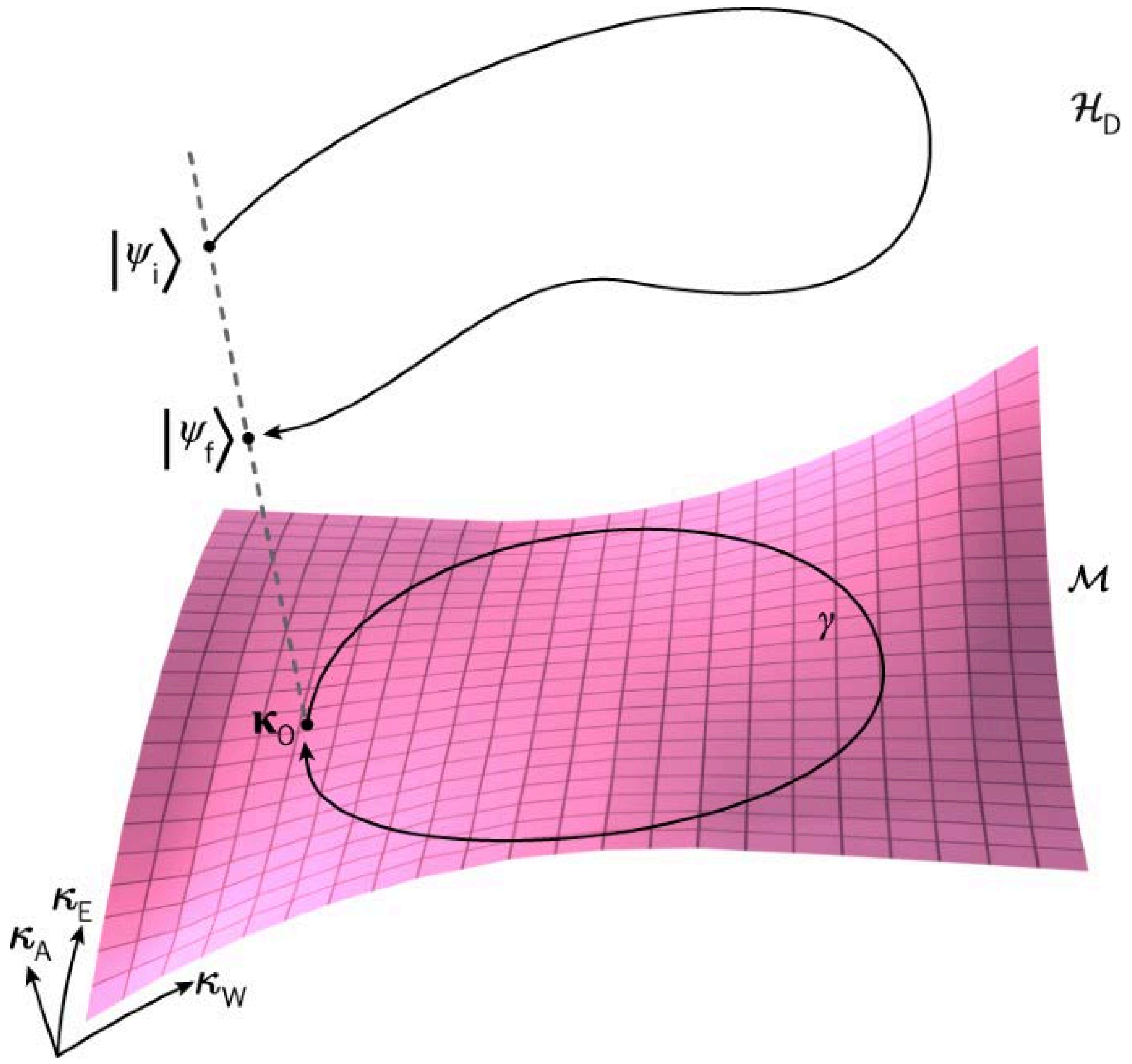

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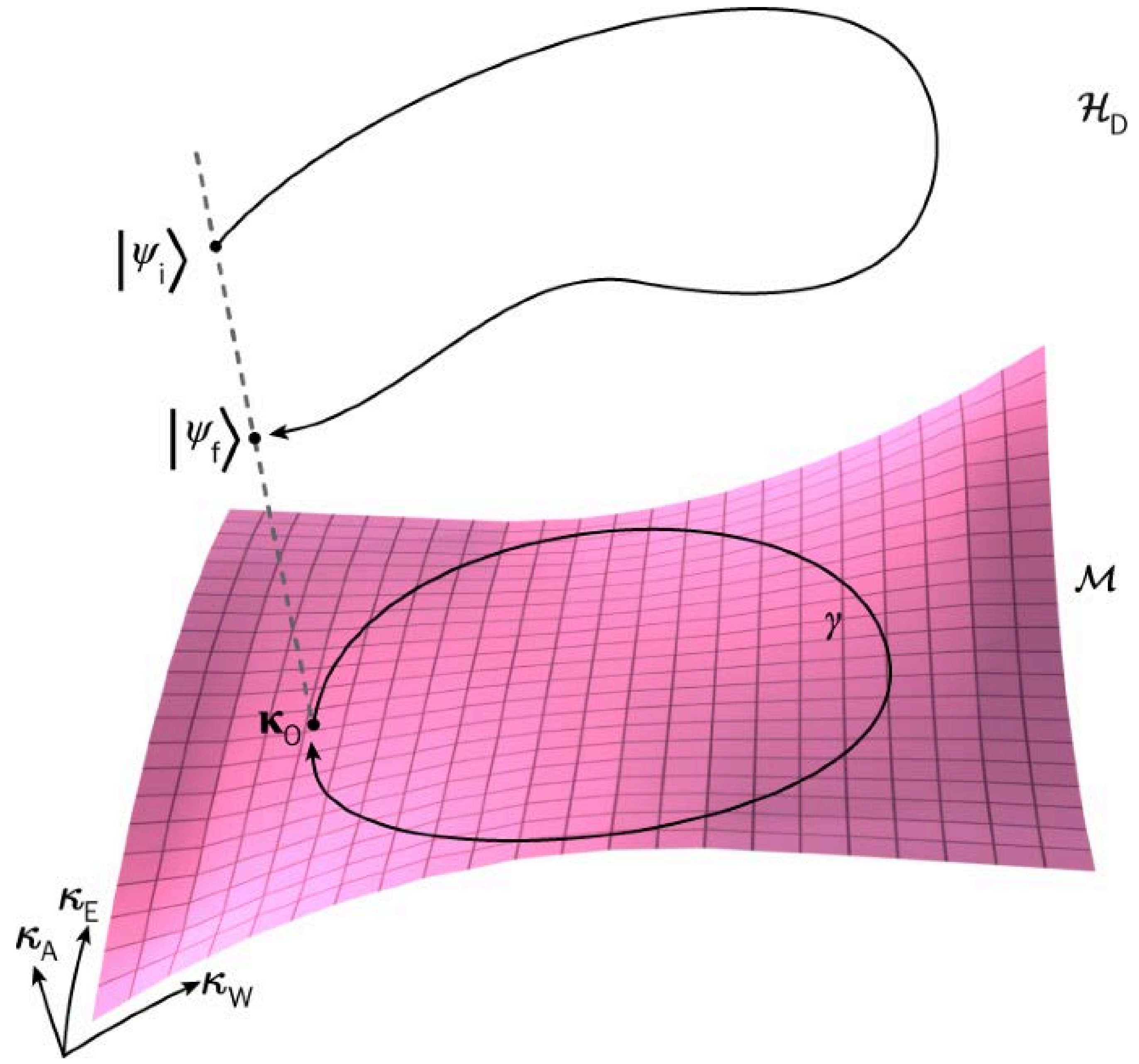
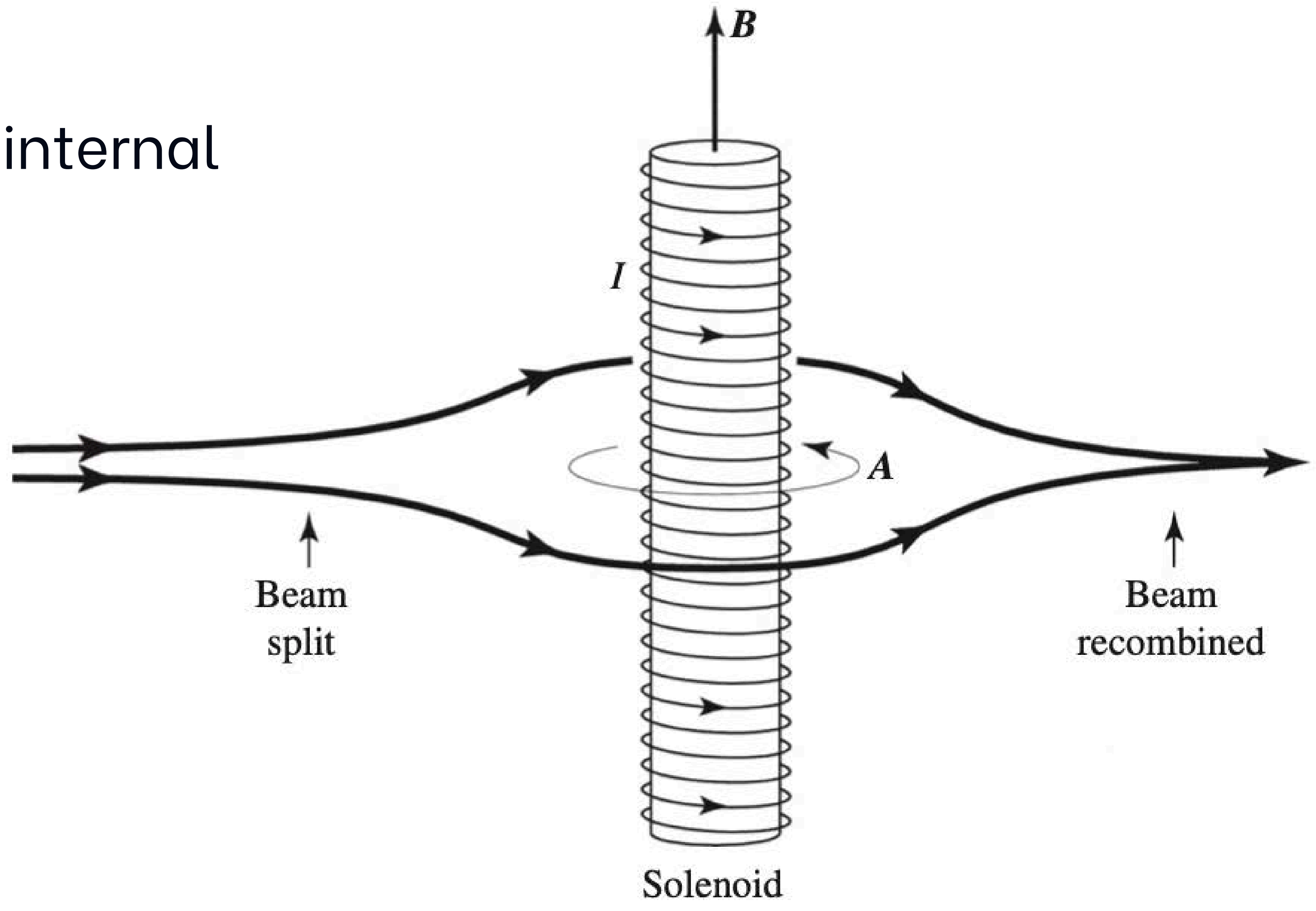
The most direct application of this concept appears in the **Aharonov-Bohm effect**



Gauge theories and topology

Aharonov-Bohm effect

Consider a solenoid with current I that produces an internal magnetic field $\vec{B}$ pointing in the $\hat{z}$ direction



Gauge theories and topology

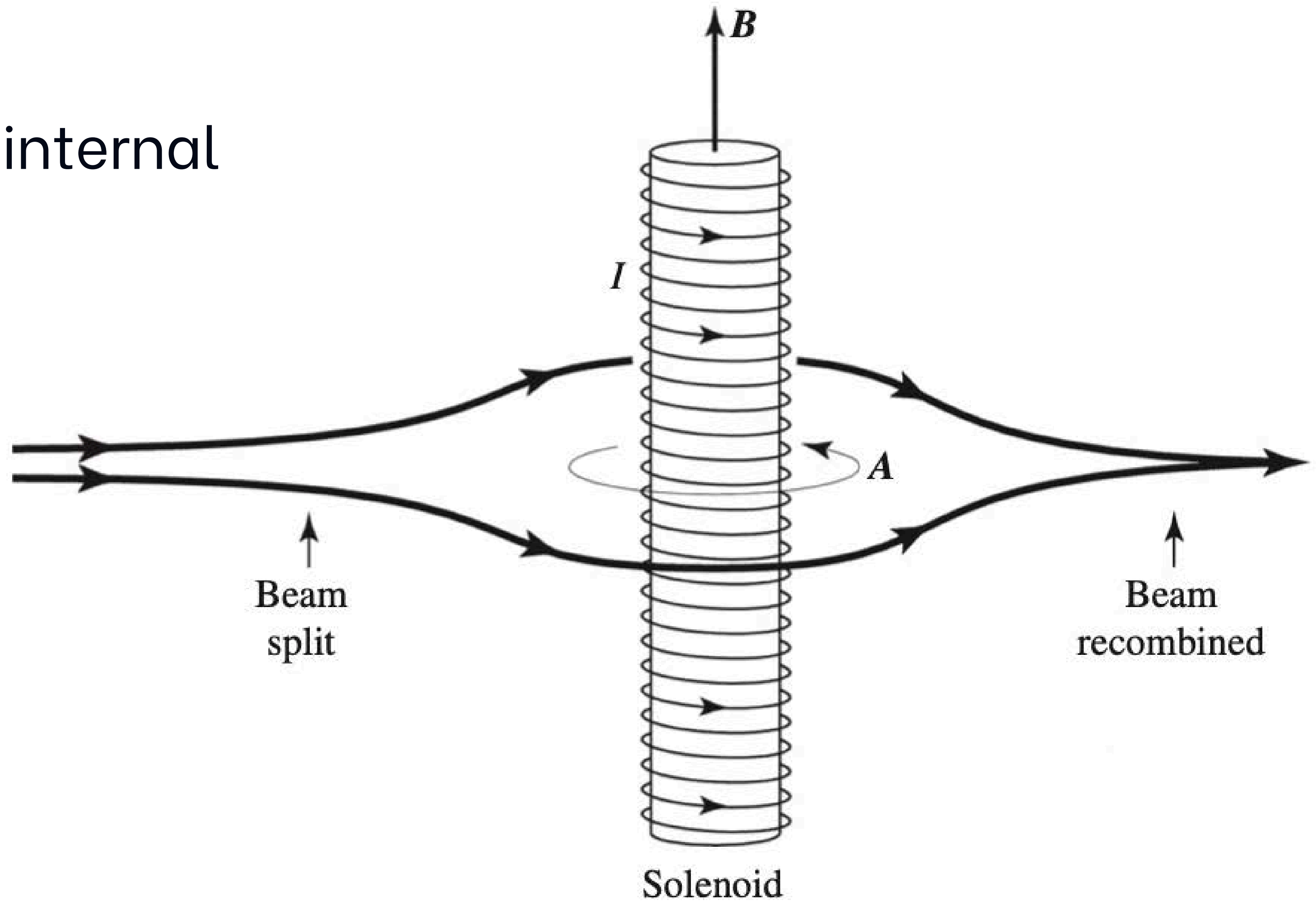
Aharonov-Bohm effect

Consider a solenoid with current I that produces an internal magnetic field $\vec{B}$ pointing in the $\hat{z}$ direction

If we constrain the electron to remain outside the solenoid (for instance, surrounding it with an insulator), we have $\vec{B}=0$, $\vec{A}\neq 0$ and $\phi=0$, then the Schrödinger equation becomes

$$\frac{1}{2m}(-i\nabla+e\vec{A})^2\psi=i\frac{\partial\psi}{\partial t}$$

with $\hbar=1$.

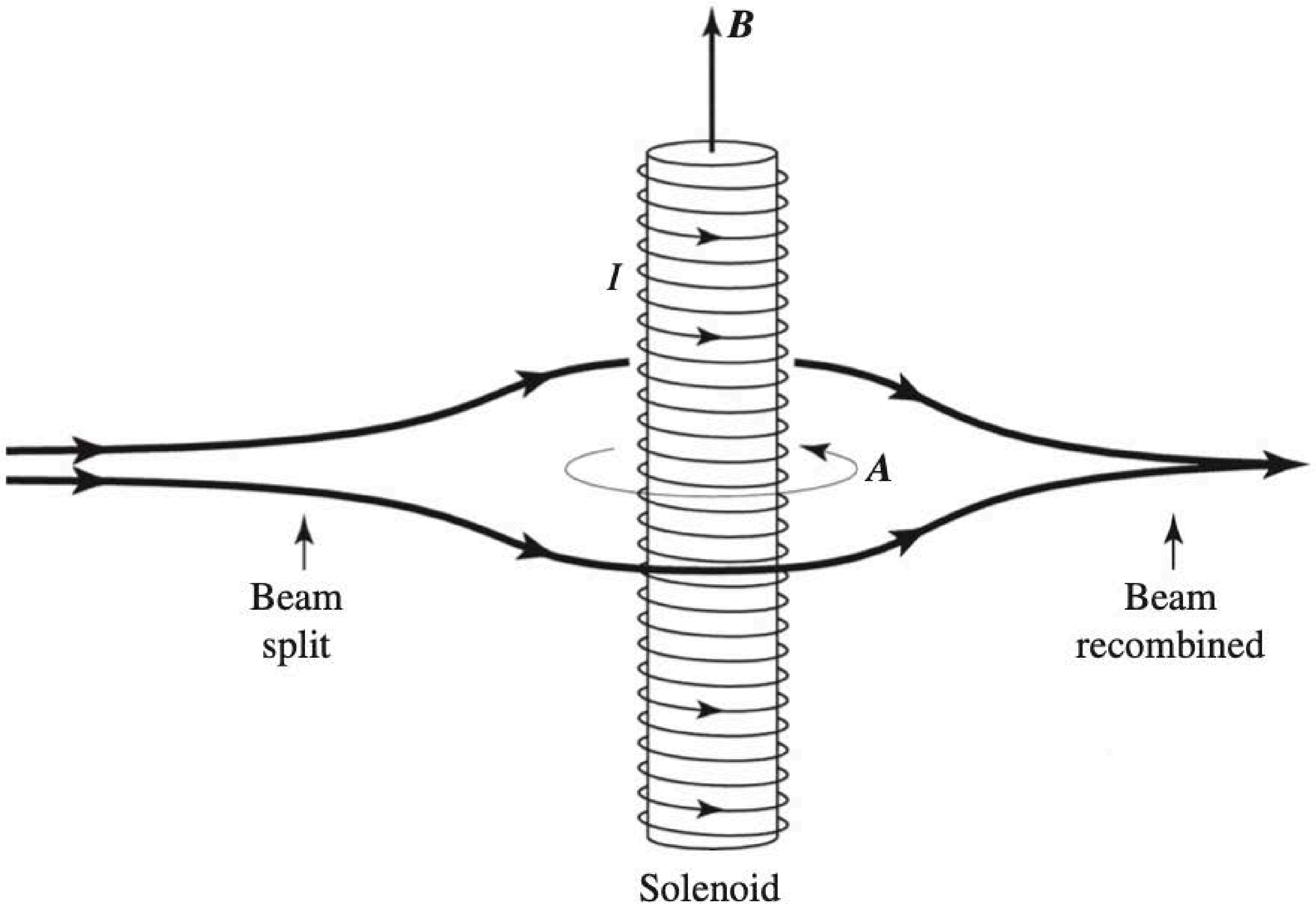


Gauge theories and topology

Aharonov-Bohm effect

It's well know that this equation has the solution

$$\psi(\vec{r}) = \psi(\vec{r}_0)\exp\left(ie\int_{\vec{r}_0}^{\vec{r}} d\vec{r}\,' \cdot \vec{A}\right)$$



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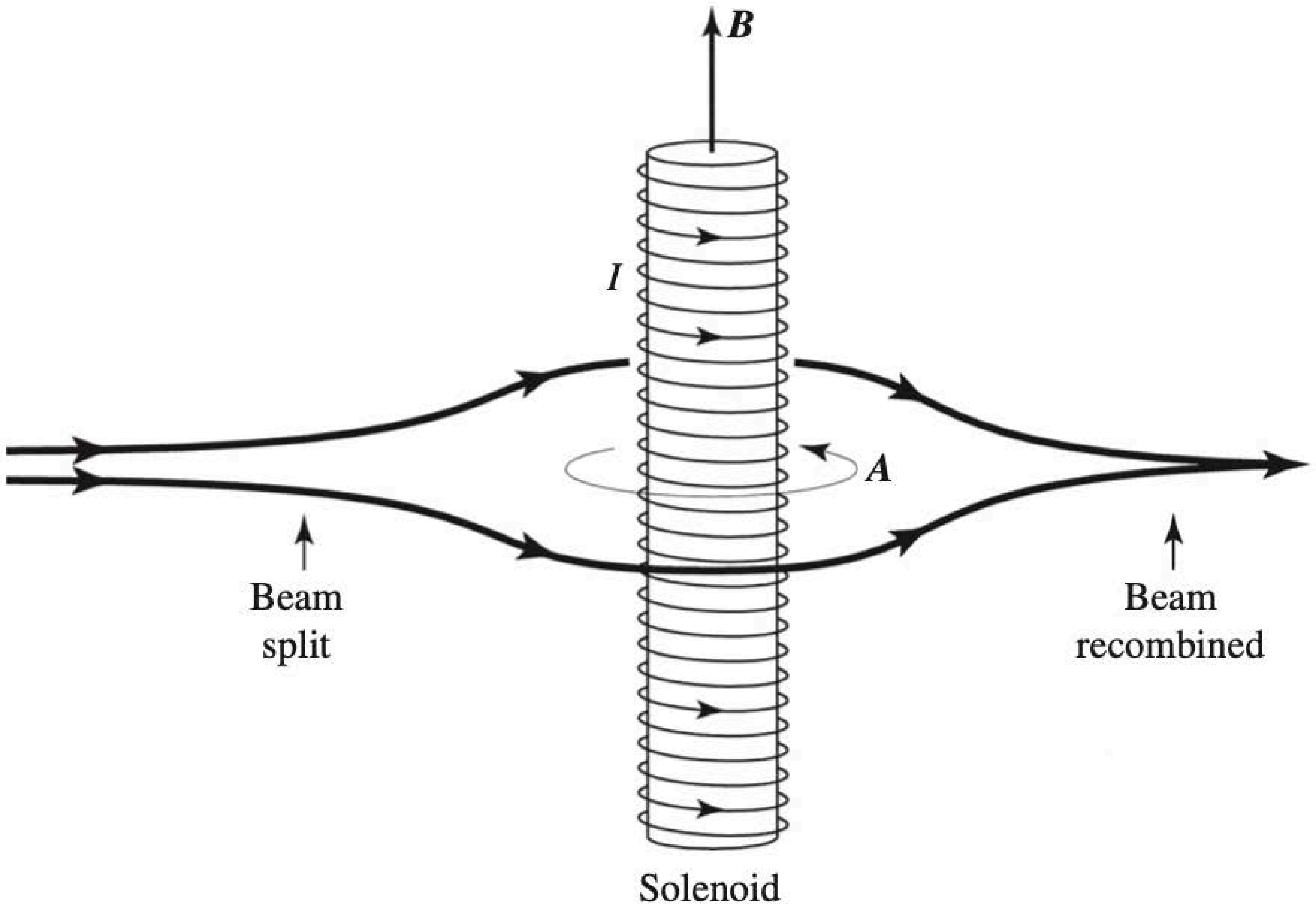
$$\psi(\vec{r}) = \psi(\vec{r}_0)\exp\left(ie\int_{\vec{r}_0}^{\vec{r}} d\vec{r}\,'\cdot\vec{A}\right)$$

So if the electron completes a loop around the solenoid, its wavefunction acquires a phase

$$\exp\left(ie\oint_C d\vec{r}\cdot\vec{A}\right)$$



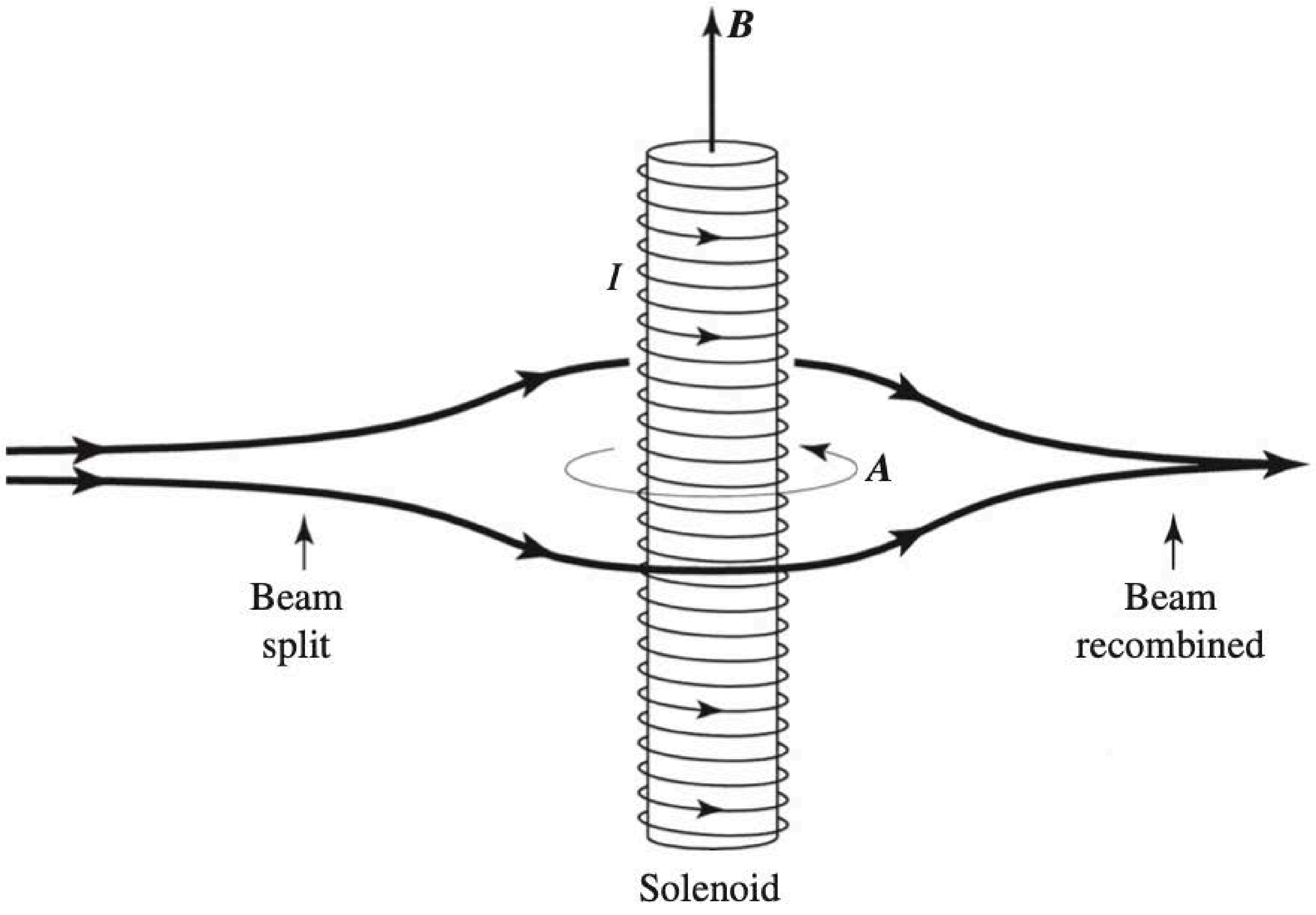
Gauge theories and topology

Aharonov-Bohm effect

But since we have $A_0 = 0$, this is simply equal to

$$\exp\left(ie\oint_C d\vec{r}\cdot\vec{A}\right) = \exp\left(ie\oint_C dx^\mu A_\mu\right) = W[C]$$

the Wilson loop around this path!



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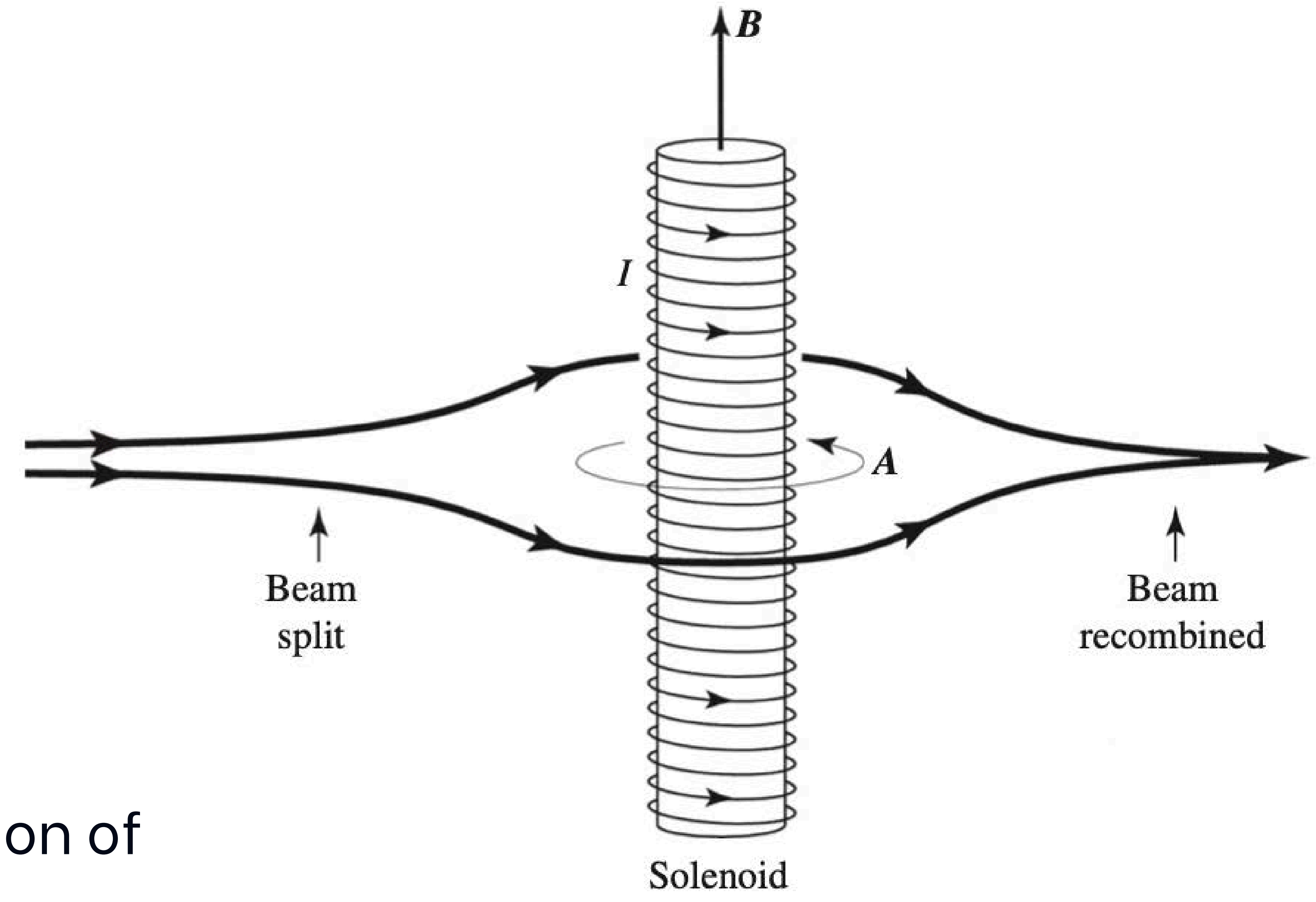
the Wilson loop around this path! This is not a surprise if we notice that the Schrödinger equation

$$\frac{1}{2m}(-i\nabla + e\vec{A})^2\psi = i\frac{\partial\psi}{\partial t}$$

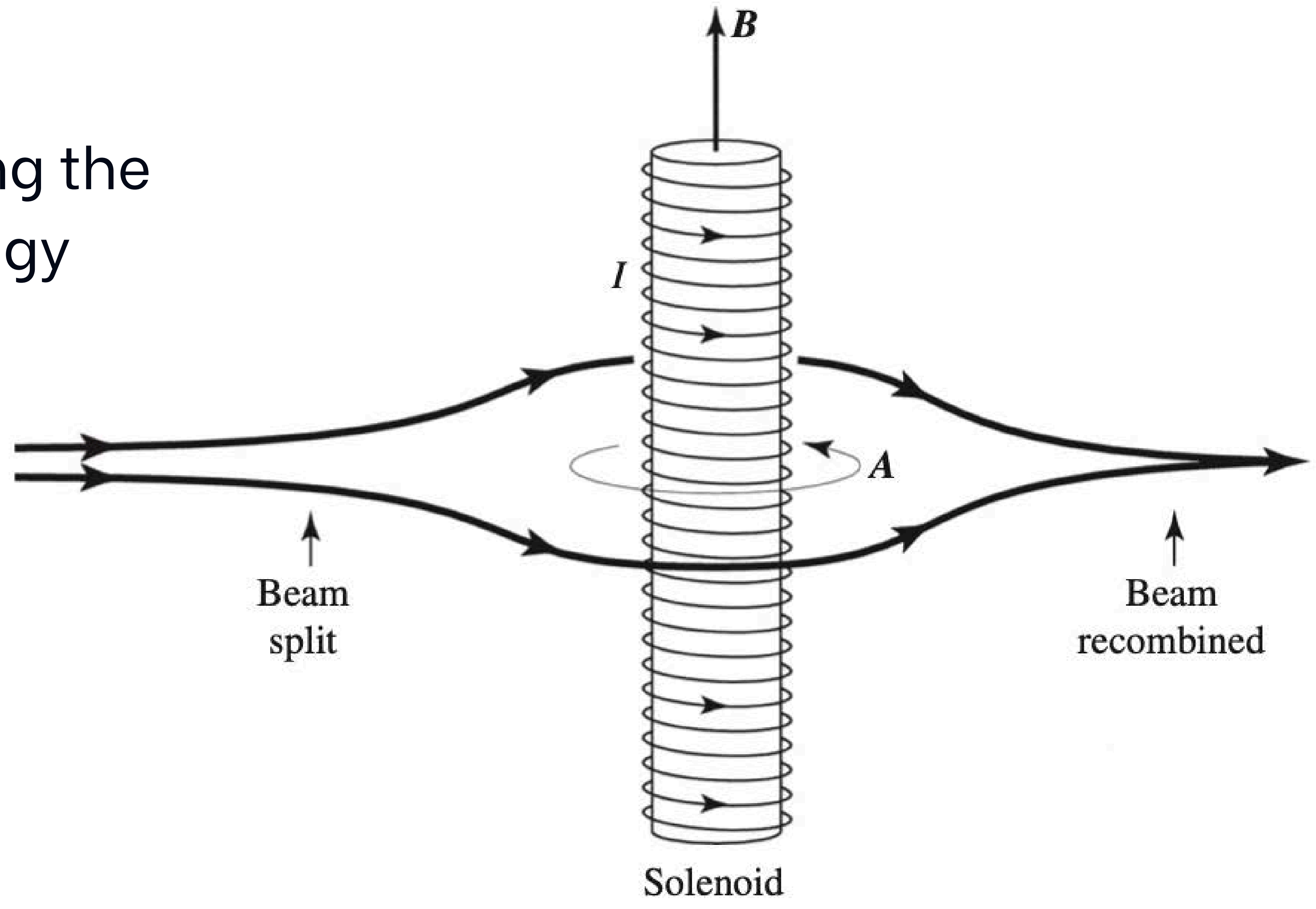
is nothing more than the nonrelativistic approximation of the parallel transport equation: $\partial_\mu\psi - ieA_\mu\psi = 0$



Gauge theories and topology

Aharonov-Bohm effect

This is a simple example of the holonomy measuring the **non-triviality** of the underlying spacetime's topology



Gauge theories and topology

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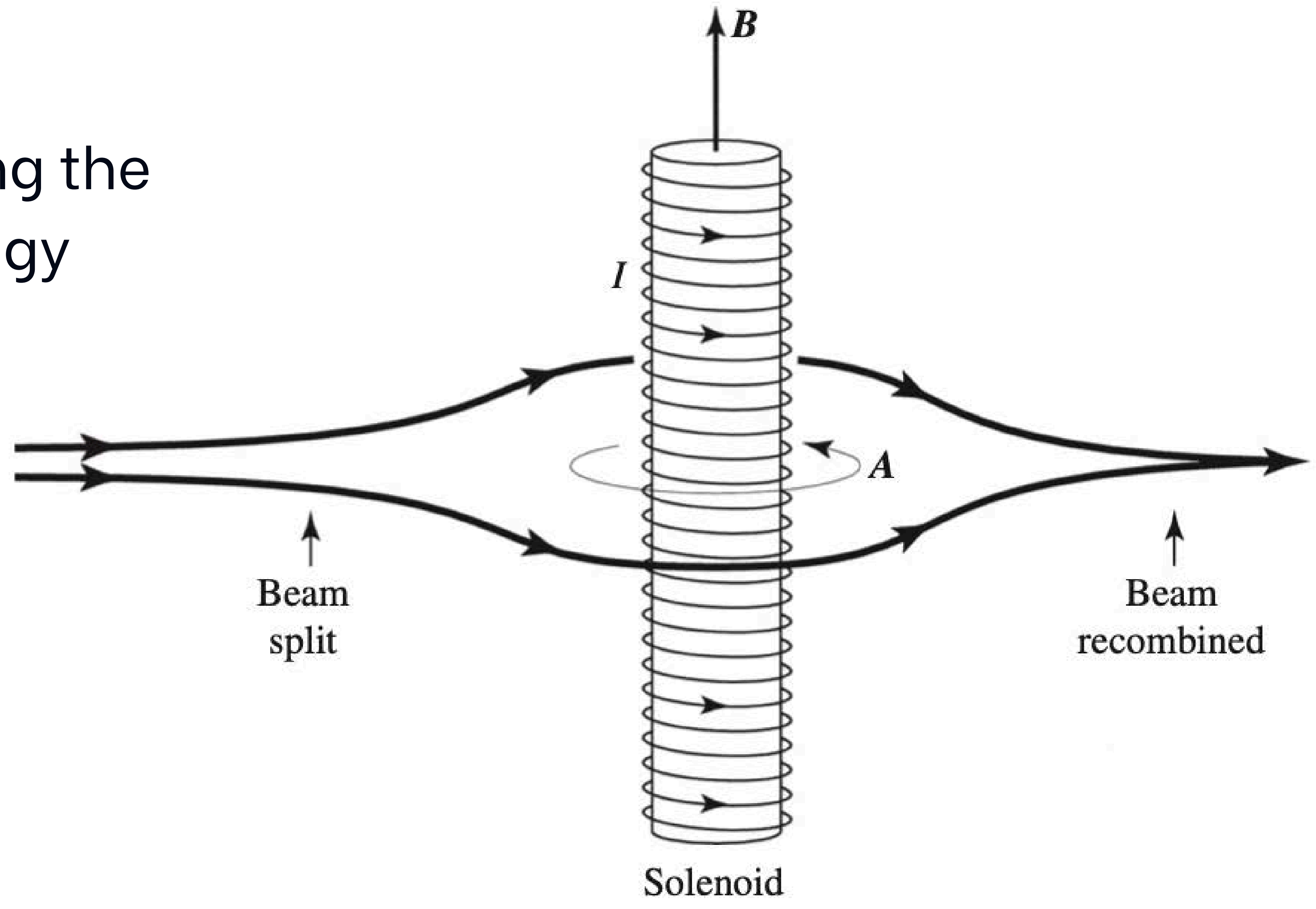
In fact, when we consider the space available for the electron to propagate being only outside the solenoid, we're imposing it to live in a space that is **not simply connected**



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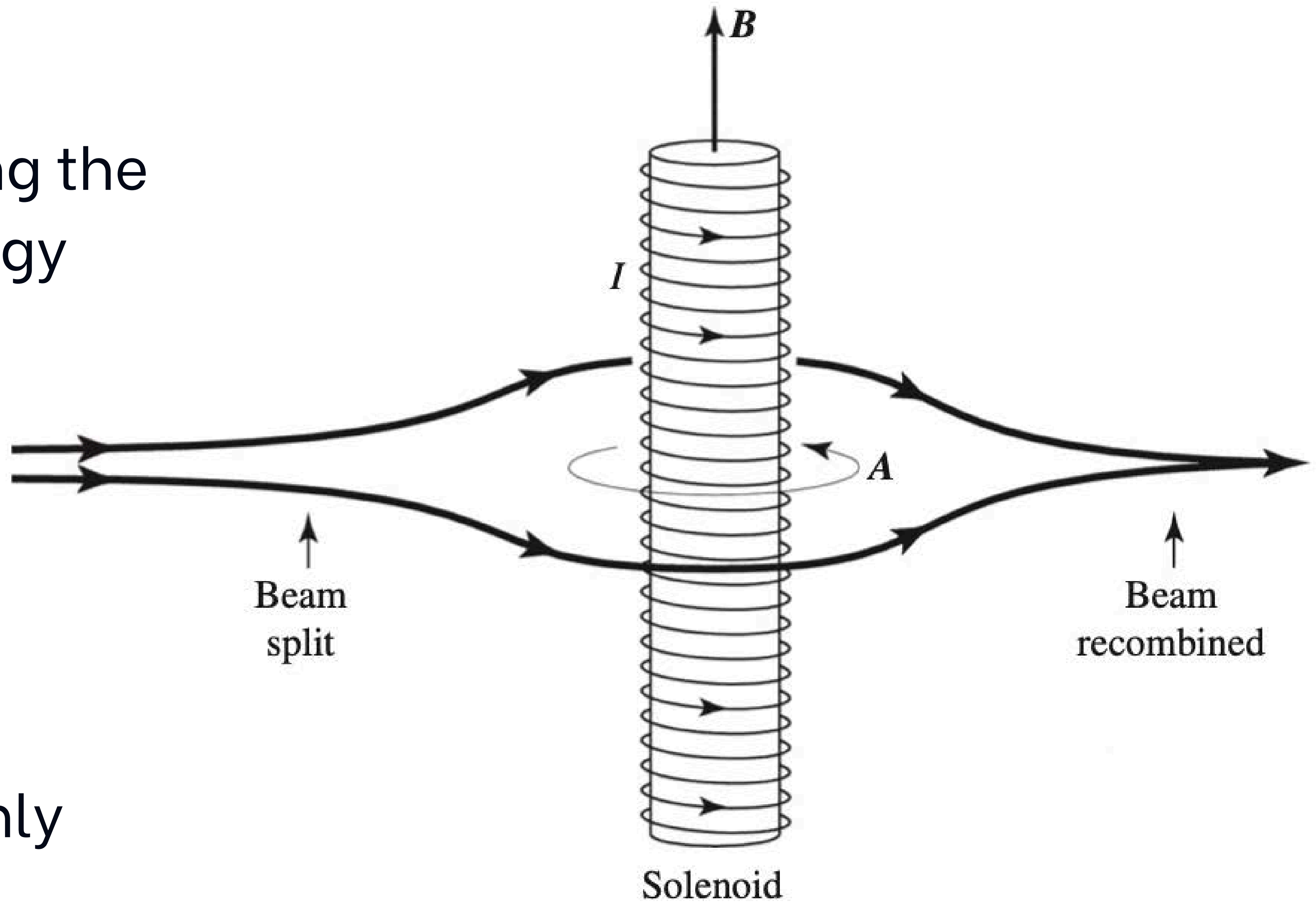
Loops that surround the solenoid **cannot** be smoothly deformed into a point



Topological QFTs

Overview of BRST quantization

Topological quantum field theories are, mostly, gauge theories. Consequently, before dealing with the topological features of the theory we need to deal first with the problems of quantizing a gauge theory

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In this work, we assume those problems are already dealt with using the formalism of **BRST quantization**

In the following, we summarize the most important notions about a gauge theory quantized using this method

Topological QFTs

Overview of BRST quantization

We consider our theory with a collective field content denoted by Φ. Inside this field multiplet we have **gauge fields**, **ghosts** and **Lagrange multipliers** used in the Dirac quantization method

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Using Noether's theorem, we can use the invariance on BRST transformations of the theory to deduce a conserved charge which is the eingenvalue of an operator Q

We can prove that this operator is **nilpotent**: $Q^2 = 0$, and so it defines the **cohomology classes**

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We also have that all physical states must be BRST-invariant, so

$$Q|\psi\rangle = 0$$

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This symmetry transformation also induces the following variation for a general operator $\mathcal{O}(\Phi)$ of the fields

$$\delta\mathcal{O} = \{Q, \mathcal{O}\}$$

where $\{\cdot,\cdot\}$ is the graded commutator: $\{Q, \mathcal{O}\} = Q\mathcal{O} - (-1)^{|\mathcal{O}|}\mathcal{O}Q$

Topological QFTs

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Not only the physical states, but the vacuum needs to be also BRST-invariant

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2. A nilpotent operator Q , the BRST charge
3. Physical states defined to be the Q-cohomology classes
4. A Q-exact energy momentum tensor

$$T_{\mu\nu} = \{Q, V_{\mu\nu}\}$$

for some functional $V_{\mu\nu}[\Phi, g]$ of the fields and the metric

Topological QFTs

We assume our theory is defined on M by the action $S = \int_M dx \sqrt{g}\, \mathcal{L}$

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The dependence of the action on the geometry of spacetime is determined by how it changes due to small variations of the local geometry → these are given by variations of the metric

$$\delta_g S = \frac{1}{2}\int_M dx\sqrt{g}\,T_{\mu\nu}\delta g^{\mu\nu}$$

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but since $T_{\mu\nu}$ is Q-exact,

$$T_{\mu\nu} = \{Q, V_{\mu\nu}\}$$

Topological QFTs

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by the BRST-invariance of the vacuum

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by the BRST-invariance of the vacuum. The partition function of a topological theory is **metric-independent** → it does not depend on the local structure of spacetime, but only on global properties: Z_0 is a **topological invariant**

Topological QFTs

Witten-type theories

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This is possible when the metric dependency happens in a very specific way → the action functional needs to be Q-exact, meaning that

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Witten's model used to describe Donaldson-Floer theory starts from pure Yang-Mills theory

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with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where $[\cdot,\cdot]$ is the usual commutator. He then adds a bunch of fields, adjusts some coupling and constructs a supersymmetric theory with the Lagrangian

$$S' = \int d^4x\sqrt{g}\,\mathrm{tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\phi D_\mu D^\mu\lambda - i\eta D_\mu\psi^\mu + iD_\mu\psi_\nu\chi^{\mu\nu} - \frac{i}{8}\phi[\chi_{\mu\nu},\chi^{\mu\nu}] - \frac{i}{2}\phi[\eta,\eta] - \frac{1}{8}[\phi,\lambda]\right)$$

Topological QFTs

Witten-type theories

The field content is therefore the bosons (A_μ, ϕ, λ) and the fermions $(\psi_\mu, \chi_{\mu\nu}, \eta)$. This Lagrangian preserves the BRST symmetry of the original Yang-Mills action

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One can compute the energy-momentum tensor, obtaining the enormous expression:

$$T_{\mu\nu} = \mathrm{tr}\Big\{F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \frac{i}{2}\Big[(D_\mu\psi_\rho - D_\rho\psi_\mu)\chi_\nu^\rho + (D_\nu\psi_\rho - D_\rho\psi_\nu)\chi_\mu^\rho$$
$$-\frac{1}{2}g_{\mu\nu}(D_\sigma\psi_\rho - D_\rho\psi_\sigma)\chi^{\sigma\rho}\Big] - \frac{1}{2}(D_\mu\phi D_\nu\lambda + D_\nu\phi D_\mu\lambda - g_{\mu\nu}D_\rho\phi D^\rho\lambda) - i(D_\mu\eta\psi_\nu$$
$$+D_\nu\eta\psi_\mu - g_{\mu\nu}D_\rho\eta\psi^\rho) - 2i\Big(\lambda\psi_\mu\psi_\nu - \frac{1}{2}g_{\mu\nu}\lambda\psi_\rho\psi^\rho\Big) + \frac{i}{2}g_{\mu\nu}\phi[\eta,\eta] + \frac{1}{8}g_{\mu\nu}[\phi,\lambda]^2$$

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and this object is Q-exact

Topological QFTs

Witten-type theories

In fact, we can find that $T_{\mu\nu} = \{Q, \lambda_{\mu\nu}\}$, with

$$\lambda_{\mu\nu} = \frac{1}{2}\mathrm{tr}\left(F_{\mu\rho}\chi_{\nu}^{\rho} + F_{\nu\rho}\chi_{\mu}^{\rho} - \frac{1}{2}g_{\mu\nu}F_{\rho\sigma}\chi^{\rho\sigma}\right) + \frac{1}{2}\mathrm{tr}(\psi_{\mu}D_{\nu}\lambda + \psi_{\nu}D_{\mu}\lambda - g_{\mu\nu}\psi_{\rho}D^{\rho}\lambda) + \frac{1}{4}g_{\mu\nu}\,\mathrm{Tr}(\eta[\phi,\lambda])$$

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This is the relation that makes it a topological theory. Moreover, if we define

$$V \equiv \frac{1}{4}\mathrm{tr}\,F_{\mu\nu}\chi^{\mu\nu} + \frac{1}{2}\mathrm{tr}\,\psi_{\mu}D^{\mu}\lambda - \frac{1}{4}\mathrm{tr}(\eta[\phi,\lambda])$$

it's possible to show that

$$S' = \{Q, V\} - \frac{1}{8}\int d^4x\,\mathrm{tr}(F_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma})$$

Topological QFTs

Witten-type theories

This last term can be written as

$$\frac{1}{8}\int \mathrm{tr}(F_{\mu\nu}F_{\rho\sigma}\underbrace{dx^{\mu}\wedge dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}}_{=\varepsilon^{\mu\nu\rho\sigma}\,d^4x})$$

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so it is a topological invariant (in fact, it measures the **first Pontryagin class** of the fibre bundle)

Topological QFTs

Witten-type theories

Maxwell action

$$-\frac{1}{2}\int F\wedge\star F = -\frac{1}{4}\int F_{\mu\nu}F_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}\sqrt{g}\,d^4x$$

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Since it is a topological invariant, we can add this term to the original theory and thus obtain a Q-exact action, which is a Witten-type topological theory

Topological QFTs

Instantons and Donaldson polynomials

The connection with Donaldson-Floer theory appears when we consider that the gauge part of this new action consists of

$$\frac{1}{4}\int d^4x\,\mathrm{tr}\left(\sqrt{g}F_{\mu\nu}F^{\mu\nu} + 2F_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right)$$

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$$\frac{1}{4}\int d^4x\,\mathrm{tr}\left(\sqrt{g}F_{\mu\nu}F^{\mu\nu}+2F_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right)$$

and defining the **dual field strength tensor** $\tilde{F}^{\mu\nu}\equiv\frac{1}{2\sqrt{g}}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}$

Topological QFTs

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The classical solutions are of extreme importance to topological theories. This happens because the only place where the coupling constant e appears is outside the action

$$S = -\frac{1}{e^2}S'$$

Topological QFTs

Instantons and Donaldson polynomials

The consequence of this is the **coupling independence** of the quantum theory, since

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therefore we can take the coupling constant to indefinitely small values and determine the generating functional as a perturbative expansion around the classical action

Topological QFTs

Instantons and Donaldson polynomials

Doing this expansion, Witten proves that Z_0 is the first **Donaldson invariant**

$$Z_0 = \sum_i (-1)^{n_i}$$

where n_i is determined by the eigenvalues of the operators acting on the fields of the theory

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This invariant was obtained by Donaldson when studying the geometry of fibre bundles and their relation to gauge theories, moduli spaces and instantons

Topological QFTs

Schwarz-type theories

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This automatically implies that $\mathcal{H} = T^0_0 = 0$

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The gauge fixing and ghost terms, however, may end up depending on the metric, but on Schwarz-type theories, this dependence is Q-exact, which yields the quantum action

$$S[\Phi, g] = S_c[\Phi] + \{Q, V[\Phi, g]\}$$

for some functional $V[\Phi, g]$

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The most common theory that satisfies these properties is **Chern-Simons** theory

Chern-Simons theory

Classical action

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The abelian action is defined on a 3-dimesional manifold as the integral ot the Chern-SImons three form:

$$S = \frac{k}{4\pi} \int A \wedge dA$$

where $A = A_\mu dx^\mu$ is the vector potential 1-form and k is an integer called the **level** of the theory

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Since this object has no mention to the metric of the space, it is a topological invariant

Chern-Simons theory

Classical action

In terms of the gauge fields A_μ, this action reads

$$\int A \wedge dA = \int A_\mu \partial_\nu A_\rho dx^\mu \wedge dx^\nu \wedge dx^\rho$$

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and that's the well know Chern-Simons term

$$S = \frac{k}{4\pi} \int \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho d^3x$$

Chern-Simons theory

Classical action

notice that this term breaks parity, which in $(2+1)d$ is the transformation

$$x^0 \to x^0,\ x^1 \to -x^1,\ x^2 \to x^2$$
$$A_0 \to A_0,\ A_1 \to -A_1,\ A_2 \to A_2$$

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that induces

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and since every term in the sum $\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$ is multiplied exatcly once by A_1 or ∂_1, we have

$$\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \to -\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

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Classical action

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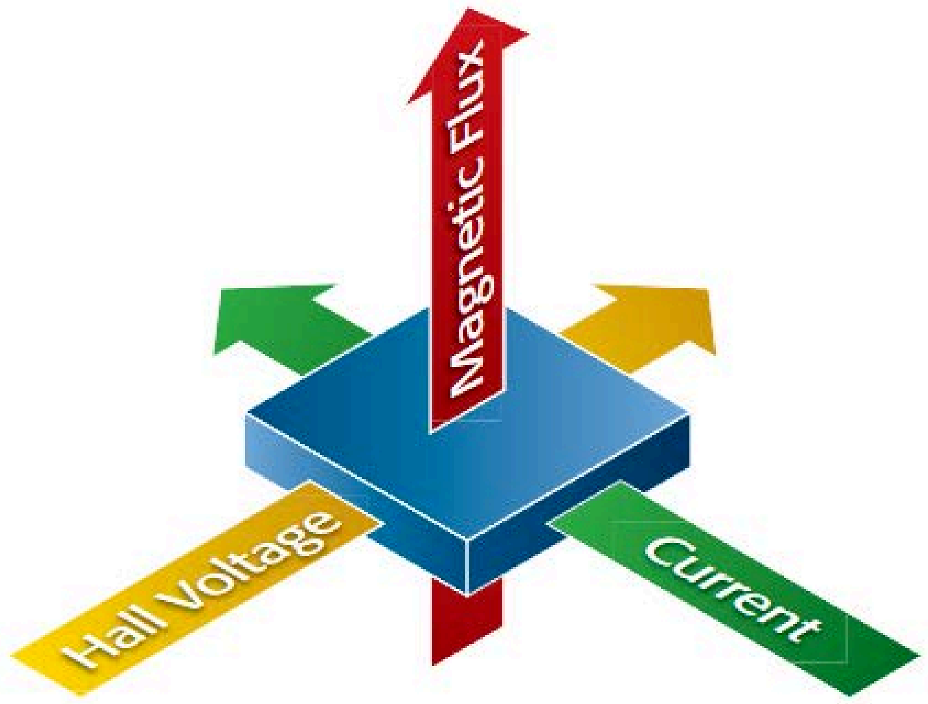
In fact, when we couple this term with the magnetic field of the electrons on the material, the interaction yields the Hall conductivity

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\frac{1}{m}$$



Chern-Simons theory

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results in an extra factor if i, which yields the euclidean partition function

$$Z_0 = \int \mathcal{D}A \exp(S) = \int \mathcal{D}A \exp\left[\frac{ik}{4\pi}\int \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)\right]$$

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Studying the semiclassical approximation of this system shows a similar relation of this object with other topological invariant, the **Ray-Singer torsion**, which is related to simply connectedness of manifolds

Chern-Simons theory

Quantum theory

Our aim at his section, however, is to look at other topological invariants of the quantum theory, namely, the VEV's of the Wilson loops:

$$\langle W[\gamma] \rangle = \int \mathcal{D}A \exp(iS) P \exp\left(i \oint_\gamma dx^\mu A_\mu\right)$$

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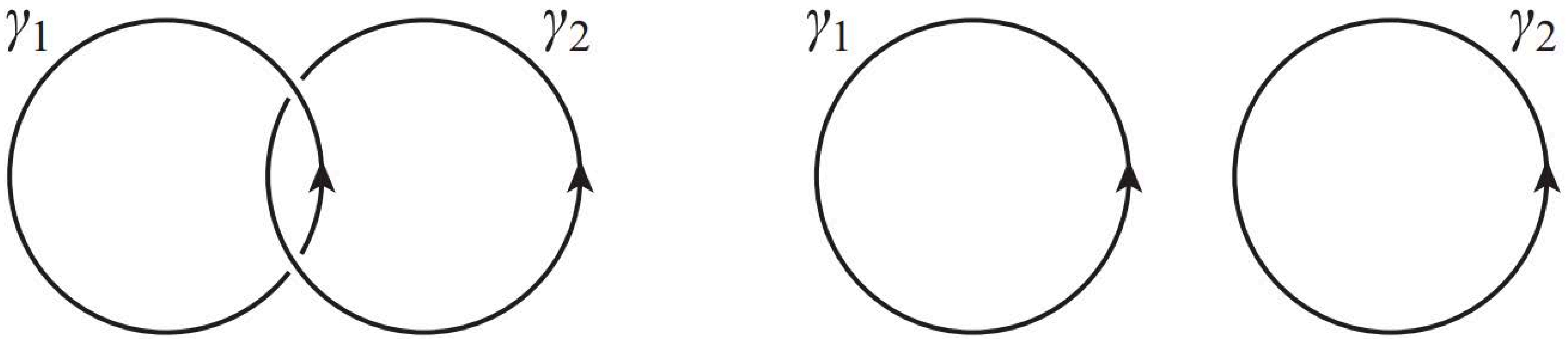
and consider the Wilson loop along a composite curve $\gamma = \gamma_1 \cup \gamma_2 \cup \ldots$ given by the union of an arbitrary number of loops

Chern-Simons theory

Wilson loops, knots and links

Each one of these loop can be interpreted as a **knot**, and their union is called a **link** → these curves may of may not be intertwinned, and we sill se that this information is contained in the Wilson loop operator of this theory



Chern-Simons theory

Wilson loops, knots and links

Parametrizing this curve as $\gamma(t) = (z^\mu(t))$ we can rewrite the Wilson loop as

$$\exp\left(i \oint_\gamma dx^\mu A_\mu\right) = \exp\left(i \int ds \frac{dz^\mu(s)}{ds} A_\mu(s)\right)$$

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where

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$$\langle W[\gamma]\rangle = \int \mathcal{D}A \exp\left[i\int d^3x\left(A_\mu[G^{-1}]^{\mu\nu}A_\nu + J^\mu A_\mu\right)\right]$$

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→ $$[G^{-1}]^{\mu\nu} = \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\partial_\rho$$

the inverse of the propagator

Chern-Simons theory

Wilson loops, knots and links

and this integral can be easily evaluated

$$\langle W[\gamma]\rangle = \exp\Big(- \frac{i}{2} \int d^3x \int d^3y J^{\mu}(x) G_{\mu\nu}(x-y) J^{\nu}(y)\Big)$$

Chern-Simons theory

Wilson loops, knots and links

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where

$$\tilde{G}_{\mu\nu}(p) = \frac{2\pi}{k}\frac{\varepsilon_{\mu\nu\rho}p^\rho}{p^2} \quad \leftrightarrow G_{\mu\nu}(x-y) = \frac{2\pi}{k}\frac{\varepsilon_{\mu\nu\rho}\partial^\rho}{\Box}\delta(x-y)$$

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$$I[\gamma] = \int d^3x\int d^3y\oint dz_1^\mu\oint dz_2^\nu\delta(x-z_1)\delta(y-z_2)\frac{2\pi}{k}\frac{\varepsilon_{\mu\nu\rho}\partial^\rho}{\Box}\delta(x-y)$$

Chern-Simons theory

Wilson loops, knots and links

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Moreover, since the loop is closed, this current must be conserved, $\partial_\mu J^\mu = 0$, and this implies that it is the curl of another vector

$$J^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu K_\rho$$

Chern-Simons theory

Wilson loops, knots and links

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hence,

$$J^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu \varepsilon_{\rho\sigma\lambda}\partial^\sigma\phi^\lambda = (\delta^\mu_\sigma\delta^\nu_\lambda - \delta^\mu_\lambda\delta^\nu_\sigma)\phi^\lambda = \partial^\mu \underbrace{\partial_\lambda\phi^\lambda}_{\text{Lorentz again}} - \Box\phi^\mu = -\Box\phi^\mu$$

Wilson loops, knots and links

So we have

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Chern-Simons theory

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$$\oint_{\gamma=\partial\Sigma} dz^\mu K_\mu = \oint_{\gamma=\partial\Sigma} d\vec{z}\cdot\vec{K} = \int_\Sigma d\vec{S}\cdot(\nabla\times\vec{K}) = \int dS_\mu \varepsilon^{\mu\nu\rho}\partial_\nu K_\rho$$

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this integral is precisely the flux of the current J^μ through the surface Σ, so it counts the number of times the curve γ pierces through Σ

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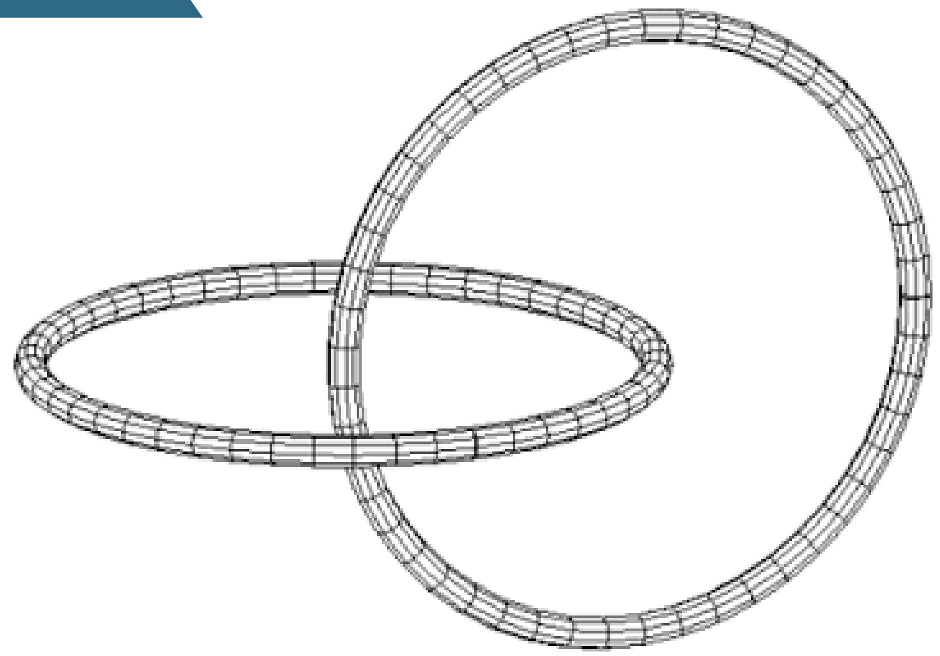

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If γ is composed of just 1 knot, then this is the **self-linking number** of this knot. This is a topological invariant, so is the Wilson loop operator

$$W[\gamma] = e^{\frac{2\pi i n_\gamma}{k}}$$

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This result has other implications, such as the fact that the particles of this theory are **anyons**, which are particles with both fermionic and bosonic prioperties

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and this leads directly to the **Jones polynomials**

Chern-Simons theory

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We can see, therefore, that it's a very hot topic, with new things being recently developed

Thank You

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