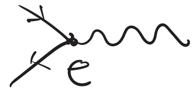
$$\psi \xrightarrow{U(1)} e^{i\theta(x)}\psi$$

$$\mathcal{L}_{QED} = \bar\psi(i\gamma^\mu D_\mu)\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} =$$

$$= \bar\psi(i\gamma^\mu\partial_\mu)\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar\psi\gamma^\mu\psi A_\mu$$



THESE "PHASE" ROTATIONS CAN BE MORE COMPLICATED:

(ISOSPIN) $N = \begin{pmatrix} p \\ n \end{pmatrix}_A \xrightarrow[\text{GLOBAL}]{SU(2)} N' = e^{i\alpha_a \frac{\sigma^a_{AB}}{2}} \begin{pmatrix} p \\ n \end{pmatrix}_B$

$A = 1, 2$ $\qquad a = 1, 2, 3$

("PROTON" AND "NEUTRON" ARE TWO STATES OF THE "NUCLEON" PARTICLE)

(QCD) $q = \begin{pmatrix} q_R \\ q_G \\ q_B \end{pmatrix}_A \xrightarrow[\text{LOCAL}]{SU(3)} q' = e^{i\alpha_a t^a_{AB}} \begin{pmatrix} q_R \\ q_G \\ q_B \end{pmatrix}_B$

$A = 1, 2, 3$ $\qquad a = 1, \dots, 8$

THE LOCAL VERSION ALSO DEMANDS A COVARIANT DERIVATIVE AND NEW GAUGE BOSONS:

$$\psi(x) \xrightarrow[\text{LOCAL}]{\tilde{N}\text{-ABEL.}} \underbrace{U(x)}_{\to\ \exp(i\alpha_a t^a)}\psi(x)$$

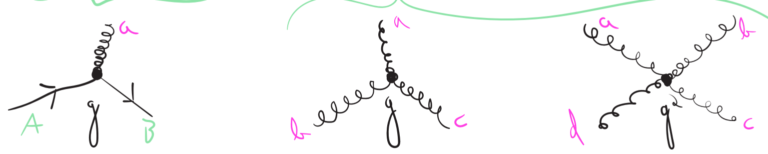
$D_\mu = \partial_\mu - ig\underbrace{A^a_\mu t^a}_{\text{MATRIX}} \longrightarrow$ ONE FOR EACH "DIRECTION" (GENERATOR)

$$A^a_\mu = (A^1_\mu, A^2_\mu, A^3_\mu)$$

$$A^m_\mu = \frac{1}{2}\begin{pmatrix} A^3_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & -A^3_\mu \end{pmatrix}$$

$$A^a_\mu t^a \xrightarrow[\text{LOCAL}]{\tilde{N}\text{-ABEL.}} U(x)\cdot A^a_\mu t^a \cdot U^\dagger(x) = \left(A^a_\mu + \frac{1}{g}(\partial_\mu\alpha^a) + f_{abc}A^b_\mu\alpha^c\right)t^a + O(\alpha^2)$$

$$\mathcal{L}_{YM} = \underbrace{\bar\psi(i\not{D} - m)\psi}\ - \underbrace{\frac{1}{4}(F^a_{\mu\nu})^2} \longrightarrow F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc}A^b_\mu A^c_\nu$$



IMPORTANT POINT: $\cancel{m^2 A^a_\mu A^{\mu a}}$

The continuous symmetries also imply, by Noether's Theorem, in conserved quantities:

$$\left.\begin{aligned} \phi &\xrightarrow[\alpha]{\text{SYM}} \phi + \alpha\Delta\phi \\ S &\xrightarrow[\alpha]{\text{SYM}} S \\ \mathcal{L} &\xrightarrow[\alpha]{\text{SYM}} \mathcal{L} + \alpha \underbrace{\partial_\mu J^\mu}_{\Delta\mathcal{L}} \end{aligned}\right\} \Delta S = 0$$

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu$$

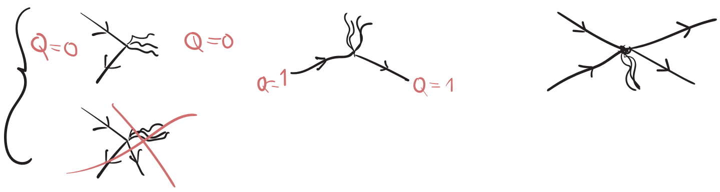
$$\partial_\mu j^\mu = 0 \to \partial_0 j^0 + \vec{\nabla}\cdot\vec{j} = 0 \Rightarrow \frac{d}{dt}\underbrace{\int d^3x\, j^0}_{Q} = -\oint d\vec{S}\cdot\vec{j}$$

For symmetries of the Lagrangian itself $J^\mu = 0$

	Symmetry	"Charge"
Time trans.:	$t \to t + a$	E
Translation:	$\vec{x} \to \vec{x} + \vec{a}$	$\vec{p}$
	Boosts	$t\vec{p} - \vec{x}E$
Rotation:	$x_i \to R_{ij}x_j$	$\vec{L}$
U(1) Phase:	$\phi(x_\mu) \to e^{i\theta}\phi(x_\mu)$	Q

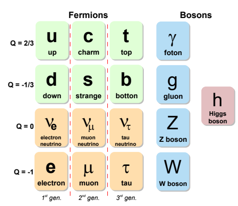
The simplest way (though not the only) this manifests in quantum theory is by selection rules and vertex-by-vertex conservation

U(1): $\phi^*\phi$, $\bar{\psi}\psi \subset \mathcal{L} \Rightarrow$



$\hookrightarrow$ This one is true even without the U(1)

Now, the Standard Model: $SU(3)_c \times SU(2)_L \times U(1)_Y$



Chirality matters

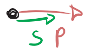
Chirality: $\{\gamma_5, \gamma_\mu\} = 0$ $\quad (\gamma_5)^2 = \hat{1}$ $\quad P_L \equiv \frac{1-\gamma_5}{2}$ $\quad P_R \equiv \frac{1+\gamma_5}{2}$

Exercise 5: show

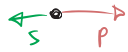
$$\begin{cases} P_L + P_R = \hat{1} \\ P_L^2 = P_L \qquad P_R^2 = P_R \\ P_L P_R = P_R P_L = 0 \end{cases} \Longrightarrow \Psi = \underbrace{P_L \Psi}_{\Psi_L} + \underbrace{P_R \Psi}_{\Psi_R}$$

CHIRALITY IS "HANDEDNESS"

MASSLESS PARTICLES: CHIRALITY-STATES = HELICITY-STATES



$H = + \rightleftharpoons \Psi = \Psi_R$



$H = - \rightleftharpoons \Psi = \Psi_L$

MASSIVE PARTICLES: CHIRALITY-STATES $\xrightarrow{\text{NOT}}$ HELICITY-STATES

$\xrightarrow{\text{NOT}}$ MASS-STATES

Exercise 6: show

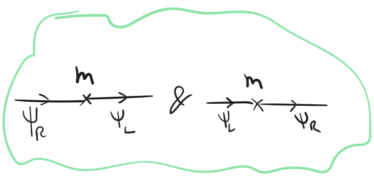
$$\begin{cases} \bar{\Psi}\gamma_\mu \Psi = \bar{\Psi}_L \gamma_\mu \Psi_L + \bar{\Psi}_R \gamma_\mu \Psi_R \\ \underbrace{m\bar{\Psi}\Psi}_{\text{MASS BASIS}} = m\,\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L \end{cases}$$

CHIRALITY BASIS ...

$\bar{\Psi} = \Psi^\dagger \gamma_0$

$\bar{\Psi}_{L/R} = (P_{L/R}\Psi)^\dagger \gamma_0$

$\gamma_5^\dagger = \gamma_5$

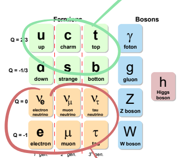


Pauli: "I cannot believe God is a weak left-hander." (L. Lederman book)

FIELDS WILL HAVE DIFFERENT U(1) "HYPER CHARGES" (WE WILL SEE THE REASONING FOR THE CHOICES SOON)

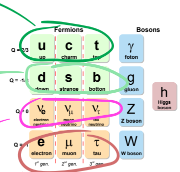
$U(1)_Y$: $\Psi \xrightarrow{U(1)_Y} e^{iY\theta(x)}\Psi$ → $\Psi^*\Psi \to \Psi^*\Psi$

$\Psi_1^*\Psi_2\Psi_3^*\Psi_4 \to \Psi_1^*\Psi_2\Psi_3^*\Psi_4 \longleftrightarrow -Y_1 + Y_2 - Y_3 + Y_4 = 0$



	LEFT-HANDED	RIGHT-HANDED
QUARKS:	$Y = +1/6$	$Y = 2/3$
		$Y = -1/3$
LEPTONS:	$Y = -1/2$	ν_R? $Y = 0$
		$Y = -1$

"UP"
"DOWN"
"UP"
"DOWN"



NOTE THE LACK OF DISTINCTION BETWEEN UP AND DOWN ON THE LEFT-HANDED SECTOR!

$SU(2)_L$: $\psi_L \xrightarrow{SU(2)_L} e^{i\frac{\sigma_a}{2}\theta_a(x)}\,\psi_L = e^{i t_a \theta_a}\,\psi_L$

$\psi_R \xrightarrow{SU(2)_L} \psi_R$

THIS PARITY VIOLATION "FORBIDS" MASSES! TAKE THE ELECTRON, FOR INSTANCE:

$$m\,\bar\psi^e\psi^e = m\,\bar\psi^e_L\psi^e_R + m\,\bar\psi^e_R\psi^e_L \xrightarrow{SU(2)_L\times U(1)_Y} m\,\bar\psi^e_L\, e^{-it_a\theta_a - i(y^e_L-y^e_R)\theta}\,\psi^e_R \;+$$

$$m\,\bar\psi^e_R\, e^{+it_a\theta_a+i(y^e_L-y^e_R)\theta}\,\psi^e_L$$

FORBIDDEN BY U(1): $\underbrace{y^e_L}_{-1} \neq \underbrace{y^e_R}_{-2}$

FORBIDEN BY SU(2)

IN FACT

$$\underbrace{e^{i t_a\theta_a(x)}}_{2\times 2\ \text{MATRIX}}\,\psi_L \qquad \begin{pmatrix}\text{"UP"}_L\\ \text{"DOWN"}_L\end{pmatrix} \longrightarrow m\,\overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L}\cdot e_R$$

PRODUCT MAKES NO SENSE (THE LAGRANGIAN CANNOT BE A MATRIX)

SU(2) DOUBLETS:
$$\underbrace{L^e = \begin{pmatrix}\nu_{eL}\\ e_L\end{pmatrix} \qquad L^\mu = \begin{pmatrix}\nu_{\mu L}\\ \mu_L\end{pmatrix} \qquad L^\tau = \begin{pmatrix}\nu_{\tau L}\\ \tau_L\end{pmatrix}}_{L^\ell}\,,\quad \ell = 1,2,3$$

$$Q^1 = \begin{pmatrix}u_L\\ d_L\end{pmatrix} \qquad Q^2 = \begin{pmatrix}c_L\\ s_L\end{pmatrix} \qquad Q^3 = \begin{pmatrix}t_L\\ b_L\end{pmatrix}$$

CONCLUSION: MASSES WILL BE COMPLICATED TO MODEL!!!

"COLOR" IS SIMPLER, ONLY QUARKS TRANSFORM (AND PARITY IS PRESERVED):

SU(3): $\psi \to e^{i t_a \theta_a}\,\psi$

↳ FUNDAMENTAL REP. OF SU(3): $\begin{pmatrix}\psi_R\\ \psi_G\\ \psi_B\end{pmatrix}$

↳ GELL-MANN MATRICES (8, 3X3 MATRICES) (SLIDE)

↳ FORBIDS "QUARK-LEPTON" TERMS: $QL \to Q\,e^{-it_a\theta}\,L$

GAUGE SECTOR:

	THEORY		EXPERIMENT	
U(1):	B_μ	ALL MASSLESS	γ	MASSLESS P-PRESERVING
SU(2):	$W^1_\mu, W^2_\mu, W^3_\mu$	$(F^a_{\mu\nu})^2$ ✓	W^+, W^-, Z	MASSIVE ?!
SU(3):	$G^1_\mu, \dots, G^8_\mu$	$\cancel{A^a_\mu A^{a\mu}}$	$G^1_\mu, \dots, G^8_\mu$	MASSLESS ✓

SOMETHING ELSE IS CLEARLY GOING ON!

Spontaneous Symmetry Breaking (SSB).

U(1) example:
- 1 scalar: $\phi \xrightarrow{U(1)} e^{i\Theta(x)}\phi$
- 1 gauge-boson: $A_\mu(x) \xrightarrow{U(1)} A_\mu(x) - \frac{1}{e}\partial_\mu \Theta(x)$

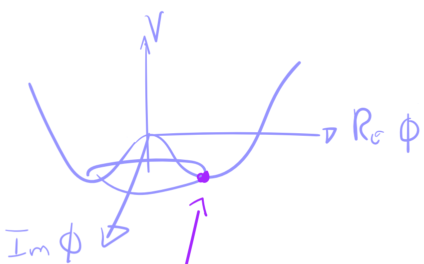
$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu\phi|^2 \underbrace{+\mu^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2}$$

$$V(\phi) = -\mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2$$

Particles are excitations around the vacuum (i.e. the minimal energy of the system)

The vacuum is usually given by $\langle\phi\rangle = 0$ but lets for this pot:

$$\frac{\partial V(\phi)}{\partial\phi} = 0 \Longrightarrow |\phi| = \underbrace{\phi_0}_{\text{Vacuum Expectation Value (VEV)}} = \left(\frac{\mu^2}{\lambda}\right)^{1/2}$$



Lets make a change of variables:

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

Now the vacuum is at $\langle\phi\rangle = \phi_0$ $\quad \langle\phi_1\rangle = \langle\phi_2\rangle = 0$, and ϕ_1 & ϕ_2 create excitations around that vacuum. Now:

$$V(\phi) = -\frac{1}{2\lambda}\mu^4 + \underbrace{\frac{1}{2}2\mu^2\phi_1^2}_{} + \mathcal{O}(\phi_i^3)$$

Exercise 7: get the full potential in terms of (ϕ_1 and ϕ_2)

ϕ_1 excitations: mass $\sqrt{2}\mu$

ϕ_2 excitations: massless (no ϕ_2^2 term)

What happens to $D_\mu\phi$?

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \sqrt{2}e\phi_0 A_\mu\partial^\mu\phi_2 + \underbrace{e^2\phi_0^2 A_\mu A^\mu}_{!!!} + \underbrace{\ldots}_{\text{cubic and quartic terms}}$$

Exercise 8 (the full thing)

$$e^2 \phi_0^2 A_\nu A^\nu = \frac{1}{2} m_A^2 A_\nu A^\nu \qquad m_A = \sqrt{2}\, e\, \phi_0$$

↳ VEV OF THE SCALAR

↳ COUPLING WITH THE SCALAR

This is GUARANTEED BY THEOREMS

GOLDSTONE THEOREM :

Global $G \xrightarrow{SSB}$ Global $H \implies$ 1 MASSLESS $\underbrace{\text{"GOLDSTONE BOSON"}}_{\text{NGB}}$ FOR EACH GENERATOR BROKEN

E.G. $U(1) \xrightarrow{SSB}$ NOTHING $\implies \phi_2$ MASSLESS

HIGGS MECHANISM :

Local $G \xrightarrow{SSB}$ Local $H \implies$ NGBs MIX WITH GAUGE BOSONS TO GIVE THEM MASS

E.G. $U(1) \xrightarrow{SSB}$ NOTHING $\implies \phi_2$ AND A_ν MIX:

$$\sqrt{2}\, e\, \phi_0\, A_\nu\, \partial^\nu \phi_2 \implies \underset{\sqrt{2}\, e\phi_0}{\longrightarrow\!\times\!\sim\sim\sim}$$

(A BASIS COULD BE CHOSEN SO THAT $\cancel{\phi_2}$ A_ν HAS 3 POLARIZATIONS)

A SECOND EXAMPLE :
$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \qquad \phi_i \in \mathbb{R} \qquad |\phi|^2 = \phi^i \phi^i$$

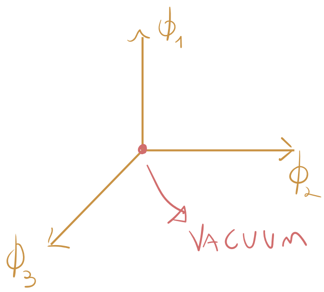
$$\mathcal{L} = \frac{1}{2}(\partial_\nu \phi^i)(\partial^\nu \phi^i) - V(|\phi|^2)$$

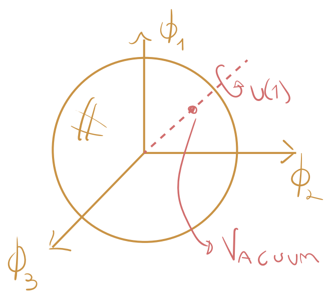
Symmetric under $SO(3)$

$$\phi^i \xrightarrow{SO(3)} R^{ij} \phi^j$$

$$R^T R = R R^T = \mathbb{1}_{3\times 3}$$

$$V(|\phi|^2) = \frac{1}{2}\mu^2 |\phi|^2 + \frac{\lambda}{4}(|\phi|^2)^2 \qquad V(|\phi|^2) = -\frac{1}{2}\mu^2 |\phi|^2 + \frac{\lambda}{4}(|\phi|^2)^2$$





$$SO(3) \longrightarrow U(1)$$

3 ROTATIONS → 1 ROTATION

THERE SHOULD BE 2 NGBs!

Exercise 9: re-write the potential in terms of variables that allow us to see that there are two massless scalars and a third massive one. What is the mass of the massive one?

NOW WE WANT TO DO THE SAME FOR THE STANDARD MODEL, ENTER THE HIGGS DOUBLET:

$$H = \begin{pmatrix} \phi \\ \tilde{\phi} \end{pmatrix} = \underbrace{\begin{pmatrix} Re[\phi] + i\, Im[\phi] \\ Re[\tilde\phi] + i\, Im[\tilde\phi] \end{pmatrix}}_{4\ D.o.F} \xrightarrow{SU(2)_L \times U(1)_y} \underbrace{e^{i t_a \theta_a}}_{SU(2)} \underbrace{e^{i y_H \theta}}_{U(1)_y} H$$

$\phi, \tilde\phi \in \mathbb{C}$

$$\boxed{y_H = \frac{1}{2}}$$

$$V(H^\dagger H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2 \Rightarrow \underbrace{SU(2)_L}_{3} \times \underbrace{U(1)_y}_{+\ 1\ \text{GEN.}} \xrightarrow{SSB} \underbrace{U(1)_{em}}_{1\ \text{GEN.}}$$

GOLDSTONE (GLOBAL): $\begin{cases} 3\ \text{NGB}_s \\ \boxed{1\ \text{massive scalar}}\quad h \in \mathbb{R} \end{cases}$ $\Rightarrow$ HIGGS (LOCAL): $\begin{cases} \underbrace{3\ \text{NGB} + B_\mu + W^1_\mu, W^2_\mu, W^3_\mu}_{\Downarrow} \\ \underbrace{3\ \text{MASSIVE GAUGE Bs}}_{W^+,\, W^-,\, Z} + \underbrace{1\ \text{MASSLESS G.B.}}_{\gamma} \end{cases}$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \underline{v + h} + i\phi_3 \end{pmatrix} \qquad \langle H \rangle = \frac{1}{2}\begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\langle \phi_1 \rangle = \langle \phi_2 \rangle = \langle \phi_3 \rangle = \langle h \rangle = 0$$

$$\boxed{v = \left(\frac{\mu^2}{\lambda}\right)^{1/2}}$$

MASSES IN GAUGE SECTOR:

$$\mathcal{L}_{HG} = -\frac{1}{4}(B_{\mu\nu})^2 - \frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}(G^a_{\mu\nu})^2 + |D_\mu H|^2 + \mu^2 H^\dagger H - \lambda (H^\dagger H)^2$$

$$D_\mu H = (\partial_\mu - i \underbrace{g}_{\substack{\text{COUPLING} \\ \text{OF} \\ SU(2)_L}} W^a_\mu t^a - \frac{i}{2} \underbrace{g'}_{\substack{\text{COUPLING} \\ \text{OF} \\ U(1)_y}} B_\mu) H$$

$Y_H = 1/2$

← NO GLUONS HERE

$t^a = \frac{\sigma^a}{2}$ $\qquad \mathbb{1}_{2\times 2}$

$$H = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}} + \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2\\ h + i\phi_3\end{pmatrix}$$

$$|D_\mu H|^2 = (D_\mu H)^\dagger (D^\mu H) \supset \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}\left(g W_\mu^a t^a + \frac{1}{2} g' B_\mu\right)\left(g W^{\mu b} t^b + \frac{1}{2} g' B^\mu\right)\begin{pmatrix}0\\ v\end{pmatrix} =$$

$$= \frac{1}{2}\frac{v^2}{4}\left[g^2 (\underbrace{W_\mu^1}) ^2 + g^2(\underbrace{W_\mu^2})^2 + (\underbrace{-g W_\mu^3 + g' B_\mu})^2\right]$$

Exercise 10

3 MASSIVE G. B.

$$W_\mu^\pm \equiv \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right) \qquad Z_\mu^0 \equiv \frac{1}{\sqrt{g^2 + g'^2}}\left(g W_\mu^3 - g' B_\mu\right)$$

$$A_\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}}\left(g' W_\mu^3 + g B_\mu\right)$$

$$\mathcal{L}_{HG} = -\frac{1}{4}(B_{\mu\nu})^2 - \frac{1}{4}(W_{\mu\nu}^a)^2 - \frac{1}{4}(G_{\mu\nu}^a)^2 + |D_\mu H|^2 + \mu^2 H^\dagger H - \lambda (H^\dagger H)^2$$

$$= \underbrace{\ldots}_{\text{SUPER COMPLICATED}} \underbrace{-\frac{1}{4}(F_{\mu\nu})^2}_{\text{QED}} + \frac{m_W^2}{2}\left(W_\mu^\pm W^{\pm\mu}\right) + \frac{m_Z^2}{2}\left(Z_\mu^0 Z^{0\mu}\right)$$

$$m_W \equiv \underbrace{g \frac{v}{2}} \qquad m_Z \equiv \underbrace{\sqrt{g^2 + g'^2}\,\frac{v}{2}}$$

COUPLINGS AND VEVs !!

Exercise 11: The following Lagrangian does not make explicit the mass of particles:

$$\mathcal{L} = (\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 - \frac{m_1^2}{2}\phi_1^2 - \frac{m_2^2}{2}\phi_2^2 - m_{12}^2\,\phi_1\phi_2 \qquad \phi_{1,2} \in \mathbb{R}$$

find the field redefinition: $\{\phi_1, \phi_2\} \rightarrow \{\phi_A, \phi_B\}$

that makes it explicit: $\mathcal{L} = (\partial_\mu \phi_A)^2 + (\partial_\mu \phi_B)^2 - \frac{m_A^2}{2}\phi_A^2 - \frac{m_B^2}{2}\phi_B^2$

and write m_A and m_B in terms of m_1, m_2 and m_{12}