Ex: $\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2$

Exercise 2: show that Euler-Lagrange eq. will give us: $(\partial_\mu\partial^\mu + m^2)\phi = 0$

SOLUTIONS ARE OF THE FORM: $e^{\pm i(kx - w_k t)}$; $w_k = \sqrt{k^2 + m^2}$

GENERAL SOLUTION:

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2w_k}}\left[a_k\, e^{i(kx - w_k t)} + a_k^\dagger\, e^{-i(kx - w_k t)}\right]$$

AFTER QUANTIZATION:

$$[\hat\phi(\vec x,t), \hat\pi(\vec x',t)] = i\delta(\vec x - \vec x') \Rightarrow [\overset{\text{ANNIHILATION}}{\hat a_k}, \overset{\text{CREATION}}{\hat a^\dagger_{k'}}] = (2\pi)^3\,\delta^3(k - k')$$

$$\hat H = \int \frac{d^3k}{(2\pi)^3} w_k\left[\underbrace{\hat a^\dagger_k\,\hat a_k} + \frac{1}{2}\right] = \overbrace{\int \frac{d^3k}{(2\pi)^3} w_k\,\hat N_k}^{:\hat H:} + \text{VACUUM ENERGY}$$

number operator (just like the harmonic oscilator), counts the number of excitations, with momentum k.

$$\hat P^i = \int d^3x\, T^{0i} = \int \frac{d^3k}{(2\pi)^3} k^i\, \hat N_k$$

SINCE $\hat H$, $\hat P$ AND $\hat N$ CAN BE DIAGONAL AT THE SAME TIME, THE THEORY WILL HAVE STATIONARY STATES WHICH HAVE A DEFINITE NUMBER OF EXCITATIONS AND EACH OF THESE HAS
$$\begin{cases} \text{MOMENTUM: } \vec k \\ \text{ENERGY: } E_k = w_k = \sqrt{k^2 + m^2} \end{cases}$$

WE CALL THESE EXCITATIONS "PARTICLES"!

THE SITUATION IS VERY SIMILAR FOR THEORIES WITH ONLY "QUADRATIC" TERMS IN SPINOR OR VECTOR FIELDS

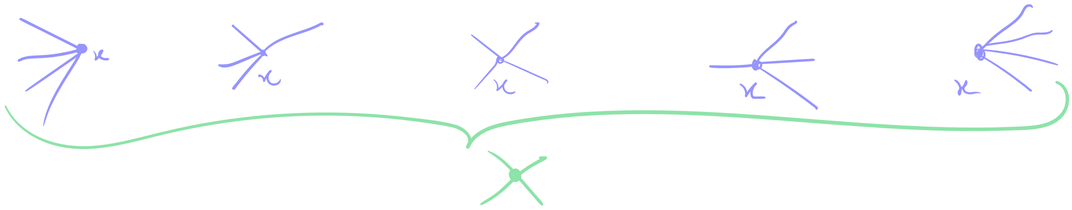
$$\mathcal{L}_\psi = i\,\bar\psi \gamma^\mu \partial_\mu \psi - m\bar\psi\psi$$

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

SO ALL OF THESE ARE FREE THEORIES, THERE ARE NO TERMS ON THE HAMILTONIAN THAT CHANGE ENERGY, MOMENTUM OR NUMBER OF EXCITATIONS

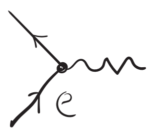
INTERACTIONS WILL COME FROM TERMS WITH 3 OR MORE POWERS OF THE FIELDS IN $\mathcal{L}$ AND $\mathcal{H}$

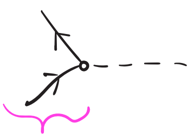
EG: $\lambda\phi^4(x) \sim aaaa + a^\dagger aaa + a^\dagger a^\dagger aa + a^\dagger a^\dagger a^\dagger a + a^\dagger a^\dagger a^\dagger a^\dagger$



→ COUPLING CONSTANT CONTROLS THE "STRENGHT" OF THE INTERACTION (THE THEORY BECOMES FREE AS $\lambda \to 0$)

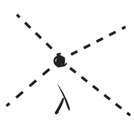
WE DON'T KNOW EXACT SOLUTIONS TO MOST INTERACTING THEORIES, SO WE HAVE TO TREAT INTERACTIONS PERTURBATIVELLY, ENTER THE FEYNMAN DIAGRAMS:

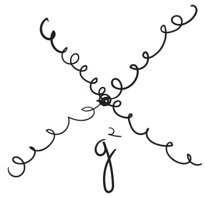
QED: $e\,\bar\psi A_\mu \gamma^\mu \psi \Rightarrow$ 

YUKAWA: $y\,\phi\bar\psi\psi \Rightarrow$ 

COMPLEX FIELDS CONTAIN PARTICLES AND ANTI-PARTICLES

$$\begin{pmatrix} \psi \sim a + b^\dagger \\ \psi^\dagger \sim b + a^\dagger \end{pmatrix}$$

$\lambda\phi^4$: $\frac{\lambda}{4!}\phi^4 \Rightarrow$ 

QCD: $-\frac{g^2}{4} f^{abc} f^{ab'c'} G^b_\mu G^c_\nu G^{b'\mu} G^{c'\nu} \Rightarrow$ 

THERE ARE ALSO DISCRETE SYMMETRIES OF THE LORENTZ GROUP:

$$S^2 = x_\mu x^\mu \longrightarrow S^2 = x'_\mu x'^\mu \begin{cases} C: & \text{PARTICLES} \rightleftarrows \text{ANTIPARTICLES} \\ P: & \text{CHIRALITY} \\ T: & \text{TIME REVERSAL} \end{cases}$$

AND INTERNAL SYMMETRIES:

COMPLEX FIELD: $\phi = \phi_1 + i\phi_2 \qquad \phi_{1,2} \in \mathbb{R}$

$$\phi \xrightarrow{U} \underbrace{e^{i\theta}}_{U} \phi \qquad (\text{GLOBAL } U(1))$$

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi + \cancel{m^2 \phi \phi} + \cancel{\frac{\lambda_3}{2} \phi \phi \phi^*} + \lambda_n (\phi^* \phi)^n$$

$$\phi \xrightarrow{U(1)} e^{i\theta(x)} \phi \qquad (\text{LOCAL OR GAUGE } U(1))$$

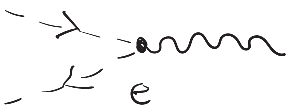
$$A_\mu \xrightarrow{U(1)} A_\mu - \frac{1}{e} \partial_\mu \theta(x)$$

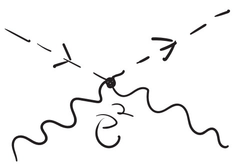
Exercise 3: check: $D_\mu \phi = (\partial_\mu + ieA_\mu)\phi \xrightarrow{U(1)} e^{i\theta(x)} D_\mu \phi$

(SCALAR QED) $$\mathcal{L} = |D_\mu \phi|^2 + m^2 \phi^* \phi \underbrace{- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}} + \cancel{m_\gamma A_\mu A^\mu} =$$

Exercise 4: check that this is Gauge invariant

$$= |\partial_\mu \phi|^2 + m \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + ie A^\mu (\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi) + e^2 A_\mu A^\mu \phi^* \phi$$



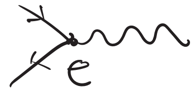


QED IS BASED ON THE SAME SYMMETRY, BUT WITH A FERMION:

$$\psi \xrightarrow{U(1)} e^{i\theta(x)}\,\psi$$

$$\mathcal{L}_{QED} = \bar\psi(i\gamma^\mu D_\mu)\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} =$$

$$= \bar\psi(i\gamma^\mu \partial_\mu)\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar\psi\gamma^\mu\psi A_\mu$$



THESE "PHASE" ROTATIONS CAN BE MORE COMPLICATED:

(ISOSPIN) $$N = \begin{pmatrix} p \\ n \end{pmatrix}_A \xrightarrow[\text{GLOBAL}]{SU(2)} N' = e^{i\alpha_a \frac{\sigma^a_{AB}}{2}} \begin{pmatrix} p \\ n \end{pmatrix}_B$$

$A = 1,2$ $\quad a = 1,2,3$

("PROTON" AND "NEUTRON" ARE TWO STATES OF THE "NUCLEON" PARTICLE)

(QCD) $$q = \begin{pmatrix} q_R \\ q_G \\ q_B \end{pmatrix}_A \xrightarrow[\text{LOCAL}]{SU(3)} q' = e^{i\alpha_a t^a_{AB}} \begin{pmatrix} q_R \\ q_G \\ q_B \end{pmatrix}_B$$

$A = 1,2,3$ $\quad a = 1,\dots,8$

THE LOCAL VERSION ALSO DEMANDS A COVARIANT DERIVATIVE AND NEW GAUGE BOSONS:

$$\psi(x) \xrightarrow[\text{LOCAL}]{\tilde{\text{N}}\text{-ABEL.}} \underbrace{U(x)}_{\to\ \text{EXP}(i\alpha_a t^a)}\,\psi(x)$$

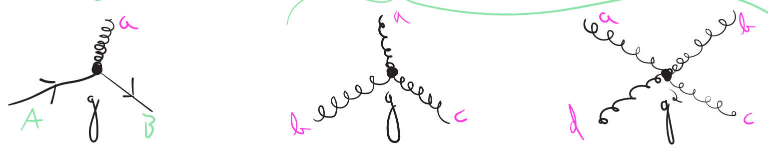
$$D_\mu = \partial_\mu - ig\underbrace{A^a_\mu t^a}_{\text{MATRIX}} \longrightarrow \text{ONE FOR EACH "DIRECTION" (GENERATOR)}$$

$$A^a_\mu = (A^1_\mu, A^2_\mu, A^3_\mu)$$

$$A^m_\mu = \frac{1}{2}\begin{pmatrix} A^3_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & -A^3_\mu \end{pmatrix}$$

$$A^a_\mu t^a \xrightarrow[\text{LOCAL}]{\tilde{\text{N}}\text{-ABEL.}} U(x)\cdot A^a_\mu t^a \cdot U^\dagger(x) = \left(A^a_\mu + \frac{1}{g}(\partial_\mu\alpha^a) + f_{abc}A^b_\mu\alpha^c\right)t^a + O(\alpha^2)$$

$$\mathcal{L}_{YM} = \underbrace{\bar\psi(i\not{D} - m)\psi}_{} - \underbrace{\frac{1}{4}(F^a_{\mu\nu})^2}_{} \longrightarrow F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc}A^b_\mu A^c_\nu$$



IMPORTANT POINT: $\cancel{m^2 A^a_\mu A^{\mu a}}$